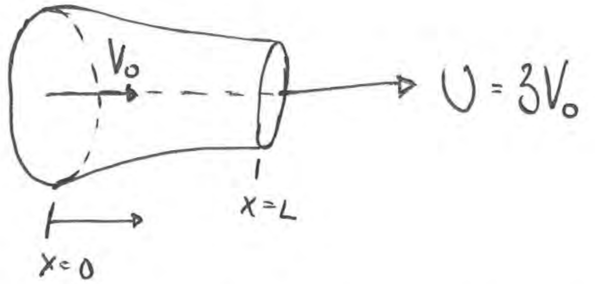
4.1



Given: $u = V_0\left(1 + \frac{2x}{L}\right)$, $v \approx w \approx 0$

$V_0 = 3\,m/s$, $L = 0.15\,m$ (for the specific case in b)

Ask: Expression of fluid acceleration $a(x)$.

$a([redacted])_{x=0}$, $a([redacted])_{x=L}$ for given V_0 and L.

a) Acceleration given by eq 4.2

$$\underbrace{a = \frac{dV}{dt}}_{\substack{\text{total} \\ \text{acceleration} \\ \left(\frac{D}{Dt}\right)}} = \underbrace{\frac{\partial V}{\partial t}}_{\text{local}} + \underbrace{u\frac{\partial V}{\partial x} + v\cancel{\frac{\partial V}{\partial y}}^{0} + w\cancel{\frac{\partial V}{\partial z}}^{0}}_{\text{convective}}$$

Comes from: $V = V(x, y, z, t) = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

Chainrule: $\frac{\partial V}{\partial t}\underbrace{\cancel{\frac{\partial t}{\partial t}}}_{1} + \frac{\partial V}{\partial x}\underbrace{\frac{\partial x}{\partial t}}_{u} + \cancel{\frac{\partial V}{\partial y}\frac{\partial y}{\partial t}}_{v \approx 0} + \cancel{\frac{\partial V}{\partial z}\frac{\partial z}{\partial t}}_{w \approx 0}$

Stationary flow, does not vary in time

$$a = u\frac{\partial V}{\partial x} = V_0\left(1 + \frac{2x}{L}\right)\begin{bmatrix} \frac{2V_0}{L} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2V_0^2}{L}\left(1 + \frac{2x}{L}\right) \\ 0 \\ 0 \end{bmatrix}$$

$$a(x) = \frac{2V_0^2}{L}\left(1 + \frac{2x}{L}\right)$$

b) $a(x=0) = \frac{2 \cdot 3^2}{0.15}(1 + 0) = 120\ m/s^2 \approx 12g$

$a(x=L) = \frac{2 \cdot 3^2}{0.15}\left(1 + \frac{2L}{L}\right) = 120 \cdot 3 = 360\ m/s^2 \approx 37g$

4.2

$$\bar{V} = V_0 \left[1 + \frac{2x}{L}\right]\hat{x} = u\,\hat{x}$$

Soln acc.

$$\bar{a} = \frac{d\bar{V}}{dt} = \underbrace{\frac{\partial \bar{V}}{\partial t}}_{=0} + (\bar{V}\cdot\nabla)\bar{V} = u\frac{\partial \bar{V}}{\partial x} + \underbrace{v\frac{\partial \bar{V}}{\partial y}}_{=0} + \underbrace{w\frac{\partial \bar{V}}{\partial z}}_{=0}$$

$$\bar{a} = u\frac{\partial u}{\partial x}\hat{x} = V_0\left(1+\frac{2x}{L}\right)V_0\cdot\frac{2}{L}\,\hat{x} = \frac{2V_0^2}{L}\left(1+\frac{2x}{L}\right)\hat{x}$$

a) $$\bar{a}(x=L) = \frac{6V_0^2}{L}\hat{x}$$

b   <u>sökt</u>: t för en partikel att röra sig mellan 0 och L

<u>givet</u>: $V = V_0 (1 + 2x/L)$ $\hat{i}$

Löst

$$V = \frac{dx}{dt} = V_0 (1 + 2x/L)$$

separera variablerna:

$$\Rightarrow \frac{dx}{V_0 (1 + 2x/L)} = dt$$

integrera

$$\frac{1}{V_0} \int_0^L \frac{1}{1 + 2x/L} \, dx = \int_0^t dt$$

$$\Rightarrow \frac{1}{V_0} \left[\frac{L}{2} \ln(1 + 2x/L) \right]_0^L = \left[t \right]_0^t$$

$$\Rightarrow \frac{L}{2V_0} \ln(3) = t$$

Svar: $t = \frac{L \ln(3)}{2 V_0} = \frac{0{,}152}{2 \cdot 3{,}048} \ln(3) = \underline{\underline{0{,}0274 \text{ s.}}}$

4.3

$$\bar{V} = (x^2-y^2+x)\hat{x} - (2xy+y)\hat{y} = u\hat{x} + v\hat{y}$$

$$\bar{a} = \frac{d\bar{V}}{dt} = \underbrace{\frac{\partial \bar{V}}{\partial t}}_{=0} + (\bar{V}\cdot\nabla)\bar{V}$$

$$\bar{a} = \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\hat{x} + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)\hat{y} = a_x\hat{x} + a_y\hat{y}$$

$$\frac{\partial u}{\partial x} = 2x+1 \qquad \frac{\partial v}{\partial x} = -2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = -2x-1 = -(2x+1)$$

$$a_x = (x^2-y^2+x)(2x+1) + (2xy+y)2y$$

$$a_y = -(x^2-y^2+x)2y + (2xy+y)(2x+1)$$

a) $a_x(1,2) = (1-4+1)(2+1) + (4+2)\cdot 4 = \underline{18\ m/s^2}$

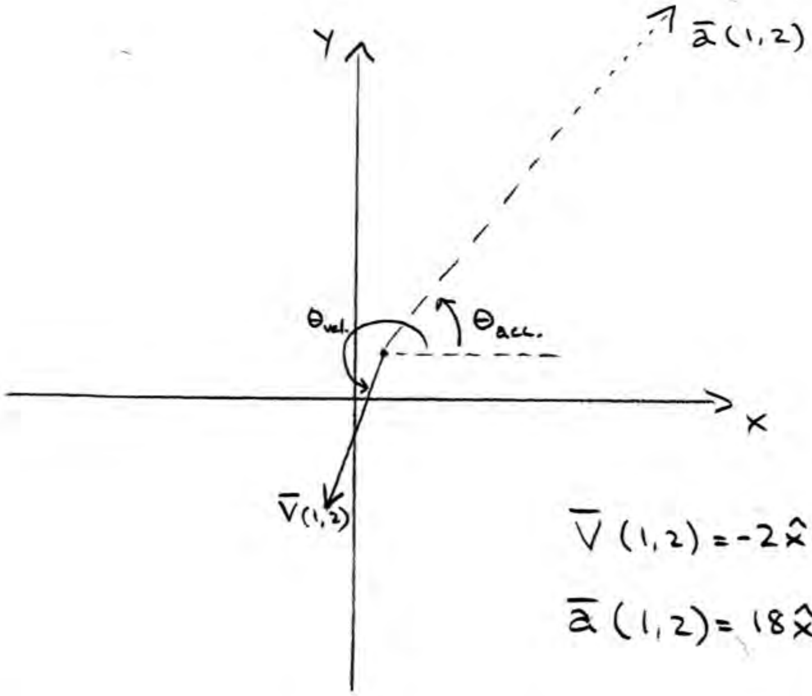
$a_y(1,2) = -(1-4+1)\cdot 4 + (4+2)(2+1) = \underline{26\ m/s^2}$

b) Projicera $\bar{V}$ på $\hat{n}_{40°} = \cos 40°\,\hat{x} + \sin 40°\,\hat{y}$

$$\bar{V}(1,2) = -2\hat{x} - 6\hat{y}$$

$$\bar{V}_{40°}(1,2) = \bar{V}(1,2)\cdot\hat{n}_{40°} = -2\cos 40° - 6\sin 40°$$

$$\approx \underline{-5{,}4\ m/s}$$



$\bar{V}(1,2) = -2\hat{x} - 6\hat{y}$

$\bar{a}(1,2) = 18\hat{x} + 26\hat{y}$

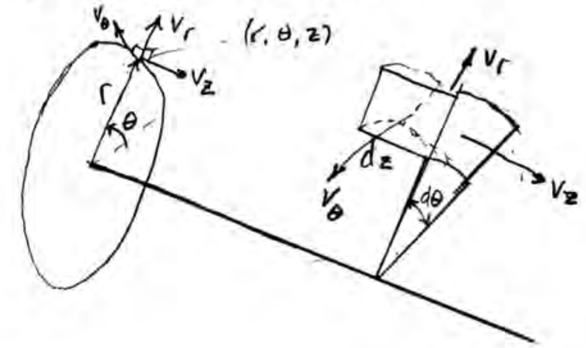
c) $\theta_{vel.} = 180° + \arctan\left(\frac{6}{2}\right) \approx \underline{251.6°}$

d) $\theta_{acc.} = \arctan\left(\frac{26}{18}\right) \approx \underline{55.3°}$

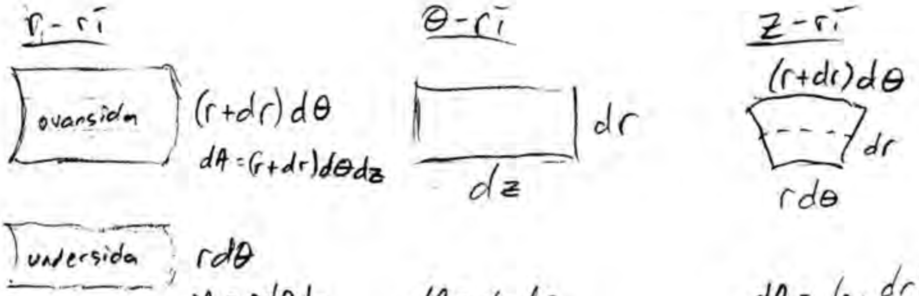
4.4

4.4

Cylindriska koordinater:



<u>Areor</u>



$dA = (r+dr)d\theta dz$

$dA = r d\theta dz$ $\quad dA = dz dr$ $\quad dA = \left(r + \frac{dr}{2}\right) d\theta \cdot dr$

<u>Sökt</u>: Härled ekv 4.12b genom att betrakta strömningen av en inkompressibel fluid genom kontrollvolymen ovan.

<u>Lösning</u> RT, B = m $\Rightarrow$ $0 = \frac{d}{dt}\left(\int_{cv} \rho \, dV\right) + \int_{cs} \rho (V \cdot n) dA$ (1)

Förenklingar:

(I) inkompressibel, fix cv $\Rightarrow$ $\frac{d}{dt}\left(\int_{cv} \rho \, dV\right) = \int_{cv} \frac{\partial \rho}{\partial t} dV = 0$

(II) infinitesimal cv $\Rightarrow$ 1-D in- och utlopp

(III) Inkompressibel $\Rightarrow$ $\Sigma(\rho VA)_{ut} - \Sigma(\rho VA)_{in} = \rho\left(\Sigma(VA)_{ut} - \Sigma(VA)_{in}\right)$

$\Rightarrow$ $0 = \Sigma(VA)_{ut} - \Sigma(VA)_{in}$ (2)

4.4 forts

Hastigheter:

Ut ur CV: V_{r+dr}, $V_{\theta+d\theta}$, V_{z+dz}

In i CV: V_r, V_θ, V

OBS! välj rätt area för in- och utflöden.

Sätt in i (2):

$$0 = V_{r+dr}(r+dr)\,d\theta dz + V_{\theta+d\theta}\,dz dr + V_{z+dz}\left(r+\frac{dr}{2}\right)d\theta dr$$

$$- v_r \cdot r d\theta dz - V_\theta\, dz dr - V_z\left(r+\frac{dr}{2}\right)d\theta dr$$

$$= \left(V_r + \frac{\partial V_r}{\partial r}dr\right)(r+dr)d\theta dz + \left(V_\theta + \frac{\partial V_\theta}{\partial \theta}d\theta\right)dz dr + \left(V_z + \frac{\partial V_z}{\partial z}dz\right)\left(r+\frac{dr}{2}\right)d\theta dr$$

$$- V_r\, r d\theta dz - V_\theta\, dz dr - v_z\left(r+\frac{dr}{2}\right)d\theta dr$$

$$= \left(V_r + \frac{\partial V_r}{\partial r} r + \frac{\partial V_r}{\partial r} dr\right) dr d\theta dz + \frac{\partial V_\theta}{\partial \theta} dr d\theta dz + \left(\frac{\partial V_z}{\partial z} r + \frac{\partial V_z}{\partial z} dr\right) dr d\theta dr$$

förkorta bort $dr d\theta dz$

Låt $dr, d\theta, dz \Rightarrow 0$

$$\Rightarrow 0 = \underbrace{V_r + r\frac{\partial V_r}{\partial r}}_{=\frac{\partial}{\partial r}(V_r \cdot r)} + \frac{\partial V_r}{\partial r}\cdot 0 + \frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}\cdot r + \frac{\partial V_z}{\partial z}\cdot 0$$

dividera med r:

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(V_r \cdot r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0 \qquad (4.12b)$$

Inkompressibel KE på cylindriska koordinater

4.05

4.5

$$V_r = K\cos\theta\left(1 - \frac{b}{r^2}\right)$$

$$V_\theta = -K\sin\theta\left(1 + \frac{b}{r^2}\right)$$

a) Kontrollera om K.E. uppfylls:

$$\underbrace{\frac{\partial \rho}{\partial t}}_{=0} + \nabla\cdot(\rho\bar{V}) = \underset{\text{ty inkompr.}}{\rho\,\nabla\cdot\bar{V}} =$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} =$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(K\cos\theta\left(r - \frac{b}{r}\right)\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(-K\sin\theta\left(1+\frac{b}{r^2}\right)\right)$$

$$\frac{1}{r}K\cos\theta\left(1 + \frac{b}{r^2}\right) - \frac{1}{r}K\cos\theta\left(1+\frac{b}{r^2}\right) = 0$$

Alltså är kontinuitet uppfylld!

b) $[b] = L^2 \quad = [r^2]$

$[K] = LT^{-1} \quad = [V]$

4.6 Given: an approx of the 2D, incomp, laminar BL on a flat surface

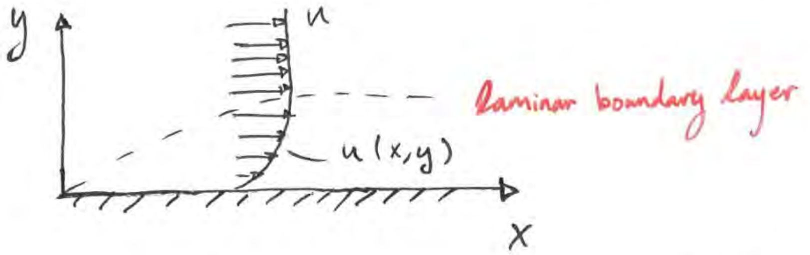
$$u \approx U\left(2\frac{y}{\delta} - 2\frac{y^3}{\delta^3} + \frac{y^4}{\delta^4}\right) \text{ for } y \leq \delta$$

$\delta = Cx^{1/2}$ C = constant

no-slip wall

Asked: Find an expression for $v(x,y)$ for $y \leq \delta$

and $\max(v(x=1m, y))$ for $U = 3m/s$ and $\delta = 1.1cm$



a) Continuity eq for incompressible flow in xy?

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

flow in x-dir and in y-dir as BL grows

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = \{\text{chainrule}\} = -\underbrace{\frac{\partial u \partial \delta}{\partial \delta\, \partial x}} = -U\left(-\frac{2y}{\delta^2} + \frac{6y^3}{\delta^4} - \frac{4y^4}{\delta^5}\right)\frac{1}{2}Cx^{-1/2}$$

$$\left\{\begin{aligned} \frac{\partial u}{\partial \delta} &= U\left(-\frac{2y}{\delta^2} + \frac{6y^3}{\delta^4} - \frac{4y^4}{\delta^5}\right) \\ \frac{\partial \delta}{\partial x} &= \frac{1}{2}Cx^{-1/2} \end{aligned}\right\}$$

$$\int_y \frac{\partial v}{\partial y}dy = v = -U\left(-\frac{y^2}{\delta^2} + \frac{\frac{3}{2}y^4}{\delta^4} - \frac{\frac{4}{5}y^5}{\delta^5}\right)\frac{1}{2}Cx^{-1/2} = U\left(\frac{y^2}{\delta^2} - \frac{3y^4}{2\delta^4} + \frac{4y^5}{5\delta^5}\right)\frac{C}{2\sqrt{x}}$$

b) $$V_{max} = [y=\delta] = U\underbrace{\left(1 - \frac{3}{2} + \frac{4}{5}\right)}_{\frac{10}{10} - \frac{15}{10} + \frac{8}{10} = \frac{3}{10}}\frac{C}{2\sqrt{x}} = \frac{3UC}{20\sqrt{x}} = \frac{3UC\sqrt{x}}{20\sqrt{x}\sqrt{x}} = \frac{3U\delta}{20x} \approx 0.005 m/s$$

4.7

How is the Continuity eq written for

a) stationary, compressible flow in xz?

b) instationary, incompressible flow in xz?

c) instationary, compressible flow in y?

d) stationary, compressible in plane polar coordinates?

$\underbrace{\text{Eq 4.4}}_{\text{Continuity eq}}$: $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$

	$\frac{\partial \rho}{\partial t}$	$\frac{\partial}{\partial x}(\rho u)$	$\frac{\partial}{\partial y}(\rho v)$	$\frac{\partial}{\partial z}(\rho w)$
stat	0	$\frac{\partial}{\partial x}(\rho u)$	$\frac{\partial}{\partial y}(\rho v)$	$\frac{\partial}{\partial z}(\rho w)$
instat	$\frac{\partial \rho}{\partial t}$	$\frac{\partial}{\partial x}(\rho u)$	$\frac{\partial}{\partial y}(\rho v)$	$\frac{\partial}{\partial z}(\rho w)$
comp	$\frac{\partial \rho}{\partial t}$	$\frac{\partial}{\partial x}(\rho u)$	$\frac{\partial}{\partial y}(\rho v)$	$\frac{\partial}{\partial z}(\rho w)$
incomp	0	$\frac{\partial u}{\partial x}$	$\frac{\partial v}{\partial y}$	$\frac{\partial w}{\partial z}$

note no ρ

$\underbrace{\frac{\partial \rho}{\partial x}}_{0} u + \frac{\partial u}{\partial x} \rho$

constant can be divided away

a) $\frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$

b) $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$

c) $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial y}(\rho v) = 0$

d) Eq D.2 Continuity eq in polar coord

$\frac{\partial}{\partial r}(r \rho V_r) + \frac{\partial}{\partial \theta}(\rho V_\theta) = 0$

no z, as plane polar coordinates

4.8

Asked: V_θ for which continuity is satisfied.

Given: Plane polar coordinates

$$V_r = r^3\cos\theta + r^2\sin\theta$$

Continuity is satisfied if:

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta) + \cancel{\frac{\partial}{\partial z}(V_z)} = 0 \quad (D.2)$$

Plane coord

note! for incompressible, stationary flow.

$$\cancel{\frac{1}{r}}\frac{\partial}{\partial \theta}(V_\theta) = -\cancel{\frac{1}{r}}\frac{\partial}{\partial r}(rV_r)$$

$$rV_r = r^4\cos\theta + r^3\sin\theta$$

$$\frac{\partial}{\partial r}(rV_r) = 4r^3\cos\theta + 3r^2\sin\theta$$

$$\frac{\partial}{\partial \theta}(V_\theta) = -(4r^3\cos\theta + 3r^2\sin\theta)$$

$$V_\theta = \int_\theta -(4r^3\cos\theta + 3r^2\sin\theta)\,d\theta = -(4r^3\sin\theta - 3r^2\cos\theta) + C$$

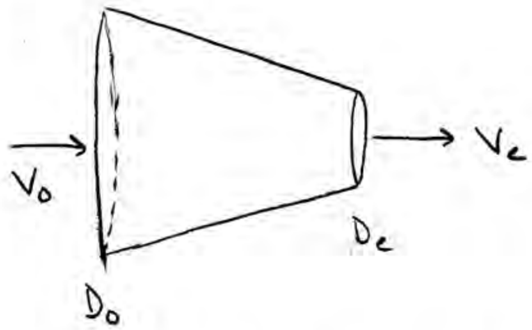
$$V_\theta = 3r^2\cos\theta - 4r^3\sin\theta (+C)$$

note! C is constant in regards to θ but can vary with r and z

$\because C = f(r, z)$

4.9

4.09



Kompressibla effekter kan anses försumbara om $M < 0.3$

$V_e > V_0 \Rightarrow \dfrac{V_e}{a} < 0.3$ sökes

Kontinuitet ger: $Q_0 = Q_e$

$$\Rightarrow V_0 A_0 = V_e A_e \Rightarrow V_e = \frac{D_0^2}{D_e^2} V_0$$

Alltså sök minsta $\dfrac{D_e}{D_0}$ som uppfyller:

$$\frac{V_0}{a\left(\frac{D_e}{D_0}\right)^2} < 0.3$$

$$\frac{D_e}{D_0} > \sqrt{\frac{V_0}{0.3a}}$$

a) $\dfrac{D_e}{D_0} > 0.31$ b) $\dfrac{D_e}{D_0} > 0.54$

4.10

4:10 Givet: Friktionslöst, inkompressibelt flöde utan gravitationseffekter

$$\mathbb{V} = 2xy\hat{i} - y^2\hat{j} = (2xy, -y^2, 0)$$

Sökt: $\frac{dp}{dx}$

Ställ upp Navier-Stokes i x-led: Anta stationärt fall (givet gm hastighetsfältet

$$u\frac{du}{dx} + v\frac{du}{dy} + \underbrace{w\frac{du}{dz}}_{=0 \text{ då } w=0} = -\frac{1}{\rho}\frac{dp}{dx} + \underbrace{\mu\left(\frac{\partial^2}{\partial x^2}u \ldots\right.}_{=0 \text{ (friktions}}$$

$$\Rightarrow u\frac{du}{dx} + v\frac{du}{dy} = -\frac{1}{\rho}\frac{dp}{dx}$$

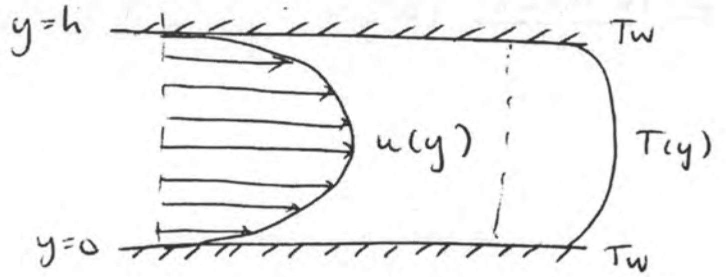
bryt ut $\frac{dp}{dx}$ och använd (1) :

$$\frac{dp}{dx} = -\rho\left(u\frac{du}{dx} + v\frac{du}{dy}\right)$$

$$= -\rho\left(2xy(2y) + (-y^2)2x\right) = -\rho 2xy^2$$

//

4.11



$$u(y) = \frac{4 \cdot u_{max}\, y(h-y)}{h^2} \quad ; \quad v = w = 0$$

inkompressibel, stat. str.

Använd ekv. (4.75) för att bestämma $T(y)$!

- - - - - - - - - - - - - - - - - - - -

Energiekvationen [4.75] $(c_v = c_p$ fel i boken$)$

$$\underbrace{\rho c_p \frac{dT}{dt}}_{\text{materiell derivata}} = k \underbrace{\nabla^2}_{\substack{\text{laplace} \\ \frac{\partial^2}{\partial x^2}() + \frac{\partial^2}{\partial y^2}()}} T + \underbrace{\Phi}_{\substack{\text{viskös dissipation} \\ \text{ekv. (4.50) s. 233}}}$$

skriv ut som (2D med $w = 0$, $\partial/\partial z = 0$)

$$\rho c_p \Big(\underbrace{\cancel{\frac{\partial T}{\partial t}}}_{=0} + u \underbrace{\cancel{\frac{\partial T}{\partial x}}}_{=0} + \underbrace{\cancel{v}}_{=0} \frac{\partial T}{\partial y} \Big) = k \Big(\underbrace{\cancel{\frac{\partial^2 T}{\partial x^2}}}_{=0} + \frac{\partial^2 T}{\partial y^2} \Big) + \mu \left(\frac{\partial u}{\partial y} \right)^2$$

$$\rightarrow \quad k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial u}{\partial y} \right)^2 = 0 \quad , \quad T = T(y)$$

Lös:

$$k \frac{\partial^2 T}{\partial y^2} = -\mu \left(\frac{\partial u}{\partial y} \right)^2 = -\mu \left[\frac{4 u_{max}}{h^2} (h - 2y) \right]^2$$

$$= -\mu \frac{16\, u_{max}^2}{h^4} (h^2 + 4y^2 - 4hy]$$

integrera 2 ggr map y, sätt in RV : $\begin{cases} T(0) = T_w & \text{(väggtemp)} \\ \left. \dfrac{dT}{dy} \right|_{y=\frac{h}{2}} = 0 & \text{(symmetri)} \end{cases}$

Integrera 1 gång

$$\Rightarrow \frac{dT}{dy} = -\frac{\mu}{k}\frac{16\,u_{max}^2}{h^4}\left(h^2 y + \frac{4y^3}{3} - \frac{4hy^2}{2}\right) + C_1$$

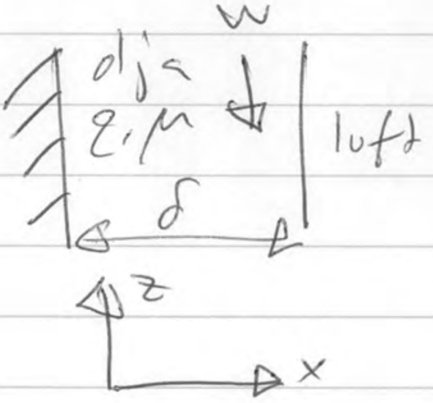
Integrera igen

$$\rightarrow T = -\frac{\mu}{k}\frac{16\,u_{max}^2}{h^4}\left(\frac{h^2y^2}{2} + \frac{4y^4}{12} - \frac{4hy^3}{6}\right) + C_1 y + C_2$$

Bestäm C_1 och C_2 mha randvillkoren

$$\rightarrow \underline{\underline{T(y) = T_w + \frac{8\,u_{max}^2 \mu}{k}\left(\frac{y}{3h} - \frac{y^2}{h^2} + \frac{4y^3}{3h^3} - 2\frac{y^4}{3h^4}\right)}}$$

4.12



givet: Fullt utbildad, ingen friktion mot luften, ϱ, μ

sökt: a) $w(x)$

b) $\mu = f\left(\varrho \delta, g, \frac{dw}{dx}\Big|_{x=\text{vägg}}\right)$

ställ upp N-S ; z-led:

$$\underbrace{\cancel{u \frac{\partial w}{\partial x}}}_{u=0} + \underbrace{\cancel{v \frac{\partial w}{\partial y}}}_{v=0} + \underbrace{\cancel{w \frac{\partial w}{\partial z}}}_{\frac{\partial}{\partial z}=0 \text{ (fullt utv.)}} = \underbrace{\cancel{-\frac{1}{\varrho}\frac{dp}{dz}}}_{\frac{\partial p}{\partial z}=0 \text{ (lufttryck konst.)}} + g_z$$

$$+ \frac{\mu}{\varrho}\left(\frac{\partial^2 w}{\partial x^2} + \underbrace{\cancel{\frac{\partial^2 w}{\partial y^2}}}_{\frac{\partial}{\partial y}=0} + \underbrace{\cancel{\frac{\partial^2 w}{\partial z^2}}}_{\frac{\partial}{\partial z}=0}\right)$$

$$\Rightarrow \quad \mu \frac{\partial^2 w}{\partial x^2} = \varrho g \qquad \text{integrera 2ggr}$$

$$w = \frac{g}{\mu}\frac{x^2}{2} + D + C$$

R.V. $\frac{dw}{dx}\Big|_{x=\delta} = 0$ (ingen friktion)

$w(x=0) = 0$ (no slip)

$$\Rightarrow \quad w = \frac{\varrho g}{\mu}\left(\frac{x^2}{2} - \delta x\right)$$

Svar (a): $w = \frac{\varrho g}{\mu}\left(\frac{x^2}{2} - \delta x\right)$

forts.

(b) sök μ som funktion av $\rho, \delta, g, \frac{dw}{dx}\Big|_{\text{vägg}}$

vet från (a) att

$$\frac{dw}{dx} = \frac{\rho g}{\mu}(x - \delta)$$

så att

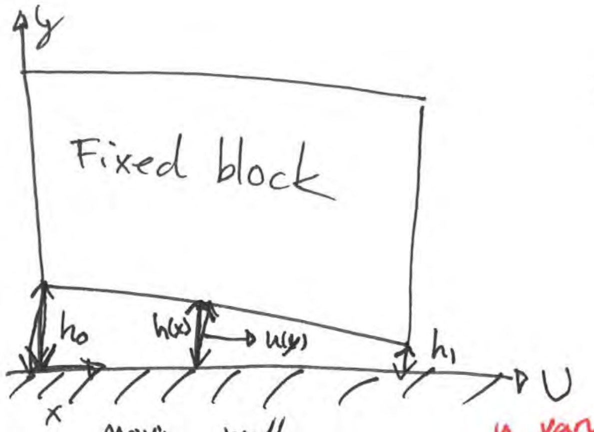
$$\frac{dw}{dx}\Big|_{x=0} = \frac{-\rho g \delta}{\mu}$$

bryt ut μ ...

$$\mu = \frac{-\rho g \delta}{\frac{dw}{dx}\Big|_{x=0}} = f\left(\rho, \delta, g, \frac{dw}{dx}\Big|_{x=0}\right)$$

//

4.13



Asked: show that

$$u = \frac{1}{2\mu}\frac{dp}{dx}(y^2 - yh) + U\left(1 - \frac{y}{h}\right)$$

Given: thin gap $h \ll L$, $p = p(x)$, $u = u(y)$

$v \approx w \approx 0$, $g = 0$

u varys only in y – u = u(y)

Start: $$\underbrace{\cancel{\rho g_x}^{\,0} - \frac{\partial p}{\partial x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right) = \cancel{\rho \frac{du}{dt}}}_{\text{x-momentum}} \quad (4.38)$$

Stationary

$$-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} = 0 \implies \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} = \text{constant over } y$$

Integration over y twice:

$$\int_y \frac{\partial^2 u}{\partial y^2} dy = \int_y \frac{1}{\mu}\frac{\partial p}{\partial x} dy \implies \frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{\partial p}{\partial x} y + C_1$$

$$\int_y \frac{\partial u}{\partial y} dy = \int_y \frac{1}{\mu}\frac{\partial p}{\partial x} y + C_1 \, dy \implies u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + C_1 y + C_2$$

two unknowns, two boundary conditions needed

$u(0) = U$ ← moving wall, no slip

$u(h) = 0$ ← fixed wall, no slip

$$u(0) = \frac{1}{2\mu}\frac{\partial p}{\partial x} 0^2 + C_1 0 + C_2 = \boxed{C_2 = U}$$

$$u(h) = \frac{1}{2\mu}\frac{\partial p}{\partial x} h^2 + C_1 h + C_2 = 0$$

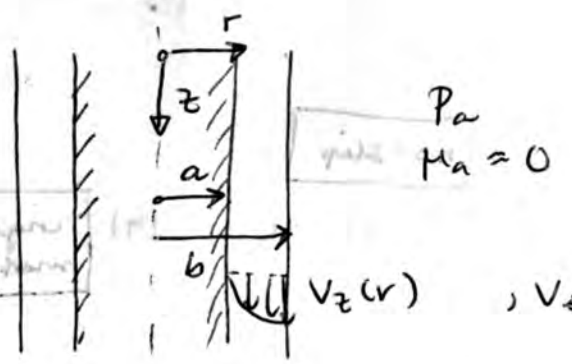
$$C_1 h = -\frac{1}{2\mu}\frac{\partial p}{\partial x} h^2 - C_2$$

$$\boxed{C_1 = -\frac{1}{2\mu}\frac{\partial p}{\partial x} h - \frac{C_2}{h}}$$

$$u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - hy\right) + U\left(1 - \frac{y}{h}\right)$$

inserted C_1 & C_2

4.14



$, v_\theta = v_r = 0$

sökt: a) Ta fram en ekvation för $v_z(r)$

b) Hur förhåller sig b till Q_{tot} ?

N.S. i cylindriska koordinater z-riktning
(Appendix D, ekv. D7.)
samt D.3

$$\frac{\partial v_z}{\partial t} + (\mathbb{V}\cdot\nabla)v_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z + \nu\nabla^2 v_z$$

skriv ut mha D.3

pos z-led nedåt $\Rightarrow g_z = g$

$$\underbrace{\frac{\partial \cancel{v_z}}{\partial t}}_{\substack{=0\\a)}} + \underbrace{\cancel{v_r}}_{\substack{=0\\c)}}\frac{\partial}{\partial r}v_z + \frac{1}{r}\underbrace{\cancel{v_\theta}}_{\substack{=0\\c)}}\frac{\partial}{\partial\theta}v_z + v_z\underbrace{\frac{\partial \cancel{v_z}}{\partial z}}_{=0\ d)} = g + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}v_z\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 \cancel{v_z}}{\partial\theta^2}}_{=0,\ e)} + \underbrace{\frac{\partial^2 \cancel{v_z}}{\partial z^2}}_{=0,\ d)}\right]$$

$$\Rightarrow \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}v_z\right) = -g$$

$$v_z = v_z(r)$$

$$\Leftrightarrow \frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = -\frac{g}{\nu}r$$

Förenklingar
a) stationärt
b) inkompressibelt
c) $v_\theta = v_r = 0$
d) fullt utb. str.
e) ingen variation i θ-led,

Integrera

$$r\frac{dv_z}{dr} = -\frac{g}{\nu}\frac{r^2}{2} + C_1$$

$$\frac{dv_z}{dr} = \frac{-gr}{\nu 2} + \frac{C_1}{r} \qquad (1)$$

$$v_z = -\frac{gr^2}{4\nu} + C_1\ln r + C_2 \qquad (2)$$

Randvillkoren:

$$\begin{cases} v_z(r=a) = 0 & (3)\ \boxed{\text{no-slip}} \\ \tau_{rz}(r=b) = 0 = \mu \dfrac{\partial v_z}{\partial r}(r=b) = 0 & (4)\ \boxed{\text{ingen friktion mot luften}} \end{cases}$$

(4) & (1) $\Rightarrow$

$$-\frac{gb}{2\nu} + \frac{C_1}{b} = 0$$

$$C_1 = \frac{gb^2}{2\nu} \quad (5)$$

(3), (2) & (5)

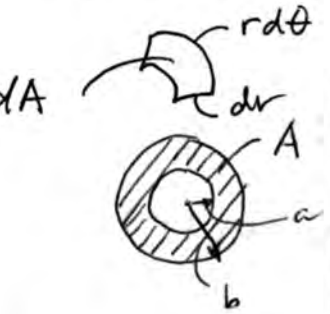
$$0 = -\frac{ga^2}{4\nu} + \frac{gb^2}{2\nu}\ln a + C_2$$

$$\Rightarrow C_2 = \frac{ga^2}{4\nu} - \frac{gb^2}{2\nu}\ln a \quad (6)$$

(5) & (6) i (2)

$$\underline{\underline{v_z}} = \frac{gb^2}{2\nu}\ln\left(\frac{r}{a}\right) - \frac{g}{4\nu}(r^2 - a^2)$$

b) Flödet Q:



$$\boxed{dA = dr \cdot r d\theta \qquad \int_{r_1}^{r_2}\int_0^{2\pi} r\,d\theta\,dr = \int_{r_1}^{r_2} 2\pi r\,dr}$$

$$Q = \int_A v_z\,dA =$$

$$= \int_a^b v_z\, 2\pi r\,dr =$$

$$= 2\pi \int_a^b \left[\frac{gb^2}{2\nu} r \ln\left(\frac{r}{a}\right) - \frac{g}{4\nu}(r^3 - a^2 r)\right] dr =$$

$$= \underset{\text{P.I.}}{\{\beta\}} = 2\pi \left\{ \frac{gb^2}{2\nu}\left[r^2\left(\frac{1}{2}\ln\frac{r}{a} - \frac{1}{4}\right)\right]_a^b - \right.$$

$$\left. - \frac{g}{4\nu}\left[\frac{r^4}{4} - \frac{a^2 r^2}{2}\right]_a^b \right\} = \left\{\begin{matrix}\text{insättning \&} \\ \text{förenkling}\end{matrix}\right\} =$$

$$= \underline{\underline{\frac{\pi g a^4}{8\nu}\left(\frac{4b^4}{a^4}\ln\frac{b}{a} - \frac{3b^4}{a^4} + \frac{4b^2}{a^2} - 1\right)}}.$$