Fluid Mechanics - MTF053

Lecture 8

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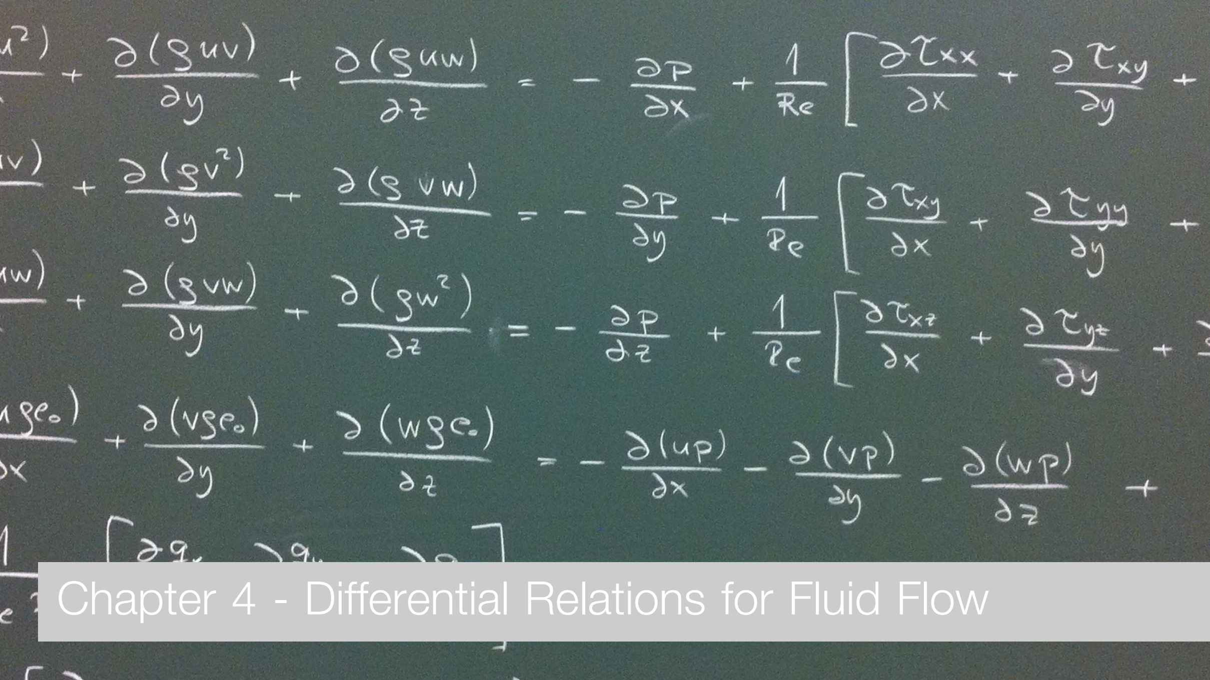


Chapter 4 - Differential Relations for Fluid Flow

Overview

rotation
stream function
conser-vation relations
Differential Relations
The Pi theorem
non-dimensional equations
modeling and similarity
Dimen-sional Analysis
Flow Relations
Bernoulli
Integral Relations
conser-vation relations
Reynolds transport
buoyancy
hydrostatic forces

Learning Outcomes

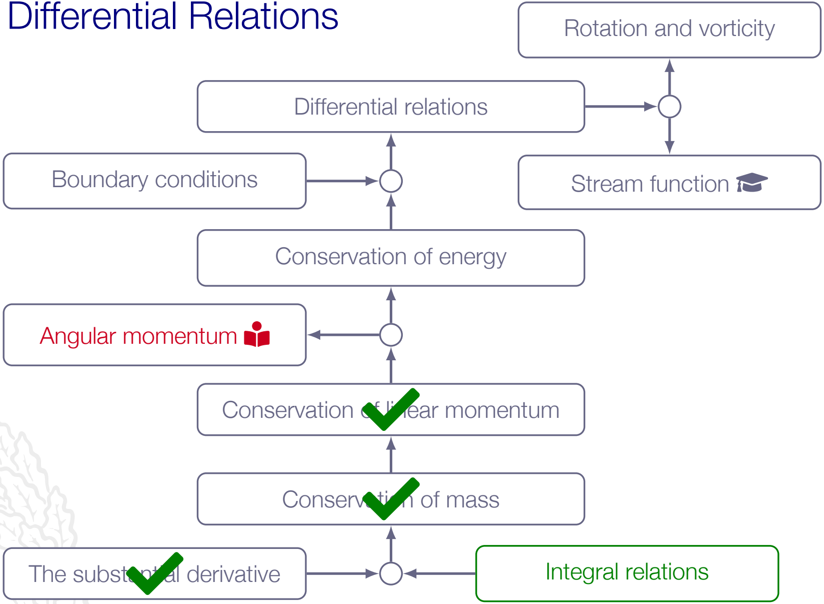
4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

14 **Derive** the continuity, momentum and energy equations on differential form

36 **Define** and explain vorticity

let's push the control volume approach to the limit ...

Roadmap - Differential Relations

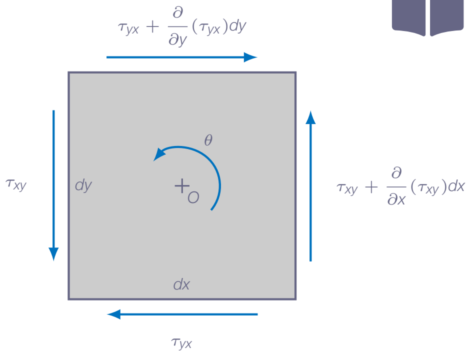


Angular Momentum

$$\sum \mathbf{M}_o = \frac{d}{dt}\left(\int_{CV} \rho(\mathbf{r} \times \mathbf{V}) d\mathcal{V}\right) + \int_{CS} (\mathbf{r} \times \mathbf{V}) \rho(\mathbf{V} \cdot \mathbf{n}) dA$$

Angular Momentum

- axis through o parallel to the z-axis
- axis through the centroid of the element
- θ angle of rotation about o



$$\left[\tau_{xy} - \tau_{yx} + \frac{1}{2}\frac{\partial}{\partial x}(\tau_{xy})dx - \frac{1}{2}\frac{\partial}{\partial y}(\tau_{yx})dy\right] dxdydz =$$

$$\frac{1}{12}\rho(dxdydz)(dx^2 + dy^2)\frac{d^2\theta}{dt^2}$$

Angular Momentum

$$\left[\tau_{xy} - \tau_{yx} + \frac{1}{2}\frac{\partial}{\partial x}(\tau_{xy})dx - \frac{1}{2}\frac{\partial}{\partial y}(\tau_{yx})dy\right] dxdydz =$$

$$\frac{1}{12}\rho(dxdydz)(dx^2 + dy^2)\frac{d^2\theta}{dt^2}$$

Neglect higher-order differential terms gives

$$\boxed{\tau_{xy} \approx \tau_{yx}}$$

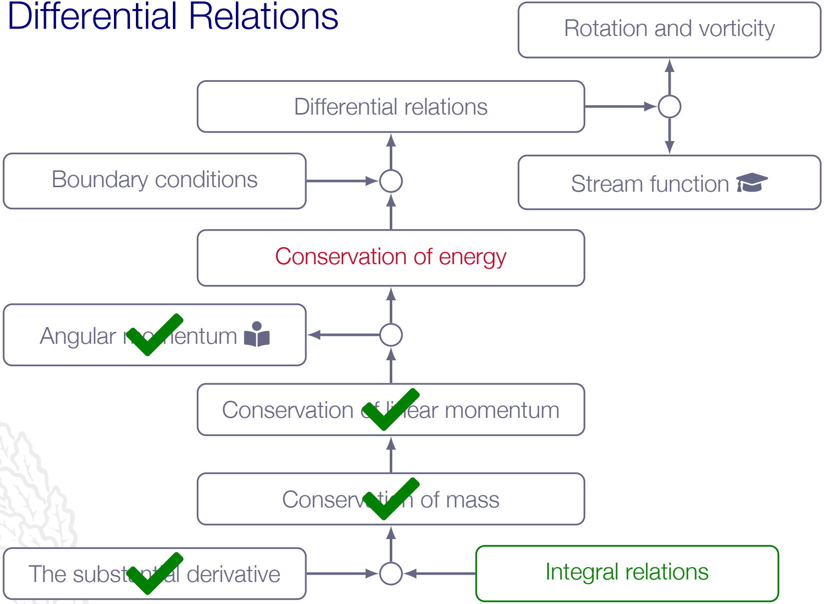
Analogously, we may obtain $\tau_{xz} \approx \tau_{zx}$ and $\tau_{zy} \approx \tau_{yz}$

Angular Momentum

Note! there is no differential angular momentum equation ...

the only result from this section is that shear stresses are symmetric: $\tau_{ij} = \tau_{ji}$

Roadmap - Differential Relations



The Energy Equation

Integral formulation:

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt}\left(\int_{CV} e\rho d\mathcal{V}\right) + \int_{CS}\left(e + \frac{p}{\rho}\right)\rho(\mathbf{V}\cdot\mathbf{n})dA$$

$$h = e + p/\rho$$

Differential form:

$$\dot{Q} - \dot{W}_\nu = \left[\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h) + \frac{\partial}{\partial z}(\rho w h)\right] dxdydz$$

$\dot{W}_s = 0$ we can not have a infinitesimal shaft protruding the control volume

The Energy Equation

$$\dot{Q} - \dot{W}_\nu = \left[\underbrace{\frac{\partial}{\partial t}(\rho e)}_{I} + \underbrace{\frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h) + \frac{\partial}{\partial z}(\rho w h)}_{II}\right] dxdydz$$

Part I.

$$\frac{\partial}{\partial t}(\rho e) = e\frac{\partial \rho}{\partial t} + \rho\frac{\partial e}{\partial t}$$

The Energy Equation

Part II.

$$\frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h) + \frac{\partial}{\partial z}(\rho w h) =$$

$$\underbrace{\frac{\partial}{\partial x}(\rho u e) + \frac{\partial}{\partial y}(\rho v e) + \frac{\partial}{\partial z}(\rho w e)}_{*} + \underbrace{\frac{\partial}{\partial x}(up) + \frac{\partial}{\partial y}(vp) + \frac{\partial}{\partial z}(wp)}_{**}$$

Part II*

$$\frac{\partial}{\partial x}(\rho u e) + \frac{\partial}{\partial y}(\rho v e) + \frac{\partial}{\partial z}(\rho w e) =$$

$$e\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] + \rho\left[u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y} + w\frac{\partial e}{\partial z}\right]$$

The Energy Equation

Part II**

$$\frac{\partial}{\partial x}(up) + \frac{\partial}{\partial y}(vp) + \frac{\partial}{\partial z}(wp) =$$

$$p\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} =$$

$$p\nabla\cdot\mathbf{V} + \mathbf{V}\cdot\nabla p$$

The Energy Equation

reassemble and collect terms:

$$\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h) + \frac{\partial}{\partial z}(\rho w h) =$$

$$\underbrace{\rho \left[\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y} + w\frac{\partial e}{\partial z} \right]}_{\frac{\partial e}{\partial t} + (\mathbf{V} \cdot \nabla) e = \frac{De}{Dt}} +$$

$$\underbrace{e \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \right]}_{continuity\ equation} +$$

$$p \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p$$

The Energy Equation

$$\dot{Q} - \dot{W}_\nu = \left[\rho \frac{De}{Dt} + p\nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p\right] dxdydz$$

The Energy Equation - Added Heat

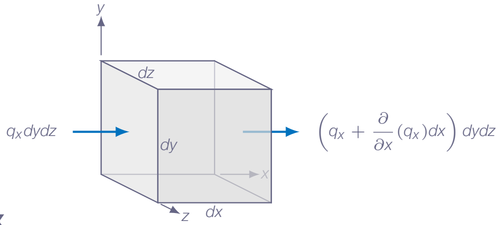
Now, let's have a look at the added heat term $\dot{Q}$

- Only **conduction** will be considered (no radiation)
- According the **Fourier's law** of conduction, the **heat flux** is proportional to the **temperature gradient**

$$\mathbf{q} = -k\nabla T$$

where k is the **thermal conductivity** and $\mathbf{q}$ is heat transfer per unit area

The Energy Equation - Added Heat



Face	Inlet heat flux	Outlet heat flux	
x	$q_x dydz$	$\left[q_x + \frac{\partial q_x}{\partial x} dx\right] dydz$	where $q_x = -k\frac{\partial T}{\partial x}$
y	$q_y dxdz$	$\left[q_y + \frac{\partial q_y}{\partial y} dy\right] dxdz$	where $q_y = -k\frac{\partial T}{\partial y}$
z	$q_z dxdy$	$\left[q_z + \frac{\partial q_z}{\partial z} dz\right] dxdy$	where $q_z = -k\frac{\partial T}{\partial z}$

The Energy Equation - Added Heat

net added heat:

$$\dot{Q} = -\left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right] dxdydz = -\nabla \cdot \mathbf{q} dxdydz$$

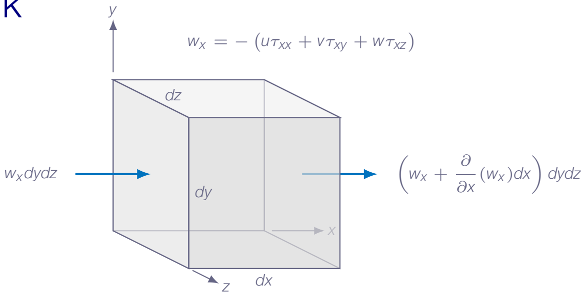
or

$$\dot{Q} = \nabla \cdot (k\nabla T) dxdydz$$

The Energy Equation - Viscous Work

The rate of work done by viscous stresses equals the product of the **stress component**, its corresponding **velocity component** and **surface area**

The Energy Equation - Viscous Work



$$\dot{W}_\nu = -\left[\frac{\partial}{\partial x}(u\tau_{xx} + v\tau_{xy} + w\tau_{xz}) + \frac{\partial}{\partial y}(u\tau_{yx} + v\tau_{yy} + w\tau_{yz}) + \frac{\partial}{\partial z}(u\tau_{zx} + v\tau_{zy} + w\tau_{zz})\right] dxdydz =$$

$$-\nabla \cdot (\mathbf{V} \cdot \tau_{ij})dxdydz$$

The Energy Equation

with the derived expressions for heat and viscous work we end up with

$$\nabla \cdot (k \nabla T) + \nabla \cdot (\mathbf{V} \cdot \tau_{ij}) = \rho \frac{De}{Dt} + p \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p$$

The Energy Equation

Now, introducing the viscous-dissipation function ϕ for Newtonian fluids and incompressible flows

$$\nabla \cdot (\mathbf{V} \cdot \tau_{ij}) = \mathbf{V} \cdot (\nabla \cdot \tau_{ij}) + \phi$$

where

$$\phi = \mu \left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 \right]$$

The Energy Equation

Note!

”All terms in the viscous-dissipation function are quadratic which means that in a viscous flow there will always be losses, which is in line with the second law of thermodynamics”

The Energy Equation

$$\nabla \cdot (k\nabla T) + \mathbf{V} \cdot (\nabla \cdot \tau_{ij}) + \phi = \rho \frac{De}{Dt} + p\nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p$$

Now, let's eliminate the term $\mathbf{V} \cdot (\nabla \cdot \tau_{ij})$ in the energy equation:

Momentum equation:

$$\rho \frac{D\mathbf{V}}{Dt} = \rho\mathbf{g} - \nabla p + \nabla \cdot \tau_{ij}$$

Multiply the momentum equation with the velocity vector (scalar product)

$$\mathbf{V} \cdot (\nabla \cdot \tau_{ij}) = \rho\mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \rho\mathbf{V} \cdot \mathbf{g} + \mathbf{V} \cdot \nabla p$$

The Energy Equation

Energy equation:

$$\rho\frac{De}{Dt} + \mathbf{V}\cdot\nabla p + p\nabla\cdot\mathbf{V} = \nabla\cdot(k\nabla T) + \mathbf{V}\cdot(\nabla\cdot\tau_{ij}) + \phi$$

eliminate $\mathbf{V}\cdot(\nabla\cdot\tau_{ij})$ using the result from previous slide

$$\rho\frac{De}{Dt} + \mathbf{V}\cdot\nabla p + p\nabla\cdot\mathbf{V} = \nabla\cdot(k\nabla T) + \rho\mathbf{V}\cdot\frac{D\mathbf{V}}{Dt} - \rho\mathbf{V}\cdot\mathbf{g} + \mathbf{V}\cdot\nabla p + \phi$$

Doesn't seem like a very wise move at this stage ...

The Energy Equation

As the next step, express energy per unit mass (e) as the sum of internal energy, kinetic energy, and potential energy (*as we did in Chapter 3*)

$$e = \hat{u} + \frac{1}{2}V^2 + gz$$

or in vector form:

$$e = \hat{u} + \frac{1}{2}\mathbf{V} \cdot \mathbf{V} - \mathbf{g}\mathbf{r}$$

where $\mathbf{g} = -(g_x, g_y, g_z)$ is the gravity vector and $\mathbf{r} = (x, y, z)$ is the location vector

The Energy Equation

$$e = \hat{u} + \frac{1}{2}\mathbf{V}\cdot\mathbf{V} - \mathbf{gr}$$

Now, apply the substantial derivative to e

$$\frac{De}{Dt} = \frac{D\hat{u}}{Dt} + \underbrace{\frac{1}{2}\frac{D}{Dt}(\mathbf{V}\cdot\mathbf{V})}_{=\mathbf{V}\cdot\frac{D\mathbf{V}}{Dt}*} - \underbrace{\frac{D}{Dt}(\mathbf{gr})}_{=\mathbf{V}\cdot\mathbf{g}*} = \frac{D\hat{u}}{Dt} + \mathbf{V}\cdot\frac{D\mathbf{V}}{Dt} - \mathbf{V}\cdot\mathbf{g}$$

* *details on the following slides*

The Energy Equation

$$\frac{1}{2}\frac{D}{Dt}(\mathbf{V}\cdot\mathbf{V}) = \frac{1}{2}\left[\underbrace{\frac{\partial}{\partial t}(\mathbf{V}\cdot\mathbf{V})}_{I} + \underbrace{(\mathbf{V}\cdot\nabla)(\mathbf{V}\cdot\mathbf{V})}_{II}\right]$$

I:

$$\frac{\partial}{\partial t}(\mathbf{V}\cdot\mathbf{V}) = \frac{\partial}{\partial t}(u^2+v^2+w^2) = \frac{\partial u^2}{\partial t} + \frac{\partial v^2}{\partial t} + \frac{\partial w^2}{\partial t} =$$

$$2u\frac{\partial u}{\partial t} + 2v\frac{\partial v}{\partial t} + 2w\frac{\partial w}{\partial t} = 2\mathbf{V}\cdot\frac{\partial \mathbf{V}}{\partial t}$$

The Energy Equation

$$\frac{1}{2}\frac{D}{Dt}(\mathbf{V}\cdot\mathbf{V}) = \frac{1}{2}\left[2\mathbf{V}\cdot\frac{\partial\mathbf{V}}{\partial t} + \underbrace{(\mathbf{V}\cdot\nabla)(\mathbf{V}\cdot\mathbf{V})}_{II}\right]$$

II:

$$(\mathbf{V}\cdot\nabla)(\mathbf{V}\cdot\mathbf{V}) = (\mathbf{V}\cdot\nabla)(u^2+v^2+w^2) =$$

$$= u\frac{\partial u^2}{\partial x} + u\frac{\partial v^2}{\partial x} + u\frac{\partial w^2}{\partial x} + v\frac{\partial u^2}{\partial x} + v\frac{\partial v^2}{\partial x} + v\frac{\partial w^2}{\partial x} + w\frac{\partial u^2}{\partial x} + w\frac{\partial v^2}{\partial x} + w\frac{\partial w^2}{\partial x} =$$

$$= 2\left[u^2\frac{\partial u}{\partial x} + uv\frac{\partial v}{\partial x} + uw\frac{\partial w}{\partial x} + uv\frac{\partial u}{\partial x} + v^2\frac{\partial v}{\partial x} + vw\frac{\partial w}{\partial x} + uw\frac{\partial u}{\partial x} + vw\frac{\partial v}{\partial x} + w^2\frac{\partial w}{\partial x}\right] =$$

$$= 2\mathbf{V}\cdot(\mathbf{V}\cdot\nabla)\mathbf{V}$$

The Energy Equation

$$\frac{1}{2}\frac{D}{Dt}(\mathbf{V}\cdot\mathbf{V}) = \frac{1}{2}\left[2\mathbf{V}\cdot\frac{\partial\mathbf{V}}{\partial t} + 2\mathbf{V}\cdot(\mathbf{V}\cdot\nabla)\mathbf{V}\right] = \mathbf{V}\cdot\frac{D\mathbf{V}}{Dt}$$

The Energy Equation

$$\frac{D}{Dt}(\mathbf{gr}) = \frac{\partial}{\partial t}(\mathbf{gr}) + (\mathbf{V}\cdot\nabla)(\mathbf{gr}) = \underbrace{\frac{\partial}{\partial t}(g_x x,\ g_y y,\ g_z z)}_{=(0,0,0)} + (\mathbf{V}\cdot\nabla)(g_x x,\ g_y y,\ g_z z) =$$

$$\underbrace{\mathbf{V}\cdot\left(x\frac{\partial g_x}{\partial x} + g_x\frac{\partial x}{\partial x},\ y\frac{\partial g_y}{\partial y} + g_y\frac{\partial y}{\partial y},\ z\frac{\partial g_z}{\partial z} + g_z\frac{\partial z}{\partial z}\right)}_{\frac{\partial g_x}{\partial x}=\frac{\partial g_y}{\partial y}=\frac{\partial g_z}{\partial z}=0 \text{ and } \frac{\partial x}{\partial x}=\frac{\partial y}{\partial y}=\frac{\partial z}{\partial z}=1} = \mathbf{V}\cdot(g_x,\ g_y,\ g_z) = \mathbf{V}\cdot\mathbf{g}$$

The Energy Equation

Now, insert

$$\frac{De}{Dt} = \frac{D\hat{u}}{Dt} + \mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \mathbf{V} \cdot \mathbf{g}$$

in the energy equation

$$\rho\frac{D\hat{u}}{Dt} + \rho\mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \rho\mathbf{V} \cdot \mathbf{g} + \mathbf{V} \cdot \nabla p + p\nabla \cdot \mathbf{V} =$$

$$\nabla \cdot (k\nabla T) + \rho\mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \rho\mathbf{V} \cdot \mathbf{g} + \mathbf{V} \cdot \nabla p + \phi$$

The highlighted terms cancel each other

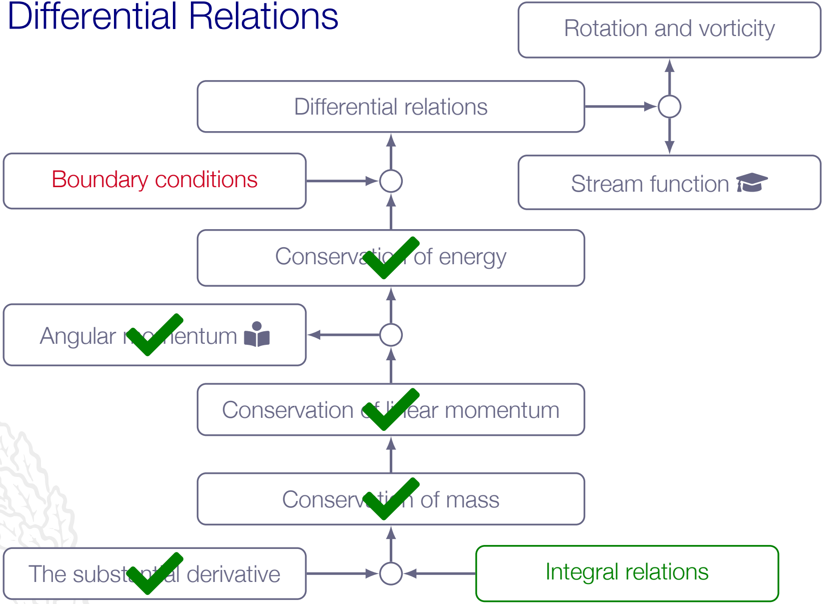
Ok, this was why momentum equation was used here ...

The Energy Equation

$$\rho \frac{D\hat{u}}{Dt} + p\nabla \cdot \mathbf{V} = \nabla \cdot (k\nabla T) + \phi$$

Local and convective changes of internal energy are balanced by pressure work, heat addition and viscous dissipation – viscous dissipation will always increase the internal energy of the fluid

Roadmap - Differential Relations



Flow Equations on Differential Form

Continuity: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

Momentum: $$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla p + \nabla \cdot \tau_{ij}$$

Energy: $$\rho \frac{D\hat{u}}{Dt} + p \nabla \cdot \mathbf{V} = \nabla \cdot (k \nabla T) + \phi$$

five equations and seven unknowns $(\rho, u, v, w, p, \hat{u}, T) \Rightarrow$ two additional relations needed:

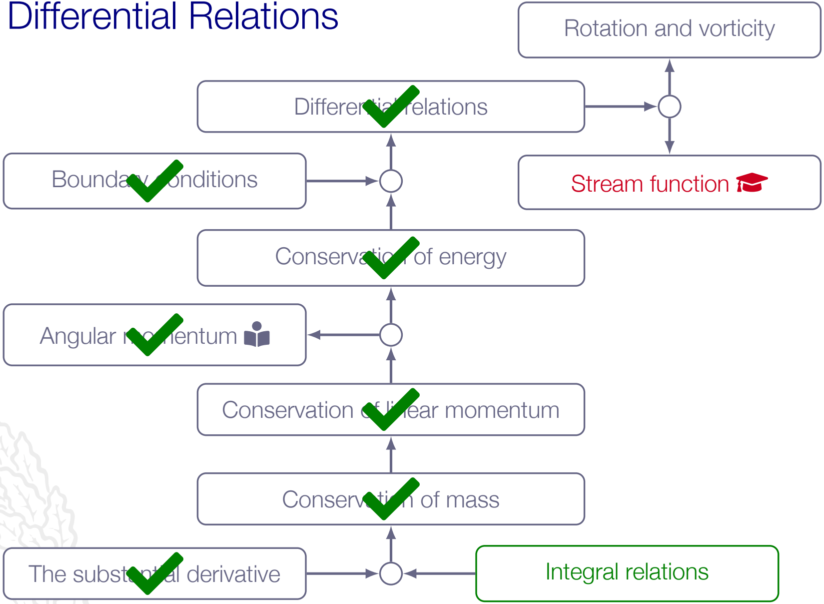
$$\rho = \rho(p, T),\ \hat{u} = \hat{u}(p, T)$$

Flow Equations on Differential Form

Boundary conditions:

- solid wall: no slip, no temperature jump
- inlet, outlet
- liquid-gas interface

Roadmap - Differential Relations



The Stream Function (*for the interested*)

fulfill the continuity equation and solve the momentum equation directly for the single variable ψ

The Stream Function (*for the interested*)

incompressible, two-dimensional flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

define $\psi(x, y)$ such that

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0$$

and thus

$$u = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial x}$$

or

$$\mathbf{V} = \left[\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right]$$

The Stream Function (*for the interested*)

The rotation of the flow field is calculated using the curl operator

$$\mathit{curl}(\mathbf{V}) = \nabla \times \mathbf{V} = -\nabla^2 \psi \mathbf{e}_z$$

The Stream Function (*for the interested*)

Now, apply the curl operator to the momentum equation

$$\nabla \times \frac{D\mathbf{V}}{Dt} = \underbrace{\nabla \times \mathbf{g}}_{=0} - \frac{1}{\rho}\underbrace{\nabla \times \nabla p}_{=0} + \nu\nabla \times \nabla^2\mathbf{V} = \nu\nabla \times \nabla^2\mathbf{V}$$

$$\nabla \times \frac{\partial \mathbf{V}}{\partial t} + \nabla \times (\mathbf{V} \cdot \nabla)\mathbf{V} = \nu\nabla \times \nabla^2\mathbf{V}$$

$$\left.\begin{array}{l} \dfrac{\partial \mathbf{V}}{\partial t} = 0 \ (\textit{steady}) \\[2em] \nu\nabla \times \nabla^2\mathbf{V} = \nu\nabla^2(\nabla \times \mathbf{V}) \end{array}\right\} \Rightarrow \nabla \times (\mathbf{V} \cdot \nabla)\mathbf{V} = \nu\nabla^2(\nabla \times \mathbf{V})$$

The Stream Function (*for the interested*)

$$(\mathbf{V}\cdot\nabla)\mathbf{V}=\frac{1}{2}\nabla(\mathbf{V}\cdot\mathbf{V})-\mathbf{V}\times(\nabla\times\mathbf{V})=\nabla\left(\frac{V^2}{2}\right)-\mathbf{V}\times(\nabla\times\mathbf{V})$$

and thus

$$\nabla\times(\mathbf{V}\cdot\nabla)\mathbf{V}=\underbrace{\nabla\times\nabla\left(\frac{V^2}{2}\right)}_{=0}-\nabla\times\mathbf{V}\times(\nabla\times\mathbf{V})=\nabla\times(\nabla\times\mathbf{V})\times\mathbf{V}$$

$$\nabla\times(\nabla\times\mathbf{V})\times\mathbf{V}=$$

$$(\mathbf{V}\cdot\nabla)(\nabla\times\mathbf{V})-((\nabla\times\mathbf{V})\cdot\nabla)\mathbf{V}+\underbrace{(\nabla\times\mathbf{V})(\nabla\cdot\mathbf{V})}_{=0\ \textit{(incompressible)}}+\mathbf{V}\underbrace{(\nabla\cdot(\nabla\times\mathbf{V}))}_{=0}=$$

$$(\mathbf{V}\cdot\nabla)(\nabla\times\mathbf{V})-((\nabla\times\mathbf{V})\cdot\nabla)\mathbf{V}$$

The Stream Function (*for the interested*)

$$(\mathbf{V}\cdot\nabla)(\nabla\times\mathbf{V}) - ((\nabla\times\mathbf{V})\cdot\nabla)\mathbf{V} = \nu\nabla^2(\nabla\times\mathbf{V})$$

insert the stream function

$$(\mathbf{V}\cdot\nabla)(\nabla\times\mathbf{V}) = (\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0)\cdot(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(0, 0, -\nabla^2\psi)$$

$$((\nabla\times\mathbf{V})\cdot\nabla)\mathbf{V} = (0, 0, -\nabla^2\psi)\cdot(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0) = 0$$

$$\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi) - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi) = \nu\nabla^2(\nabla^2\psi)$$

Stream Function (*for the interested*)

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi) = \nu\nabla^2(\nabla^2\psi)$$

+ one equation for ψ that fulfills both the momentum and continuity equations
+ scalar equation
- contains fourth-order derivatives

Stream Function (*for the interested*)

Definition of a streamline in two dimensions:

$$\frac{dx}{u} = \frac{dy}{v}$$

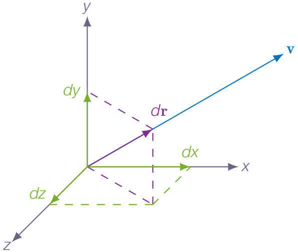
or

$$udy - vdx = 0$$

and thus

$$\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = d\psi = 0$$

or ψ is constant along a streamline ...



$d\mathbf{r}$ is aligned with $\mathbf{v}$

$dx \propto u$

$dy \propto v$

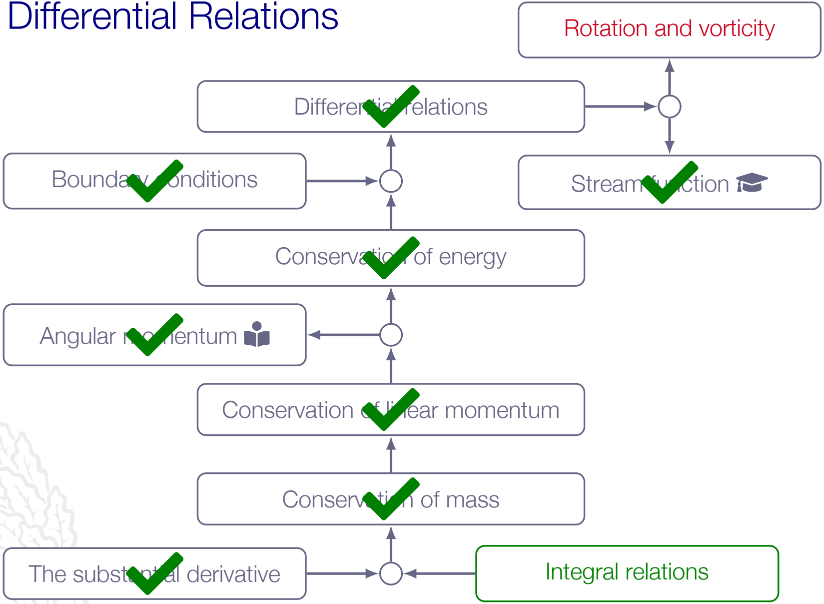
$dz \propto w$

Stream Function (*for the interested*)

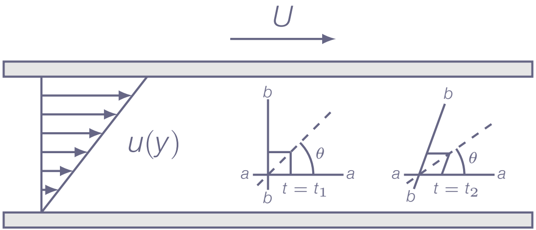
Implication:

- Lines of constant ψ are streamlines of the flow
- If we know $\psi(x, y)$, lines of constant ψ will be streamlines of the flow

Roadmap - Differential Relations

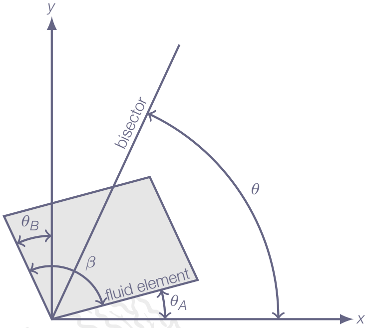


Flow Rotation



- Is the Couette flow irrotational?
- Note the change of the fluid element bisector angle θ

Flow Rotation



$$\beta = \frac{\pi}{2} + \theta_B - \theta_A$$

$$\theta = \frac{\beta}{2} + \theta_A = \frac{\pi}{4} + \frac{1}{2}(\theta_A + \theta_B)$$

the angular velocity of the bisector:

$$\dot{\theta} = \frac{1}{2}\left(\dot{\theta}_A + \dot{\theta}_B\right)$$

Flow Rotation

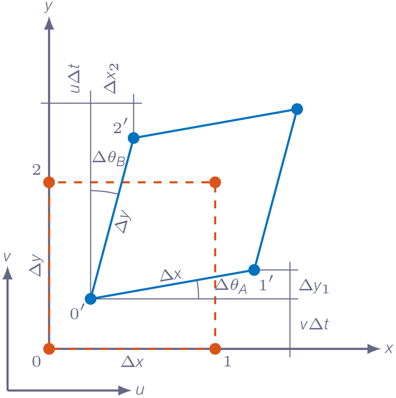
$$\sin(\Delta\theta_A) = \frac{\Delta y_1}{\Delta x} \approx \frac{\left(v + \frac{\partial v}{\partial x}\Delta x\right)\Delta t - v\Delta t}{\Delta x} = \frac{\partial v}{\partial x}\Delta t$$

$\sin(\Delta\theta_A) \approx \Delta\theta_A$ for small angles

$$\Rightarrow \underbrace{\frac{\Delta\theta_A}{\Delta t}}_{=\dot{\theta}_A} \approx \frac{\partial v}{\partial x}$$

in the same way $\dot{\theta}_B \approx -\dfrac{\partial u}{\partial y}$

the angular velocity of the bisector: $\dot{\theta} = \dfrac{1}{2}\left(\dot{\theta}_A + \dot{\theta}_B\right) = \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right)$



Flow Rotation

From previous slide we get the angular velocity about the z axis

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Using the same reasoning, we can get the angular velocities about the x and y axes

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

Flow Rotation

$$\omega = \frac{1}{2}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \frac{1}{2}curl(\mathbf{V})$$

The flow **vorticity** ζ is defined as:

$$\zeta = 2\omega = curl(\mathbf{V})$$

Flows with **zero vorticity** are called **irrotational**

Frictionless Irrotational Flow

If the flow is both **frictionless** and **irrotational**:

1. the momentum equation reduces to Euler's equation

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{g} - \nabla p$$

2. the acceleration term can be simplified

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}$$

where we can use the vector identity

$$(\mathbf{V}\cdot\nabla)\mathbf{V} = \nabla(\frac{1}{2}V^2) + \zeta\times\mathbf{V}$$

Doesn't seem like a simplification but let's try ...

Frictionless Flow

1. combine Euler's equation with the modified acceleration term
2. divide by ρ
3. dot product between the entire equation and an arbitrary displacement vector $d\mathbf{r}$

$$\left[\frac{\partial \mathbf{V}}{\partial t} + \nabla(\frac{1}{2}V^2) + \zeta \times \mathbf{V} + \frac{1}{\rho}\nabla p - \mathbf{g}\right] \cdot d\mathbf{r} = 0$$

Frictionless Flow

Now we want to get rid of the term $(\zeta \times \mathbf{V}) \cdot d\mathbf{r}$

1. $\mathbf{V} = 0$; no flow - not interesting
2. $\zeta = 0$; irrotational flow
3. $d\mathbf{r}$ perpendicular to $(\zeta \times \mathbf{V})$; strange
4. $d\mathbf{r}$ parallel to $\mathbf{V}$; integrate along a streamline

Frictionless Flow

Fourth alternative: integrate along a streamline:

$$\left[\frac{\partial \mathbf{V}}{\partial t} + \nabla(\frac{1}{2}V^2) + \frac{1}{\rho}\nabla p - \mathbf{g}\right] \cdot d\mathbf{r} = 0$$

performing the scalar products gives

$$-\mathbf{g} \cdot d\mathbf{r} = \{\mathbf{g} = -g\mathbf{e}_z\} = gdz$$

$$\nabla p \cdot d\mathbf{r} = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = dp$$

$$\nabla(\frac{1}{2}V^2) \cdot d\mathbf{r} = \frac{1}{2}\left(\frac{\partial V^2}{\partial x}dx + \frac{\partial V^2}{\partial y}dy + \frac{\partial V^2}{\partial z}dz\right) = \frac{1}{2}d\left(V^2\right)$$

Frictionless Flow

$$\frac{\partial \mathbf{V}}{\partial t} \cdot d\mathbf{r} + \frac{1}{2} d\left(V^2\right) + \frac{dp}{\rho} + g dz = 0$$

Frictionless Flow

Integrate between any two points along the streamline

$$\int_1^2 \frac{\partial V}{\partial t} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0$$

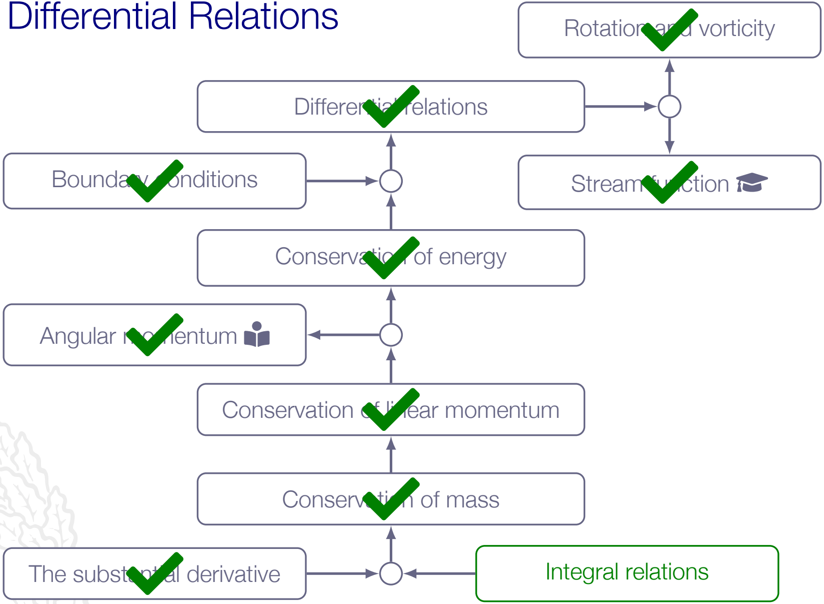
The Bernoulli equation for frictionless unsteady flow

Steady incompressible flow gives

$$\frac{p}{\rho} + \frac{1}{2}V^2 + gz = constant$$

Note! for irrotational flow this last results holds in the entire flow field with the same constant

Roadmap - Differential Relations



Millennium Problems