Fluid Mechanics - MTF053

Lecture 5

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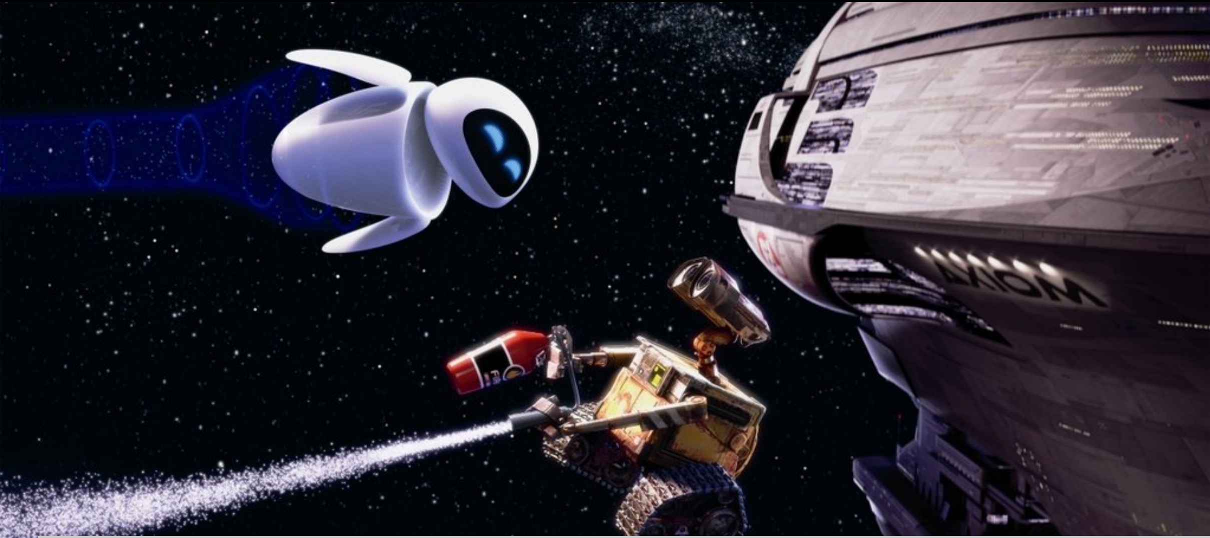


Chapter 3 - Integral Relations for a Control Volume

Overview

conser-vation relations
Differential Relations
sional Analysis
Bernoulli
Flow Relations
Integral Relations
conser-vation relations
Reynolds transport theorem
hydrostatic forces
buoyancy
Pressure
velocity field
Reynolds number

Learning Outcomes

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

12 **Define** Reynolds transport theorem using the concepts control volume and system

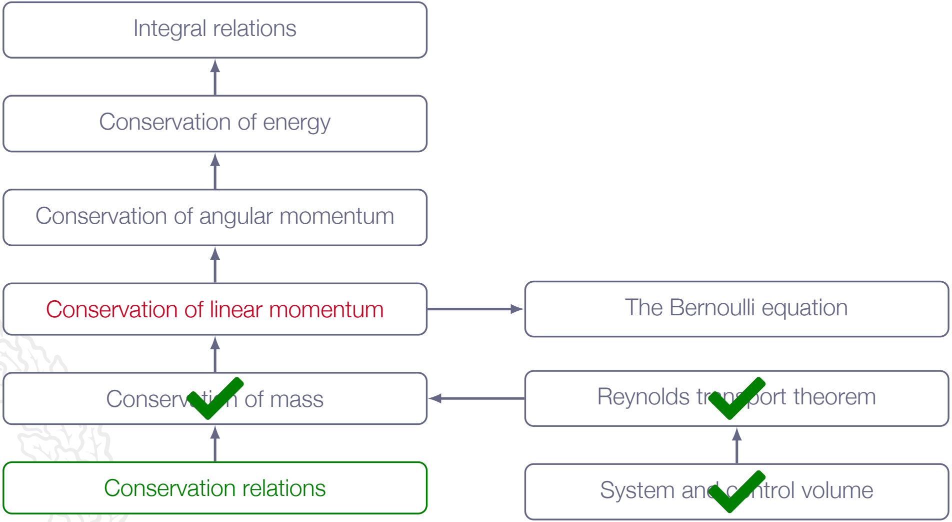
13 **Derive** the control volume formulation of the continuity, momentum, and energy equations using Reynolds transport theorem and solving problems using those relations

15 **Derive** and use the Bernoulli equation (using the relation includes having knowledge about its limitations)

we will derive methods suitable for estimation of forces and system analysis

fluid flow finally ...

Roadmap - Integral Relations



Conservation of Linear Momentum

Linear Momentum

Reynolds transport theorem with $B = m\mathbf{V}$ and $\beta = dB/dm = d(m\mathbf{V})/dm = \mathbf{V}$

$$\boxed{\frac{d}{dt}(m\mathbf{V})_{sys} = \sum \mathbf{F} = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho d\mathcal{V}\right) + \int_{CS} \mathbf{V}\rho(\mathbf{V_r}\cdot\mathbf{n})dA}$$

1. $\mathbf{V}$ is the velocity relative to an inertial (non-accelerating) coordinate system
2. $\sum \mathbf{F}$ is the vector sum of all forces on the system (surface forces and body forces)
3. the relation is a vector relation (three components)

Linear Momentum

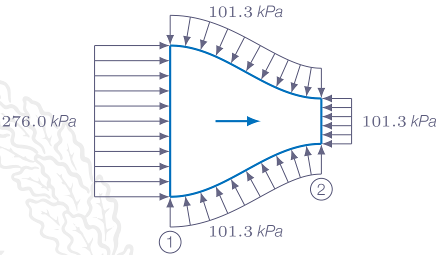
Forces:

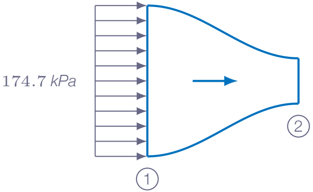
1. solid bodies that protrude through the control volume surface
2. forces due to pressure and viscous stresses of the surrounding fluid

Surface Pressure Force

$$\mathbf{F}_p = \int_{CS} p(-\mathbf{n})dA$$

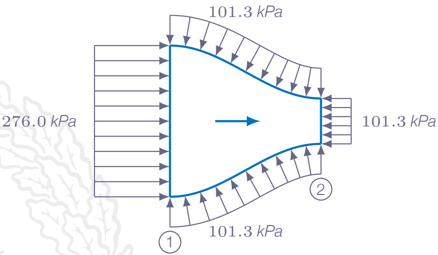
$$\mathbf{F}_p = \int_{CS} (p - p_{atm})(-\mathbf{n})dA = \int_{CS} p_{gage}(-\mathbf{n})dA$$

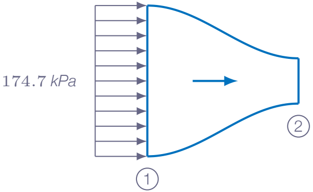




Surface Pressure Force

A free jet leaving a confined duct and exits into the ambient atmosphere will be at atmospheric pressure





Linear Momentum - Example

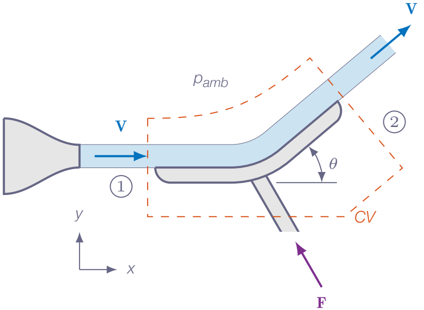
Steady-state flow: deflection av a water jet without changing its velocity magnitude

- steady-state
- water $\Rightarrow$ incompressible
- atmospheric pressure on all control volume surfaces
- neglect friction

$$\mathbf{F} = \dot{m}_2\mathbf{V}_2 - \dot{m}_1\mathbf{V}_1$$

$$|\mathbf{V}_1| = |\mathbf{V}_2| = V$$

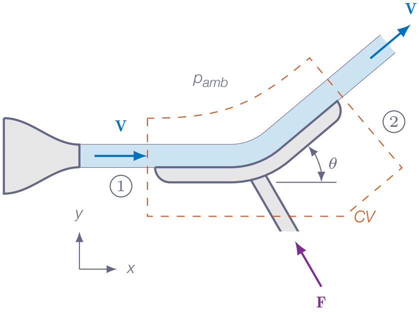
- mass conservation: $\dot{m}_1 = \dot{m}_2 = \dot{m} = \rho AV$



Linear Momentum - Example

$$F_x = \dot{m}V(\cos\theta - 1)$$
$$F_y = \dot{m}V\sin\theta$$

$$\mathbf{F} = \dot{m}V(\cos\theta - 1, \sin\theta, 0)$$



Momentum Flux Correction Factor

One-dimensional flow through inlets and outlets is of course not true in reality

Introducing the correction factor ζ

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \zeta V_{av}\dot{m}$$

where (for incompressible flow)

$$V_{av} = \frac{1}{A}\int u dA$$

Momentum Flux Correction Factor

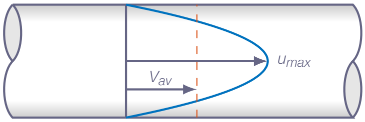
Laminar pipe flow:

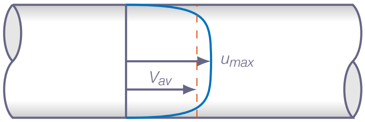
Velocity profile:

$$u(r) = U_{max} \left(1 - \left(\frac{r}{R}\right)^2\right)$$

From example 2 in the conservation-of-mass section we have that the average velocity for a laminar pipe flow is obtained as:

$$V_{av} = \frac{1}{2} U_{max}$$





Momentum Flux Correction Factor

$$u(r) = U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \int_0^R U_{max}^2\left(1 - \left(\frac{r}{R}\right)^2\right)\rho U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)2\pi r dr$$

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = 2\pi\rho U_{max}^2\int_0^R\left(1 - \left(\frac{r}{R}\right)^2\right)^2 r dr$$

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{1}{3}\pi R^2\rho U_{max}^2$$

Momentum Flux Correction Factor

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{1}{3}\rho\pi R^2 U_{max}^2$$

the mass flow $\dot{m}$ can be obtained as:

$$\dot{m} = V_{av}\rho\pi R^2 = \frac{1}{2}U_{max}\rho\pi R^2$$

which gives

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{2U_{max}}{3}\dot{m}$$

from the definition of ζ

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \zeta V_{av}\dot{m} = \left\{V_{av} = \frac{1}{2}U_{max}\right\} = \frac{\zeta U_{max}}{2}\dot{m}$$

Momentum Flux Correction Factor

comparing the two expressions, we have that

$$\frac{2U_{max}}{3}\dot{m} = \frac{\zeta U_{max}}{2}\dot{m}$$

and thus $\zeta = 4/3$ for laminar incompressible flow

Momentum Flux Correction Factor

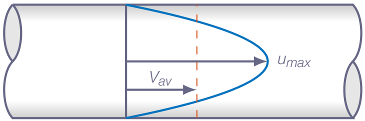
Turbulent pipe flow:

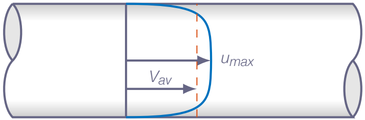
Velocity Profile:

$$u(r) \approx U_{max} \left(1 - \frac{r}{R}\right)^m, \; m \approx \frac{1}{7}$$

From example 3 in the conservation-of-mass section we have that the average velocity for a laminar pipe flow is obtained as:

$$V_{av} = \frac{2U_{max}}{(1+m)(2+m)}$$





Momentum Flux Correction Factor

$$\rho 2\pi U_{max}^2 \int_0^R \left(1 - \frac{r}{R}\right)^{2m} r dr = \zeta \rho \pi R^2 V_{av}^2 = \zeta \frac{4\rho\pi R^2 U_{max}^2}{(1+m)^2(2+m)^2} \Rightarrow$$

$$2\left[\frac{(r-R)\left(1-\frac{r}{R}\right)^{2m}(2mr+r+R)}{2(1+2m)(1+m)}\right]_0^R = \zeta \frac{4R^2}{(1+m)^2(2+m)^2} \Rightarrow$$

$$\frac{R^2}{(1+2m)(1+m)} = \zeta \frac{4R^2}{(1+m)^3(2+m)^3} \Rightarrow \zeta = \frac{(1+m)^2(2+m)^2}{4(1+2m)(1+m)}$$

Momentum Flux Correction Factor

Laminar pipe flow:

$\zeta = 4/3$ should be used

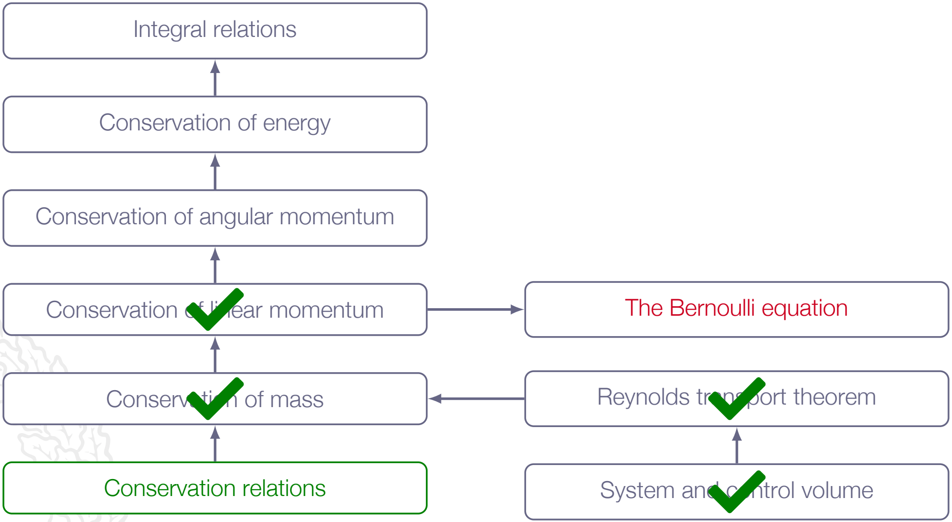
Turbulent pipe flow:

$$\zeta = \frac{(1+m)^2(2+m)^2}{4(1+2m)(1+m)}$$

m	1/5	1/6	1/7	1/8	1/9
ζ	1.037	1.027	1.020	1.016	1.013

($\zeta = 1.0$ is often a good approximation)

Roadmap - Integral Relations



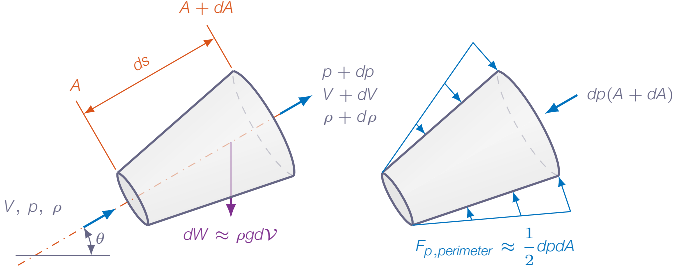
Daniel Bernoulli

The Bernoulli Equation

The relation between pressure, velocity, and elevation in a frictionless flow

The Bernoulli Equation

Frictionless flow along a streamline (streamtube with infinitesimal cross section area)



conservation of mass:

$$\frac{d}{dt}\left(\int_{CV} \rho d\mathcal{V}\right) + \dot{m}_{out} - \dot{m}_{in} = 0 \approx \frac{\partial \rho}{\partial t} d\mathcal{V} + d\dot{m}$$

where $\dot{m} = \rho A V$ and $d\mathcal{V} \approx A ds$

$$d\dot{m} = d(\rho A V) = -\frac{\partial \rho}{\partial t} A ds$$

The Bernoulli Equation

linear momentum equation in the streamwise direction:

$$\sum dF_s = \frac{d}{dt}\left(\int_{CV} V\rho d\mathcal{V}\right) + (\dot{m}V)_{out} - (\dot{m}V)_{in} \approx \frac{\partial}{\partial t}(\rho V)Ads + d(\dot{m}V)$$

frictionless flow means: only pressure and gravity forces

$$dF_{s,p} \approx \frac{1}{2}dpdA - (A + dA)dp \approx -Adp$$

$$dF_{s,grav} = -dW\sin\theta = -(g\rho A)ds\sin\theta = -g\rho Adz$$

$$\sum dF_s = -g\rho Adz - Adp = \frac{\partial}{\partial t}(\rho V)Ads + d(\dot{m}V)$$

The Bernoulli Equation

$$-g\rho A dz - A dp = \frac{\partial \rho}{\partial t} V A ds + \frac{\partial V}{\partial t} \rho A ds + \dot{m} dV + V d\dot{m}$$

the continuity equation gives

$$V \left[\frac{\partial \rho}{\partial t} A ds + d\dot{m} \right] = 0$$

and thus

$$\frac{\partial V}{\partial t} \rho A ds + A dp + \dot{m} dV + g\rho A dz = 0$$

Now, divide by ρA

$$\frac{\partial V}{\partial t} ds + \frac{dp}{\rho} + V dV + g dz = 0$$

The Bernoulli Equation

Bernoulli's equation for unsteady frictionless flow along a streamline (the relation just derived) can be integrated between any two points along the streamline

$$\int_1^2 \frac{\partial V}{\partial t} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}\left(V_2^2 - V_1^2\right) + g\left(z_2 - z_1\right) = 0$$

The Bernoulli Equation

Steady ($\partial V/\partial t = 0$), incompressible (constant density) flow:

$$\boxed{p_1 + \frac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g z_2 = const}$$

The Bernoulli Equation

Note! the following restrictive assumptions have been made in the derivation

1. steady flow

   many flows can be treated as steady at least when doing control volume type of analysis

2. incompressible flow

   low velocity gas flow without significant changes in pressure, liquid flow

3. frictionless flow

   friction is in general important

4. flow along a single streamline

   different streamlines in general have different constants, we shall see later that under specific circumstances all streamlines have the same constant

One should be aware of these restrictions when using the Bernoulli relation

Relation to the Energy Equation

$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g z_2 = const$$

- Derived from the momentum equation
- May be interpreted as a idealized energy equation (changes from 1 to 2)
  - reversible pressure work
  - kinetic energy change
  - potential energy change
  - no exchange due to viscous dissipation

Stagnation, Static, and Dynamic Pressures

In many flows, elevation changes are negligible

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2 = p_o$$

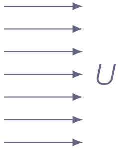
- Static pressure: p_1 and p_2
- Dynamic pressure: $\frac{1}{2}\rho V_1^2$ and $\frac{1}{2}\rho V_2^2$
- Stagnation (total) pressure: p_o

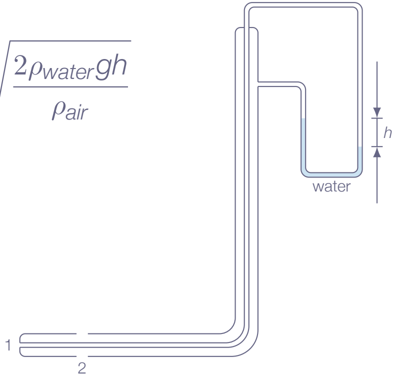
Pitot Static Tube

Pitot Static Tube

$$p_1 + \frac{1}{2}\rho_{air}U_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho_{air}U_2^2 + \rho g z_2$$

$$\left.\begin{aligned} U_1 &= 0. \\ U_2 &= U \\ z_1 &\approx z_2 \\ p_1 - p_2 &= \rho_{water} g h \end{aligned}\right\} \Rightarrow U = \sqrt{\frac{2\rho_{water} g h}{\rho_{air}}}$$





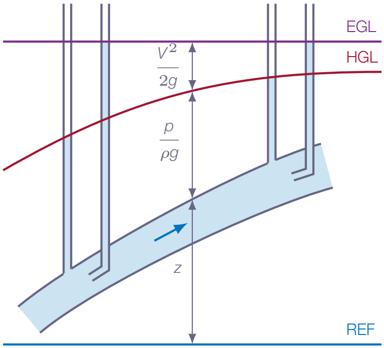
Hydraulic and Energy Grade Lines

EGL: $\dfrac{p}{\rho g} + \dfrac{V^2}{2g} + z$

constant if:

- no friction
- no heat transfer
- no work

HGL: $\dfrac{p}{\rho g} + z =$ EGL - $\dfrac{V^2}{2g}$

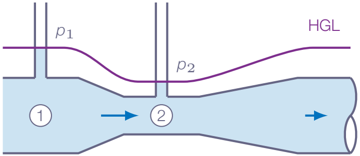


Venturi Tube

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2$$

$z_1 = z_2$ gives

$$V_2^2 - V_1^2 = \frac{2\Delta p}{\rho}$$



continuity:

$$A_1V_1 = A_2V_2 \Rightarrow V_1 = \frac{A_2}{A_1}V_2 = \frac{D_2^2}{D_1^2}V_2$$

inserted in the Bernoulli equation, this gives

$$V_2 = \left[\frac{2D_1^4\Delta p}{\rho(D_1^4 - D_2^4)}\right]^{1/2} \Rightarrow \dot{m} = \rho A_2 V_2 = \frac{\pi D_1^2 D_2^2}{4}\left[\frac{2\rho\Delta p}{D_1^4 - D_2^4}\right]^{1/2}$$

Tank Problem - Solution 1

conservation of mass:

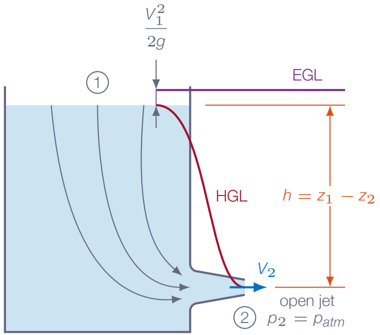
$$A_1V_1 = A_2V_2 \Rightarrow V_1 = \frac{A_2}{A_1}V_2$$

Bernoulli:

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2$$

$$p_1 = p_2 = p_{atm}$$

$$V_2^2 - V_1^2 = 2g(z_1 - z_2) = 2gh$$

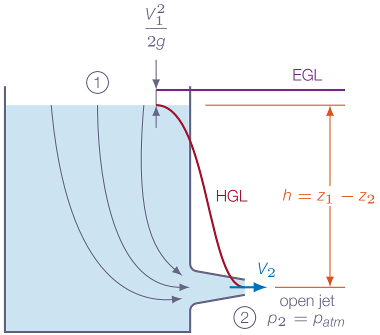


$$V_2 = \sqrt{\frac{2gh}{1 - \left(\frac{A_2}{A_1}\right)^2}}$$

$$A_2 \ll A_1 \Rightarrow V_2 \approx \sqrt{2gh}$$

Tank Problem - Solution 2

The outflow is very small in compared to the tank volume and thus the water surface hardly moves at all, *i.e.* $V_1 \approx 0$



Bernoulli:

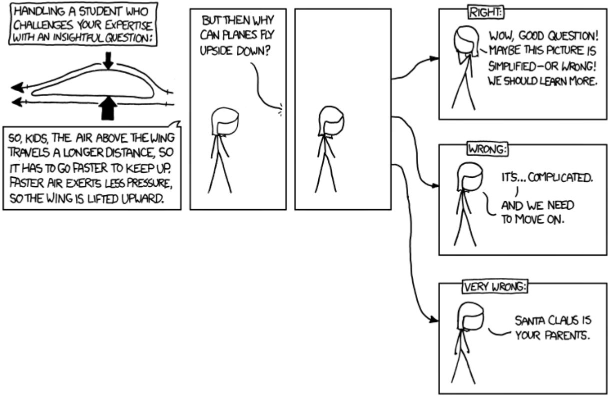
$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2$$

$$V_2^2 = 2g(z_1 - z_2) = 2gh$$

$$V_1 \approx 0, \; p_1 = p_2 = p_{atm}$$

$$V_2 = \sqrt{2gh}$$

Airfoil


HANDLING A STUDENT WHO CHALLENGES YOUR EXPERTISE WITH AN INSIGHTFUL QUESTION:
SO, KIDS, THE AIR ABOVE THE WING TRAVELS A LONGER DISTANCE, SO IT HAS TO GO FASTER TO KEEP UP. FASTER AIR EXERTS LESS PRESSURE, SO THE WING IS LIFTED UPWARD.
BUT THEN WHY CAN PLANES FLY UPSIDE DOWN?
RIGHT:
WOW, GOOD QUESTION! MAYBE THIS PICTURE IS SIMPLIFIED—OR WRONG! WE SHOULD LEARN MORE.
WRONG:
IT'S... COMPLICATED.
AND WE NEED TO MOVE ON.
VERY WRONG:
SANTA CLAUS IS YOUR PARENTS.