Fluid Mechanics - MTF053

Lecture 4

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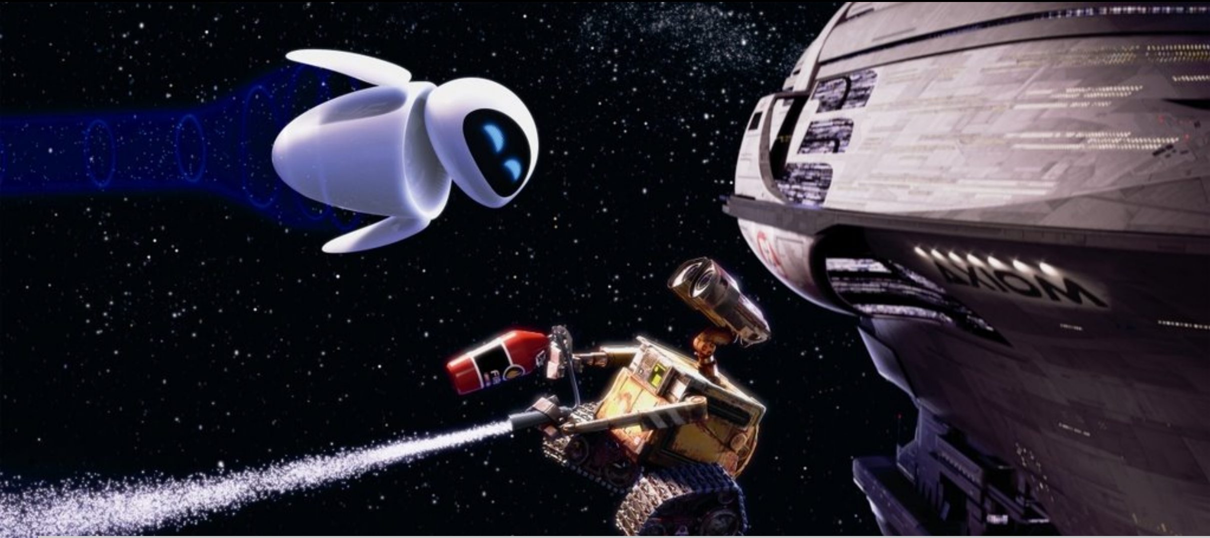


Chapter 3 - Integral Relations for a Control Volume

Overview

sional
Analysis
Differential Relations
conser-vation relations
Flow Relations
Integral Relations
Bernoulli
conser-vation relations
Reynolds transport theorem
hydrostatic forces
buoyancy
Pressure
velocity field
Reynolds number

Learning Outcomes

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

12 **Define** Reynolds transport theorem using the concepts control volume and system

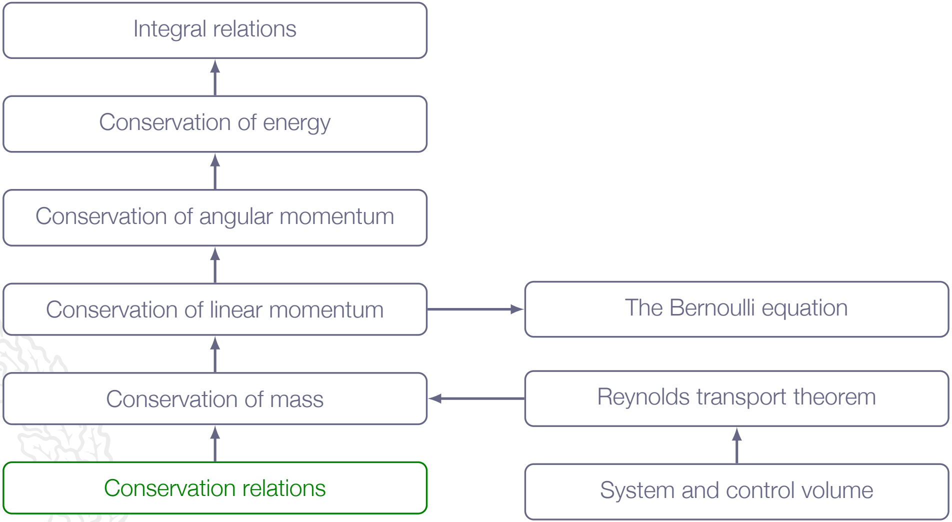
13 **Derive** the control volume formulation of the continuity, momentum, and energy equations using Reynolds transport theorem and solving problems using those relations

15 **Derive** and use the Bernoulli equation (using the relation includes having knowledge about its limitations)

we will derive methods suitable for estimation of forces and system analysis

fluid flow finally ...

Roadmap - Integral Relations

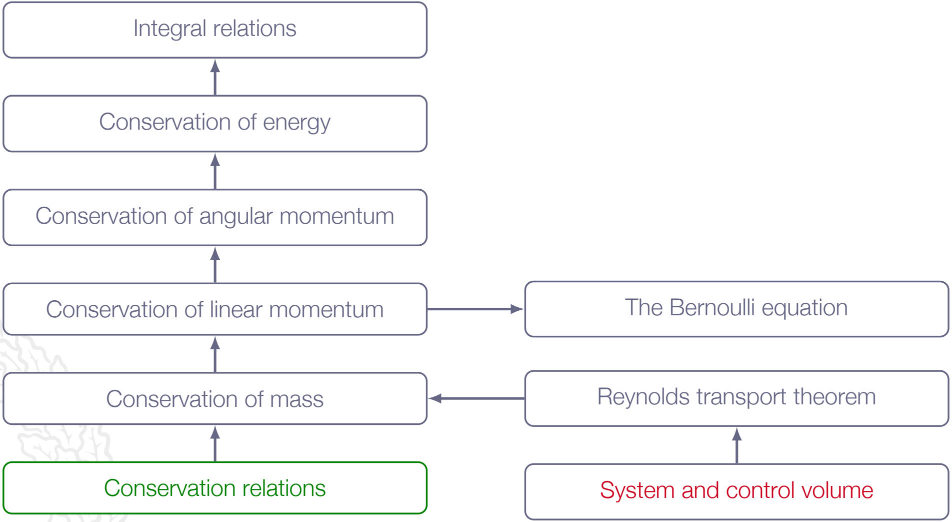


Motivation

Fluid motion analysis:

- differential approach (chapter 4):
  - describe the detailed flow pattern at every point in the flow
- control volume approach (chapter 3):
  - working with a finite region, balance in and out flow and determine gross flow effects (force, torque, energy exchange, ...)
  - gives useful engineering estimates

Roadmap - Integral Relations



System vs Control Volume

All laws of mechanics are written for a system:

- A system is an arbitrary quantity of mass of fixed identity m
- The system is separated from its surroundings by its boundaries
- Interaction between the system and its surroundings

System Mass

$$m_{syst} = const$$

$$\frac{dm}{dt} = 0$$

Obvious in solid mechanics but needs attention in fluid mechanics

Conservation Relations

1. Mass
2. Linear momentum
3. Angular momentum
4. Energy

Linear Momentum

If the surroundings exert a net force $\mathbf{F}$ on the system, the mass in the system will begin to accelerate

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{V}}{dt} = \frac{d}{dt}(m\mathbf{V})$$

Angular Momentum

If the surroundings exert a net moment $\mathbf{M}$ about the center of mass of the system, there will be a rotation effect

$$\mathbf{M} = \frac{d\mathbf{H}}{dt}$$

where $\mathbf{H} = \Sigma(\mathbf{r} \times \mathbf{V})\delta m$ is the angular momentum of the system about its center of mass

Energy

First law of thermodynamics

$$\delta Q - \delta W = dE$$

Second law of thermodynamics

$$dS \leq \frac{\delta Q}{T}$$

State Relations

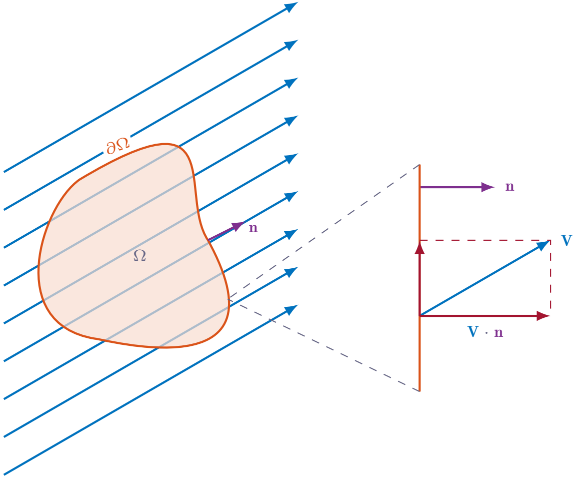
- The above-listed relations includes thermodynamic properties
- Needs to be supplemented by a **state relation**
- Remember: a thermodynamic property can be calculated from **any two other thermodynamics properties**

$$p = p(\rho, T),\ e = e(\rho, T)$$

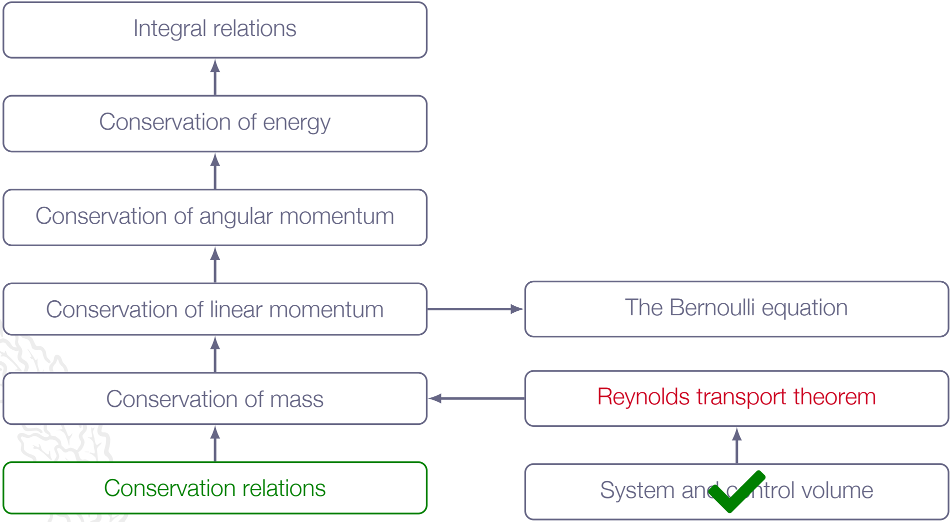
Volume and Mass Flow Rate

$$Q = \int_{\partial\Omega} (\mathbf{V} \cdot \mathbf{n}) dA$$

$$\dot{m} = \int_{\partial\Omega} \rho(\mathbf{V} \cdot \mathbf{n}) dA$$



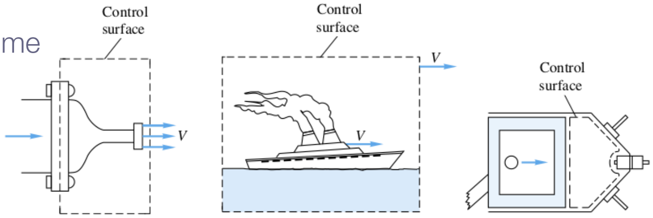
Roadmap - Integral Relations



Reynolds Transport Theorem

Converts mathematical relations for a specific system to relations for a specific region

- fixed control volume
- moving control volume
- deformable control volume



Reynolds Transport Theorem

Let B be any **extensive** property of the fluid (energy, momentum, enthalpy, ...)

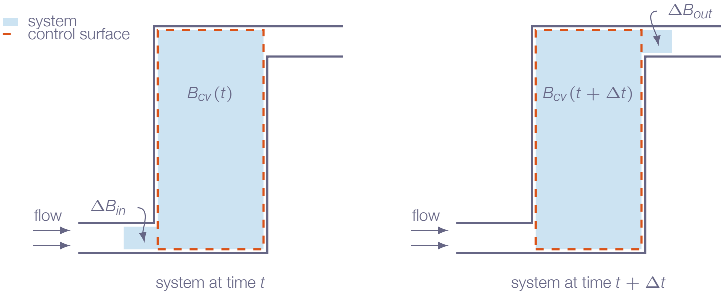
β is the corresponding **intensive** value (*the amount B per unit mass*)

The total amount of B in the control volume is

$$B_{CV} = \int_{CV} \beta dm = \int_{CV} \beta \rho d\mathcal{V}$$

where $\beta = \dfrac{dB}{dm}$

Reynolds Transport Theorem



$$B_{sys}(t) = B_{cv}(t) + \Delta B_{in}$$

$$B_{sys}(t + \Delta t) = B_{cv}(t + \Delta t) + \Delta B_{out}$$

Reynolds Transport Theorem

The rate of change of B for the system:

$$\frac{dB_{sys}}{dt} = \lim_{\Delta t \to 0} \left[\frac{B_{sys}(t + \Delta t) - B_{sys}(t)}{\Delta t} \right]$$

Apply relations from previous slide $\Rightarrow$

$$\frac{dB_{sys}}{dt} = \lim_{\Delta t \to 0} \left[\frac{B_{cv}(t + \Delta t) + \Delta B_{out} - B_{cv}(t) - \Delta B_{in}}{\Delta t} \right]$$

Reynolds Transport Theorem

Rewriting $\Rightarrow$

$$\frac{dB_{sys}}{dt} = \underbrace{\lim_{\Delta t \to 0}\left[\frac{B_{cv}(t+\Delta t) - B_{cv}(t)}{\Delta t}\right]}_{\frac{dB_{cv}}{dt}} + \underbrace{\underbrace{\lim_{\Delta t \to 0}\frac{\Delta B_{out}}{\Delta t}}_{\dot{B}_{out}} - \underbrace{\lim_{\Delta t \to 0}\frac{\Delta B_{in}}{\Delta t}}_{\dot{B}_{in}}}_{\dot{B}_{net}}$$

$$\frac{dB_{sys}}{dt} = \frac{dB_{cv}}{dt} + \dot{B}_{net}$$

Reynolds Transport Theorem

Rate of change of *B* within the control volume

$$\frac{d}{dt}\left(\int_{CV}\beta\rho d\mathcal{V}\right)$$

Net flux of *B* over the control volume surface

$$\int_{CS}\beta\rho(\mathbf{V}\cdot\mathbf{n})dA$$

Reynolds Transport Theorem

$$\underbrace{\frac{d}{dt}(B_{sys})}_{Lagrange} = \underbrace{\frac{d}{dt}\left(\int_{CV}\beta\rho d\mathcal{V}\right) + \int_{CS}\beta\rho(\mathbf{V}\cdot\mathbf{n})dA}_{Euler}$$

Reynolds Transport Theorem

For a fixed control volume (the volume does not change in time)

$$\frac{d}{dt}\left(\int_{CV}\beta\rho d\mathcal{V}\right)=\int_{CV}\frac{\partial}{\partial t}(\beta\rho)d\mathcal{V}$$

Reynolds Transport Theorem

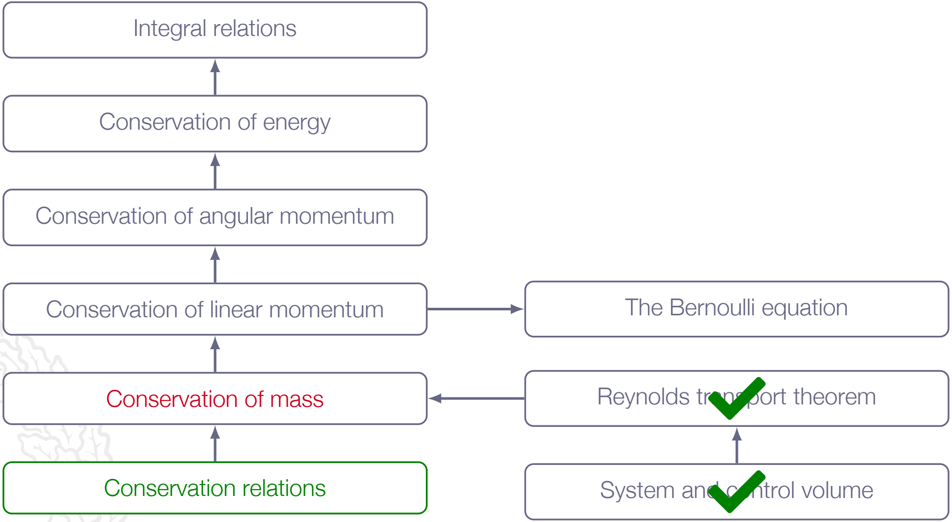
If the control volume moves with the constant velocity $\mathbf{V_s}$, the relative velocity of the fluid crossing the control volume surface $\mathbf{V_r}$ is

$$\mathbf{V_r} = \mathbf{V} - \mathbf{V_s}$$

and thus

$$\frac{d}{dt}(B_{sys}) = \frac{d}{dt}\left(\int_{CV} \beta\rho d\mathcal{V}\right) + \int_{CS} \beta\rho(\mathbf{V_r}\cdot\mathbf{n})dA$$

Roadmap - Integral Relations



Conservation of Mass

Reynolds transport theorem with $B = m$ and $\beta = dB/dm = dm/dm = 1$

$$\frac{d}{dt}(m_{sys}) = 0 = \frac{d}{dt}\left(\int_{CV} \rho d\mathcal{V}\right) + \int_{CS} \rho(\mathbf{V_r} \cdot \mathbf{n})dA$$

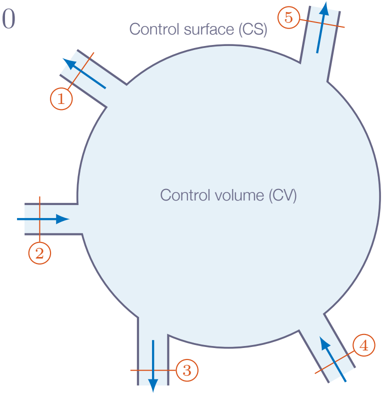
for a fixed control volume

$$\boxed{\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \int_{CS} \rho(\mathbf{V_r} \cdot \mathbf{n})dA = 0}$$

Conservation of Mass

for a control volume with a number of one-dimensional inlets and outlets

$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$



Conservation of Mass

Steady state $\Rightarrow \partial\rho/\partial t = 0$

$$\int_{CS} \rho(\mathbf{V_r} \cdot \mathbf{n})dA = 0$$

or

$$\sum_i (\rho_i A_i V_i)_{out} = \sum_i (\rho_i A_i V_i)_{in}$$

Conservation of Mass

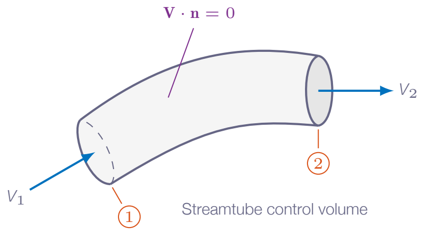
Incompressible flow $\Rightarrow \partial\rho/\partial t = 0$

$$\int_{CS} (\mathbf{V_r} \cdot \mathbf{n}) dA = 0$$

or

$$\sum_i (A_i V_i)_{out} = \sum_i (A_i V_i)_{in}$$

Conservation of Mass - Example 1



Steady flow through a streamtube

- steady state ⇒ no changes in time
- streamtube ⇒ only flow through the surfaces 1 and 2

$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2 = const$$

if the density is constant (incompressible flow)

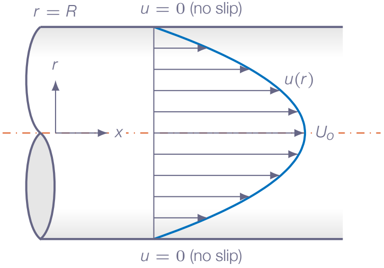
$$Q = A_1 V_1 = A_2 V_2 = const \Rightarrow V_2 = \frac{A_1}{A_2} V_1$$

Remember: a streamtube is constructed from a set of streamlines

Conservation of Mass - Example 2

Compute the average velocity for a steady **laminar** incompressible viscous flow through a circular tube with given axial velocity profile

$$u = U_o \left(1 - \left(\frac{r}{R}\right)^2\right)$$



Assumptions:

1. Laminar flow
2. Steady state $\Rightarrow$ no changes in time
3. Incompressible $\Rightarrow$ constant density

$$V_{av} = \frac{1}{A}\int u dA = \frac{1}{\pi R^2}\int_0^R U_o \left(1 - \left(\frac{r}{R}\right)^2\right) 2\pi r dr = \frac{2U_o}{R^2}\int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right) r dr$$

Conservation of Mass - Example 2

$$V_{av} = \frac{2U_o}{R^2} \int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right) rdr = \frac{2U_o}{R^2} \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = \frac{U_o}{2}$$

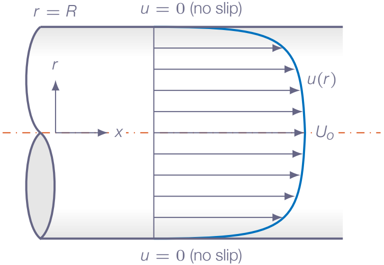
Thus, for **laminar** pipe flow

$$V_{av} = \frac{U_o}{2}$$

Conservation of Mass - Example 3

Compute the average velocity for a steady **turbulent** incompressible viscous flow through a circular tube with given axial velocity profile

$$u \approx U_o \left(1 - \frac{r}{R}\right)^m$$



Assumptions:

1. Turbulent flow: $1/5 \geq m \geq 1/9$
2. Steady state $\Rightarrow$ no changes in time
3. Incompressible $\Rightarrow$ constant density

$$V_{av} \approx \frac{1}{A}\int u dA = \frac{1}{\pi R^2}\int_0^R U_o \left(1 - \frac{r}{R}\right)^m 2\pi r dr = \frac{2U_o}{R^2}\int_0^R \left(1 - \frac{r}{R}\right)^m r dr$$

Conservation of Mass - Example 3

$$V_{av} \approx \frac{2U_o}{R^2}\int_0^R \left(1-\frac{r}{R}\right)^m r dr = \frac{2U_o}{R^2}\left[\frac{(r-R)\left(1-\frac{r}{R}\right)^m (mr+r+R)}{(m+1)(m+2)}\right]_0^R$$

Thus, for **turbulent** pipe flow

$$V_{av} \approx \frac{2U_o}{(m+1)(m+2)}$$

$$m = 1/7 \Rightarrow V_{av} \approx 49U_o/60 \approx 0.82U_o$$