Fluid Mechanics - MTF053

Lecture 15

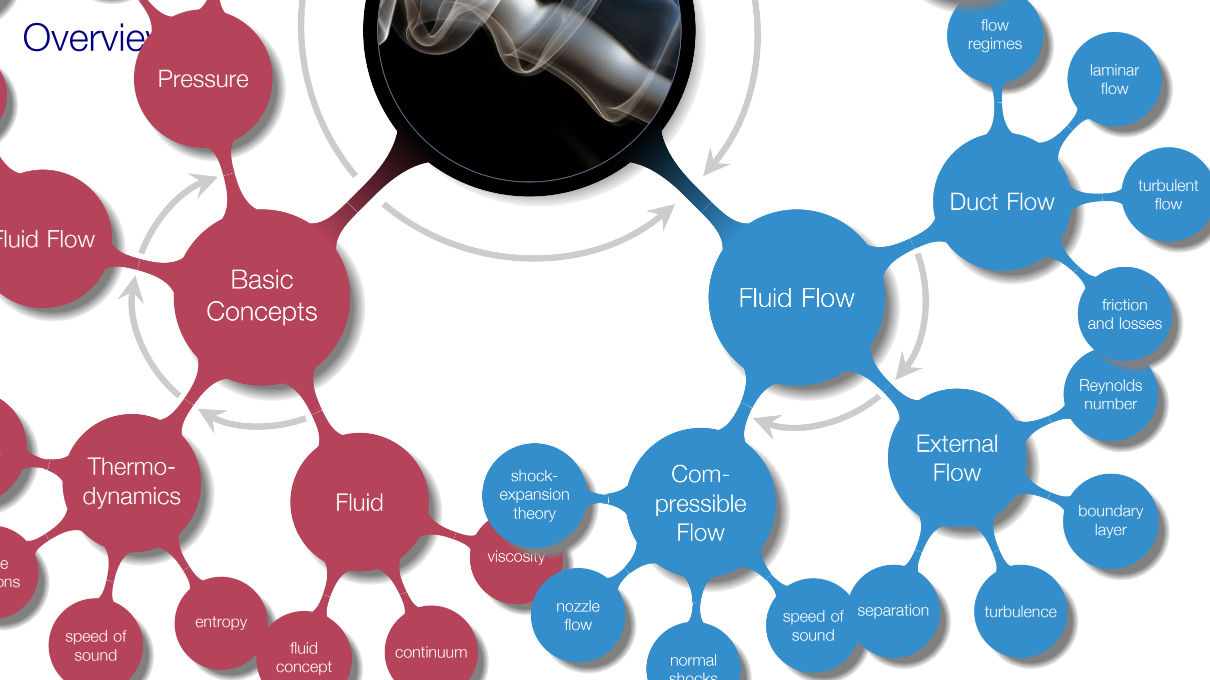
Niklas Andersson

Chalmers University of Technology
Department of Mechanics and Maritime Sciences
Division of Fluid Mechanics
Gothenburg, Sweden

niklas.andersson@chalmers.se

Chapter 7 - Flow Past Immersed Bodies

Overvie


Pressure
Basic Concepts
Thermo-dynamics
Fluid
viscosity
speed of sound
entropy
fluid concept
continuum
flow regimes
laminar flow
Duct Flow
turbulent flow
friction and losses
Fluid Flow
Reynolds number
External Flow
boundary layer
Com-pressible Flow
shock-expansion theory
nozzle flow
normal
speed of sound
separation
turbulence

Learning Outcomes

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

6 **Explain** what a boundary layer is and when/where/why it appears

21 **Explain** how the flat plate boundary layer is developed (transition from laminar to turbulent flow)

22 **Explain** and use the Blasius equation

23 **Define** the Reynolds number for a flat plate boundary layer

24 **Explain** what is characteristic for a turbulent flow

29 **Explain** flow separation (separated cylinder flow)

30 **Explain** how to delay or avoid separation

31 **Derive** the boundary layer formulation of the Navier-Stokes equations

32 **Understand** and explain displacement thickness and momentum thickness

33 **Understand**, **explain** and **use** the concepts drag, friction drag, pressure drag, and lift

Let's take a deep dive into boundary-layer theory

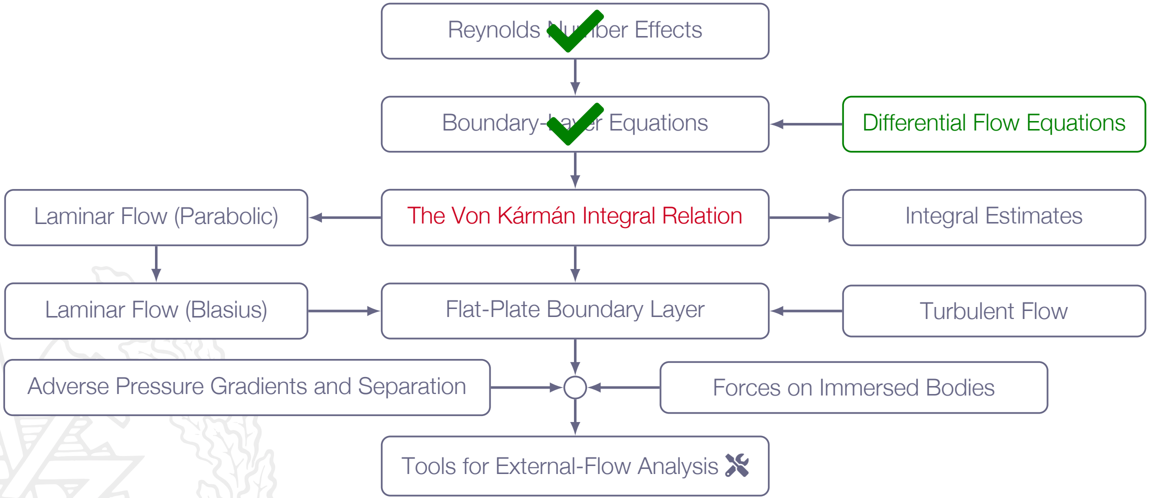
Complementary Course Material

These lecture notes covers chapter 7 in the course book and additional course material that you can find in the following documents

MTF053_Equation-for-Boundary-Layer-Flows.pdf

MTF053_Turbulence.pdf

Roadmap - Flow Past Immersed Bodies



The Von Kármán Integral Relation

Approximate solutions for $\delta(x)$ and $\tau_w(x)$

Control volume approach applied to a boundary layer

Assuming steady-state incompressible flow

$$\sum \mathbf{F} = \int_{CS} \mathbf{V}\rho(\mathbf{V} \cdot \mathbf{n})dA$$

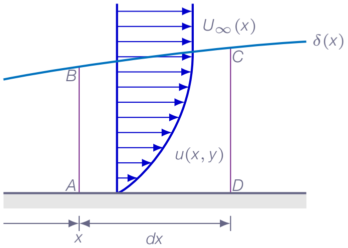
The Von Kármán Integral Relation

Massflow

$$\dot{m}_{AB} = \rho \int_0^{\delta} u dy$$

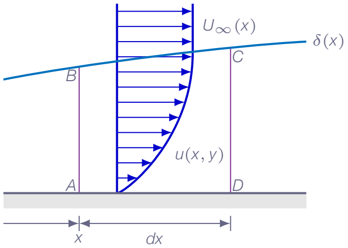
$$\dot{m}_{CD} = \rho \int_0^{\delta} u dy + \frac{d}{dx}\left[\rho \int_0^{\delta} u dy\right] dx$$

$$\dot{m}_{BC} = \rho \frac{d}{dx}\left[\int_0^{\delta} u dy\right] dx$$



The Von Kármán Integral Relation

Momentum



$$I_{AB} = \rho \int_0^\delta u^2 dy$$

$$I_{CD} = \rho \int_0^\delta u^2 dy + \frac{d}{dx}\left[\rho \int_0^\delta u^2 dy\right] dx$$

$$I_{BC} = U\dot{m}_{BC} = \rho U_\infty \frac{d}{dx}\left[\int_0^\delta u dy\right] dx$$

$$I_{CD} - I_{AB} - I_{BC} = \rho \frac{d}{dx}\left[\int_0^\delta u^2 dy\right] dx - \rho U_\infty \frac{d}{dx}\left[\int_0^\delta u dy\right] dx$$

The Von Kármán Integral Relation

Pressure forces in the x-direction

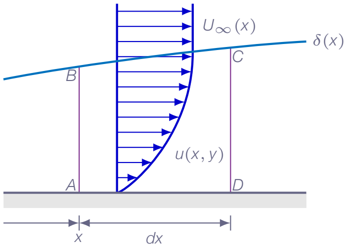
AB: $p\delta$

CD: $-\left(p + \frac{dp}{dx}dx\right)\left(\delta + \frac{d\delta}{dx}dx\right)$

BC: $\approx \left(p + \frac{1}{2}\frac{dp}{dx}dx\right)\frac{d\delta}{dx}dx$

Shear forces in the x-direction

AD: $-\tau_w dx$



The Von Kármán Integral Relation

Forces

$$dF_x =$$

$$-\tau_w dx + p\delta - \left[p\delta + p\frac{d\delta}{dx}dx + \delta\frac{dp}{dx}dx + \frac{dp}{dx}\frac{d\delta}{dx}dxdx\right] + p\frac{d\delta}{dx}dx + \frac{1}{2}\frac{dp}{dx}\frac{d\delta}{dx}dxdx$$

products of infinitesimal quantities can be regarded to be zero and thus

$$dF_x = -\tau_w dx - \delta\frac{dp}{dx}dx$$

The Von Kármán Integral Relation

Momentum equation

Now we have all components of the momentum equation defined

$$\rho \frac{d}{dx}\left[\int_0^{\delta} u^2 dy\right] - \rho U_\infty \frac{d}{dx}\left[\int_0^{\delta} u dy\right] = -\tau_w - \delta \frac{dp}{dx}$$

The momentum equation for boundary layers or **Von Kármán's integral relation**

Note! the relation is valid for laminar and turbulent flows (for turbulent flows use time-averaged quantities)

The Von Kármán Integral Relation

Outside of the boundary layer the flow is inviscid $\Rightarrow$ we can use Bernoulli

$$p + \frac{1}{2}\rho U_\infty^2 = const \Rightarrow \frac{dp}{dx} + \rho U_\infty \frac{dU_\infty}{dx} = 0 \Rightarrow -\frac{1}{\rho}\frac{dp}{dx} = U_\infty \frac{dU_\infty}{dx}$$

The Von Kármán Integral Relation

$$\frac{1}{\rho}\frac{dp}{dx} = -U_\infty\frac{dU_\infty}{dx} \Rightarrow \frac{\tau_w}{\rho} - \delta U_\infty\frac{dU_\infty}{dx} = U_\infty\frac{d}{dx}\left[\int_0^\delta u dy\right] - \frac{d}{dx}\left[\int_0^\delta u^2 dy\right]$$

$$\delta U_\infty\frac{dU_\infty}{dx} = U_\infty\frac{dU_\infty}{dx}\int_0^\delta dy$$

$$U_\infty\frac{d}{dx}\left[\int_0^\delta u dy\right] = \frac{d}{dx}\left[U_\infty\int_0^\delta u dy\right] - \frac{dU_\infty}{dx}\int_0^\delta u dy$$

$$\frac{\tau_w}{\rho} - U_\infty\frac{dU_\infty}{dx}\int_0^\delta dy = \frac{d}{dx}\left[U_\infty\int_0^\delta u dy\right] - \frac{dU_\infty}{dx}\int_0^\delta u dy - \frac{d}{dx}\left[\int_0^\delta u^2 dy\right]$$

The Von Kármán Integral Relation

$$\frac{\tau_w}{\rho} - U_\infty \frac{dU_\infty}{dx} \int_0^\delta dy = \frac{d}{dx}\left[U_\infty \int_0^\delta u dy\right] - \frac{dU_\infty}{dx} \int_0^\delta u dy - \frac{d}{dx}\left[\int_0^\delta u^2 dy\right]$$

$$U_\infty \frac{dU_\infty}{dx} \int_0^\delta dy - \frac{dU_\infty}{dx} \int_0^\delta u dy = \frac{dU_\infty}{dx} \int_0^\delta (U_\infty - u) dy$$

$$\frac{d}{dx}\left[U_\infty \int_0^\delta u dy\right] - \frac{d}{dx}\left[\int_0^\delta u^2 dy\right] = \frac{d}{dx} \int_0^\delta u(U_\infty - u) dy$$

The Von Kármán Integral Relation

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^{\delta} u(U_\infty - u)dy + \frac{dU_\infty}{dx}\int_0^{\delta}(U_\infty - u)dy$$

The Von Kármán Integral Relation

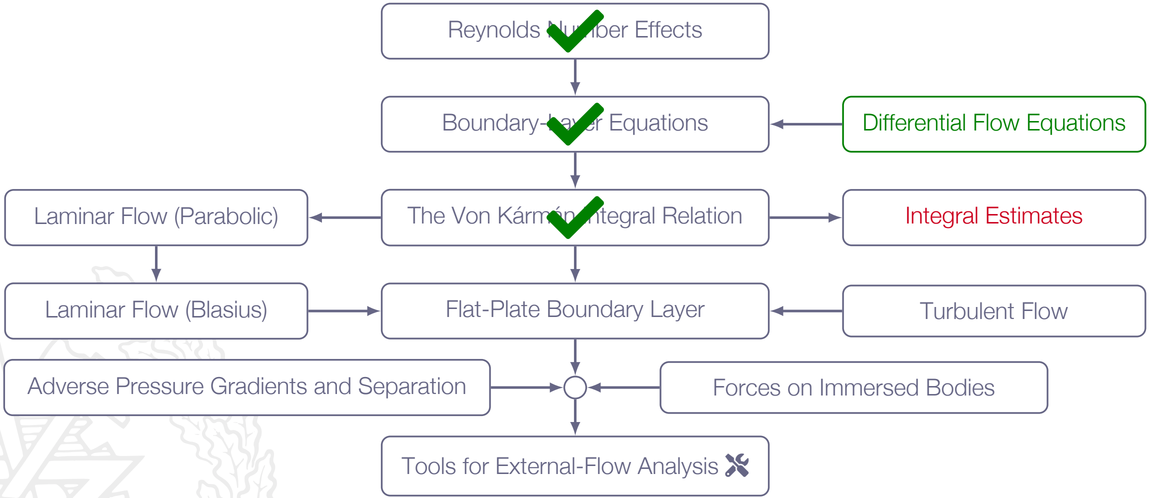
$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^\delta u(U_\infty - u)dy + \frac{dU_\infty}{dx}\int_0^\delta (U_\infty - u)dy$$

Constant freestream velocity gives

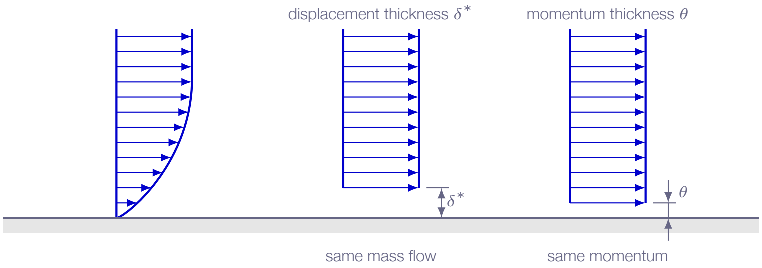
$$\frac{dU_\infty}{dx} = 0 \Rightarrow \frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^\delta u(U_\infty - u)dy$$

Ok, but what does this mean??

Roadmap - Flow Past Immersed Bodies

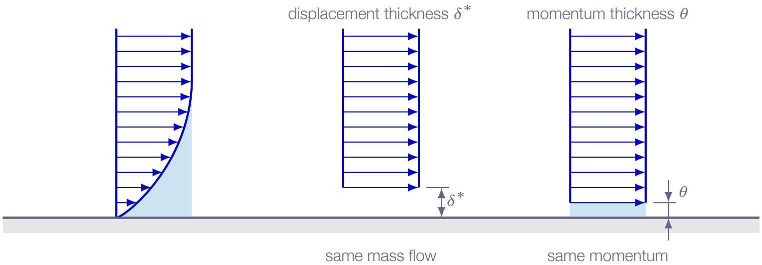


Momentum Integral Estimates



"The presence of a boundary layer will result in a small but finite displacement of the flow streamlines"

Momentum Thickness



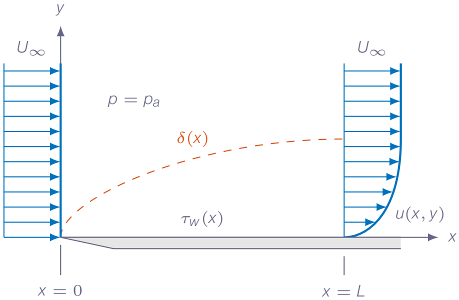
$$\int_0^{\delta} \rho u(U_\infty - u) b dy = \rho U_\infty^2 b\theta \Rightarrow \theta = \int_0^{\delta} \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

Note! *b is the width of the flat plate*

Momentum Thickness

The drag D for a plate of width b

$$D(x) = b\int_0^x \tau_w(x)dx \Rightarrow \frac{dD}{dx} = b\tau_w$$



from before we have

$$\underbrace{\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^\delta u(U_\infty - u)dy}_{\text{the Von Kármán integral relation}} = \frac{d}{dx}U_\infty^2 \underbrace{\int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy}_{\text{the displacement thickness } \theta} = U_\infty^2 \frac{d\theta}{dx}$$

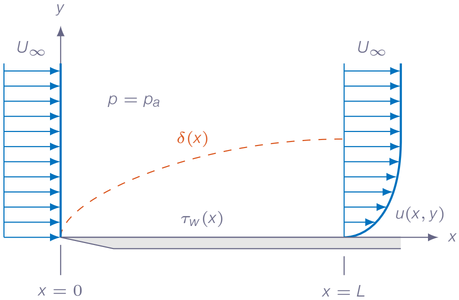
and thus

$$\frac{dD}{dx} = b\rho U_\infty^2 \frac{d\theta}{dx} \Rightarrow D(x) = \rho b U_\infty^2 \theta$$

Momentum Thickness

$$D(x) = \rho b U_\infty^2 \theta, \ \tau_w = \rho U_\infty^2 \frac{d\theta}{dx}$$

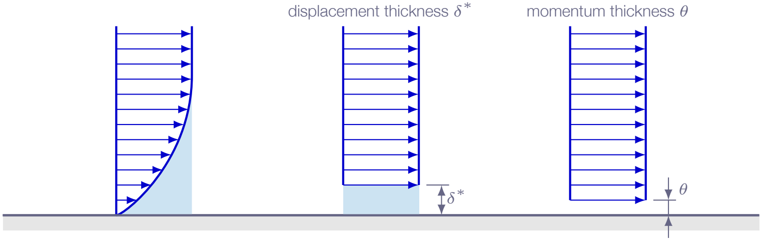
$$\theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$



Note!

1. the momentum thickness θ is a measure of the total drag
2. can be used both for laminar and turbulent flows
3. no assumption about velocity profile shape made

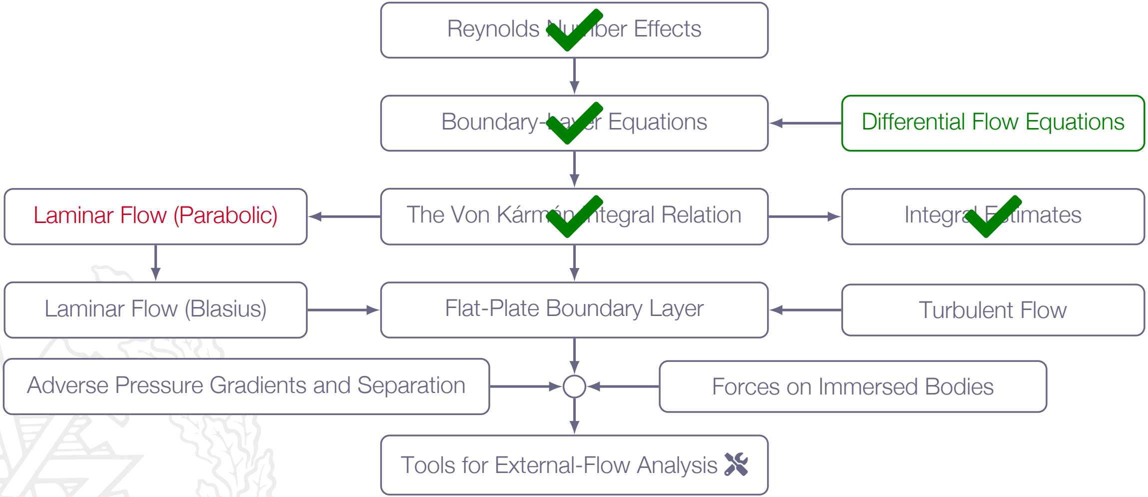
Displacement Thickness



$$\int_0^{\delta} \rho(U_\infty - u)bdy = \rho U_\infty b\delta^* \Rightarrow \delta^* = \int_0^{\delta} \left(1 - \frac{u}{U_\infty}\right) dy$$

δ^* is an estimate of the displacement in the wall-normal direction of streamlines in the outer part of the boundary layer due to the deficit of massflow caused by the no-slip condition at the wall - a measure of the boundary-layer thickness

Roadmap - Flow Past Immersed Bodies



Laminar Boundary Layer

The Von Kármán integral relation gives us the wall shear stress (τ_w) as a function of the velocity profile ($u(y)$) and the boundary-layer thickness (δ)

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^{\delta} u(U_\infty - u)dy$$

So now we need a velocity profile $u = u(y)$ to continue ...

Laminar Boundary Layer

Assumptions:

1. Boundary layer over a flat plate
2. Constant freestream velocity $U_\infty = const \Rightarrow \frac{dU_\infty}{dx} = 0$
3. Laminar flow
4. Parabolic velocity profile

Laminar Boundary Layer - Parabolic Velocity Profile

$$u(y) = A + By + Cy^2$$

The constants A, B, and C are defined using boundary conditions

1. no slip:
   $u(0) = 0 \Rightarrow A = 0$

2. constant velocity at $y = \delta$:
   $\left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0 \Rightarrow B + 2C\delta = 0 \Rightarrow B = -2\delta C$

3. freestream velocity:
   $u(\delta) = U_\infty \Rightarrow B\delta + C\delta^2 = U_\infty \Rightarrow \{B = -2\delta C\} \Rightarrow -C\delta^2 = U_\infty \Rightarrow C = -\frac{U_\infty}{\delta^2}$

Laminar Boundary Layer - Parabolic Velocity Profile

$$u(y) = A + By + Cy^2$$

$$A = 0, \quad B = \frac{2U_\infty}{\delta}, \quad C = -\frac{U_\infty}{\delta^2}$$

$$u(y) = U_\infty \left(\frac{2}{\delta} y - \frac{1}{\delta^2} y^2 \right)$$

Laminar Boundary Layer - Parabolic Velocity Profile

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^{\delta} u(U_\infty - u)dy$$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu \frac{2U_\infty}{\delta}$$

$$\int_0^{\delta} u(U_\infty - u)dy = \int_0^{\delta} U_\infty^2 \left(\frac{2}{\delta}y - \frac{1}{\delta^2}y^2\right) - U_\infty^2 \left(\frac{4}{\delta^2}y^2 - \frac{4}{\delta^3}y^3 + \frac{1}{\delta^4}y^4\right) dy = \frac{2}{15}U_\infty^2\delta$$

$$\frac{\mu}{\rho}\frac{2U_\infty}{\delta} = \frac{d}{dx}\left(\frac{2}{15}U_\infty^2\delta\right) \Rightarrow \frac{\nu}{\delta} = \frac{U_\infty}{15}\frac{d\delta}{dx} \Rightarrow \delta d\delta = 15\frac{\nu}{U_\infty}dx$$

Laminar Boundary Layer - Parabolic Velocity Profile

$$\delta d\delta = 15\frac{\nu}{U_\infty}dx$$

$$\frac{\delta^2}{2} = 15\frac{\nu}{U_\infty}x + C = \{x = 0 \Rightarrow \delta = 0 \Rightarrow C = 0\} = 15\frac{\nu}{U_\infty}x$$

$$\delta = \sqrt{\frac{30\nu x}{U_\infty}} \Rightarrow \frac{\delta}{x} = \sqrt{\frac{30\nu}{U_\infty x}} \approx \frac{5.5}{\sqrt{Re_x}}$$

Laminar Boundary Layer - Parabolic Velocity Profile

$$\tau_w = \mu \frac{2U_\infty}{\delta} = \frac{2\mu U_\infty}{\sqrt{\dfrac{30\nu x}{U_\infty}}} = \frac{2}{\sqrt{30}} \frac{\rho U_\infty^2}{\sqrt{\dfrac{U_\infty x}{\nu}}} \approx \frac{0.365}{\sqrt{Re_x}} \rho U_\infty^2$$

Introducing the **skin friction coefficient** c_f

$$c_f = \frac{2\tau_w}{\rho U_\infty^2} \approx \frac{0.73}{\sqrt{Re_x}}$$

Laminar Boundary Layer - Parabolic Velocity Profile

Note! *more accurate solutions for laminar flat plate boundary layers exists:*

$$c_f \approx \frac{0.664}{\sqrt{Re_x}}, \quad \frac{\delta}{x} \approx \frac{5.0}{\sqrt{Re_x}}$$

Ok, so where did we go wrong?

For external (unconfined) boundary layers, the velocity profile is not parabolic – but quite close to parabolic ...