Fluid Mechanics - MTF053

Lecture 12

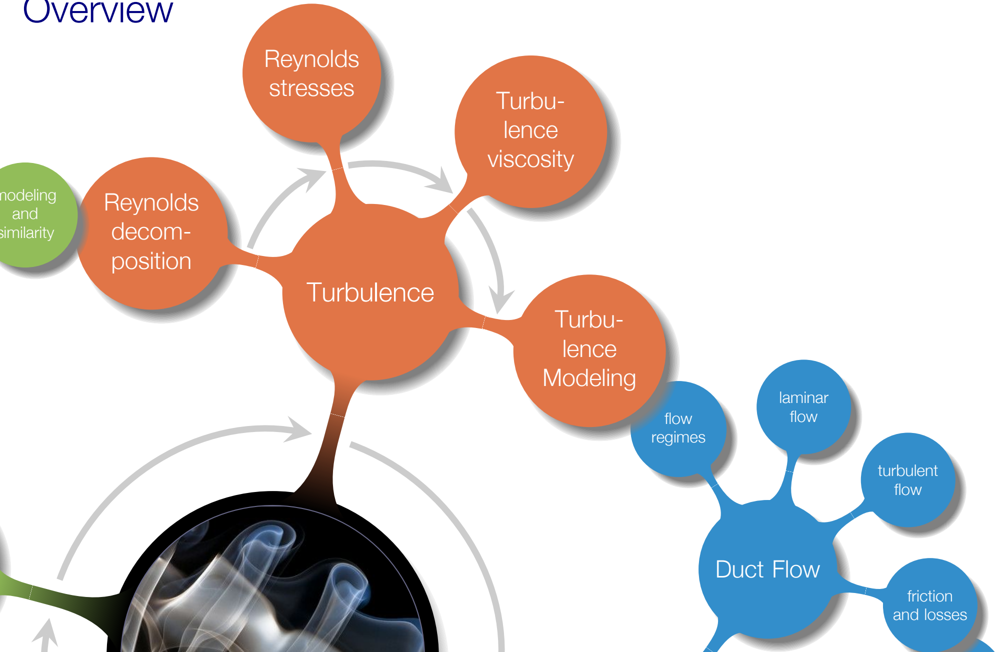
Niklas Andersson

Chalmers University of Technology
Department of Mechanics and Maritime Sciences
Division of Fluid Mechanics
Gothenburg, Sweden

niklas.andersson@chalmers.se

Chapter 6 - Viscous Flow in Ducts

Overview


Reynolds stresses
Turbu- lence viscosity
modeling and similarity
Reynolds decom- position
Turbulence
Turbu- lence Modeling
flow regimes
laminar flow
turbulent flow
Duct Flow
friction and losses

Learning Outcomes

3 **Define** the Reynolds number

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

6 **Explain** what a boundary layer is and when/where/why it appears

8 **Understand** and be able to **explain** the concept shear stress

18 **Explain** losses appearing in pipe flows

19 **Explain** the difference between laminar and turbulent pipe flow

20 **Solve** pipe flow problems using Moody charts

24 **Explain** what is characteristic for a turbulent flow

25 **Explain** Reynolds decomposition and derive the RANS equations

26 **Understand** and **explain** the Boussinesq assumption and turbulent viscosity

27 **Explain** the difference between the regions in a boundary layer and what is characteristic for each of the regions (viscous sub layer, buffer region, log region)

if you think about it, pipe flows are everywhere (a pipe flow is not a flow of pipes)

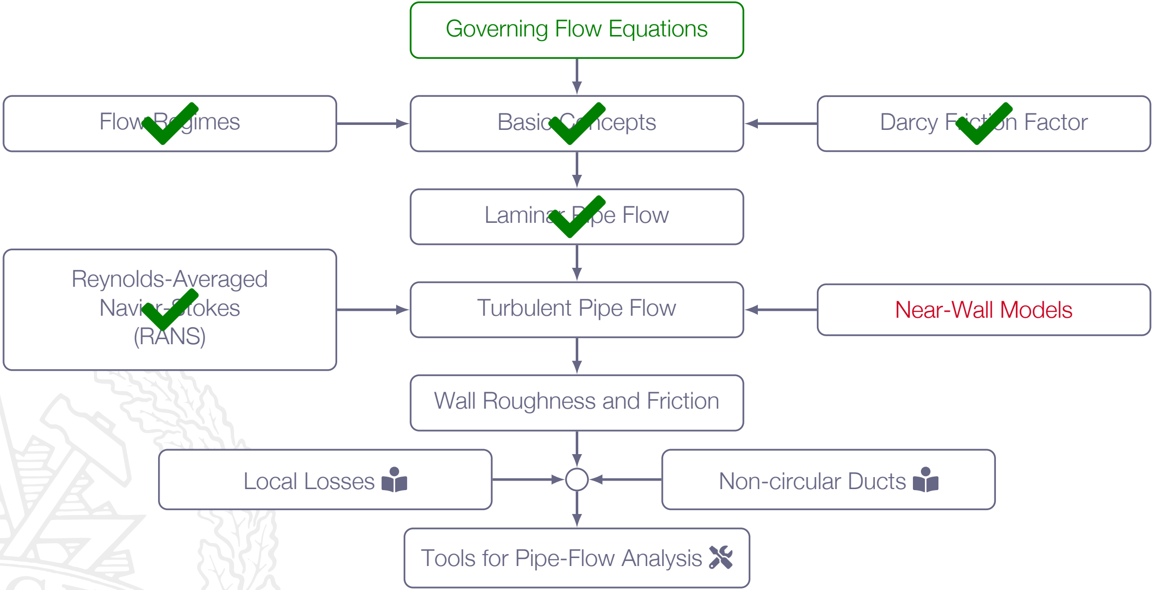
Complementary Course Material

These lecture notes covers chapter 6 in the course book and additional course material that you can find in the following documents

MTF053_Equation-for-Boundary-Layer-Flows.pdf

MTF053_Turbulence.pdf

Roadmap - Viscous Flow in Ducts



Turbulent Pipe Flow - Boundary-Layer Equations

Momentum equation (x-component)

$$\rho \frac{D\overline{u}}{Dt} \approx -\frac{\partial \overline{p}}{\partial x} + \rho g_x + \frac{\partial \tau}{\partial y}$$

where

$$\tau = \mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{u'v'} = (\mu + \mu_t)\frac{\partial \overline{u}}{\partial y}$$

For boundary-layer flows

$$\rho \left(\overline{u} \frac{\partial \overline{u}}{dx} + \overline{v} \frac{\partial \overline{u}}{dy} \right) = -\frac{d\overline{p}}{dx} + \rho g_x + (\mu + \mu_t)\frac{\partial \overline{u}}{\partial y}$$

(will be discussed in more detail in later lectures)

Turbulent Pipe Flow - Boundary-Layer Equations

$$\rho \left(\bar{u}\frac{\partial \bar{u}}{dx} + \bar{v}\frac{\partial \bar{u}}{dy} \right) = -\frac{d\bar{p}}{dx} + \rho g_x + \frac{\partial \tau}{\partial y}$$

$$y \to 0 \Rightarrow \begin{cases} \bar{u} \to 0 \\ \bar{v} \to 0 \end{cases} \Rightarrow$$

$$\frac{\partial \tau}{\partial y} = \frac{d\bar{p}}{dx} - \rho g_x$$

Turbulent Pipe Flow - Boundary-Layer Equations

$$\frac{\partial \tau}{\partial y} = \frac{d\overline{p}}{dx} - \rho g_x$$

$$\tau(y) = \left(\frac{d\overline{p}}{dx} - \rho g_x\right) y + C$$

$$\tau(0) = C = \tau_w \Rightarrow \tau(y) = \left(\frac{d\overline{p}}{dx} - \rho g_x\right) y + \tau_w$$

Note! with a negative pressure gradient, the shear stress will reduce with increasing distance from the wall

Turbulent Pipe Flow - Boundary-Layer Equations

$$\tau(y) = \left(\frac{d\overline{p}}{dx} - \rho g_x \right) y + \tau_w$$

At the wall, the shear stress is equal to the wall-shear stress

$$y \to 0 \Rightarrow \tau(y) \to \tau_w$$

In fact, assuming that the **shear stress** (τ) is **constant** and equal to the wall-shear stress (τ_w) is a valid assumption in the **near-wall region** (some distance from the wall but still close) as long as the pressure gradient is moderate.

Outside of the near-wall region, inertial effects has to be accounted for, i.e., $D\bar{u}/Dt$ will not be zero and thus the shear stress will not be equal to the wall-shear stress.

Turbulent Boundary Layers

A turbulent boundary layer may be divided into different regions where the physical processes leading to shear stress are clearly distinguishable

The viscous sublayer
: the shear stress is dominated by molecular viscosity (μ)

The buffer region
: molecular viscosity (μ) and turbulent viscosity (μ_t) are equally important

The log layer
: the shear stress is dominated by turbulent viscosity (μ_t)

The outer region
: inertial effects must be accounted for

Turbulent Boundary Layers

In the following we will discuss two turbulent boundary layer regions in detail:

- **The viscous sublayer** - the region closest to the wall
- **The log region** - outside of the viscous sublayer but still in the near-wall region

Viscous Sublayer

At the wall

$$\tau = \tau_w = \mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{u'v'}$$

$$y \rightarrow 0 \Rightarrow \begin{cases} u' \rightarrow 0 \\ v' \rightarrow 0 \end{cases} \Rightarrow$$

$$\tau = \mu \frac{\partial \overline{u}}{\partial y}$$

Viscous Sublayer

$$\tau = \mu \frac{\partial \overline{u}}{\partial y} \Rightarrow \overline{u}(y) = \frac{\tau_w}{\mu} y + C$$

$$\overline{u}(0) = 0 \Rightarrow C = 0 \Rightarrow$$

$$\overline{u}(y) = \frac{\tau_w}{\mu} y$$

Note! in the viscous sublayer, the average velocity increase linearly with the wall distance

Viscous Sublayer

Introducing friction velocity defined as

$$\boxed{u^* = \sqrt{\frac{\tau_w}{\rho}}}$$

and thus

$$\overline{u}(y) = \frac{\tau_w}{\mu} y = \frac{\rho u^{*2} y}{\mu} = \frac{u^{*2} y}{\nu}$$

which can be rewritten as:

$$\underbrace{\frac{\overline{u}}{u^*}}_{u^+} = \underbrace{\frac{u^* y}{\nu}}_{y^+} \text{ valid for } y^+ \leq 5 - 10$$

The Log Region

Now, let's move a bit further out from the wall

1. $\tau = const = \tau_w$ still (*we have not moved that far out from the wall*)

2. outside of the viscous sublayer $\mu_t \gg \mu$ and thus

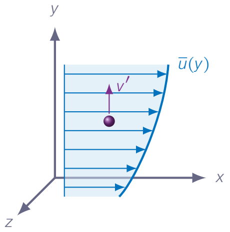
$$\tau = \tau_w = \mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{u'v'} \approx -\rho \overline{u'v'} = \mu_t \frac{\partial \overline{u}}{\partial y}$$

We need an estimate of μ_t to be able to solve this ...

The Log Region

Let's first examine the relation between u' and v' (the velocity fluctuations in the x and y directions)

The illustration below shows a fluid particle in a boundary-layer flow

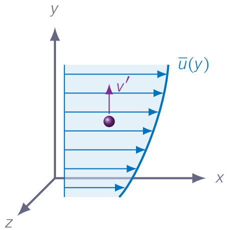


The Log Region

A positive v' fluctuation will lead to a vertical transport of the fluid particle

The fluid particle will end up in a position in the flow where the axial velocity is higher than where it came from, thus leading to a negative fluctuation in the axial velocity at that position $(u' < 0)$

In the same way, a negative v' fluctuation will lead to $u' > 0$

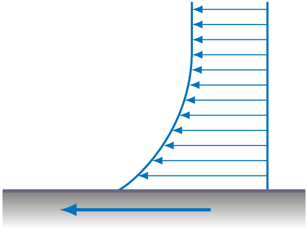


The product $u'v'$ will **always** be negative if $\partial\bar{u}/\partial y$ is positive in the wall-normal direction

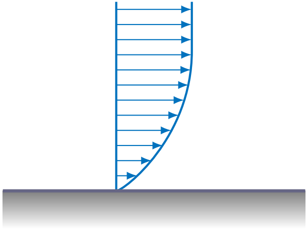
Thus $\tau_{turb} = -\rho\overline{u'v'} = \mu_t \dfrac{\partial\bar{u}}{\partial y}$ is positive

The Log Region

What about other type of boundary layers such as for example the flow over a moving surface



moving wall

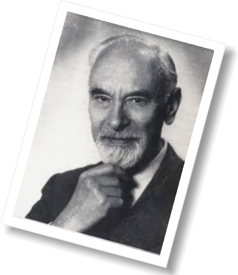


frame of reference of the wall

The Log Region

Prandtl's mixing length concept

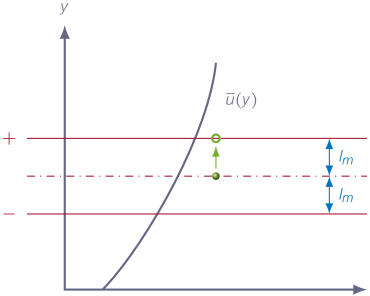
"the average distance that a small mass of fluid will travel before it exchanges its momentum with another mass of fluid"



Ludwig Prandtl 1875-1953

$$\bar{u}(y + l_m) = \bar{u}(y) + l_m \frac{\partial \bar{u}}{\partial y}$$

$$\bar{u}(y - l_m) = \bar{u}(y) - l_m \frac{\partial \bar{u}}{\partial y}$$



Prandtl assumed $u' \approx l_m \dfrac{\partial \bar{u}}{\partial y}$

He further assumed v' to be of the same size as u'

The Log Region

Prandtl's mixing length concept

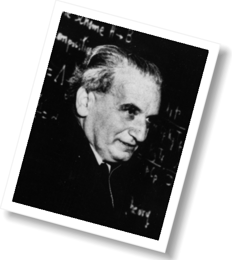
$$\tau_t = -\rho\overline{u'v'} \approx \rho l_m^2 \left(\frac{\partial \overline{u}}{\partial y}\right)^2$$

$$-\rho\overline{u'v'} \approx \mu_t \frac{\partial \overline{u}}{\partial y} \Rightarrow \mu_t \approx \rho l_m^2 \left|\frac{\partial \overline{u}}{\partial y}\right|$$

$$\nu_t = \frac{\mu_t}{\rho} \approx l_m^2 \left|\frac{\partial \overline{u}}{\partial y}\right|$$

The Log Region

Prandtl's mixing length concept



Theodore von Kármán 1881-1963

So, how do we estimate the mixing length l_m

$$l_m(y) = a_o + a_1 y + a_2 y^2 + \ldots$$

1. $y \to 0 \Rightarrow l_m \to 0 \Rightarrow a_o = 0$
2. small values of y (*we are still very close to the wall*) $\Rightarrow l_m = a_1 y$

$$l_m = \kappa y$$

where κ is Kármán's constant $\kappa \approx 0.41$

The Log Region

$$\mu_t \approx \rho l_m^2 \left| \frac{\partial \overline{u}}{\partial y} \right| = \rho \kappa^2 y^2 \left| \frac{\partial \overline{u}}{\partial y} \right|$$

$$\tau_w = \mu_t \frac{\partial \overline{u}}{\partial y} = \rho \kappa^2 y^2 \left(\frac{\partial \overline{u}}{\partial y} \right)^2 = \rho u^{*2}$$

$$\kappa^2 y^2 \left(\frac{\partial \overline{u}}{\partial y} \right)^2 = u^{*2} \Rightarrow$$

$$\frac{\partial \overline{u}}{\partial y} = \frac{u^*}{\kappa y}$$

The Log Region

$$\frac{\partial \overline{u}}{\partial y} = \frac{u^*}{\kappa y} \Rightarrow$$

$$\overline{u}(y) = \frac{u^*}{\kappa} \ln(y) + C$$

or in non-dimensional form

$$\underbrace{\frac{\overline{u}(y)}{u^*}}_{u^+} = \frac{1}{\kappa} \ln \underbrace{\left(\frac{y u^*}{\nu} \right)}_{y^+} + \underbrace{\frac{C}{u^*} - \ln \left(\frac{u^*}{\nu} \right)}_{B}$$

The Log Region

$$\boxed{u^+ = \frac{1}{\kappa}\ln\left(y^+\right) + B}$$

valid for $30 \lesssim y^+ \lesssim 1000$

From experiments we have:

$\kappa \approx 0.41$ and $4.9 < B < 5.5$

flow over a flat plate (external flow): $B \approx 4.9$
duct flow (internal flow): $B \approx 5.3$
White: $B \approx 5.0$

Outer region

In the outer region it has been found that

$$\frac{U - \overline{u}}{u^*} = f\left(\frac{y}{\delta}\right)$$

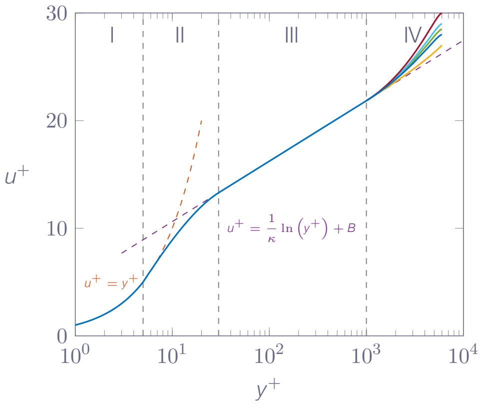
where δ is the thickness of the outer layer and U the velocity at the edge of the outer layer

Regions in a Turbulent Boundary Layer

between the viscous sublayer and the log region, none of the models works

in the outer region, inertial forces needs to be included

$$\rho \left(\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} \right) \neq 0$$



I: viscous sublayer
II: buffer layer
III: log-law region
IV: outer layer

Example - Pipe Flow Boundary Layer

Given data:

Air at 20°C flows through a 14-cm-diameter pipe. The flow is fully developed and the centerline velocity is 5.0 m/s

$$\text{Air @ 20°C} \Rightarrow \rho = 1.2\, kg/m^3,\ \mu = 1.8 \times 10^{-5}\, kg/(ms)$$

$$D = 0.14\, m$$

$$U_{max} = 5.0\, m/s$$

Assumptions:

steady-state, fully-developed, turbulent, incompressible pipe flow

Task:

From the provided data, estimate the friction velocity (u^*) and the wall-shear stress (τ_w)

Example - Pipe Flow Boundary Layer

Assume turbulent flow:

$$V_{av} = \frac{2U_{max}}{(1+m)(2+m)}$$

$m = 1/7$ gives $V_{av} = 4.08\ m/s$

$$Re_D = \frac{\rho V_{av} D}{\mu} \approx 38000 \gg Re_{D_{critical}} = 2300$$

The flow is turbulent

Example - Pipe Flow Boundary Layer

Assume that the log-law is valid all the way to the center of the pipe

$$u^+ = \frac{1}{\kappa}\ln(y^+) + B \Leftrightarrow 0 = \frac{1}{\kappa}\ln(y^+) + B - u^+$$

or (at the center of the pipe where $y = R$ and $u = U_{max}$)

$$0 = \frac{1}{\kappa}\ln\left(\frac{Ru^*}{\nu}\right) + B - \frac{U_{max}}{u^*}$$

where $\kappa = 0.41$ and $B = 5.0$

$$u^* = \sqrt{\frac{\tau_w}{\rho}}$$

Example - Pipe Flow Boundary Layer

Find estimates of u^* and τ_w using a Newton-Raphson solver

Using the definitions of y^+, u^+, and u^*, we can get a function $f(\tau_w)$

$$f(\tau_w) = \frac{1}{\kappa}\ln\left(\frac{R\sqrt{\tau_w}}{\sqrt{\rho}\nu}\right) + B - \frac{U_{max}\sqrt{\rho}}{\sqrt{\tau_w}}$$

The derivative of $f(\tau_w)$ is obtained as (*details on next slide*)

$$f'(\tau_w) = \frac{(1/\kappa)\sqrt{\tau_w} + U_{max}\sqrt{\rho}}{2\tau_w^{3/2}} = \frac{(1/\kappa) + u^+}{2\tau_w}$$

Example - Pipe Flow Boundary Layer

$$f(\tau_w) = \frac{1}{\kappa}\ln\left(\frac{R\sqrt{\tau_w}}{\sqrt{\rho}\nu}\right) + B - \frac{U_{max}\sqrt{\rho}}{\sqrt{\tau_w}}$$

$$f'(\tau_w) = \frac{\partial}{\partial\tau_w}\left(\frac{1}{\kappa}\ln\left(\frac{R\sqrt{\tau_w}}{\sqrt{\rho}\nu}\right)\right) - \frac{\partial}{\partial\tau_w}\left(\frac{U_{max}\sqrt{\rho}}{\sqrt{\tau_w}}\right) =$$

$$= \frac{\partial}{\partial\tau_w}\left(\frac{1}{\kappa}\left[\ln\left(\frac{R}{\sqrt{\rho}\nu}\right) + \ln\left(\sqrt{\tau_w}\right)\right]\right) - \left(-\frac{1}{2}\right)\frac{U_{max}\sqrt{\rho}}{\tau_w^{3/2}} =$$

$$= \frac{\partial}{\partial\tau_w}\left(\frac{1}{\kappa}\left[\ln\left(\frac{R}{\sqrt{\rho}\nu}\right) + \frac{1}{2}\ln\left(\tau_w\right)\right]\right) + \frac{U_{max}\sqrt{\rho}}{2\tau_w^{3/2}} =$$

$$= \left(\frac{1}{\kappa}\right)\frac{1}{2\tau_w} + \frac{U_{max}\sqrt{\rho}}{2\tau_w^{3/2}} = \frac{(1/\kappa)\sqrt{\tau_w} + U_{max}\sqrt{\rho}}{2\tau_w^{3/2}} = \frac{(1/\kappa) + u^+}{2\tau_w}$$

Example - Pipe Flow Boundary Layer

With the functions $f(\tau_w)$ and $f'(\tau_w)$ defined, we can set up an iterative Newton-Raphson solver to find τ_w using

$$\tau_{w_{n+1}} = \tau_{w_n} - \frac{f(\tau_{w_n})}{f'(\tau_{w_n})}$$

where $n + 1$ and n are iteration numbers. Iterate until converged with the following convergence criterium:

$$\left| \frac{f(\tau_{w_n})}{f'(\tau_{w_n})} \right| \leq \tau_w \times 10^{-4}$$

Example - Pipe Flow Boundary Layer

```python
import numpy as np

def calc_yplus_uplus(rho,mu,tau_w,y,U):
    nu=mu/rho
    ustar=np.sqrt(tau_w/rho)
    yplus=y*ustar/nu
    uplus=U/ustar
    return yplus,uplus,ustar

mu    =  1.8e-5 # fluid viscosity (dynamic viscosity)
rho   =  1.2    # fluid density
u_max =  5.0    # centerline velocity
R     =  0.07   # pipe radius
kappa =  0.41   # von Kármán constant
B     =  5.0    # integration constant in the log-law
```

Example - Pipe Flow Boundary Layer

```
tau_w = mu*u_max/R # initial guess

yplus,uplus,ustar=calc_yplus_uplus(rho,mu,tau_w,R,u_max)

dtau_w = 10.*tau_w

while( abs(dtau_w) > 0.0001*tau_w ):
  f       = (1./kappa)*np.log(yplus)-uplus+B
  df      = 0.5*((1./kappa)+uplus)/tau_w
  dtau_w = -f/df
  tau_w   = tau_w+dtau_w
  yplus,uplus,ustar=calc_yplus_uplus(rho,mu,tau_w,R,u_max)
```

Example - Pipe Flow Boundary Layer

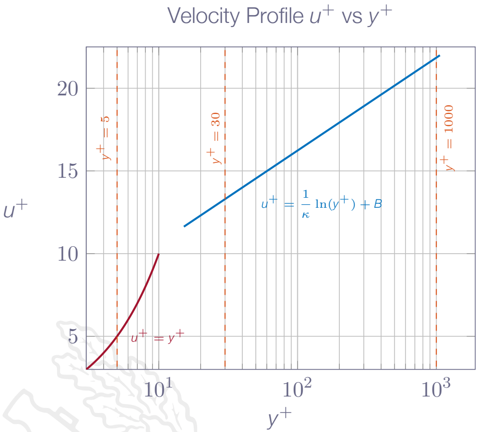
iteration	τ_w	f/f'
1	0.003531	2.244938e-03
2	0.009029	5.498838e-03
3	0.020451	1.142164e-02
4	0.038183	1.773146e-02
5	0.054798	1.661537e-02
6	0.061401	6.602591e-03
7	0.062021	6.204740e-04
8	0.062026	4.575602e-06

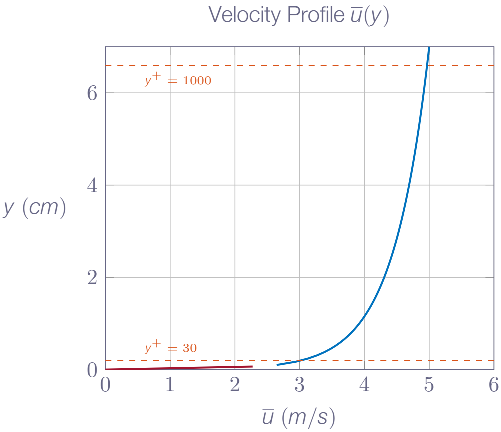
Example - Pipe Flow Boundary Layer

variable	dimension	value
y^+ (pipe center)		1061
u^*	m/s	0.227
τ_w	N/m^2	0.062

Note! $y^+ = 1061$ is actually outside the range of y^+ values for which the log-law is valid - but it is very close to the limit...

Example - Pipe Flow Boundary Layer





Note! The upper limit of the **viscous sublayer** $(y^+ = 5)$ corresponds to a distance from the wall of $y = 0.3$ *mm* or 0.2% of the pipe diameter and the lower bound for the **log region** $(y^+ = 30)$ corresponds to a wall distance of $y = 2.0$ *mm* or 1.4% of the pipe diameter.