

Fluid Mechanics - MTF053

Lecture 11

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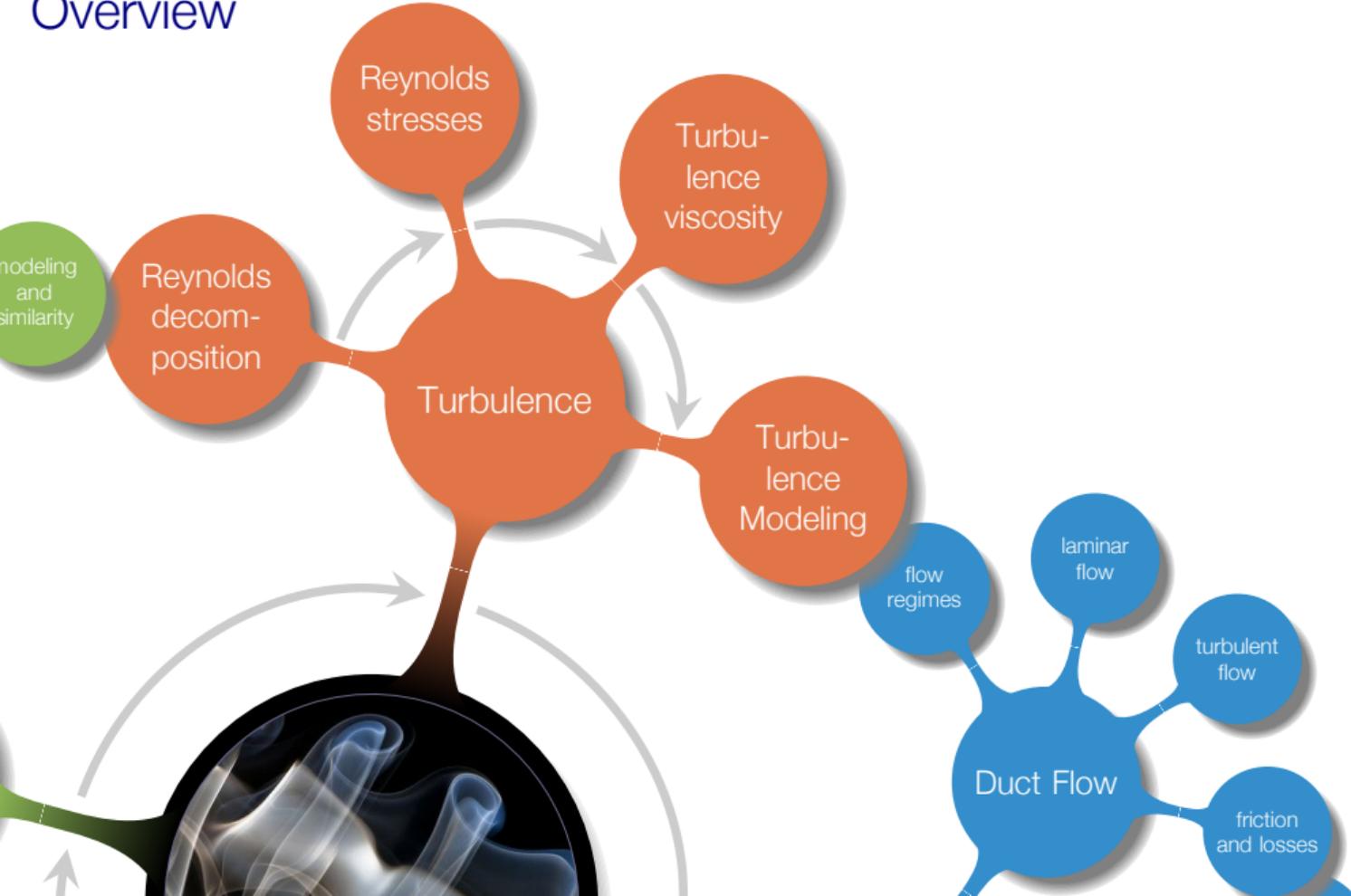
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Chapter 6 - Viscous Flow in Ducts

Overview



Learning Outcomes

- 3 **Define** the Reynolds number
- 4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category
- 6 **Explain** what a boundary layer is and when/where/why it appears
- 8 **Understand** and be able to **explain** the concept shear stress
- 18 **Explain** losses appearing in pipe flows
- 19 **Explain** the difference between laminar and turbulent pipe flow
- 20 **Solve** pipe flow problems using Moody charts
- 24 **Explain** what is characteristic for a turbulent flow
- 25 **Explain** Reynolds decomposition and derive the RANS equations
- 26 **Understand** and **explain** the Boussinesq assumption and turbulent viscosity
- 27 **Explain** the difference between the regions in a boundary layer and what is characteristic for each of the regions (viscous sub layer, buffer region, log region)

if you think about it, pipe flows are everywhere (a pipe flow is not a flow of pipes)

Complementary Course Material

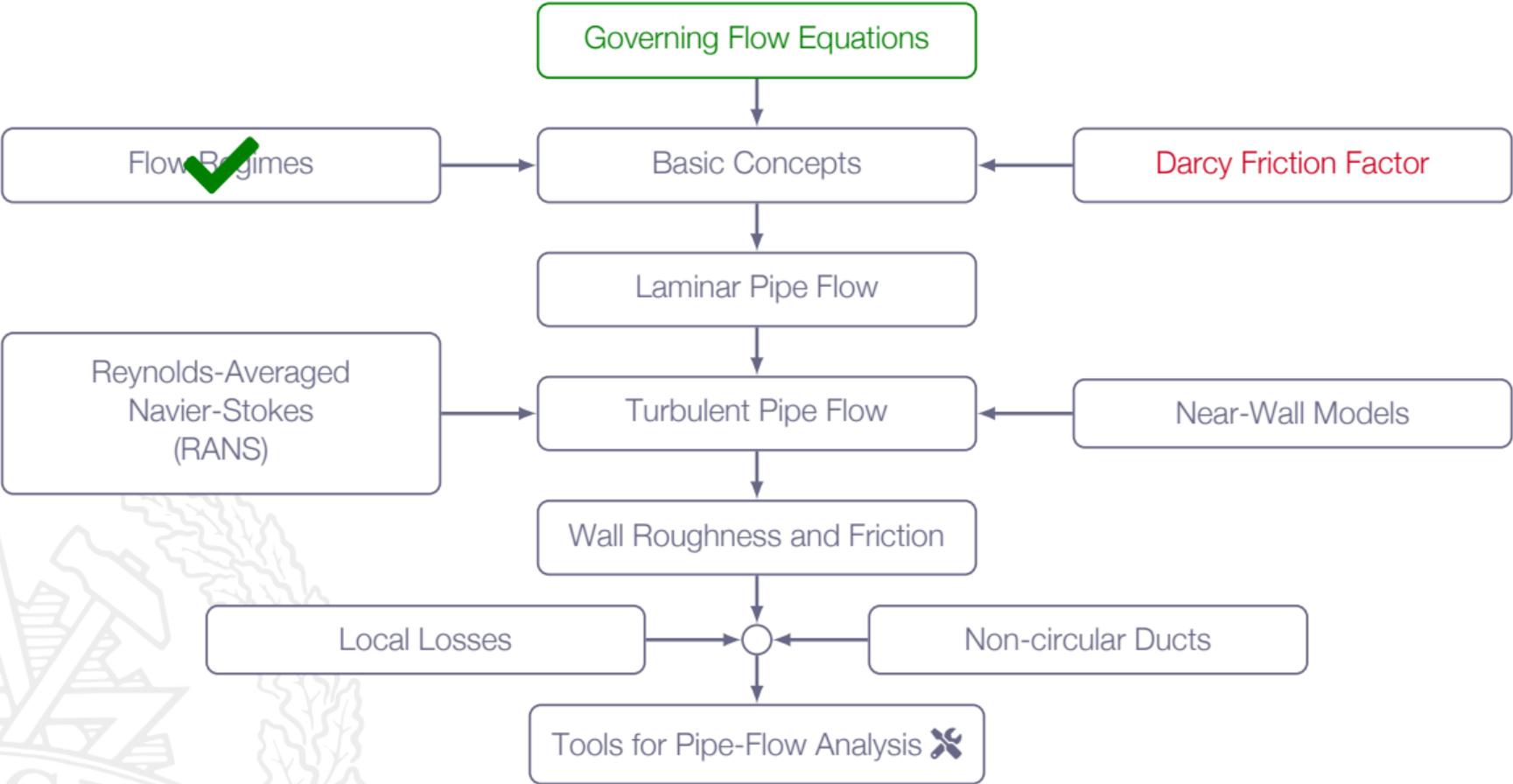
These lecture notes covers chapter 6 in the course book and additional course material that you can find in the following documents

MTF053_Equation-for-Boundary-Layer-Flows.pdf

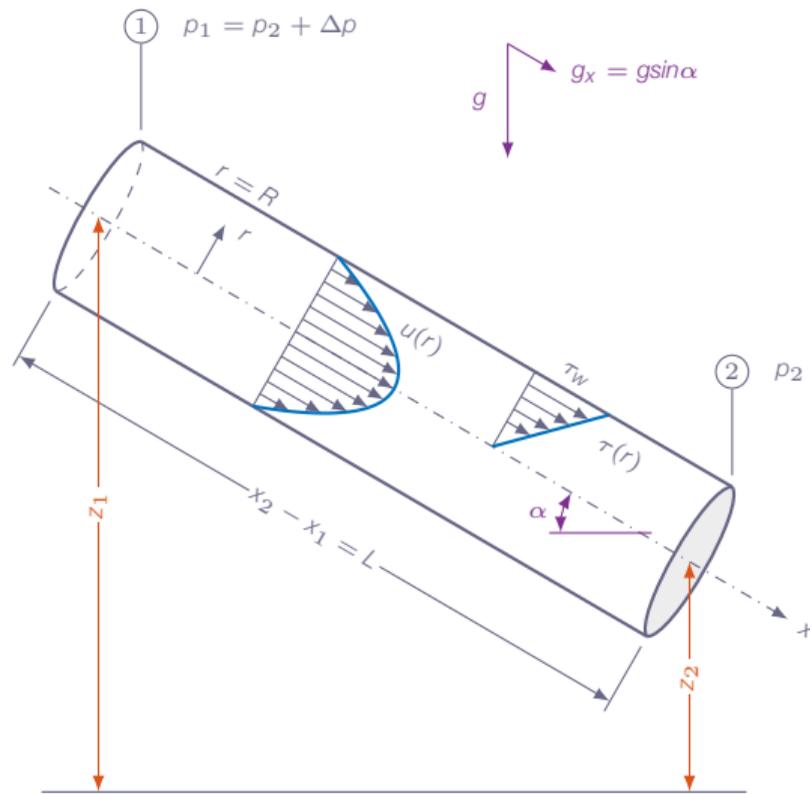
MTF053_Turbulence.pdf



Roadmap - Viscous Flow in Ducts



Head Loss



Head Loss

$$Q_1 = Q_2 = Q, \quad V_1 = V_2 = V$$

Steady-flow energy equation:

$$\left(\frac{p}{\rho g} + \alpha \frac{V^2}{2g} + z \right)_1 = \left(\frac{p}{\rho g} + \alpha \frac{V^2}{2g} + z \right)_2 + h_f$$

- ▶ No pumps or turbines between 1 and 2
- ▶ Fully developed flow ($\alpha_1 = \alpha_2$)

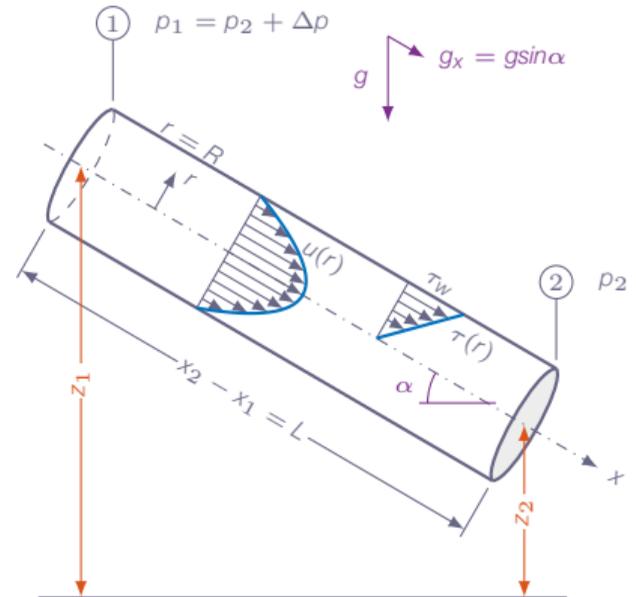
$$h_f = (z_1 - z_2) + \left(\frac{p_1 - p_2}{\rho g} \right) = \Delta z + \frac{\Delta p}{\rho g}$$

Head Loss

Apply the momentum equation along the pipe:

$$\sum F_x = \Delta p(\pi R^2) + \rho g(\pi R^2)L \sin \alpha - \tau_w(2\pi R)L$$

$$\sum F_x = \dot{m}(V_2 - V_1) = 0$$



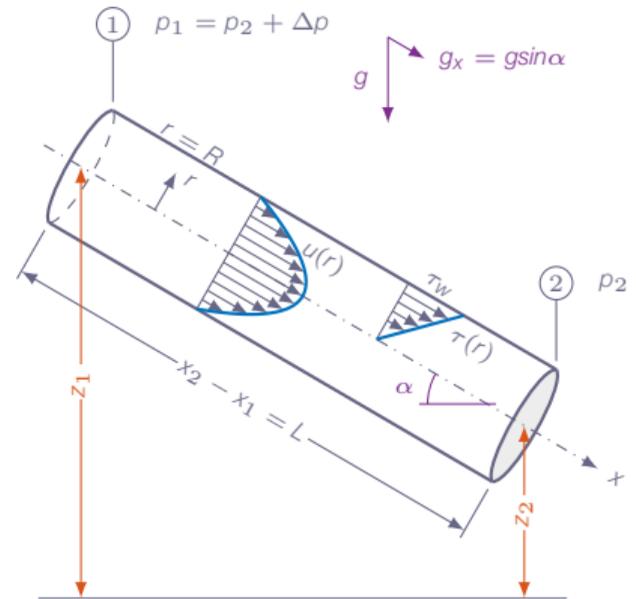
Head Loss

$$\Delta p(\pi R^2) + \rho g(\pi R^2)L \sin \alpha = \tau_w(2\pi R)L$$

$$\frac{\Delta p}{\rho g} + L \sin \alpha = \frac{2\tau_w L}{\rho g R}$$

$$\frac{\Delta p}{\rho g} + \Delta z = \frac{4\tau_w L}{\rho g d}$$

$$h_f = \frac{4\tau_w L}{\rho g d}$$



Friction Factor

$$h_f = f_D \frac{L V^2}{d 2g}$$

where

$$f_D = f(\text{Re}_d, \varepsilon/d, \text{duct shape})$$

is the **Darcy friction factor**

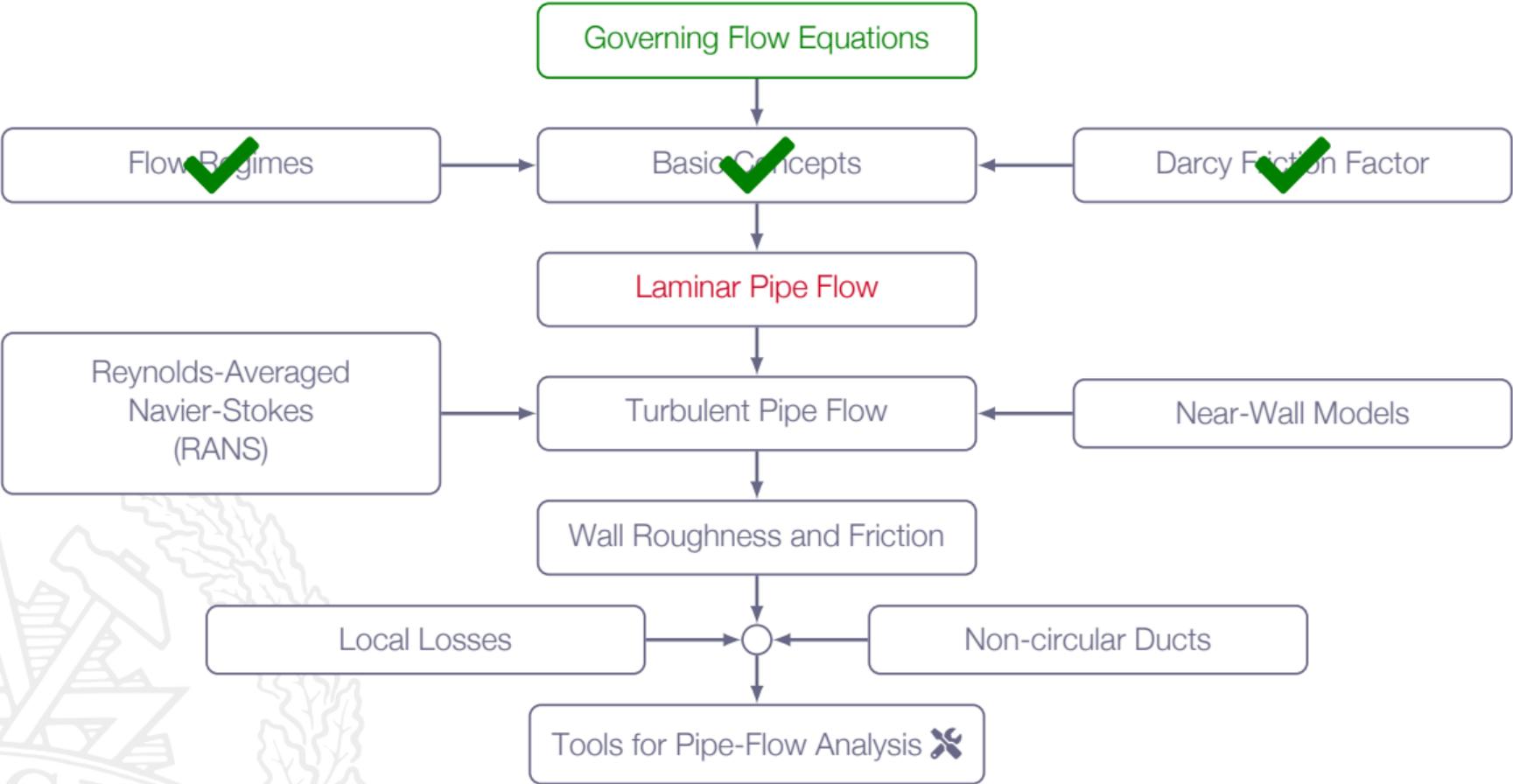
$$\frac{4\tau_w L}{\rho g d} = f_D \frac{L V^2}{d 2g} \Rightarrow f_D = \frac{8\tau_w}{\rho V^2}$$

Note! for non-circular pipes, τ_w is an average value around the duct perimeter



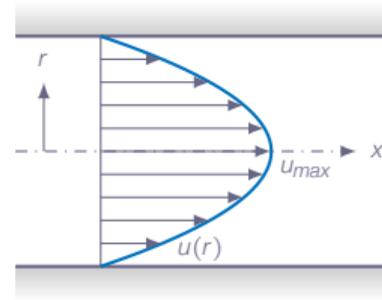
Henry Darcy 1803-1858

Roadmap - Viscous Flow in Ducts



Fully-Developed Laminar Pipe Flow

- ▶ Fully developed
- ▶ circular pipe with the diameter D and radius R
- ▶ Pressure driven (Poiseuille flow)



$$u(r) = u_{max} \left(1 - \left(\frac{r}{R} \right)^2 \right) \Rightarrow \frac{du}{dr} = -2u_{max} \frac{r}{R^2} = \left\{ V = \frac{u_{max}}{2} \right\} = -4V \frac{r}{R^2}$$

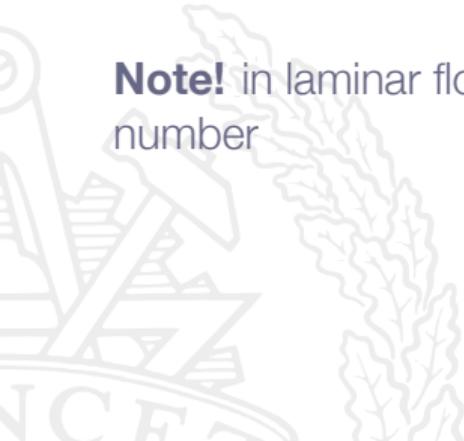
$$\tau_w = \mu \left. \frac{du}{dr} \right|_{r=R} = \frac{4\mu V}{R} = \frac{8\mu V}{D}$$

Fully-Developed Laminar Pipe Flow

For laminar flow:

$$f_D = \frac{8\tau_w}{\rho V^2} = \left\{ \tau_w = \frac{8\mu V}{D} \right\} = \frac{64\mu}{\rho V D} = \frac{64}{Re_D}$$

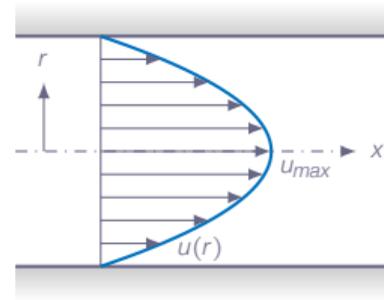
Note! in laminar flow, the friction factor is inversely proportional to the Reynolds number



Fully-Developed Laminar Pipe Flow



- ▶ Fully developed
- ▶ circular pipe with the diameter D and radius R
- ▶ Pressure driven (Poiseuille flow)



$$u(r) = u_{max} \left(1 - \left(\frac{r}{R} \right)^2 \right) \text{ where } u_{max} = -\frac{dp}{dx} \frac{R^2}{4\mu}$$





$$-\frac{dp}{dx} = \left(\frac{\Delta p + \rho g \Delta z}{L} \right) \Rightarrow u_{max} = \left(\frac{\Delta p + \rho g \Delta z}{L} \right) \frac{R^2}{4\mu}$$

$$V = \frac{u_{max}}{2} = \left(\frac{\Delta p + \rho g \Delta z}{L} \right) \frac{R^2}{8\mu}$$

$$Q = \int u dA = VA = V \frac{\pi D^2}{4} = \left(\frac{\Delta p + \rho g \Delta z}{L} \right) \frac{\pi D^4}{128\mu}$$



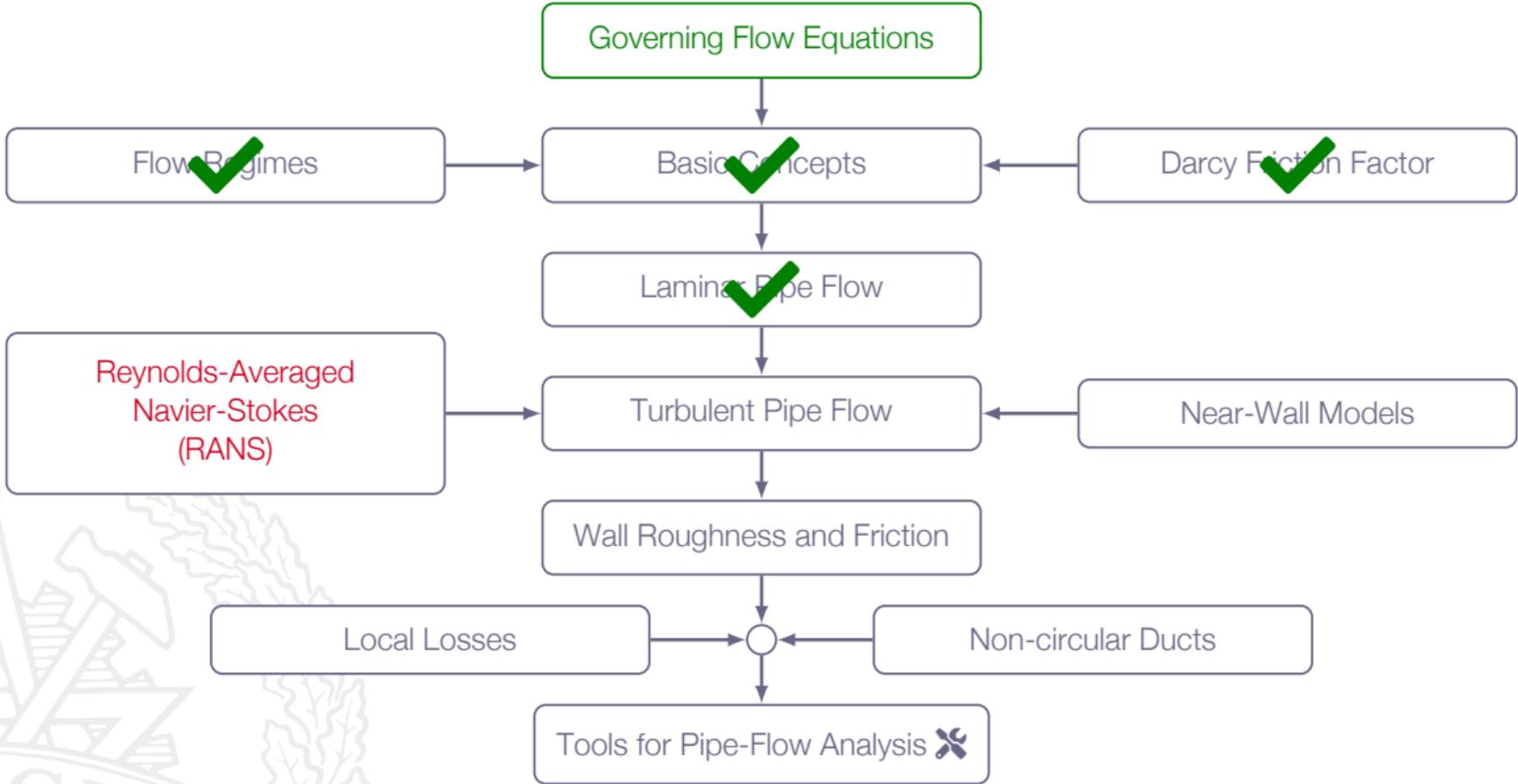


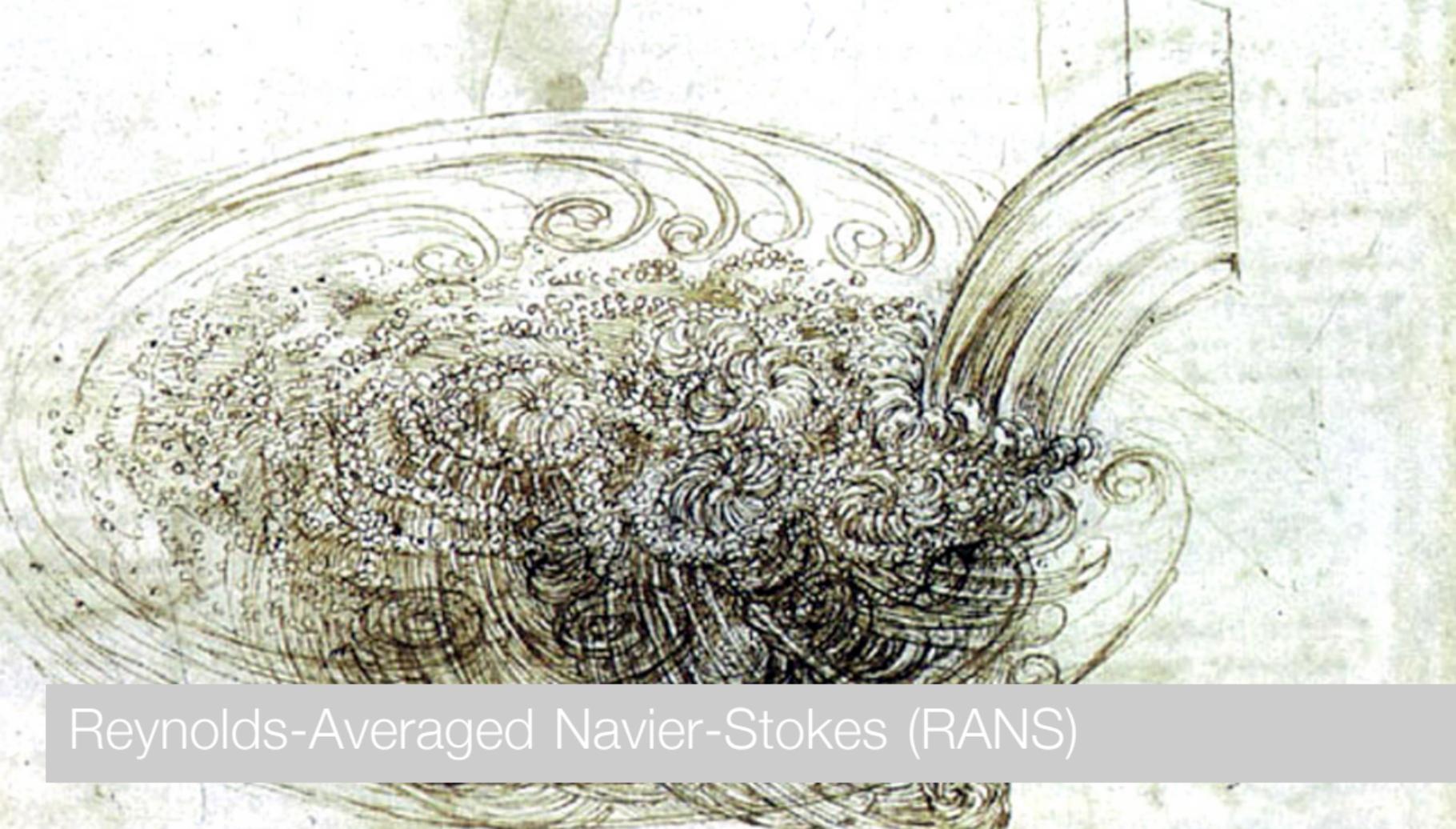
We can now calculate the head loss according to

$$h_f = f_D \frac{L}{D} \frac{V^2}{2g} \text{ where } f_D = \frac{8\tau_w}{\rho V^2}$$

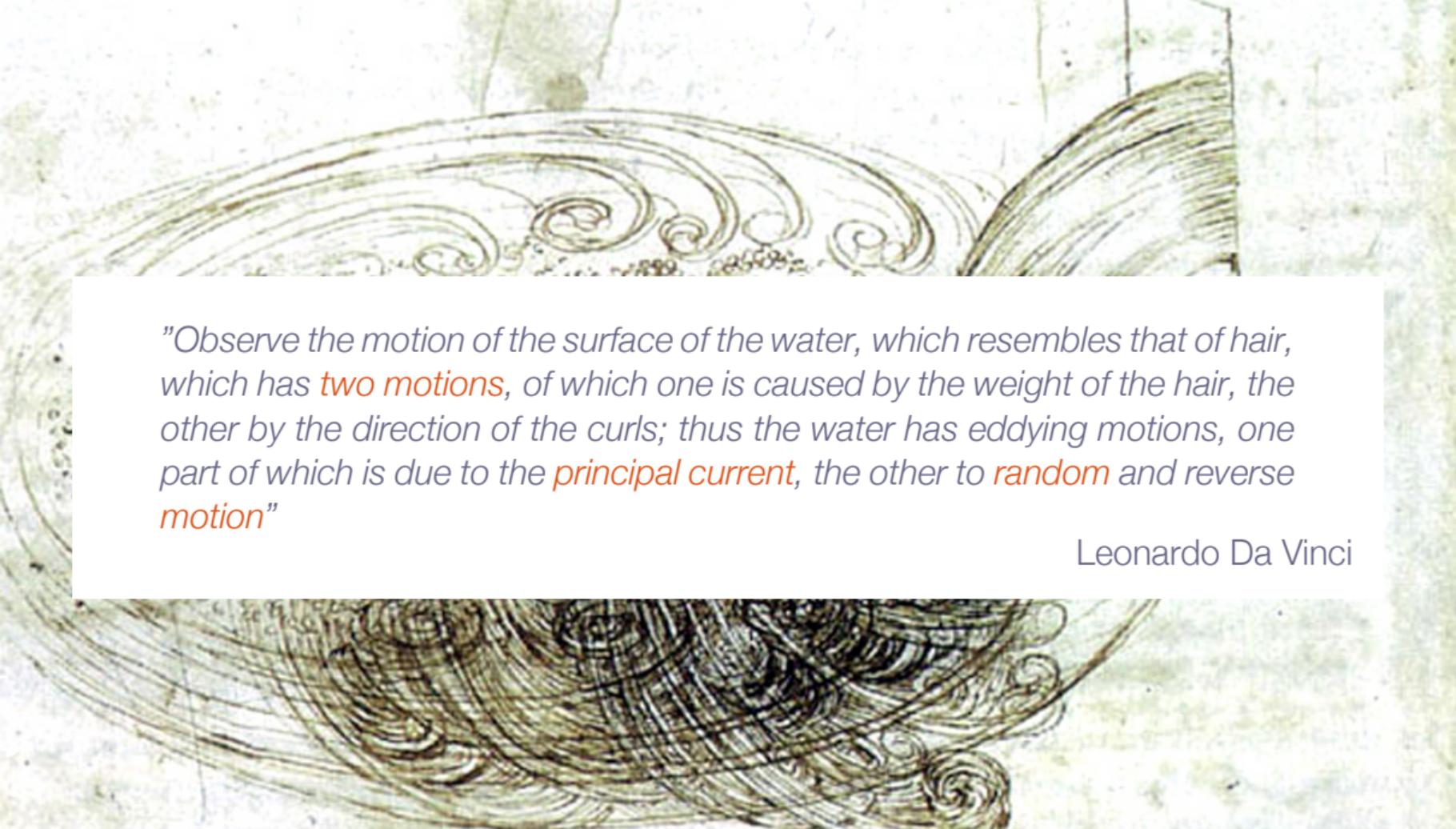
$$h_f = \frac{4\tau_w L}{\rho g D} = \left\{ \tau_w = \frac{8\mu V}{D} \right\} = \frac{16\mu VL}{\rho g D R} = \frac{32\mu VL}{\rho g D^2} = \left\{ V = \frac{4Q}{\pi D^2} \right\} = \frac{128\mu QL}{\pi \rho g D^4}$$

Roadmap - Viscous Flow in Ducts





Reynolds-Averaged Navier-Stokes (RANS)

A detailed pencil sketch by Leonardo da Vinci showing the motion of water. The drawing features several large, overlapping circular eddies with smaller, tighter curls inside them. The lines are drawn with varying thickness and direction, creating a sense of fluid, swirling movement. The background is a light, textured paper.

*"Observe the motion of the surface of the water, which resembles that of hair, which has **two motions**, of which one is caused by the weight of the hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the **principal current**, the other to **random** and reverse **motion**"*

Leonardo Da Vinci

Governing Equations

Assumptions:

1. constant density and viscosity
2. no thermal interaction

Flow equations:

continuity: $\nabla \cdot \mathbf{V} = 0$

momentum: $\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V}$

Governing Equations

The differential energy equation is not included here but let's have a look at it anyway

$$\rho \frac{D\hat{u}}{Dt} + p \nabla \cdot \mathbf{V} = \nabla \cdot (k \nabla T) + \phi$$

Pressure work:

pressure drives the flow through the duct

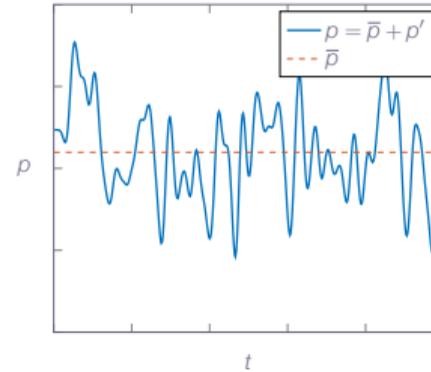
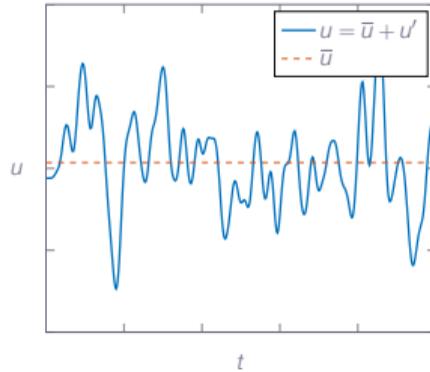
Viscous work:

no-slip condition \Rightarrow zero velocity at the walls \Rightarrow no work done by wall shear stress

So, where does the energy go?

pressure work is balanced by **viscous dissipation** in the interior of the flow

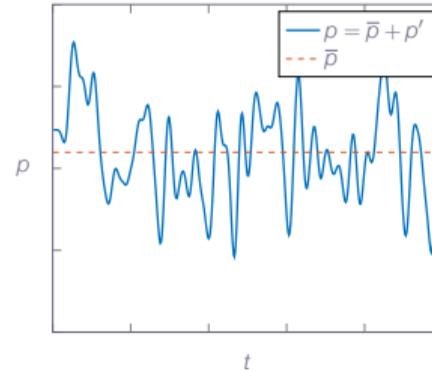
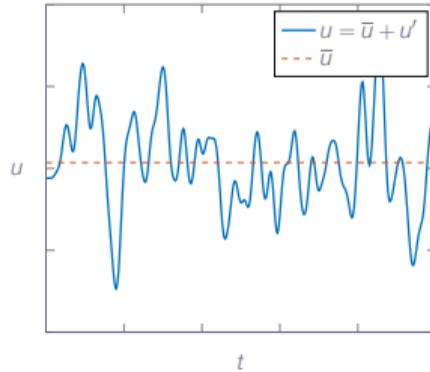
Reynolds' Decomposition



Not possible to solve analytically

Often, the time-averaged quantities are what we are looking for

Reynolds' Decomposition



$$\bar{u} = \frac{1}{T} \int_0^T u dt$$

$$u' = u - \bar{u}$$

$$\bar{u'} = \frac{1}{T} \int_0^T (u - \bar{u}) dt = \bar{u} - \bar{u} = 0$$

Reynolds' Decomposition

The mean square of the fluctuations are, however, not zero

$$\overline{u'^2} = \frac{1}{T} \int_0^T u'^2 dt \neq 0$$

measure of *turbulence intensity*

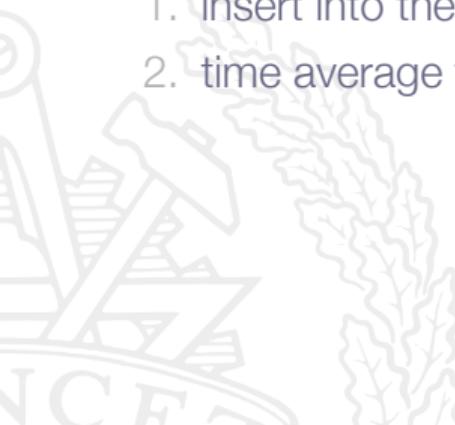
Mean of fluctuation products are generally not zero ($\overline{u'v'}$, $\overline{u'p'}$)

Reynolds' Decomposition

Reynolds' idea was to split all properties into mean and fluctuating parts:

$$u = \bar{u} + u', \quad v = \bar{v} + v', \quad w = \bar{w} + w', \quad p = \bar{p} + p'$$

1. insert into the governing equations
2. time average the equations



Reynolds' Decomposition

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Momentum (x-component):

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Reynolds' Decomposition

Continuity:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

time averaging the equation gives

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0$$

and as a consequence

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

Reynolds' Decomposition

Momentum (x-component):

$$\begin{aligned} & \rho \left(\frac{\partial \bar{u}}{\partial t} + \frac{\partial u'}{\partial t} \right) + \\ & \rho \left(\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{u} \frac{\partial u'}{\partial x} + u' \frac{\partial \bar{u}}{\partial x} + u' \frac{\partial u'}{\partial x} \right) + \\ & \rho \left(\bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{v} \frac{\partial u'}{\partial y} + v' \frac{\partial \bar{u}}{\partial y} + v' \frac{\partial u'}{\partial y} \right) + \\ & \rho \left(\bar{w} \frac{\partial \bar{u}}{\partial z} + \bar{w} \frac{\partial u'}{\partial z} + w' \frac{\partial \bar{u}}{\partial z} + w' \frac{\partial u'}{\partial z} \right) = \\ & - \frac{\partial \bar{p}}{\partial x} - \frac{\partial p'}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2} + \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2} \right) \end{aligned}$$

Reynolds' Decomposition

Momentum (x-component):

time averaging the equation gives:

$$\rho \left(\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{w} \frac{\partial \bar{u}}{\partial z} + \overline{u' \frac{\partial u'}{\partial x}} + \overline{v' \frac{\partial u'}{\partial y}} + \overline{w' \frac{\partial u'}{\partial z}} \right) =$$
$$-\frac{\partial \bar{p}}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2} \right)$$

The highlighted terms can be rewritten as:

$$\overline{u' \frac{\partial u'}{\partial x}} + \overline{v' \frac{\partial u'}{\partial y}} + \overline{w' \frac{\partial u'}{\partial z}} = \frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z} - \underbrace{\overline{u' \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right)}}_{=0}$$

Reynolds' Decomposition

the continuity equation reduces to

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0$$

the axial component of the momentum equation:

$$\rho \frac{D\bar{u}}{Dt} = -\frac{\partial \bar{p}}{\partial x} + \rho g_x + \frac{\partial}{\partial x} \left(\mu \frac{\partial \bar{u}}{\partial x} - \overline{\rho u'^2} \right) +$$
$$\frac{\partial}{\partial y} \left(\mu \frac{\partial \bar{u}}{\partial y} - \overline{\rho u'v'} \right) + \frac{\partial}{\partial z} \left(\mu \frac{\partial \bar{u}}{\partial z} - \overline{\rho u'w'} \right)$$

Reynolds' Decomposition

By applying Reynolds' decomposition to our governing equations, we have introduced a number of new unknowns

The number of equations is the same as before, which means problems

Our new problem has a name

The closure problem



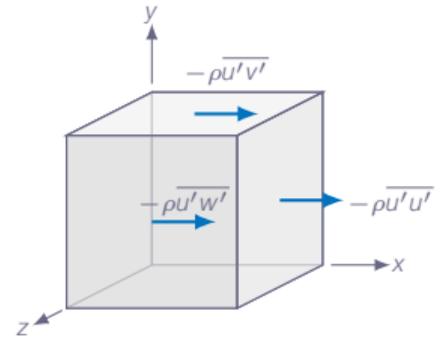
Reynolds' Decomposition

The three correlation terms $-\overline{\rho u'^2}$, $-\overline{\rho u'v'}$, and $-\overline{\rho u'w'}$ are called **Reynolds stresses** or turbulent stresses

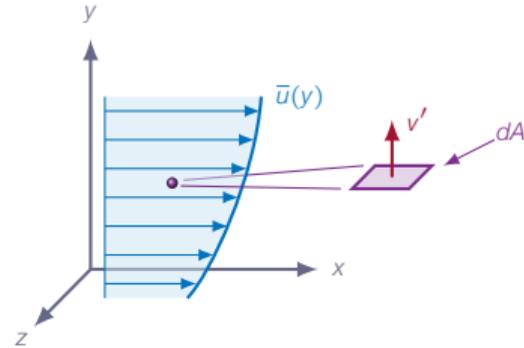
In duct and boundary layer flow, the stress $-\overline{\rho u'v'}$, associated with the direction normal to the wall, is dominant

$$\rho \frac{D\bar{u}}{Dt} \approx -\frac{\partial \bar{p}}{\partial x} + \rho g_x + \frac{\partial \tau}{\partial y}$$

$$\tau = \mu \frac{\partial \bar{u}}{\partial y} - \overline{\rho u'v'} = \tau_{lam} + \tau_{turb}$$



Reynolds' Decomposition



mass flow through surface element: $\dot{m}_y = \rho v' dA$

momentum balance in x-direction: $F_x = \dot{m}_y u = \rho v' (\bar{u} + u') dA$

$$\tau_{dA} = -\frac{\bar{F}_x}{dA} = -\overline{\rho v' (\bar{u} + u')} = -\overline{\rho v' \bar{u}} - \overline{\rho u' v'} = \{ \overline{v' \bar{u}} = \bar{v}' \bar{u} = 0 \} = -\overline{\rho u' v'}$$

$\Rightarrow -\overline{\rho u' v'}$ can be interpreted as a shear stress

Reynolds' Decomposition

Introducing **turbulent viscosity** μ_t defined such that

$$-\rho \overline{u'v'} = \mu_t \frac{\partial \bar{u}}{\partial y}$$

Boussinesq's assumption

With the turbulent viscosity, the total shear stress τ becomes:

$$\tau = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'} = (\mu + \mu_t) \frac{\partial \bar{u}}{\partial y}$$

Reynolds' Decomposition

- ▶ laminar shear (τ_{lam}) dominates in the near-wall region
- ▶ turbulent shear (τ_{turb}) dominates in the outer region
- ▶ both are important in the overlap layer

