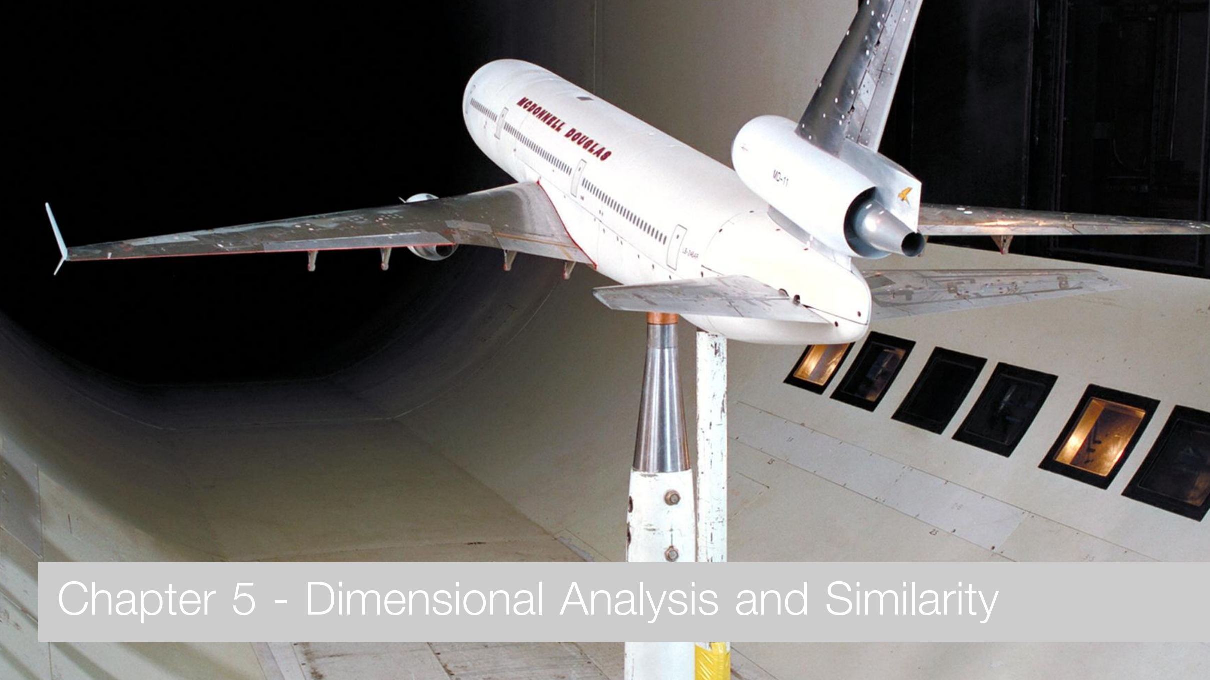
Fluid Mechanics - MTF053

Lecture 9

Niklas Andersson

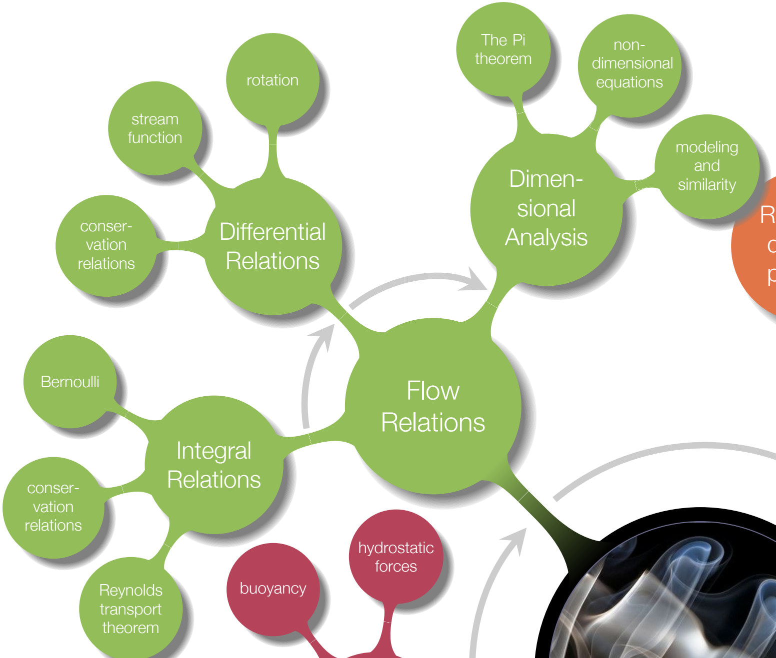
Chalmers University of Technology
Department of Mechanics and Maritime Sciences
Division of Fluid Mechanics
Gothenburg, Sweden

niklas.andersson@chalmers.se



Chapter 5 - Dimensional Analysis and Similarity

Overview


rotation
stream function
conser-vation relations
Differential Relations
The Pi theorem
non-dimensional equations
modeling and similarity
Dimen-sional Analysis
Flow Relations
Bernoulli
conser-vation relations
Integral Relations
Reynolds transport theorem
hydrostatic forces
buoyancy

Learning Outcomes

3 **Define** the Reynolds number

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

17 **Explain** about how to use non-dimensional numbers and the Π theorem

we will learn about how to plan experiments and compare experimental data using dimensionless numbers

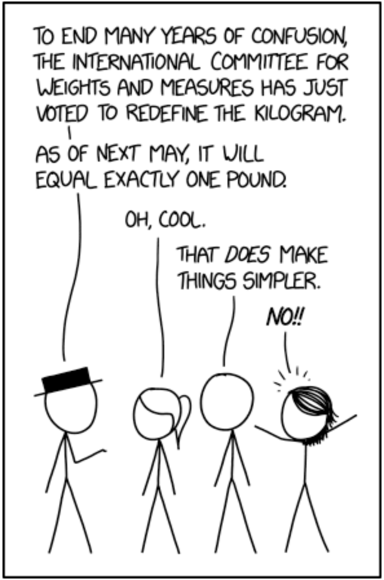
Motivation

"Most practical fluid flow problems are too complex, both geometrically and physically, to be solved analytically. They must be tested by experiments or approximated by CFD"

Dimensional analysis:

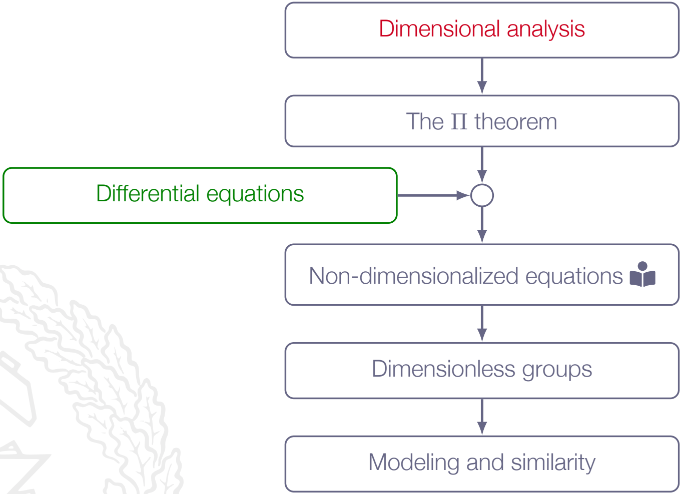
- Large data sets may be represented by a **few curves** or even a single curve
- A systematic tool for **data reduction**
- Experimental/simulation data are **more general** in **dimensionless** form

Dimensions



TO END MANY YEARS OF CONFUSION, THE INTERNATIONAL COMMITTEE FOR WEIGHTS AND MEASURES HAS JUST VOTED TO REDEFINE THE KILOGRAM.
AS OF NEXT MAY, IT WILL EQUAL EXACTLY ONE POUND.
OH, COOL.
THAT *DOES* MAKE THINGS SIMPLER.
NO!!

Roadmap - Dimensional Analysis and Similarity



Dimensional Analysis - What is it?

Dimensional analysis is a tool for systematic

1. **planning** of experiments
   similarity between model and prototype
2. **presentation** of experimental data
   insight into physical relationships
3. **interpretation** of measurements
   identify important and unimportant parameters

Dimensional Analysis - What is it?

General description:

"If a phenomenon depends on n dimensional variables, dimensional analysis will reduce the problem to only k dimensionless variables, where the reduction $n - k$ depends on the problem complexity"

"Generally, $n - k$ equals the number of primary dimensions"

Dimensional Analysis - Example Problem

Problem definition:

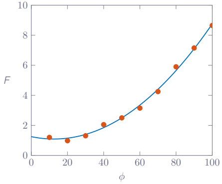
Suppose that we know that the force F on a particular body shape in a fluid flow depends on

1. The length of the body L
2. The flow freestream velocity V
3. The fluid density ρ
4. The fluid viscosity μ

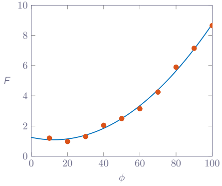
$$\Rightarrow \quad F = f(L, V, \rho, \mu)$$

Dimensional Analysis - Example Problem

Let's say that we need ten data points to define a curve



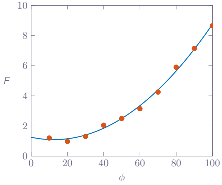
Dimensional Analysis - Example Problem



Let's say that we need ten data points to define a curve

We need to test 10 lengths and for each of those, 10 velocities, ...

Dimensional Analysis - Example Problem

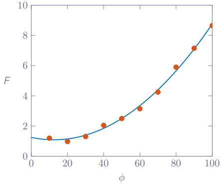


Let's say that we need ten data points to define a curve

We need to test 10 lengths and for each of those, 10 velocities, ...

For our example problem we need to do **10000 experiments!!**

Dimensional Analysis - Example Problem



Let's say that we need ten data points to define a curve

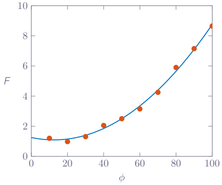
We need to test 10 lengths and for each of those, 10 velocities, ...

For our example problem we need to do **10000 experiments!!**

With dimensional analysis, the problem can be reduced as follows

$$\frac{F}{\rho V^2 L^2} = g\left(\frac{\rho V L}{\mu}\right) \text{ or } C_F = g(Re)$$ *where g is an unknown function*

Dimensional Analysis - Example Problem



Let's say that we need ten data points to define a curve

We need to test 10 lengths and for each of those, 10 velocities, ...

For our example problem we need to do **10000 experiments!!**

With dimensional analysis, the problem can be reduced as follows

$$\frac{F}{\rho V^2 L^2} = g\left(\frac{\rho V L}{\mu}\right) \text{ or } C_F = g(Re) \textit{ where g is an unknown function}$$

The number of experiments needed have been **reduced by a factor of 1000!!**

Similarity - Model and Prototype

Let's go back to the example problem from before

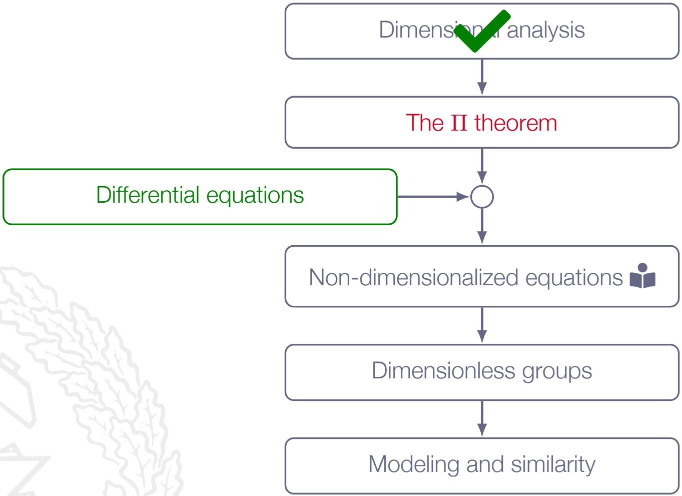
$$C_F = g(Re)$$

so if $Re_m = Re_p$ that means that $C_{F,m} = C_{F,p}$ (where m is model and p prototype)

$$C_{F,m} = \frac{F_m}{\rho_m V_m^2 L_m^2} \text{ and } C_{F,p} = \frac{F_p}{\rho_p V_p^2 L_p^2}$$

$$\frac{F_m}{\rho_m V_m^2 L_m^2} = \frac{F_p}{\rho_p V_p^2 L_p^2} \Rightarrow \frac{F_p}{F_m} = \frac{\rho_p}{\rho_m}\left(\frac{V_p}{V_m}\right)^2\left(\frac{L_p}{L_m}\right)^2$$

Roadmap - Dimensional Analysis and Similarity



The Buckingham Π-theorem

Systematic identification of non-dimensional numbers (Π-groups):

"If there is a physically meaningful equation involving a certain number n of physical variables, then the original equation can be rewritten in terms of a set of k dimensionless parameters Π_1, Π_2, ..., Π_k. The reduction, $j = n - k$, equals the number of variables that do not form a Π among themselves and is always less than or equal to the number of physical dimensions involved"

The Buckingham Π-theorem

Systematic identification of non-dimensional numbers (Π-groups):

1. List and count the **number of variables** in the problem n
2. List the **dimensions** for each of the n variables
3. Count **number of dimensions** m
4. Find the **reduction** j
   4.1 initial guess: j equals the **number of dimensions** m
   4.2 look for j variables that do not form a Π
   4.3 if not possible reduce j by one and go back to 4.2
5. Select j **scaling parameters**
6. Add one of the other variables to your j **repeating variables** and form a power product
7. Algebraically, find exponents that make the product dimensionless

The Buckingham Π-theorem - Example

$$F = f(L, U, \rho, \mu)$$

number of variables: $n = 5$

F	L	U	ρ	μ
$\{MLT^{-2}\}$	$\{L\}$	$\{LT^{-1}\}$	$\{ML^{-3}\}$	$\{ML^{-1}T^{-1}\}$

number of dimensions: m=3

reduction: $j \leq 3$

number of dimensionless groups: $k = n - j \geq 2$

The Buckingham Π-theorem - Example

1. Inspecting the variables, we see that L, U, and ρ cannot form a **Π-group**

   only ρ contains M (mass)

   only U contains T (time)

2. L, U, and ρ are selected as the j **repeating variables**

3. The **reduction** will be $j = 3$ and thus $k = n - j = 2$

4. One of the **Π-groups** will contain F and the other will contain μ

The Buckingham Π-theorem - Example

$$\Pi_1 = L^a U^b \rho^c F \Rightarrow (L)^a (LT^{-1})^b (ML^{-3})^c (MLT^{-2}) = M^0 L^0 T^0$$

$$\begin{array}{rcccccccccc} L: & a & + & b & - & 3c & + & 1 & = & 0 \\ M: & & & & & c & + & 1 & = & 0 \\ T: & & - & b & & & - & 2 & = & 0 \end{array}$$

which gives

$$a = -2,\ b = -2,\ c = -1$$

and thus

$$\Pi_1 = \frac{F}{\rho U^2 L^2} = C_F$$

The Buckingham Π-theorem - Example

$$\Pi_2 = L^a U^b \rho^c \mu^{-1} \Rightarrow (L)^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^{-1} = M^0 L^0 T^0$$

$$\begin{array}{rccccccccc} L: & a & + & b & - & 3c & + & 1 & = & 0 \\ M: & & & & & c & - & 1 & = & 0 \\ T: & & - & b & & & + & 1 & = & 0 \end{array}$$

which gives

$$a = b = c = 1$$

and thus

$$\Pi_2 = \frac{\rho U L}{\mu} = Re$$

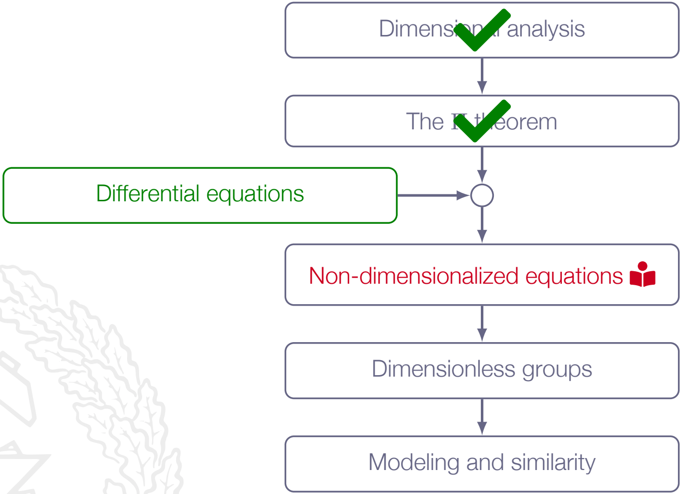
The Buckingham Π-theorem - Example

If $F = f(L, V, \rho, \mu)$, the theorem guaranties that, in this case, $\Pi_1 = g(\Pi_2)$

$$\frac{F}{\rho U^2 L^2} = g\left(\frac{\rho U L}{\mu}\right) \text{ or } C_F = g(Re)$$

where g is an unknown function

Roadmap - Dimensional Analysis and Similarity



Non-dimensionalized Equations

Why would one want to make the governing equations non-dimensional?

1. Understand flow physics
2. Gives information about under what conditions terms are negligible
3. A way to find important non-dimensional groups for a specific flow

Non-dimensionalized Equations

The incompressible flow continuity and momentum equations and corresponding boundary conditions:

Continuity: $\nabla \cdot \mathbf{V} = 0$

Navier-Stokes: $\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{V}$

Solid surface: no-slip ($\mathbf{V} = 0$ if fixed surface)

Inlet/outlet: known velocity and pressure

Non-dimensionalized Equations

The variables in the continuity and momentum equations contain **three primary dimensions**; M, L, and T

All variables included $(\rho, \mathbf{V}, p, x, y, z)$ can be made non-dimensional using three constants:

1. density: ρ
2. reference velocity: U
3. reference length: L

reference properties are constants characteristic for a specific flow

Non-dimensionalized Equations

non-dimensional variables are denoted by an asterisk:

$$\mathbf{V}^* = \frac{\mathbf{V}}{U}$$

$$t^* = \frac{tU}{L}$$

$$\nabla^* = L\nabla$$

$$p^* = \frac{p - \rho\mathbf{gr}}{\rho U^2}$$

$$(x^*, y^*, z^*) = \frac{1}{L}(x,\ y,\ z)$$

Non-dimensionalized Equations - Continuity

$$\nabla \cdot \mathbf{V} = 0$$

$$\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

introducing non-dimensional variables

$$\nabla \cdot \mathbf{V} = \frac{\partial (Uu^*)}{\partial (Lx^*)} + \frac{\partial (Uv^*)}{\partial (Ly^*)} + \frac{\partial (Uw^*)}{\partial (Lz^*)} = \frac{U}{L}\left[\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*}\right] = \frac{U}{L}\nabla^* \cdot \mathbf{V}^* \Rightarrow$$

$$\nabla^* \cdot \mathbf{V}^* = 0$$

Non-dimensionalized Equations - Momentum

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{V}$$

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial\mathbf{V}}{\partial t} + u\frac{\partial\mathbf{V}}{\partial x} + v\frac{\partial\mathbf{V}}{\partial y} + w\frac{\partial\mathbf{V}}{\partial z}$$

introducing non-dimensional variables

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial(U\mathbf{V}^*)}{\partial(t^*L/U)} + (u^*U)\frac{\partial(U\mathbf{V}^*)}{\partial(x^*L)} + (v^*U)\frac{\partial(U\mathbf{V}^*)}{\partial(y^*L)} + (w^*U)\frac{\partial(U\mathbf{V}^*)}{\partial(z^*L)}$$

$$\frac{D\mathbf{V}}{Dt} = \frac{U^2}{L}\left[\frac{\partial\mathbf{V}^*}{\partial t^*} + u^*\frac{\partial\mathbf{V}^*}{\partial x^*} + v^*\frac{\partial\mathbf{V}^*}{\partial y^*} + w^*\frac{\partial\mathbf{V}^*}{\partial z^*}\right] = \frac{U^2}{L}\frac{D\mathbf{V}^*}{Dt^*}$$

Non-dimensionalized Equations - Momentum

$$\rho\frac{U^2}{L}\frac{D\mathbf{V}^*}{Dt^*} = \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{V}$$

$$\rho\nabla\mathbf{g}\mathbf{r} = \rho\nabla\left(g_x x,\ g_y y,\ g_z z\right) =$$

$$\rho\left(g_x\frac{\partial x}{\partial x} + x\frac{\partial g_x}{\partial x},\ g_y\frac{\partial y}{\partial y} + y\frac{\partial g_y}{\partial y},\ g_z\frac{\partial z}{\partial z} + z\frac{\partial g_z}{\partial z}\right) =$$

$$\rho(g_x,\ g_y,\ g_z) = \rho\mathbf{g}$$

$$\rho\mathbf{g} - \nabla p = \nabla(\rho\mathbf{g}\mathbf{r} - p) = -\rho U^2\nabla p^* = -\rho\frac{U^2}{L}\nabla^* p^*$$

Non-dimensionalized Equations - Momentum

$$\rho\frac{U^2}{L}\frac{D\mathbf{V}^*}{Dt^*}=-\rho\frac{U^2}{L}\nabla^*p^*+\mu\nabla^2\mathbf{V}$$

$$\mu\nabla^2\mathbf{V}=\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2},\ \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2},\ \frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right]$$

introducing non-dimensional variables

$$\mu\nabla^2\mathbf{V}=\mu\left[\frac{\partial^2(Uu^*)}{\partial(L^2x^{*2})}+\frac{\partial^2(Uu^*)}{\partial(L^2y^{*2})}+\frac{\partial^2(Uu^*)}{\partial(L^2z^{*2})},\ \frac{\partial^2(Uv^*)}{\partial(L^2x^{*2})}+\frac{\partial^2(Uv^*)}{\partial(L^2y^{*2})}+\frac{\partial^2(Uv^*)}{\partial(L^2z^{*2})},\right.$$

$$\left.\frac{\partial^2(Uw^*)}{\partial(L^2x^{*2})}+\frac{\partial^2(Uw^*)}{\partial(L^2y^{*2})}+\frac{\partial^2(Uw^*)}{\partial(L^2z^{*2})}\right]=\frac{\mu U}{L^2}\nabla^{*^2}\mathbf{V}^*$$

Non-dimensionalized Equations - Momentum

$$\rho\frac{U^2}{L}\frac{D\mathbf{V}^*}{Dt^*}=-\rho\frac{U^2}{L}\nabla^*p^*+\frac{\mu U}{L^2}\nabla^{*^2}\mathbf{V}^*$$

divide by $\rho U/L^2$

$$\frac{D\mathbf{V}^*}{Dt^*}=-\nabla^*p^*+\frac{\mu}{\rho UL}\nabla^{*^2}\mathbf{V}^*$$

Non-dimensionalized Equations

Continuity: $\nabla^* \cdot \mathbf{V}^* = 0$

Navier-Stokes: $$\frac{D\mathbf{V}^*}{Dt^*} = -\nabla^* p^* + \frac{\mu}{\rho U L}\nabla^{*2}\mathbf{V}^*$$

Solid surface: no-slip ($\mathbf{V}^* = 0$ if fixed surface)

Inlet/outlet: known velocity and pressure ($\mathbf{V}^*$, p^*)

Non-dimensionalized Equations

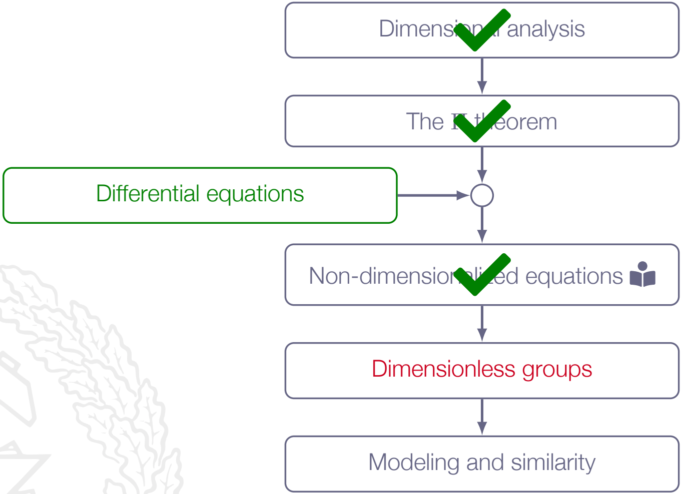
The **Reynolds number** appears in the non-dimensional Navier-Stokes equations

$$\frac{D\mathbf{V}^*}{Dt^*} = -\nabla^* p^* + \frac{\mu}{\rho U L}\nabla^{*2}\mathbf{V}^*$$

$$Re = \frac{\rho U L}{\mu}$$

Reynolds number - ratio of inertia and viscosity

Roadmap - Dimensional Analysis and Similarity



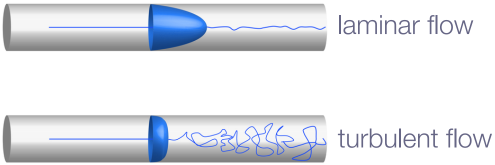
Dimensionless Groups

Definitions and interpretations of non-dimensional groups frequently used in fluid mechanics

parameter	definition	interpretation	importance
Reynolds number	$Re = \frac{\rho U L}{\mu}$	$\frac{\text{inertia}}{\text{viscosity}}$	almost always
Mach number	$M = \frac{U}{a}$	$\frac{\text{flow speed}}{\text{speed of sound}}$	compressible flow
Froude number	$Fr = \frac{U^2}{gL}$	$\frac{\text{inertia}}{\text{gravity}}$	free-surface flow
Weber number	$We = \frac{\rho U^2 L}{\Upsilon}$	$\frac{\text{inertia}}{\text{surface tension}}$	free-surface flow
Prandtl number	$Pr = \frac{\mu C_p}{k}$	$\frac{\text{dissipation}}{\text{conduction}}$	heat convection
specific heat ratio	$\gamma = \frac{C_p}{C_v}$	$\frac{\text{enthalpy}}{\text{internal energy}}$	compressible flow
Strouhal number	$St = \frac{\omega L}{U}$	$\frac{\text{oscillation}}{\text{mean flow speed}}$	oscillating flow
roughness ratio	$\frac{\varepsilon}{L}$	$\frac{\text{wall roughness}}{\text{body length}}$	turbulent flow
pressure coefficient	$C_p = \frac{p - p_\infty}{0.5\rho U^2}$	$\frac{\text{static pressure}}{\text{dynamic pressure}}$	aerodynamics
lift coefficient	$C_L = \frac{F_L}{0.5\rho U^2 A}$	$\frac{\text{lift force}}{\text{dynamic force}}$	aerodynamics
drag coefficient	$C_D = \frac{F_D}{0.5\rho U^2 A}$	$\frac{\text{drag force}}{\text{dynamic force}}$	aerodynamics
skin friction coefficient	$C_f = \frac{\tau_{wall}}{0.5\rho U^2}$	$\frac{\text{wall-shear stress}}{\text{dynamic pressure}}$	boundary layers

The Reynolds Number

$$Re = \frac{\rho U L}{\mu} = \frac{UL}{\nu}$$



Compressible Flow

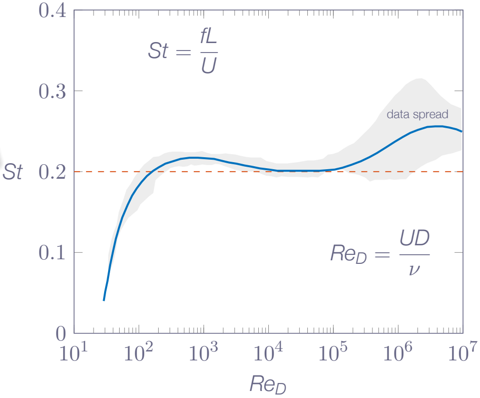
https://www.youtube.com/watch?v=wRaDPnpnx04

$$Ma = \frac{U}{a} = \frac{U}{\sqrt{\gamma R T}}$$

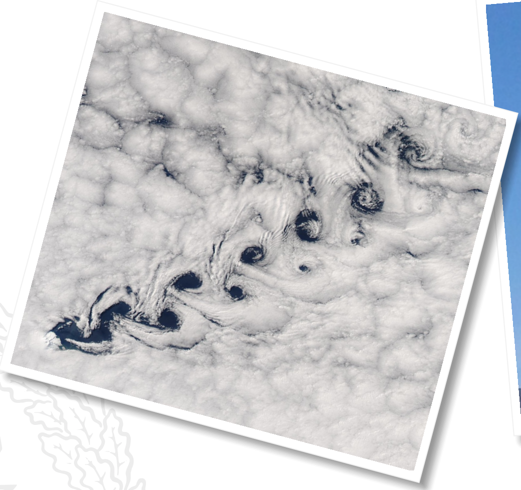
$$\gamma = \frac{C_p}{C_v}$$

Oscillating Flows

Von Kármán vortex street



Oscillating Flows



Oscillating Flows

https://www.youtube.com/watch?v=XggxeuFDaDU

Tacoma bridge collapse 1940

oscillating frequency close to the natural vibration frequency of the bridge structure

Oscillating Flows

https://www.youtube.com/watch?v=ptYrbQGk6DQ

Example of Successful Dimensional Analysis

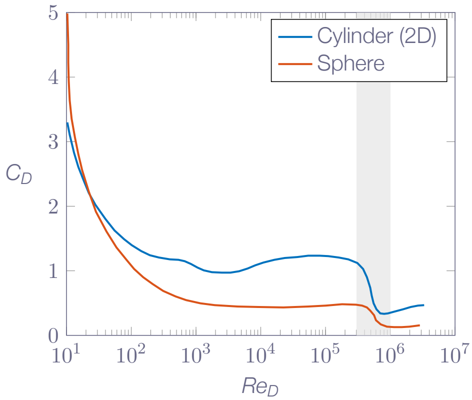
collection of data from a large number of experiments

cylinder: $C_D = \frac{F_D}{\frac{1}{2}\rho U^2 Ld}$

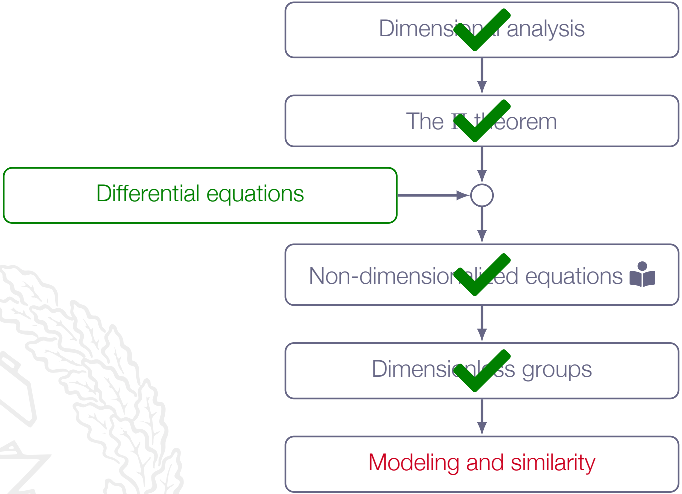
sphere: $C_D = \frac{F_D}{\frac{1}{2}\rho U^2 \frac{1}{4}\pi d^2}$

general: $C_D = \frac{F_D}{\frac{1}{2}\rho U^2 A_p}$

A_p is the projected area



Roadmap - Dimensional Analysis and Similarity



Modeling and Similarity

Scaling of experimental results from **model** scale to **prototype** scale:

> *"Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for the model and the prototype"*

Geometric Similarity

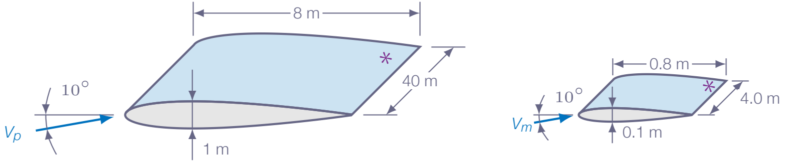
”A model and prototype are geometrically similar if and only if all body dimensions in all three coordinates have the same linear-scale ratio”

”All angles are preserved in geometric similarity. All flow directions are preserved. The orientations of model and prototype with respect to the surroundings must be identical”

Geometric Similarity

Geometric Similarity

Homologous points - points that with the same relative location



1. all dimensions should be scaled with the same linear scaling ratio
2. angle of attach should be the same
3. scaled nose radius
4. scaled surface roughness

Kinematic Similarity

”The motions of two systems are kinematically similar if homologous particles lie at homologous points at homologous times”

Geometric similarity is probably not sufficient to establish time-scale equivalence

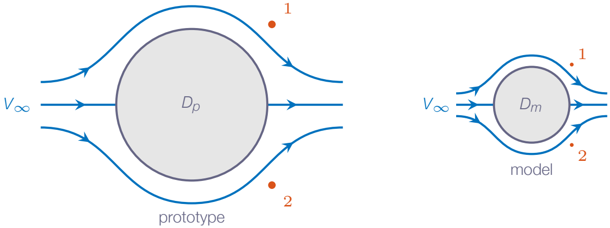
Dynamic considerations:

1. Reynolds number equivalence
2. Mach number equivalence

Kinematic Similarity

"Incompressible frictionless low-speed flows without free surfaces are kinematically similar with independent length and time scales"

$$
\begin{aligned}
D_m &= \alpha D_p \\
V_{\infty_m} &= \beta V_{\infty_p} \\
V_{1_m} &= \beta V_{1_p} \\
V_{2_m} &= \beta V_{2_p}
\end{aligned}
$$



Dynamic Similarity

"Dynamic similarity is achieved when the model and prototype have the same length scale ratio, time scale ratio, and force scale ratio"

Compressible flow:

1. Reynolds number equivalence
2. Mach number equivalence
3. specific-heat ratio equivalence

Incompressible flow without free surfaces:

1. Reynolds number equivalence

Incompressible flow with free surfaces:

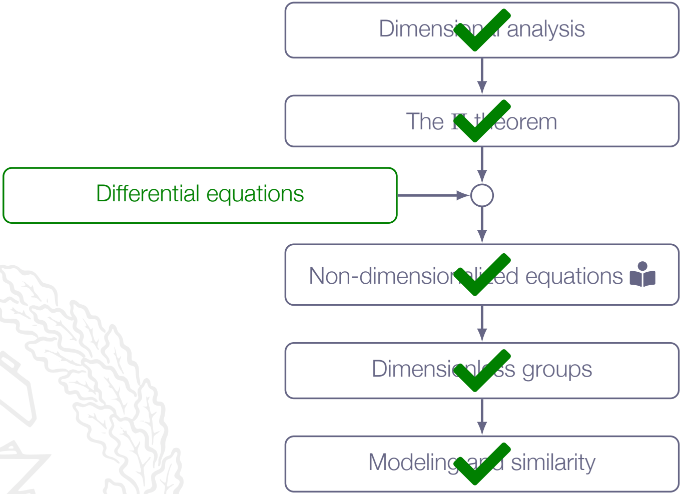
1. Reynolds number equivalence
2. Froude number equivalence (*and if necessary Weber number and/or cavitation number*)

Dynamic Similarity

$$\mathbf{F}_{inertia} = \mathbf{F}_{pressure} + \mathbf{F}_{gravity} + \mathbf{F}_{friction}$$

"Dynamic similarity ensures that each of the force components will be in the same ratio and have the same directions for model and prototype"

Roadmap - Dimensional Analysis and Similarity



Dimensional Analysis