Fluid Mechanics - MTF053

Lecture 6

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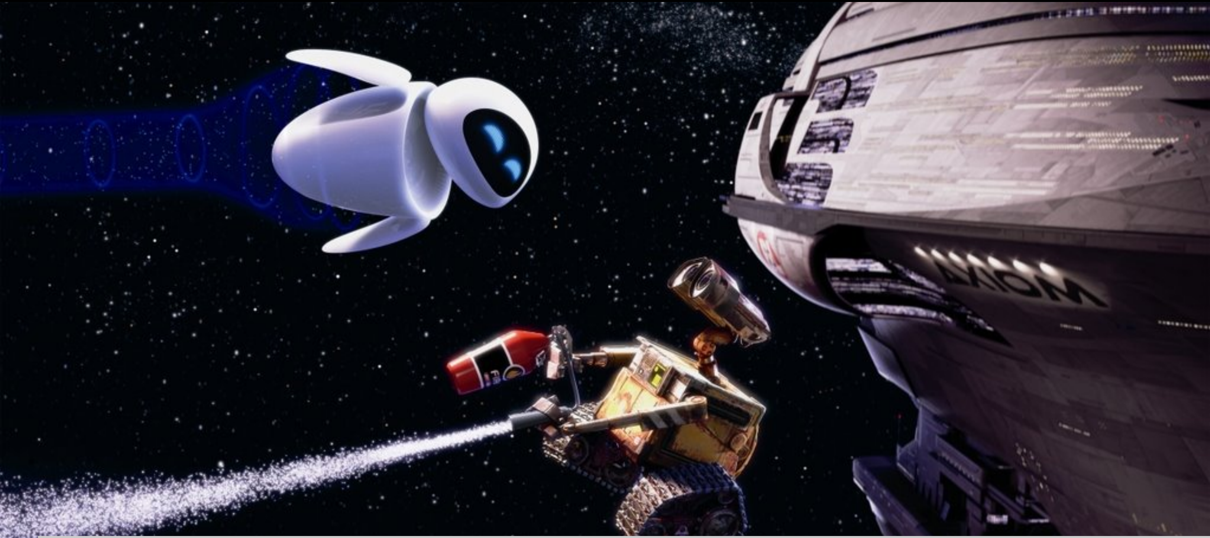


Chapter 3 - Integral Relations for a Control Volume

Overview

conser-vation relations
Differential Relations
sional Analysis
Bernoulli
Integral Relations
Flow Relations
conser-vation relations
Reynolds transport theorem
buoyancy
hydrostatic forces
Pressure
velocity field
Reynolds number

Learning Outcomes

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

12 **Define** Reynolds transport theorem using the concepts control volume and system

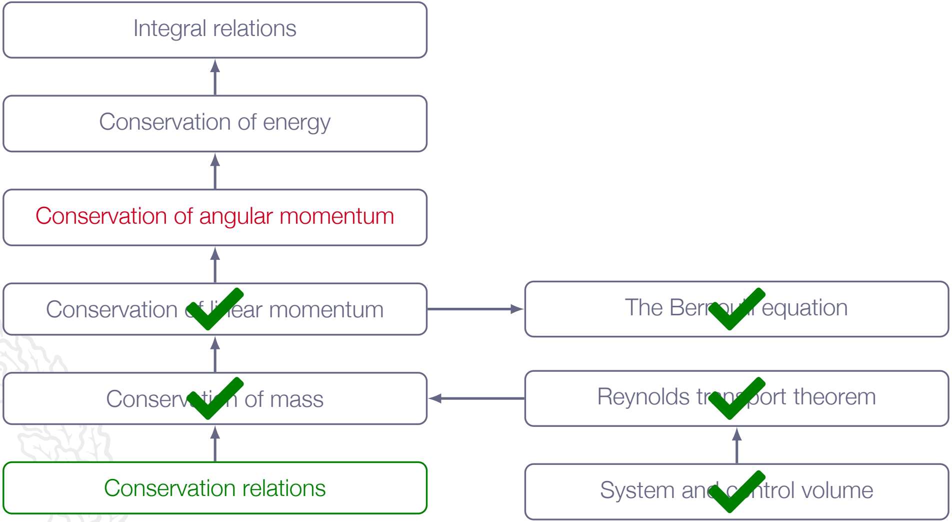
13 **Derive** the control volume formulation of the continuity, momentum, and energy equations using Reynolds transport theorem and solving problems using those relations

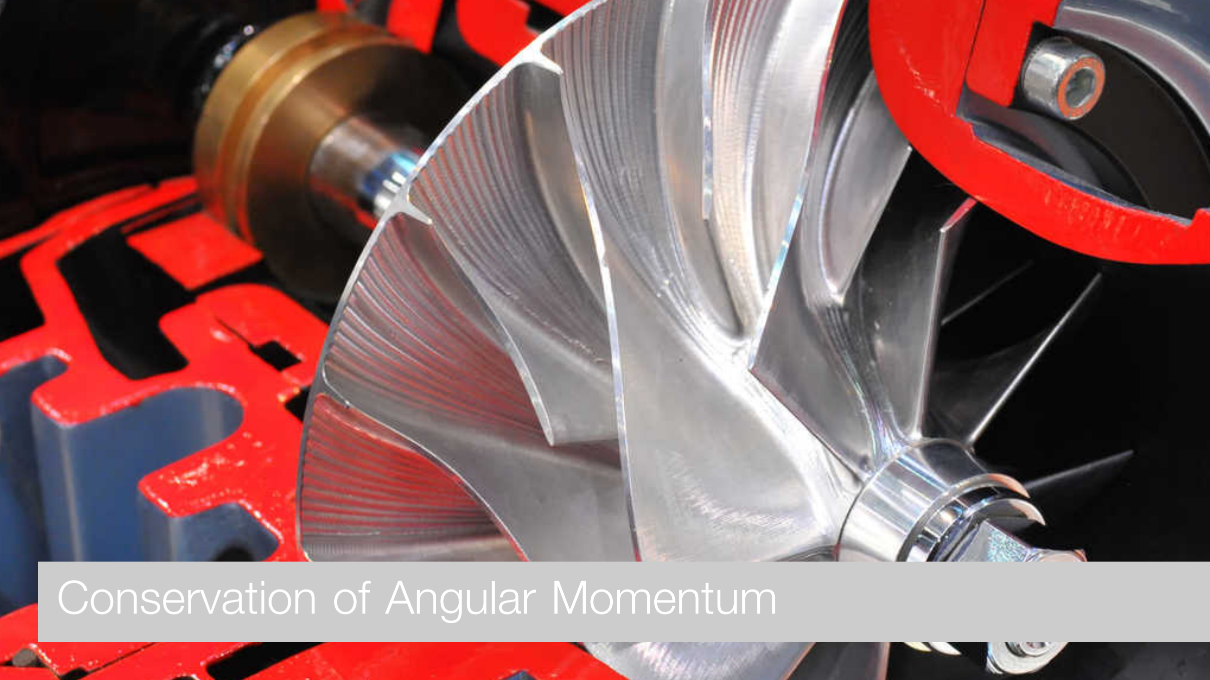
15 **Derive** and use the Bernoulli equation (using the relation includes having knowledge about its limitations)

we will derive methods suitable for estimation of forces and system analysis

fluid flow finally ...

Roadmap - Integral Relations





Conservation of Angular Momentum

Angular Momentum

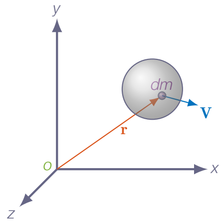
Angular momentum about a point o

$$\mathbf{H}_o = \int_{syst} (\mathbf{r} \times \mathbf{V}) dm = B$$

where $\mathbf{r}$ is the position vector from o to the element mass dm and $\mathbf{V}$ is the velocity of that element

The amount of angular momentum per unit mass

$$\beta = \frac{d\mathbf{H}_o}{dm} = \mathbf{r} \times \mathbf{V}$$



Conservation of Angular Momentum

Reynold's transport theorem:

$$\left.\frac{d\mathbf{H}_o}{dt}\right|_{syst} = \frac{d}{dt}\left[\int_{CV} (\mathbf{r} \times \mathbf{V})\rho d\mathcal{V}\right] + \int_{CS} (\mathbf{r} \times \mathbf{V})\rho(\mathbf{V}_r \cdot \mathbf{n})dA$$

for inertial coordinate systems:

$$\frac{d\mathbf{H}_o}{dt} = \sum \mathbf{M}_o = \sum (\mathbf{r} \times \mathbf{F})_o$$

Conservation of Angular Momentum

Non-deformable inertial control volume:

$$\sum \mathbf{M}_o = \frac{d}{dt}\left[\int_{CV}(\mathbf{r}\times\mathbf{V})\rho d\mathcal{V}\right] + \int_{CS}(\mathbf{r}\times\mathbf{V})\rho(\mathbf{V}\cdot\mathbf{n})dA$$

for a limited number of **one-dimensional** inlets and outlets

$$\int_{CS}(\mathbf{r}\times\mathbf{V})\rho(\mathbf{V}\cdot\mathbf{n})dA = \sum(\mathbf{r}\times\mathbf{V})_{out}\dot{m}_{out} - \sum(\mathbf{r}\times\mathbf{V})_{in}\dot{m}_{in}$$

Angular Momentum Example - Garden Sprinkler

Assumptions:

1. steady-state flow
2. incompressible flow (water)

Angular Momentum Example - Garden Sprinkler

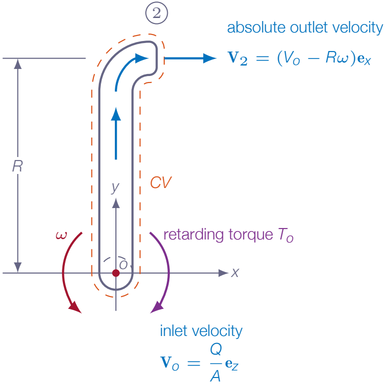
$$\sum \mathbf{M}_o = \underbrace{\frac{d}{dt}\left[\int_{CV} (\mathbf{r}\times\mathbf{V})\rho d\mathcal{V}\right]}_{=0\ (steady\ state)} + \int_{CS} (\mathbf{r}\times\mathbf{V})\rho(\mathbf{V_r}\cdot\mathbf{n})dA$$

$$\mathbf{V}_2 = (V_o - R\omega, 0, 0)$$

$$\mathbf{V}_o = (0, 0, V_o)$$

$$\mathbf{r}_2 = (0, R, 0)$$

$$\mathbf{r}_o = (0, 0, 0)$$



Angular Momentum Example - Garden Sprinkler

$$\sum \mathbf{M}_o = \int_{CS} (\mathbf{r} \times \mathbf{V})\rho(\mathbf{V_r} \cdot \mathbf{n})dA$$

inlet:

$$(\mathbf{r}_o \times \mathbf{V}_o) = (0,0,0) \times (0,0,V_o) = (0,0,0)$$

$$(\mathbf{V}_{o,r} \cdot \mathbf{n}_o) = (0,0,V_o) \cdot (0,0,-1) = -V_o$$

$$(\mathbf{r}_o \times \mathbf{V}_o)\rho(\mathbf{V}_o \cdot \mathbf{n}_o)A_o = -(0,0,0)\rho V_o A_o = (0,0,0)$$

outlet:

$$(\mathbf{r}_2 \times \mathbf{V}_2) = (0,R,0) \times (V_o - R\omega,0,0) = (0,0,R^2\omega - RV_o)$$

$$(\mathbf{V}_{2,r} \cdot \mathbf{n}_2) = (V_o,0,0) \cdot (1,0,0) = V_o$$

$$(\mathbf{r}_2 \times \mathbf{V}_2)\rho(\mathbf{V}_2 \cdot \mathbf{n}_2)A_2 = (0,0,R^2\omega - RV_o)\rho V_o A_2 = \rho Q(0,0,R^2\omega - RV_o)$$

Angular Momentum Example - Garden Sprinkler

$$\sum \mathbf{M}_o = (0, 0, -T_o) = \rho Q(0, 0, R^2\omega - RV_o)$$

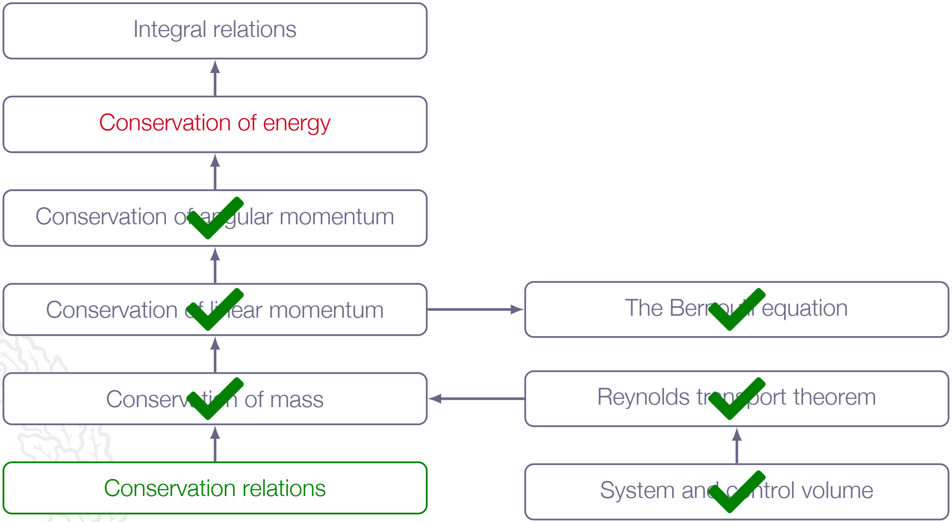
$$-T_o = \rho Q R^2 \left(\omega - \frac{V_o}{R}\right) \Rightarrow \omega = \frac{V_o}{R} - \frac{T_o}{\rho Q R^2}$$

Note:

with a negligible retarding torque, *i.e.* $T_o \approx 0.$, we get $\omega \approx \omega|_{T_o=0.} = \dfrac{V_o}{R}$ $[rad/s]$

the torque required to hold the sprinkler arm still is $T_o|_{\omega=0.} = \rho Q V_o R$ $[Nm]$

Roadmap - Integral Relations



The Energy Equation

Reynold's transport theorem applied the the **first law of thermodynamics** $(B = E, \ \beta = dE/dm = e)$

$$\frac{dQ_{sys}}{dt} - \frac{dW_{sys}}{dt} = \frac{dE_{sys}}{dt} = \frac{d}{dt}\left(\int_{CV} e\rho d\mathcal{V}\right) + \int_{CS} e\rho(\mathbf{V}\cdot\mathbf{n})dA$$

Recall:

positive Q_{sys}: heat added **to** the system

positive W_{sys}: work done **by** the system on its surroundings

The Energy Equation - Energy per Unit Mass

$$e = e_{internal} + e_{kinetic} + e_{potential} + e_{other}$$

e_{other} could be related to, for example, chemical reactions, nuclear reactions, or magnetic fields and will not be considered here

$$e = \hat{u} + \frac{1}{2}V^2 + gz$$

The Energy Equation - Work

The work term $\dot{W}$ can be divided into shaft work, pressure work, and work related to viscous forces

$$\dot{W} = \dot{W}_s + \dot{W}_p + \dot{W}_\nu$$

$$\dot{W}_p = \int_{CS} p(\mathbf{V} \cdot \mathbf{n})dA$$

$$\dot{W}_\nu = -\int_{CS} \tau \cdot \mathbf{V}dA$$

The Energy Equation - Pressure Work

$$\dot{W}_p = \int_{CS} p(\mathbf{V} \cdot \mathbf{n})dA$$

The rate of work done by **pressure forces** on the **control volume surfaces**

internal forces will always have an opposite force leading to cancelation

The Energy Equation - Viscous Work

$$\dot{W}_{\nu} = -\int_{CS} \tau \cdot \mathbf{V} dA$$

The rate of work related to **viscous stresses** on the **control volume surfaces**

important or not depending on flow situation

The Energy Equation - Control Volume Boundaries

Solid walls:

no-slip $\Rightarrow \dot{W}_\nu = 0$

Machine surfaces:

viscous work included implicitly in shaft work

Inlets/outlets:

flow aligned with surface normal (usually) and normal viscous stress components are in most cases very small

Streamlines:

viscous stresses may be significant *depending on streamline location*

The Energy Equation

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt}\left(\int_{CV} e\rho d\mathcal{V}\right) + \int_{CS} \rho e(\mathbf{V}\cdot\mathbf{n})dA + \int_{CS} p(\mathbf{V}\cdot\mathbf{n})dA$$

collecting surface integrals gives

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt}\left(\int_{CV} e\rho d\mathcal{V}\right) + \int_{CS} \left(e + \frac{p}{\rho}\right) \rho(\mathbf{V}\cdot\mathbf{n})dA$$

The Energy Equation

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt}\left(\int_{cv} e\rho d\mathcal{V}\right) + \int_{cs}\left(e + \frac{p}{\rho}\right)\rho(\mathbf{V}\cdot\mathbf{n})dA$$

or

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt}\left[\int_{cv}\left(\hat{u} + \frac{1}{2}V^2 + gz\right)\rho d\mathcal{V}\right] + \int_{cs}\left(\hat{h} + \frac{1}{2}V^2 + gz\right)\rho(\mathbf{V}\cdot\mathbf{n})dA$$

where $\hat{h}$ is the enthalpy defined as $\hat{h} = \hat{u} + p/\rho$

The Energy Equation

Steady state:

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \int_{CS} \left(\hat{h} + \frac{1}{2}V^2 + gz\right) \rho(\mathbf{V} \cdot \mathbf{n})dA$$

Special case: one inlet and one outlet (both one-dimensional)

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = -\dot{m}_1 \left(\hat{h}_1 + \frac{1}{2}V_1^2 + gz_1\right) + \dot{m}_2 \left(\hat{h}_2 + \frac{1}{2}V_2^2 + gz_2\right)$$

continuity $\Rightarrow \dot{m}_1 = \dot{m}_2 = \dot{m}$, divide by $\dot{m}$ gives

$$\hat{h}_1 + \frac{1}{2}V_1^2 + gz_1 = \hat{h}_2 + \frac{1}{2}V_2^2 + gz_2 - q + w_s + w_\nu$$

The Energy Equation

$$\hat{h}_1 + \frac{1}{2}V_1^2 + gz_1 = \hat{h}_2 + \frac{1}{2}V_2^2 + gz_2 - q + w_s + w_\nu$$

all terms has the dimension $[m^2/s^2]$, divide by g $[m/s^2]$ to get dimension $[m]$

$$\frac{p_1}{\rho g} + \frac{\hat{u}_1}{g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{\hat{u}_2}{g} + \frac{V_2^2}{2g} + z_2 - \frac{q}{g} + h_s + h_\nu$$

$p/(\rho g)$: pressure head

$V^2/2g$: velocity head

The Energy Equation

1. steady-state flow
2. incompressible (low speed)
3. pipe/duct that may or may not include turbines and pumps
4. solid walls $\Rightarrow h_\nu = 0$

$$\underbrace{\left(\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1\right)}_{h_{o_1}} = \underbrace{\left(\frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2\right)}_{h_{o_2}} + \frac{\hat{u}_2 - \hat{u}_1 - q}{g}$$

where h_o is *available head* or *total head*

The Energy Equation

friction head losses h_f (always positive)

pump head input h_p

turbine head extraction h_t

$$\left(\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1\right)_{in} = \left(\frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2\right)_{out} + h_f - h_p + h_t$$

Kinetic Energy Correction Factor

One-dimensional flow through inlets and outlets is of course not true in reality

Introducing the correction factor α

$$\int \frac{1}{2} V^2 \rho (\mathbf{V} \cdot \mathbf{n}) dA = \frac{\alpha {V_{av}}^2}{2} \dot{m}$$

where (for incompressible flow)

$$V_{av} = \frac{1}{A} \int u dA$$

Kinetic Energy Correction Factor

$$\left(\frac{p_1}{\rho g}+\frac{\alpha_1 V_1^2}{2g}+z_1\right)_{in}=\left(\frac{p_2}{\rho g}+\frac{\alpha_2 V_2^2}{2g}+z_2\right)_{out}+h_f-h_p+h_t$$

Kinetic Energy Correction Factor

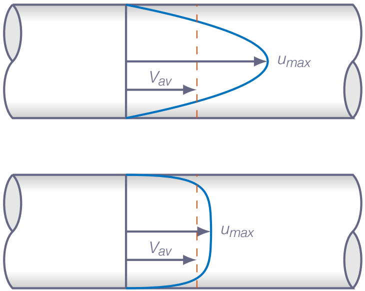
Laminar pipe flow:

Velocity profile:

$$u(r) = U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

From example 2 in the conservation-of-mass section we have that the average velocity for a laminar pipe flow is obtained as:

$$V_{av} = \frac{1}{2}U_{max}$$



Kinetic Energy Correction Factor

$$u(r) = U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

$$\int \frac{1}{2}V^2\rho(\mathbf{V}\cdot\mathbf{n})dA = \int_0^R \frac{1}{2}U_{max}^2\left(1 - \left(\frac{r}{R}\right)^2\right)^2 \rho U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right) 2\pi r dr$$

$$\int \frac{1}{2}V^2\rho(\mathbf{V}\cdot\mathbf{n})dA = \rho\pi U_{max}^3 \int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right)^3 r dr$$

$$\int \frac{1}{2}V^2\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{1}{8}\rho\pi R^2 U_{max}^3$$

Kinetic Energy Correction Factor

$$\int \frac{1}{2}V^2\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{1}{8}\rho\pi R^2 U_{max}^3$$

the mass flow $\dot{m}$ can be obtained as:

$$\dot{m} = V_{av}\rho\pi R^2 = \frac{1}{2}U_{max}\rho\pi R^2$$

which gives

$$\int \frac{1}{2}V^2\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{U_{max}^2}{4}\dot{m}$$

from the definition of α

$$\int \frac{1}{2}V^2\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{\alpha V_{av}^2}{2}\dot{m} = \left\{V_{av} = \frac{1}{2}U_{max}\right\} = \frac{\alpha U_{max}^2}{8}\dot{m}$$

Kinetic Energy Correction Factor

comparing the two expressions, we have that

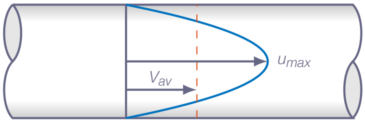
$$\frac{U_{max}^2}{4}\dot{m} = \frac{\alpha U_{max}^2}{8}\dot{m}$$

and thus $\alpha = 2$ for laminar incompressible flow

Turbulent pipe flow:

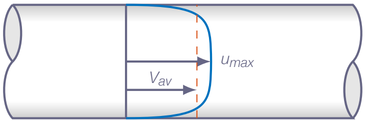
Velocity Profile:

$$u(r) \approx U_{max} \left(1 - \frac{r}{R}\right)^{m}, \; m \approx \frac{1}{7}$$



From example 3 in the conservation-of-mass section we have that the average velocity for a laminar pipe flow is obtained as:

$$V_{av} = \frac{2U_{max}}{(1+m)(2+m)}$$



Kinetic Energy Correction Factor

$$\rho\pi U_{max}^3 \int_0^R \left(1-\frac{r}{R}\right)^{3m} rdr = \alpha\frac{1}{2}\rho\pi R^2 V_{av}^3 = \alpha\frac{4\rho\pi R^2 U_{max}^3}{(1+m)^3(2+m)^3} \Rightarrow$$

$$\left[\frac{(r-R)\left(1-\frac{r}{R}\right)^{3m}(3mr+r+R)}{(1+3m)(2+3m)}\right]_0^R = \alpha\frac{4R^2}{(1+m)^3(2+m)^3} \Rightarrow$$

$$\frac{R^2}{(1+3m)(2+3m)} = \alpha\frac{4R^2}{(1+m)^3(2+m)^3} \Rightarrow \alpha = \frac{(1+m)^3(2+m)^3}{4(1+3m)(2+3m)}$$

Kinetic Energy Correction Factor

Laminar pipe flow:

$\alpha = 2.0$ should be used

Turbulent pipe flow:

$$\alpha = \frac{(1+m)^3(2+m)^3}{4(1+3m)(2+3m)}$$

m	1/5	1/6	1/7	1/8	1/9
α	1.106	1.077	1.058	1.046	1.037

$\alpha = 1.0$ is often a good approximation for turbulent pipe flows

The Energy Equation - Pump Example

Task:

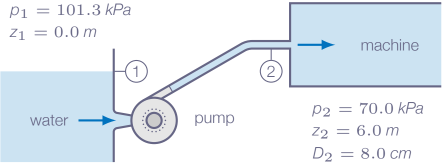
Calculate pump power if $\eta = 0.8$

Given:

1. Geometry and pressures from figure
2. The pump delivers water at a flow rate $Q = 0.04\ m^3/s$
3. Friction losses between 1 and 2 are given by $h_f = KV_2^2/(2g)$ where $K \approx 7.5$
4. $\alpha \approx 1.07$

Assumptions:

1. steady-state flow
2. negligible viscous work
3. large reservoir ($V_1 \approx 0$)



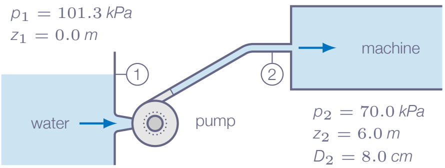
The Energy Equation - Pump Example

$$V_2 = \frac{Q}{A_2} = 7.96\ m/s$$

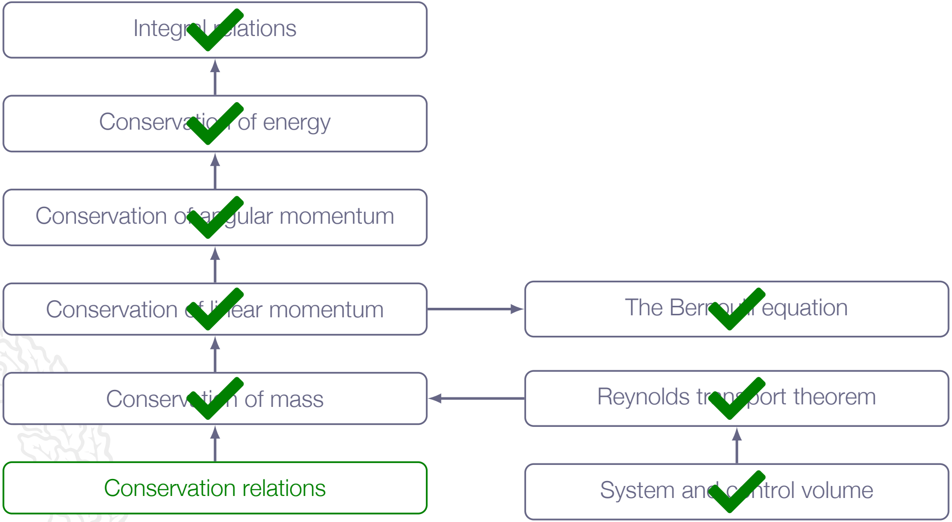
$$\left(\frac{p_1}{\rho g} + \frac{\alpha V_1^2}{2g} + z_1\right)_{in} = \left(\frac{p_2}{\rho g} + \frac{\alpha V_2^2}{2g} + z_2\right)_{out} + h_f - h_p + h_t$$

$$h_p = \frac{p_2 - p_1}{\rho g} + (z_2 - z_1) + (\alpha_2 + K)\frac{V_2^2}{2g} = 30.5\ m$$

$$P_{pump} = \frac{\rho g Q h_p}{\eta} = 14960\ W\ (or\ 20\ hp)$$



Roadmap - Integral Relations



Integral Relations - Considerations

1. control volume type: **non-deforming**?, **non-accelerating**?

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1. control volume type: **non-deforming**?, **non-accelerating**?
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3. can **friction** be neglected?
4. can the fluid be assumed to be **incompressible**?

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5. if compressible, can the **ideal gas law** be used?

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7. is there **heat transfer**, **shaft work** or **viscous work**

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2. **steady** flow? if not can the frame of reference be changed?
3. can **friction** be neglected?
4. can the fluid be assumed to be **incompressible**?
5. if compressible, can the ideal gas law be used?
6. do we need to account for **body forces** (gravity etc)?
7. is there **heat transfer**, **shaft work** or **viscous work**
8. can inlets/outlets be assumed to be **one-dimensional**

Integral Relations