

# Fluid Mechanics - MTF053

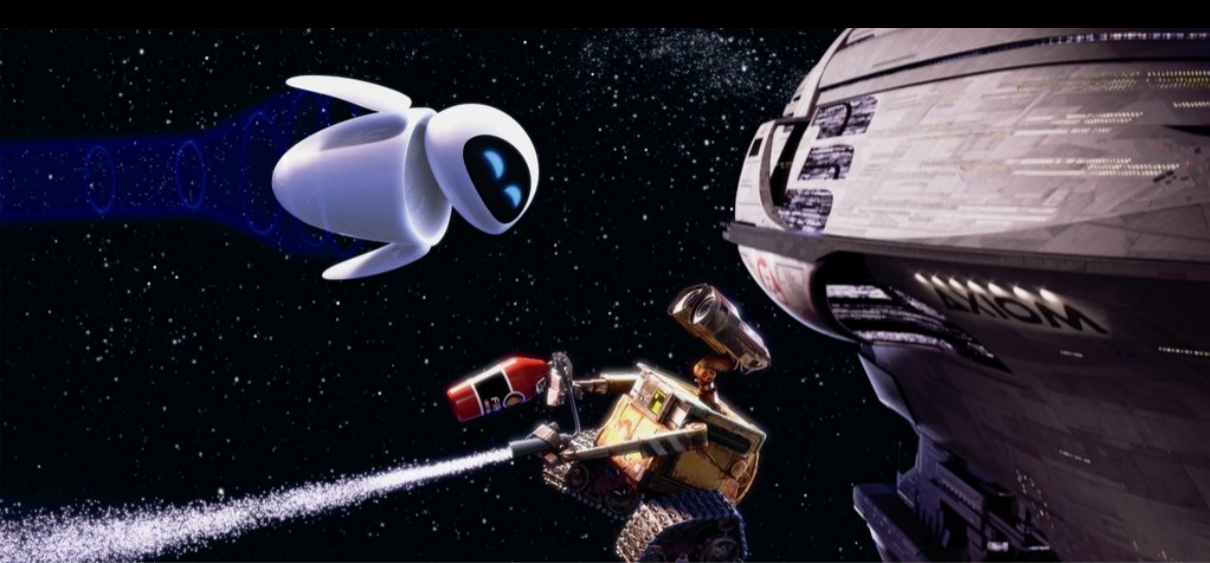
## Lecture 4

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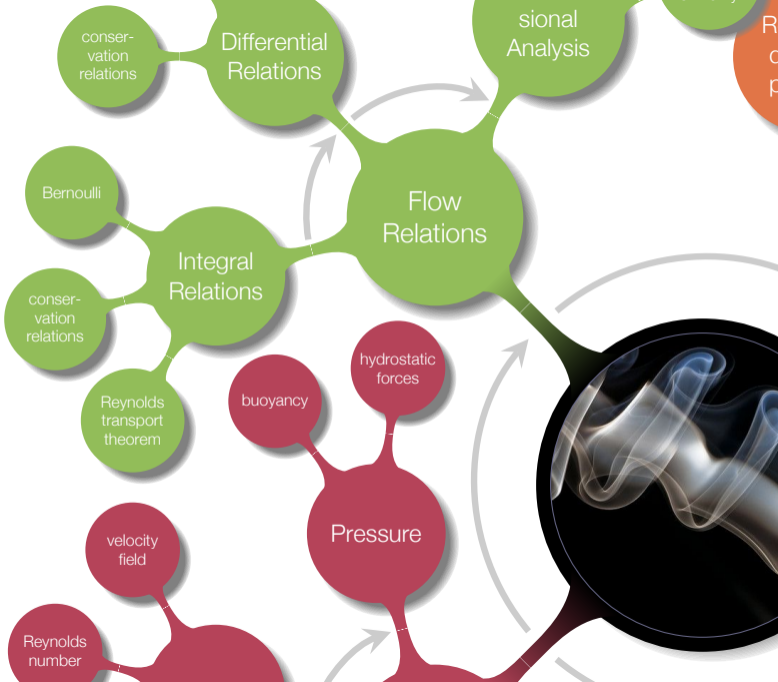
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## Chapter 3 - Integral Relations for a Control Volume

# Overview



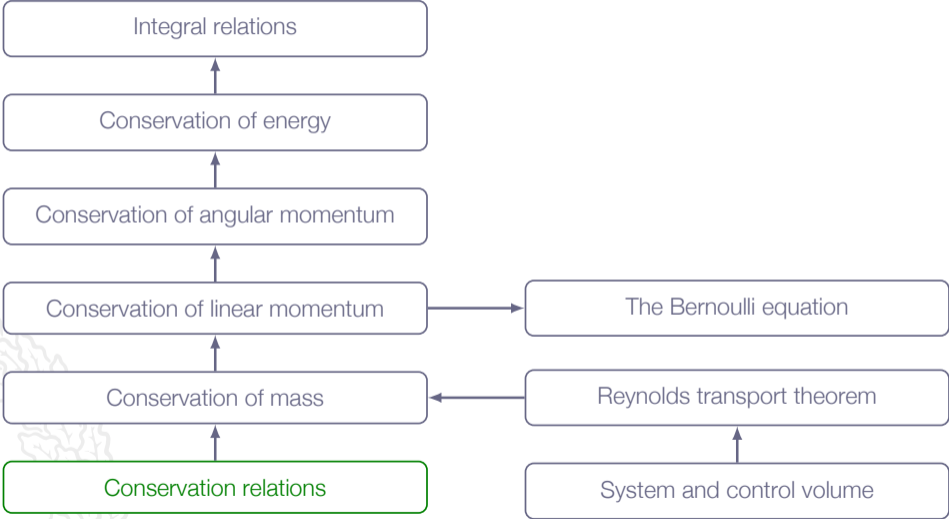
# Learning Outcomes

- 4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category
- 12 **Define** Reynolds transport theorem using the concepts control volume and system
- 13 **Derive** the control volume formulation of the continuity, momentum, and energy equations using Reynolds transport theorem and solving problems using those relations
- 15 **Derive** and use the Bernoulli equation (using the relation includes having knowledge about its limitations)

*we will derive methods suitable for estimation of forces and system analysis*

*fluid flow finally ...*

# Roadmap - Integral Relations



# Motivation

Fluid motion analysis:

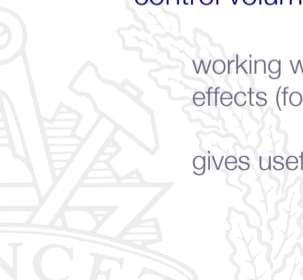
differential approach (chapter 4):

describe the detailed flow pattern at every point in the flow

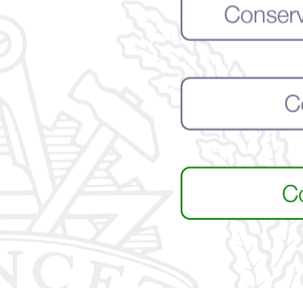
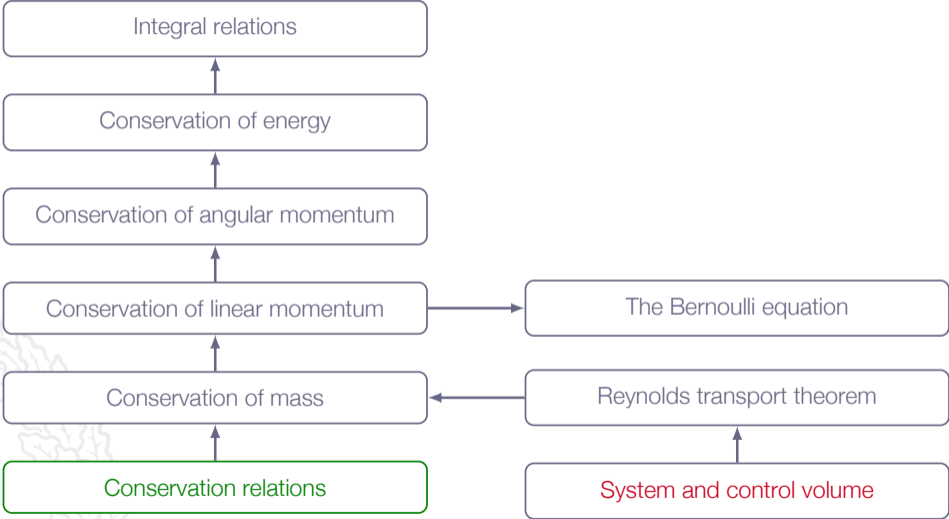
control volume approach (chapter 3):

working with a finite region, balance in and out flow and determine gross flow effects (force, torque, energy exchange, ... )

gives useful engineering estimates



# Roadmap - Integral Relations



# System vs Control Volume

All laws of mechanics are written for a **system**:

A system is an arbitrary quantity of **mass of fixed identity**  $m$

Mechanical relations describes interaction between the system and its surroundings

We need the corresponding relations expressed for a **control volume**:

A region in space defined by the control volume surface

Flow can pass through a control volume  $\Rightarrow$  a system can pass through a control volume

# Conservation Relations for a System

1. Conservation of mass
2. Conservation of linear momentum
3. Conservation of angular momentum
4. Conservation of energy

The above-listed relations includes thermodynamic properties and needs to be supplemented by a **state relation**

Remember: a thermodynamic property can be calculated from **any two other thermodynamics properties**

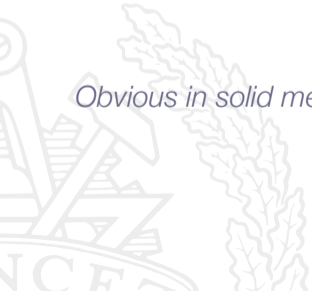
$$p = p(\rho, T), \quad e = e(\rho, T)$$

# Conservation of Mass for a System

$$m_{\text{syst}} = \text{const}$$

$$\frac{dm}{dt} = 0$$

*Obvious in solid mechanics but needs attention in fluid mechanics*



# Conservation of Linear Momentum for a System

If the surroundings exert a net force  $\mathbf{F}$  on the system, the mass in the system will begin to accelerate

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{V}}{dt} = \frac{d}{dt}(m\mathbf{V})$$

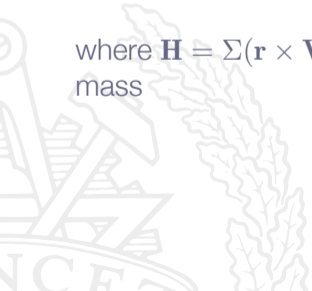


# Conservation of Angular Momentum for a System

If the surroundings exert a net moment  $\mathbf{M}$  about the center of mass of the system, there will be a rotation effect

$$\mathbf{M} = \frac{d\mathbf{H}}{dt}$$

where  $\mathbf{H} = \Sigma(\mathbf{r} \times \mathbf{V})\delta m$  is the angular momentum of the system about its center of mass



# Conservation of Energy for a System

First law of thermodynamics

$$\delta Q - \delta W = dE$$

Second law of thermodynamics

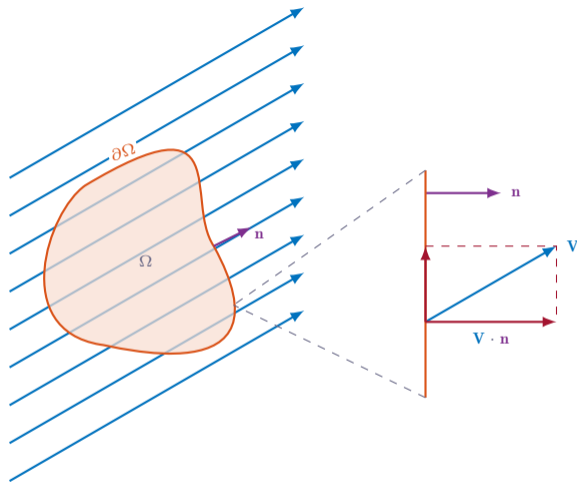
$$dS \geq \frac{\delta Q}{T}$$



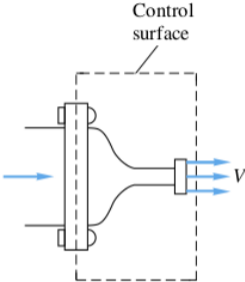
# Control Volume - Volume and Mass Flow Rate

$$Q = \int_{\partial\Omega} (\mathbf{V} \cdot \mathbf{n}) dA$$

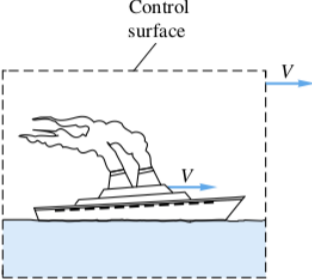
$$\dot{m} = \int_{\partial\Omega} \rho(\mathbf{V} \cdot \mathbf{n}) dA$$



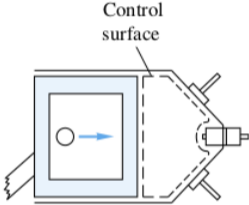
# Control Volume - Classification



fixed



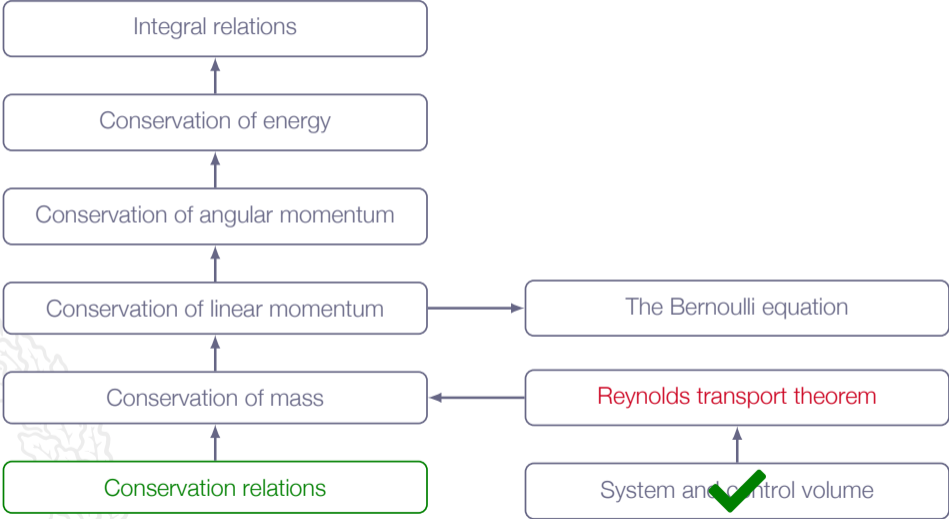
moving



deformable



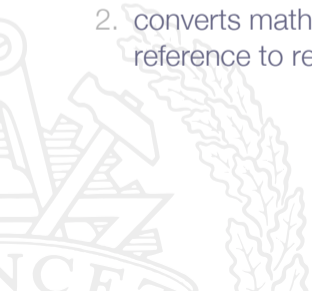
# Roadmap - Integral Relations



# Reynolds Transport Theorem

## Reynolds transport theorem

1. converts mathematical relations derived for a specific **system** to relations for a specific **region**
2. converts mathematical relations expressed in the **Lagrangian** frame of reference to relations expressed in the **Eulerian** frame of reference



# Reynolds Transport Theorem

Let  $B$  be any **extensive** property of the fluid (energy, momentum, enthalpy, ... )

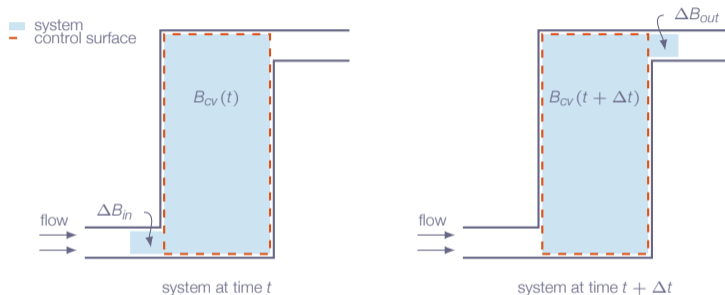
$\beta$  is the corresponding **intensive** value (*the amount  $B$  per unit mass*)

The total amount of  $B$  in the control volume is

$$B_{CV} = \int_{CV} \beta dm = \int_{CV} \beta \rho dV$$

where  $\beta = \frac{dB}{dm}$

# Reynolds Transport Theorem



$$B_{\text{sys}}(t) = B_{\text{cv}}(t) + \Delta B_{\text{in}}$$

$$B_{\text{sys}}(t + \Delta t) = B_{\text{cv}}(t + \Delta t) + \Delta B_{\text{out}}$$

# Reynolds Transport Theorem

The rate of change of  $B$  for the system:

$$\frac{dB_{\text{sys}}}{dt} = \lim_{\Delta t \rightarrow 0} \left[ \frac{B_{\text{sys}}(t + \Delta t) - B_{\text{sys}}(t)}{\Delta t} \right]$$

Apply relations from previous slide  $\Rightarrow$

$$\frac{dB_{\text{sys}}}{dt} = \lim_{\Delta t \rightarrow 0} \left[ \frac{(B_{\text{cv}}(t + \Delta t) + \Delta B_{\text{out}}) - (B_{\text{cv}}(t) + \Delta B_{\text{in}})}{\Delta t} \right]$$

# Reynolds Transport Theorem

Rewriting  $\Rightarrow$

$$\frac{dB_{\text{sys}}}{dt} = \underbrace{\lim_{\Delta t \rightarrow 0} \left[ \frac{B_{\text{cv}}(t + \Delta t) - B_{\text{cv}}(t)}{\Delta t} \right]}_{\frac{dB_{\text{cv}}}{dt}} + \underbrace{\lim_{\Delta t \rightarrow 0} \frac{\Delta B_{\text{out}}}{\Delta t} - \lim_{\Delta t \rightarrow 0} \frac{\Delta B_{\text{in}}}{\Delta t}}_{\dot{B}_{\text{net}}}$$

$$\frac{dB_{\text{sys}}}{dt} = \frac{dB_{\text{cv}}}{dt} + \dot{B}_{\text{net}}$$

# Reynolds Transport Theorem

Rate of change of  $B$  within the control volume

$$\frac{dB_{cv}}{dt} = \frac{d}{dt} \left( \int_{cv} \beta \rho d\mathcal{V} \right)$$

Net flux of  $B$  over the control volume surface

$$\dot{B}_{net} = \int_{CS} \beta \rho (\mathbf{V} \cdot \mathbf{n}) dA$$

# Reynolds Transport Theorem

$$\underbrace{\frac{d}{dt}(B_{\text{sys}})}_{\text{Lagrange}} = \underbrace{\frac{d}{dt} \left( \int_{\text{CV}} \beta \rho d\mathcal{V} \right) + \int_{\text{CS}} \beta \rho (\mathbf{V} \cdot \mathbf{n}) dA}_{\text{Euler}}$$

# Reynolds Transport Theorem

For a fixed control volume (the volume does not change in time)

$$\frac{d}{dt} \left( \int_{CV} \beta \rho d\mathcal{V} \right) = \int_{CV} \frac{\partial}{\partial t} (\beta \rho) d\mathcal{V}$$



# Reynolds Transport Theorem

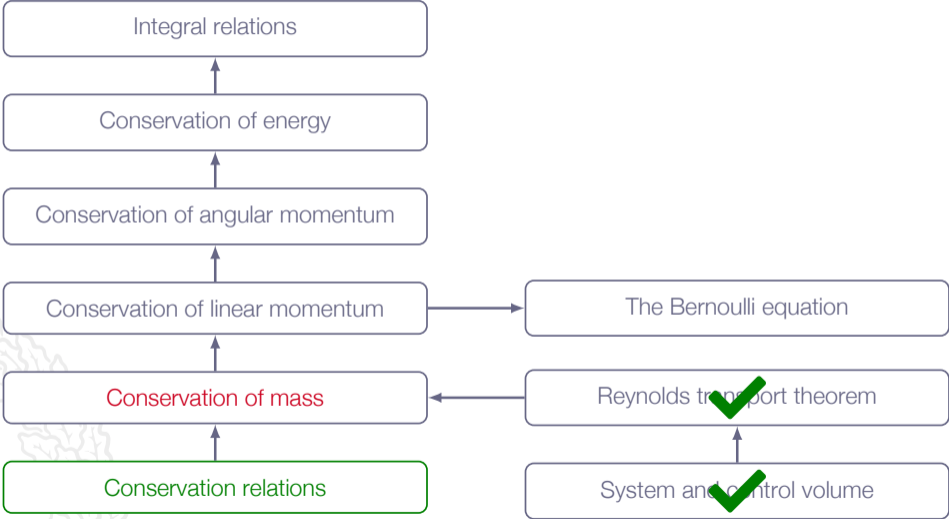
If the control volume moves with the constant velocity  $\mathbf{V}_s$ , the relative velocity of the fluid crossing the control volume surface  $\mathbf{V}_r$  is

$$\mathbf{V}_r = \mathbf{V} - \mathbf{V}_s$$

and thus

$$\frac{d}{dt}(B_{\text{sys}}) = \frac{d}{dt} \left( \int_{\text{CV}} \beta \rho d\mathcal{V} \right) + \int_{\text{CS}} \beta \rho (\mathbf{V}_r \cdot \mathbf{n}) dA$$

# Roadmap - Integral Relations



# Conservation of Mass

Reynolds transport theorem with  $B = m$  and  $\beta = dB/dm = dm/dm = 1$

$$\frac{d}{dt}(m_{\text{sys}}) = 0 = \frac{d}{dt} \left( \int_{CV} \rho d\mathcal{V} \right) + \int_{CS} \rho(\mathbf{V}_r \cdot \mathbf{n}) dA$$

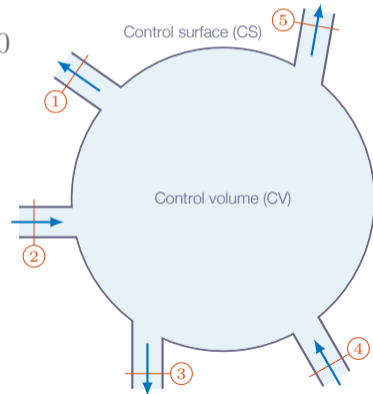
for a fixed control volume

$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \int_{CS} \rho(\mathbf{V}_r \cdot \mathbf{n}) dA = 0$$

# Conservation of Mass

for a **fixed control volume** with a limited number of **one-dimensional** inlets and outlets

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$



# Conservation of Mass

fixed control volume formulation

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \int_{CS} \rho(\mathbf{V}_r \cdot \mathbf{n}) dA = 0$$

Steady state  $\Rightarrow \partial \rho / \partial t = 0$  and thus:

$$\int_{CS} \rho(\mathbf{V}_r \cdot \mathbf{n}) dA = 0$$

for a limited number of one-dimensional inlets and outlets:

$$\sum_i (\rho_i A_i V_i)_{out} = \sum_i (\rho_i A_i V_i)_{in}$$

# Conservation of Mass

fixed control volume formulation

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \int_{CS} \rho (\mathbf{V}_r \cdot \mathbf{n}) dA = 0$$

Incompressible flow  $\Rightarrow \rho = const \Rightarrow \partial \rho / \partial t = 0$  and thus:

$$\int_{CS} (\mathbf{V}_r \cdot \mathbf{n}) dA = 0$$

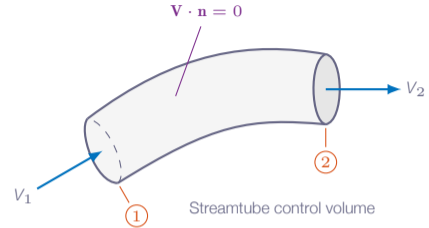
for a limited number of one-dimensional inlets and outlets:

$$\sum_i (A_i V_i)_{out} = \sum_i (A_i V_i)_{in}$$

# Conservation of Mass - Example 1

## Steady flow through a streamtube:

1. steady state  $\Rightarrow$  no changes in time
2. streamtube  $\Rightarrow$  only flow through the surfaces 1 and 2



$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \text{const}$$

if the density is constant (incompressible flow)

$$Q = A_1 V_1 = A_2 V_2 = \text{const} \Rightarrow V_2 = \frac{A_1}{A_2} V_1$$

**Remember:** a streamtube is constructed from a set of streamlines

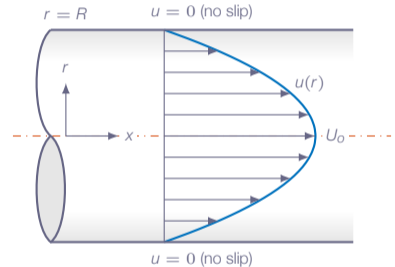
# Conservation of Mass - Example 2

Compute the average velocity for a steady **laminar** incompressible viscous flow through a circular tube with given axial velocity profile

$$u = U_o \left( 1 - \left( \frac{r}{R} \right)^2 \right)$$

## Assumptions:

1. Laminar flow
2. Steady state  $\Rightarrow$  no changes in time
3. Incompressible  $\Rightarrow$  constant density



$$V_{av} = \frac{1}{A} \int u dA = \frac{1}{\pi R^2} \int_0^R U_o \left( 1 - \left( \frac{r}{R} \right)^2 \right) 2\pi r dr = \frac{2U_o}{R^2} \int_0^R \left( 1 - \left( \frac{r}{R} \right)^2 \right) r dr$$

## Conservation of Mass - Example 2

$$V_{av} = \frac{2U_o}{R^2} \int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right) r dr = \frac{2U_o}{R^2} \left[ \frac{r^2}{2} - \frac{r^4}{4R^2} \right]_0^R = \frac{U_o}{2}$$

Thus, for **laminar** pipe flow

$$V_{av} = \frac{U_o}{2}$$



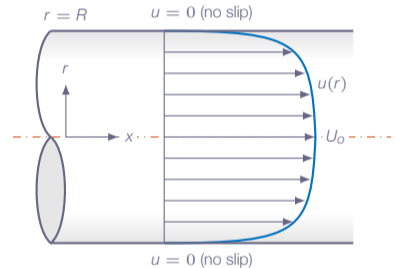
# Conservation of Mass - Example 3

Compute the average velocity for a steady **turbulent** incompressible viscous flow through a circular tube with given axial velocity profile

$$u \approx U_o \left(1 - \frac{r}{R}\right)^m$$

## Assumptions:

1. Turbulent flow:  $1/5 \geq m \geq 1/9$
2. Steady state  $\Rightarrow$  no changes in time
3. Incompressible  $\Rightarrow$  constant density



$$V_{av} \approx \frac{1}{A} \int u dA = \frac{1}{\pi R^2} \int_0^R U_o \left(1 - \frac{r}{R}\right)^m 2\pi r dr = \frac{2U_o}{R^2} \int_0^R \left(1 - \frac{r}{R}\right)^m r dr$$

## Conservation of Mass - Example 3

$$V_{av} \approx \frac{2U_o}{R^2} \int_0^R \left(1 - \frac{r}{R}\right)^m r dr = \frac{2U_o}{R^2} \left[ \frac{(r - R) \left(1 - \frac{r}{R}\right)^m (mr + r + R)}{(m + 1)(m + 2)} \right]_0^R$$

Thus, for **turbulent** pipe flow

$$V_{av} \approx \frac{2U_o}{(m + 1)(m + 2)}$$

$$m = 1/7 \Rightarrow V_{av} \approx 49U_o/60 \approx 0.82U_o$$