Fluid Mechanics - MTF053

Chapter 9

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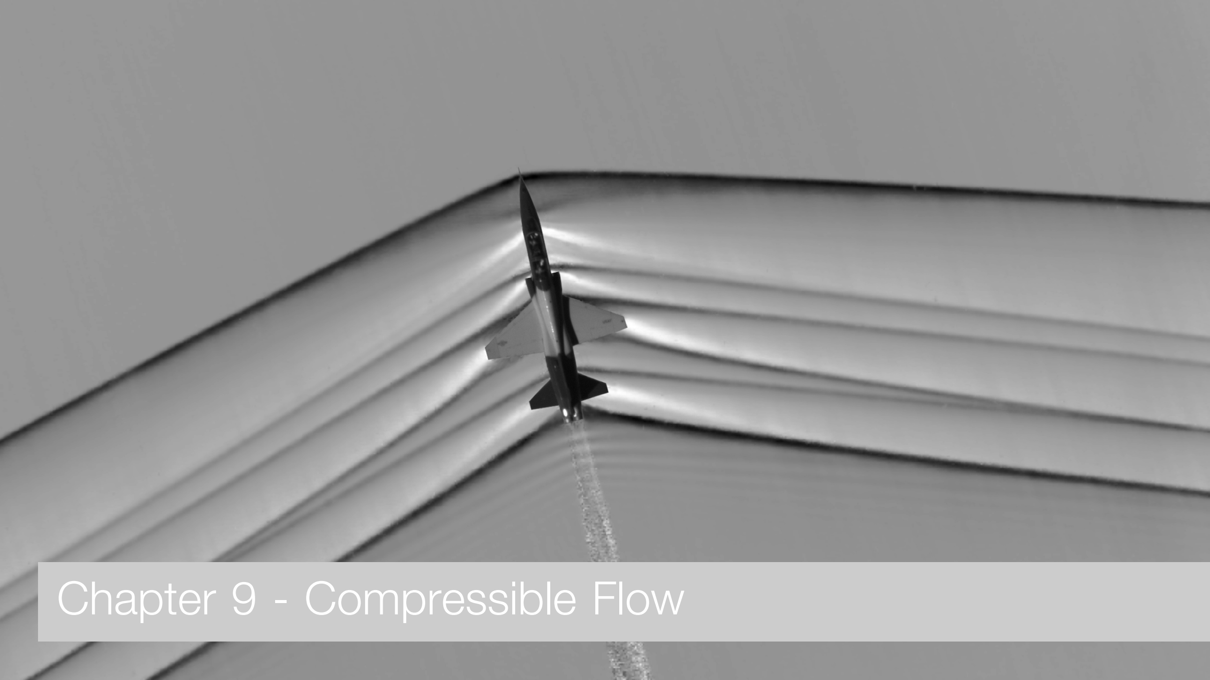
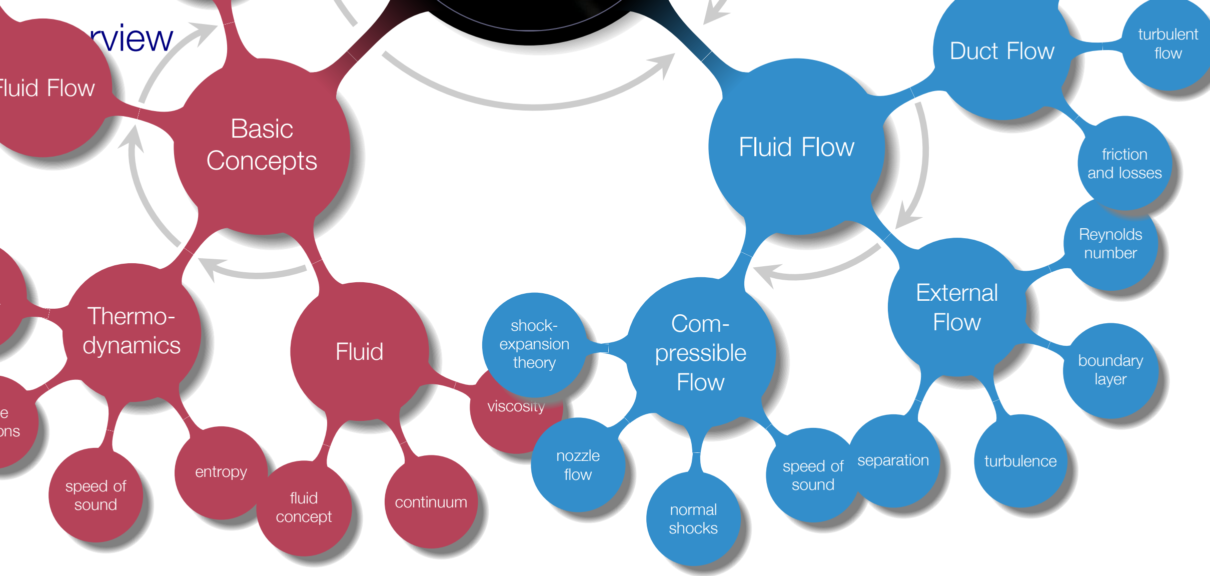


Chapter 9 - Compressible Flow


rview
Fluid Flow
Basic Concepts
Thermo-dynamics
Fluid
speed of sound
entropy
fluid concept
continuum
viscosity
Fluid Flow
Duct Flow
turbulent flow
friction and losses
Reynolds number
External Flow
boundary layer
separation
turbulence
Com-pressible Flow
shock-expansion theory
nozzle flow
normal shocks
speed of sound

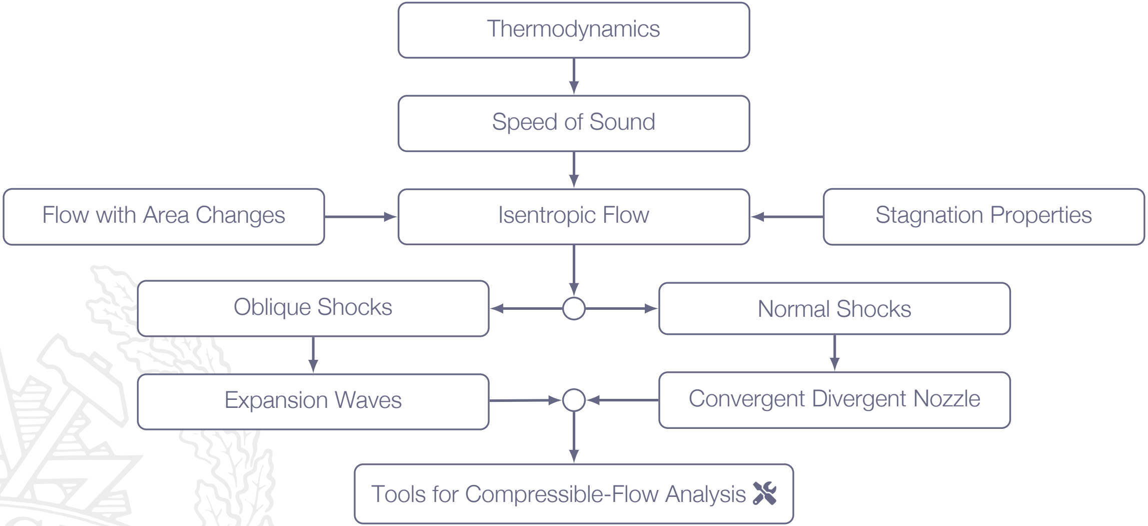
Learning Outcomes

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

37 **Understand and explain** basic concepts of compressible flows (the gas law, speed of sound, Mach number, isentropic flow with changing area, normal shocks, oblique shocks, Prandtl-Meyer expansion)

Let's go supersonic ...

Roadmap - Compressible Flow



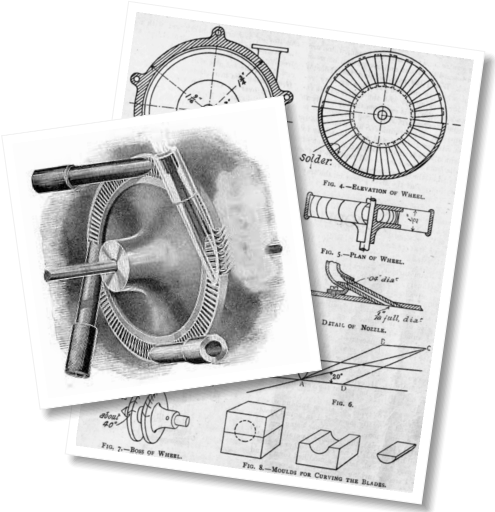
Motivation

Compressible flow:

- flows where **variations in density** are significant
- most often **high-speed gas flows** (gas dynamics)
- fluids moving at speeds comparable to the **speed of sound**
- not common in liquids (would require very high pressures)

Historical Milestones

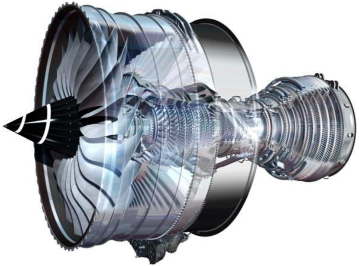
First supersonic flight - Charles Yeager 1947



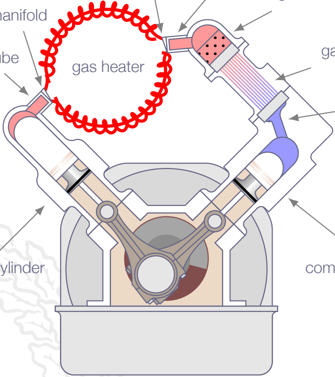
Steam turbine with convergent-divergent nozzles - Carl Gustav de Laval 1893

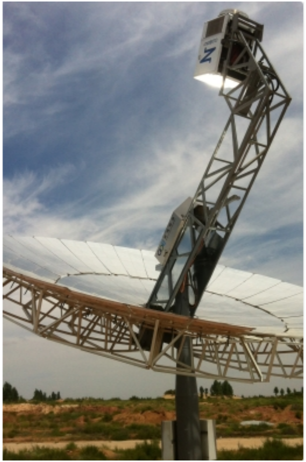
Compressible Flow Applications

Compressible Flow Applications



Compressible Flow Applications





Governing Equations

With significant density changes follows substantial changes in pressure and temperature

The **energy equation must be included**

Four equations:

1. Continuity
2. Momentum
3. Energy
4. Equation of state

Unknowns: ρ, p, T, and $\mathbf{V}$

The four equations must be solved simultaneously

Mach Number Regimes

Incompressible flow

insignificant density changes

Subsonic flow

local and global Mach number less than unity

Transonic flow

1. subsonic flow with regions of supersonic flow (local Mach number can be higher than one)
2. supersonic flow with regions of subsonic flow (local Mach number can be less than one)

Supersonic flow

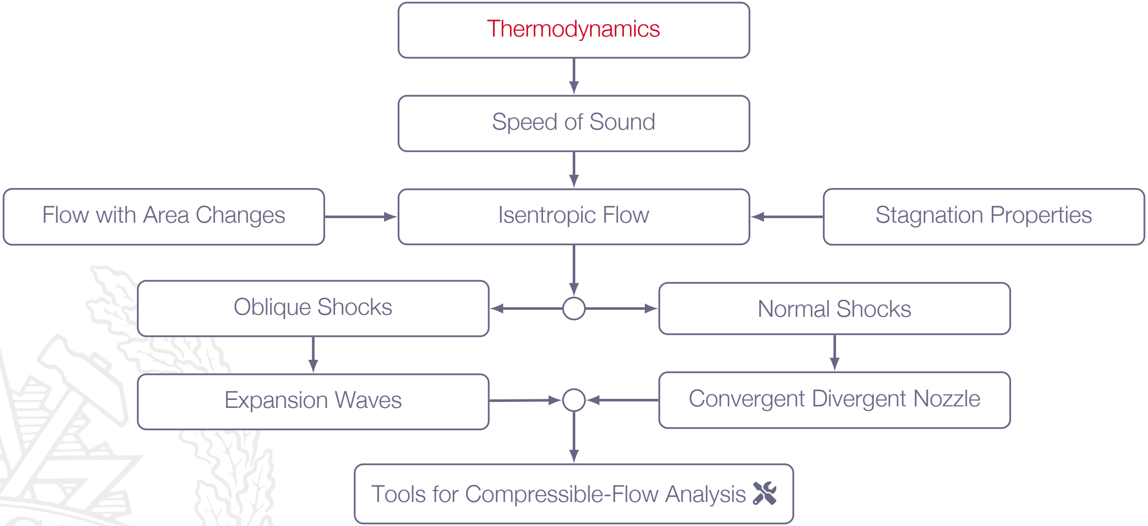
local and global Mach number higher than one

Hypersonic flow

Mach number higher than 5.0

Increasing Mach number

Roadmap - Compressible Flow



Ratio of Specific Heats

The ratio of specific heats is important in compressible flow

$$\gamma = \frac{C_p}{C_v}$$

γ is a fluid property

For **moderate temperatures** γ is a **constant**

For **higher temperatures** γ **varies** with temperature

(*for air:* $\gamma = 1.4$)

Equation of State

In the following, we will assume that the **ideal gas law** is applicable and that the **specific heats** are **constants**:

$$p = \rho R T$$

$$R = C_p - C_v = const$$

$$\gamma = \frac{C_p}{C_v} = const$$

Auxiliary relations:

$$C_v = \frac{R}{\gamma - 1}, C_p = \frac{\gamma R}{\gamma - 1}$$

Internal Energy and Enthalpy

Constant specific heats:

$$d\hat{u} = C_v dT$$

$$dh = C_p dT$$

Variable specific heats:

$$\hat{u} = \int C_v dT$$

$$h = \int C_p dT$$

Isentropic Relations

First law of thermodynamics

$$\delta q + \delta w = de$$

For reversible processes: $\delta w = -pd\upsilon$ (where $\upsilon = p/\rho$)

$$h = e + \frac{p}{\rho} = e + p\upsilon \Rightarrow dh = de + pd\upsilon + \upsilon dp$$

$$\delta q = dh - \upsilon dp$$

Second law of thermodynamics

$$ds = \frac{\delta q_{rev}}{T} = \frac{\delta q}{T} + ds_{irev} \Rightarrow ds \geq \frac{\delta q}{T}$$

Isentropic Relations

compute entropy change from the first and second law of thermodynamics (assuming reversible heat addition)

$$Tds = dh - \frac{dp}{\rho}$$

for perfect gases, $dh = C_p dT$

$$\int_1^2 ds = \int_1^2 C_p \frac{dT}{T} - R \int_1^2 \frac{dp}{p}$$

for constant specific heats (calorically perfect)

$$s_2 - s_1 = C_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} = C_v \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1}$$

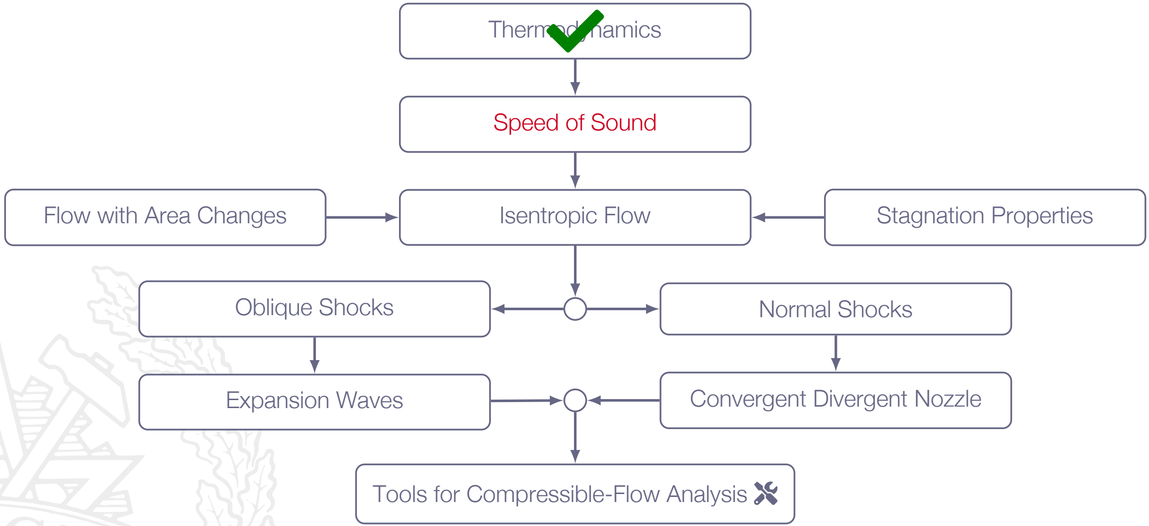
Isentropic Relations

$$s_2 - s_1 = C_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} = C_v \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1}$$

for isentropic flow ($s_2 = s_1$) we get

$$\boxed{\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}}$$

Roadmap - Compressible Flow



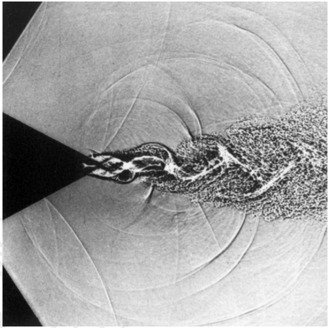
Speed of Sound

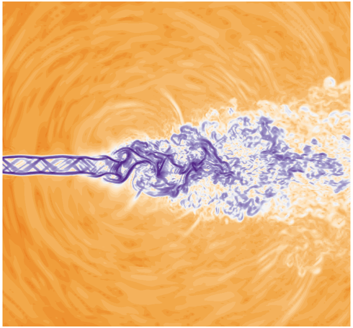
The rate of propagation of a **pressure pulse of infinitesimal strength** through a **fluid at rest**

Related to the molecular activity of the fluid

A thermodynamic property

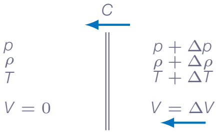
Speed of Sound



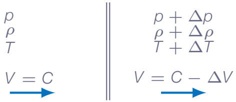


Speed of Sound

frame of reference fixed to fluid



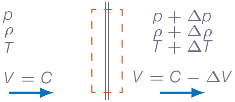
frame of reference following the wave



Speed of Sound

frame of reference following the wave

continuity:



$$\rho A C = (\rho + \Delta\rho) A (C - \Delta V)$$

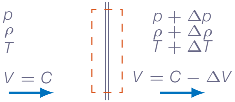
$$\Delta V = C \frac{\Delta\rho}{\rho + \Delta\rho}$$

Note! there are no gradients in the flow so viscous effects are confined to the interior of the wave

Speed of Sound

frame of reference following the wave

momentum:



$$pA - (p + \Delta p)A = (\rho A C)(C - \Delta V - C) \Rightarrow \Delta p = \rho C \Delta V$$

with ΔV from the continuity equation we get

$$C^2 = \frac{\Delta p}{\Delta \rho}\left(1 + \frac{\Delta \rho}{\rho}\right)$$

Note! the larger $\Delta\rho/\rho$, the higher the propagation velocity

Speed of Sound

In the limit of infinitesimal strength $\Delta\rho \to 0$ and thus

$$C^2 = a^2 = \frac{\partial p}{\partial \rho}$$

- There is no added heat and thus the process adiabatic
- For weak waves the process can also be assumed to be reversible

$$a^2 = \left.\frac{\partial p}{\partial \rho}\right|_s$$

Speed of Sound

$$a^2 = \left.\frac{\partial p}{\partial \rho}\right|_s$$

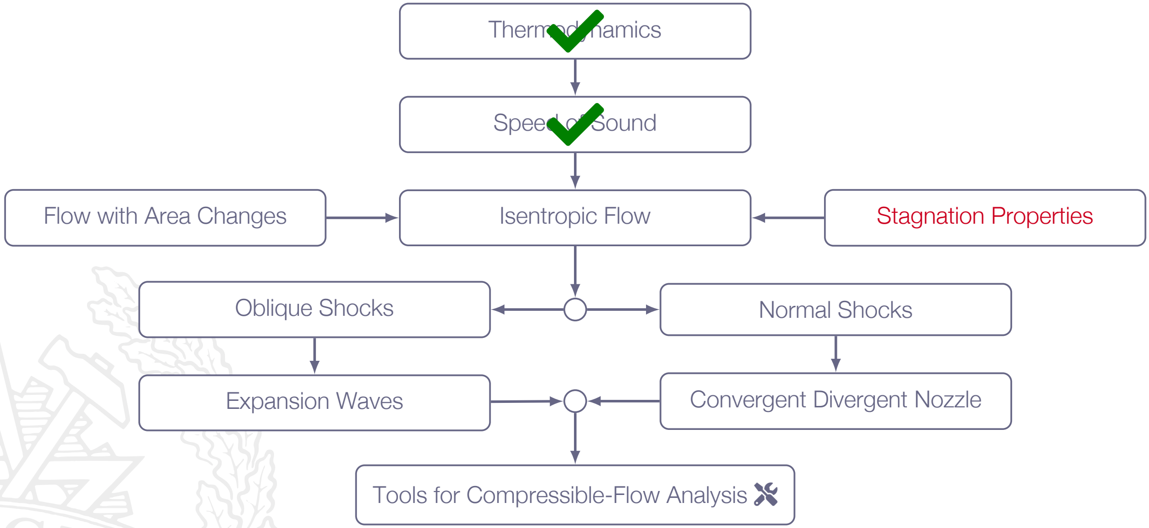
The isentropic relation gives

$$p = \rho^\gamma \Rightarrow \frac{\partial p}{\partial \rho} = \gamma \rho^{\gamma-1} = \gamma \frac{p}{\rho} = \gamma RT$$

and thus

$$\boxed{a = \sqrt{\gamma RT}}$$

Roadmap - Compressible Flow



Stagnation Enthalpy

Consider high-speed gas flow past an insulated wall

$$h_1 + \frac{1}{2}V_1^2 + gz_1 = h_2 + \frac{1}{2}V_2^2 + gz_2 - q + w_\nu$$

1. differences in potential energy extremely small
2. outside of the boundary layer, heat transfer and viscous work are zero

$$h_1 + \frac{1}{2}V_1^2 = h_2 + \frac{1}{2}V_2^2 = const$$

Stagnation Enthalpy

$$\boxed{h + \frac{1}{2}V^2 = h_o}$$

"The maximum enthalpy that the fluid would achieve if brought to rest adiabatically"

Stagnation Temperature

For a calorically perfect gas $h = C_p T$

$$h + \frac{1}{2}V^2 = h_o$$

$$C_p T + \frac{1}{2}V^2 = C_p T_o$$

Where T_o is the stagnation temperature

Mach Number Relations

$$C_pT + \frac{1}{2}V^2 = C_pT_o \Rightarrow 1 + \frac{V^2}{2C_pT} = \frac{T_o}{T}$$

$$C_pT = \frac{\gamma R}{\gamma - 1}T = \frac{\gamma RT}{\gamma - 1} = \frac{a^2}{\gamma - 1}$$

$$\boxed{\frac{T_o}{T} = 1 + \left(\frac{\gamma - 1}{2}\right)M^2}$$

Mach Number Relations

Since $a \propto T^{1/2}$ we get

$$\frac{a_o}{a} = \left(\frac{T_o}{T}\right)^{1/2} = \left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right]^{1/2}$$

Mach Number Relations

If the flow is adiabatic and reversible (isentropic), we may use the isentropic relations

$$\frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{\gamma/(\gamma-1)} = \left[1 + \left(\frac{\gamma-1}{2}\right) M^2\right]^{\gamma/(\gamma-1)}$$

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{1/(\gamma-1)} = \left[1 + \left(\frac{\gamma-1}{2}\right) M^2\right]^{1/(\gamma-1)}$$

Stagnation Properties

p_o and ρ_o - the pressure and density that the flow would achieve if brought to rest isentropically

All stagnation properties are constants in an isentropic flow

h_o, T_o, and a_o are constants in an adiabatic flow but not necessarily p_o and ρ_o

p_o and ρ_o will vary throughout an adiabatic flow as the entropy changes due to friction or shocks

Critical Properties

Another useful set of reference variables is the critical properties (sonic conditions)

$$\frac{T_o}{T} = 1 + \left(\frac{\gamma - 1}{2}\right) M^2 = \{M = 1.0\} = 1 + \left(\frac{\gamma - 1}{2}\right) = \left(\frac{2 + \gamma - 1}{2}\right) = \left(\frac{\gamma + 1}{2}\right)$$

Critical Properties

$$\frac{T^*}{T_o} = \left(\frac{2}{\gamma + 1}\right)$$

$$\frac{a^*}{a_o} = \left(\frac{2}{\gamma + 1}\right)^{1/2}$$

$$\frac{p^*}{p_o} = \left(\frac{2}{\gamma + 1}\right)^{\gamma/(\gamma-1)}$$

$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma + 1}\right)^{1/(\gamma-1)}$$

Critical Properties

Air $\gamma = 1.4$

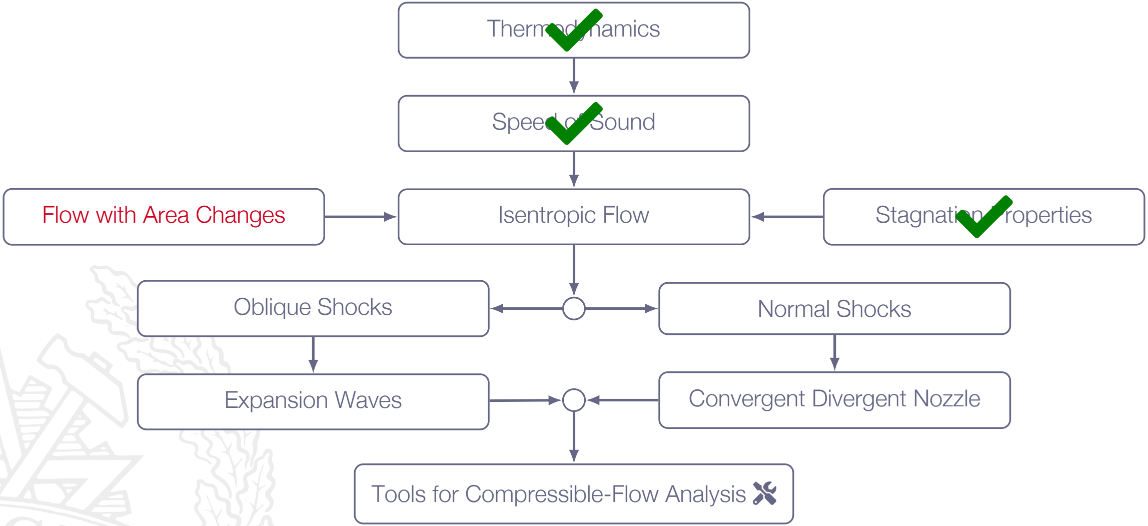
$$\frac{T^*}{T_o} = \left(\frac{2}{\gamma+1}\right) = 0.8333$$

$$\frac{a^*}{a_o} = \left(\frac{2}{\gamma+1}\right)^{1/2} = 0.9129$$

$$\frac{p^*}{p_o} = \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} = 0.5283$$

$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)} = 0.6339$$

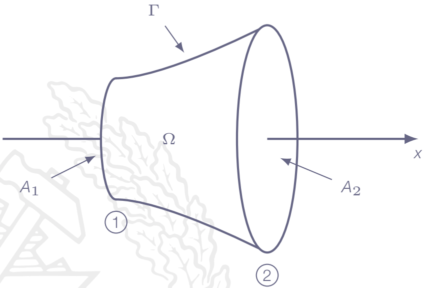
Roadmap - Compressible Flow



Isentropic Quasi-1D Flow

Quasi-1D Assumptions:

1. Flow properties varies in one direction only (x)
2. The flow area is a smooth function $A = A(x)$
3. Steady-state, inviscid and isentropic flow



The Area-Velocity Relation

Continuity:

$$\rho(x)V(x)A(x) = const \Rightarrow d(\rho VA) = 0 \Rightarrow AVd\rho + \rho AdV + \rho VdA = 0$$

divide by ρVA gives

$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0$$

The Area-Velocity Relation

In the following, isentropic flow is assumed

Stagnation enthalpy:

$$h_o = h + \frac{1}{2}V^2 = const \Rightarrow dh + VdV = 0$$

The first and second law of thermodynamics:

$$Tds = 0 = dh - \frac{dp}{\rho} \Rightarrow dh = \frac{dp}{\rho}$$

and thus

$$\frac{dp}{\rho} + VdV = 0$$

The Area-Velocity Relation

$$\frac{dp}{\rho} + VdV = 0$$

From the definition of the **speed of sound**

$$dp = a^2 d\rho \Rightarrow a^2 \frac{d\rho}{\rho} + VdV = 0 \Rightarrow \frac{d\rho}{\rho} = -\frac{1}{a^2} VdV$$

The Area-Velocity Relation

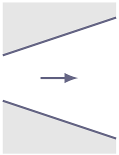
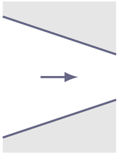
$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = \frac{dV}{V} - \frac{1}{a^2}VdV + \frac{dA}{A} = 0$$

$$\frac{dV}{V}\left(\frac{V^2}{a^2} - 1\right) = \frac{dA}{A}$$

$$\boxed{\frac{dV}{V} = \frac{1}{M^2 - 1}\frac{dA}{A} = -\frac{dp}{\rho V^2}}$$

The Area-Velocity Relation

$$\boxed{\frac{dV}{V}(M^2-1)=\frac{dA}{A}}$$

		Subsonic $M<1$	**Supersonic** $M>1$
	$dA>0$	subsonic diffuser $dV<0$ $dp>0$	supersonic nozzle $dV>0$ $dp<0$
	$dA<0$	subsonic nozzle $dV>0$ $dp<0$	supersonic diffuser $dV<0$ $dp>0$

The Area-Velocity Relation

$$\boxed{\frac{dV}{V}(M^2-1)=\frac{dA}{A}}$$

What happens when $M=1$?

The Area-Velocity Relation

$$\boxed{\frac{dV}{V}(M^2-1)=\frac{dA}{A}}$$

What happens when $M = 1$?

$M = 1$ when $dA = 0$

The Area-Velocity Relation

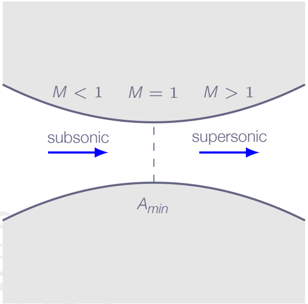
$$\boxed{\frac{dV}{V}(M^2-1)=\frac{dA}{A}}$$

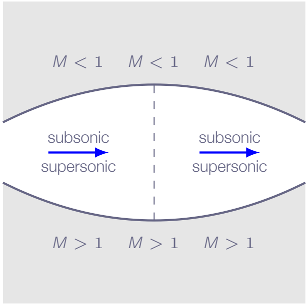
What happens when $M=1$?

$M=1$ when $dA=0$

maximum or minimum area

The Area-Velocity Relation





The Area-Mach-Number Relation

$$\rho A V = \rho^* A^* V^* \Rightarrow \frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{V^*}{V}$$

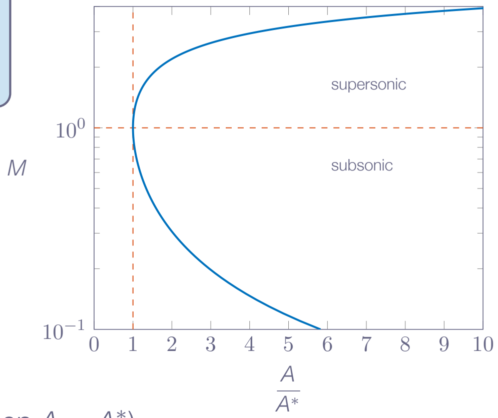
$$\frac{\rho^*}{\rho} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho} = \left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{1/(\gamma-1)}$$

$$\frac{V^*}{V} = \frac{(\gamma R T^*)^{1/2}}{V} = \frac{(\gamma R T)^{1/2}}{V}\left(\frac{T^*}{T_o}\right)^{1/2}\left(\frac{T_o}{T}\right)^{1/2} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{1/2}$$

$$\boxed{\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{(\gamma+1)/(\gamma-1)}}$$

The Area-Mach-Number Relation

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{(\gamma+1)/(\gamma-1)}$$



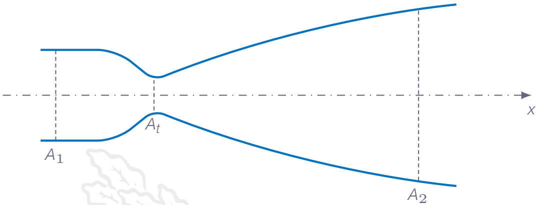
Note!

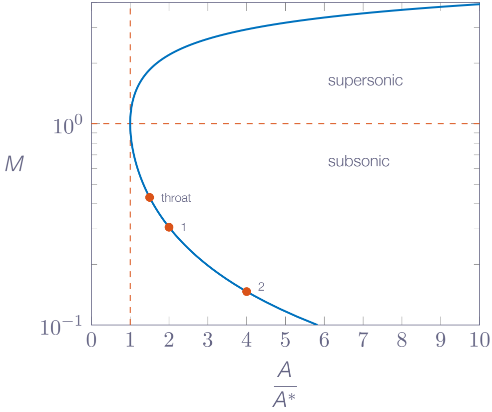
Two possible solutions for each value of $\dfrac{A}{A^*}$:

one subsonic and one supersonic (except when $A = A^*$)

The Area-Mach-Number Relation

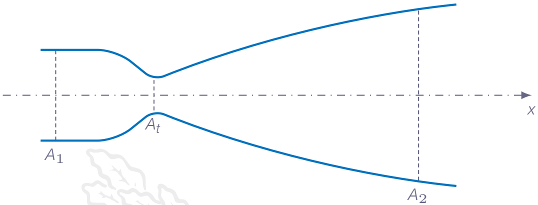
Sub-critical (**non-choked**) nozzle flow

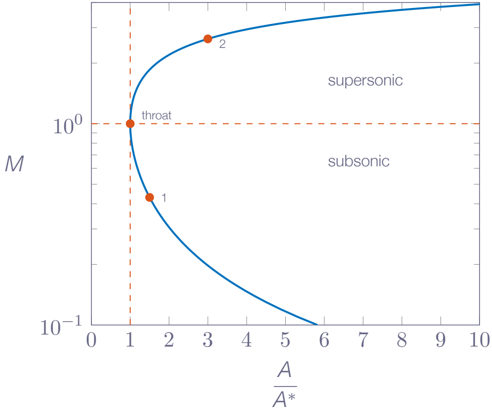




The Area-Mach-Number Relation

Critical (**choked**) nozzle flow





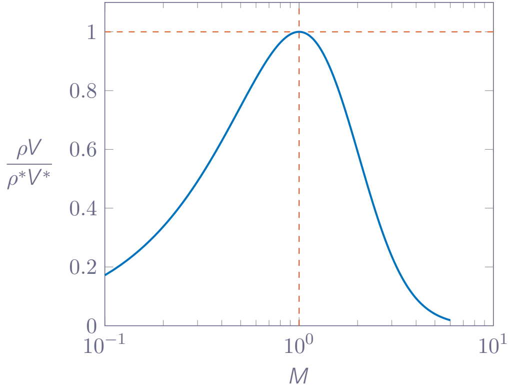
Choking

$$\rho V A = \rho^* V^* A^* \Rightarrow \frac{A^*}{A} = \frac{\rho V}{\rho^* V^*}$$

ρV – *massflow per unit area*

From the area-Mach-number relation:

$$\frac{A^*}{A} = \begin{cases} < 1 & \text{if} \quad M < 1 \\ 1 & \text{if} \quad M = 1 \\ < 1 & \text{if} \quad M > 1 \end{cases}$$



The **maximum** possible **mass flow** through a duct is achieved when its throat reaches **sonic conditions**

Choking

1. $\dot{m}_{max} = \rho^* A^* V^*$

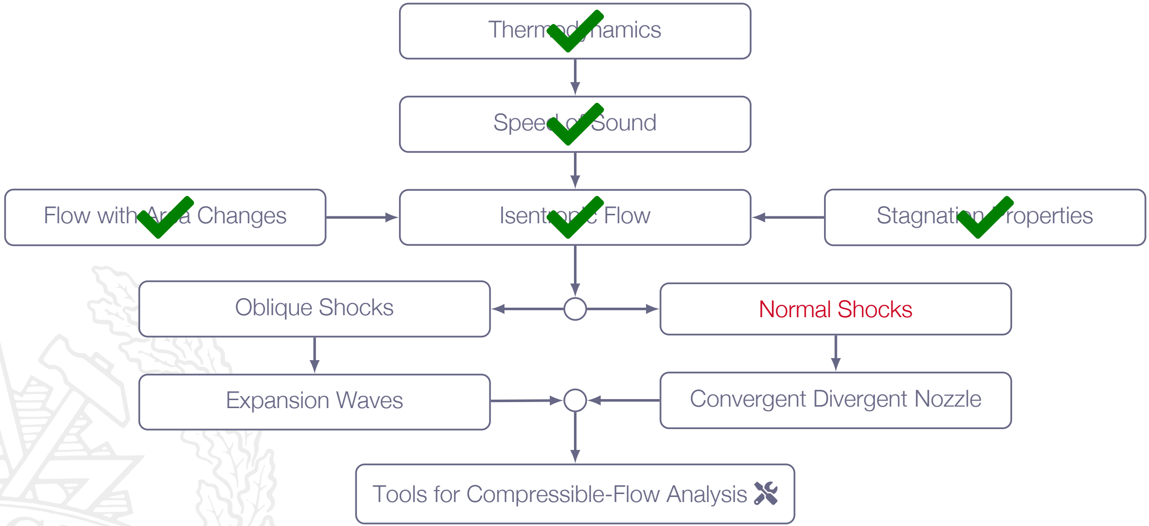
2. $\dfrac{\rho^*}{\rho_o} = \left(\dfrac{2}{\gamma+1}\right)^{1/(\gamma-1)}$

3. $V^* = \sqrt{\gamma R T^*}$

4. $\dfrac{T^*}{T_o} = \dfrac{2}{\gamma+1}$

$$\boxed{\dot{m}_{max} = \frac{A^* p_o}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{\frac{(\gamma+1)}{(\gamma-1)}}}}$$

Roadmap - Compressible Flow



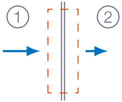
Shock Waves

"Shock waves are nearly discontinuous changes in a supersonic flow"

Reasons for the appearance of shocks in a flow can be for example:

1. higher downstream pressure
2. sudden changes in flow direction
3. blockage by a downstream body
4. explosion

Normal Shocks



Continuity:

$$\rho_1 u_1 = \rho_2 u_2$$

Momentum:

$$p_1 - p_2 = \rho_2 u_2^2 - \rho_1 u_1^2$$

Energy:

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2 = h_o$$

The Rankine-Hugoniot relation:

$$h_2 - h_1 = \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right)$$

Normal Shocks

$$h_2 - h_1 = \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right)$$

Note! The **Rankine-Hugoniot** relation **only** includes **thermodynamic properties** (no velocities) and gives a relation between the flow state upstream of the shock and the flow downstream of the shock

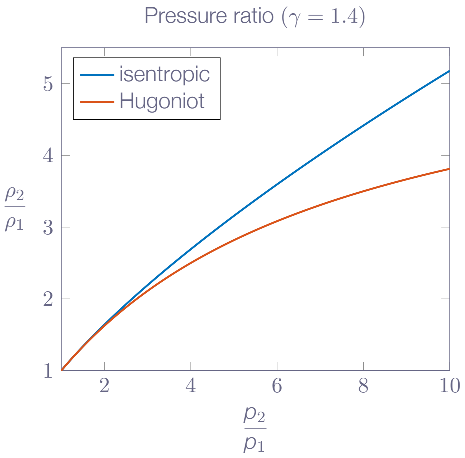
Normal Shocks

The Rankine-Hugoniot relation

$$\frac{\rho_2}{\rho_1} = \frac{1 + \left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right)}{\left(\frac{\gamma+1}{\gamma-1}\right) + \left(\frac{p_2}{p_1}\right)}$$

The isentropic relation

$$\frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{1/\gamma}$$



Normal Shocks

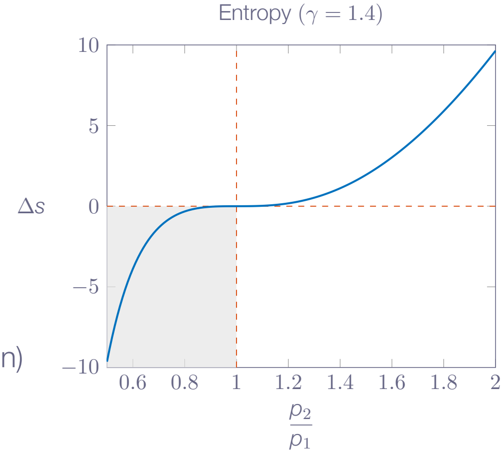
The second law of thermodynamics

$$s_2 - s_1 = C_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

can be rewritten as

$$s_2 - s_1 = C_v \ln \left[\frac{p_2}{p_1} \left(\frac{\rho_1}{\rho_2} \right)^{\gamma} \right]$$

(ρ_2/ρ_1 from the Rankine-Hugoniot relation)



Note! a reduction of entropy is a violation of the second law of thermodynamics

Normal Shocks

For a perfect gas, it is possible to obtain relations for normal shocks that only include upstream variables

Momentum equation: $p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$

$$p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2 = \{\rho_1 u_1 = \rho_2 u_2\} = \rho_1 u_1 (u_1 - u_2) = \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

divide by p_1

$$\frac{p_2}{p_1} = 1 + \frac{\rho_1 u_1^2}{p_1}\left(1 - \frac{u_2}{u_1}\right)$$

$$u_1^2 = M_1^2 a_1^2 = M_1^2 \gamma R T_1 = \gamma M_1^2 \frac{p_1}{\rho_1} \Rightarrow \frac{p_2}{p_1} = 1 + \gamma M_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

Normal Shocks

$$\frac{p_2}{p_1} = 1 + \gamma M_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

Using the energy equation its possible obtain a relation for $\frac{u_2}{u_1}$ (the derivation is quite lengthy though)

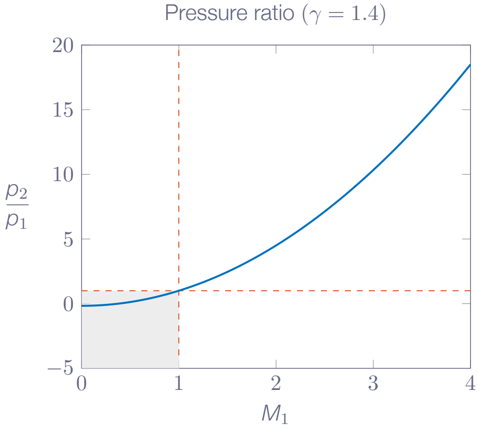
$$\frac{u_2}{u_1} = \frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2}$$

and thus

$$\boxed{\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right)}$$

Normal Shocks

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right)$$



Note! from before we know that p_2/p_1 must be greater than 1.0, which means that M_1 must be greater than 1.0

Normal Shocks

Momentum equation: $p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$

$$M = \frac{u}{a} \Rightarrow p_1 + \rho_1 M_1^2 a_1^2 = p_2 + \rho_2 M_2^2 a_2^2$$

$$a = \sqrt{\gamma R T} = \sqrt{\frac{\gamma p}{\rho}} \Rightarrow p_1 + \rho_1 M_1^2 \frac{\gamma p_1}{\rho_1} = p_2 + \rho_2 M_2^2 \frac{\gamma p_2}{\rho_2}$$

$$p_1 \left(1 + \gamma M_1^2\right) = p_2 \left(1 + \gamma M_2^2\right)$$

$$\boxed{\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}}$$

Normal Shocks

Two ways to calculate the pressure ratio over the shock

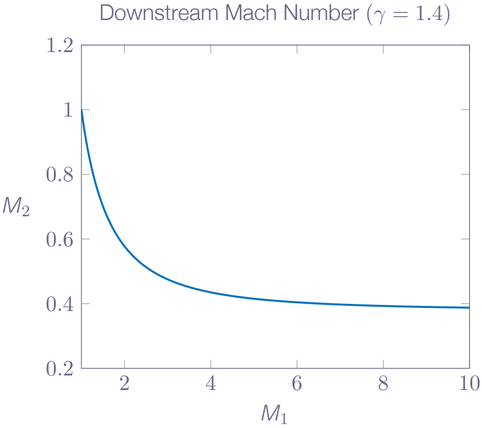
$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right) \qquad \frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}$$

Setting the relations equal gives a relation for the downstream Mach number

$$\boxed{M_2^2 = \frac{(\gamma - 1)M_1^2 + 2}{2\gamma M_1^2 - (\gamma - 1)}}$$

Normal Shocks

$$M_2^2 = \frac{(\gamma - 1)M_1^2 + 2}{2\gamma M_1^2 - (\gamma - 1)}$$

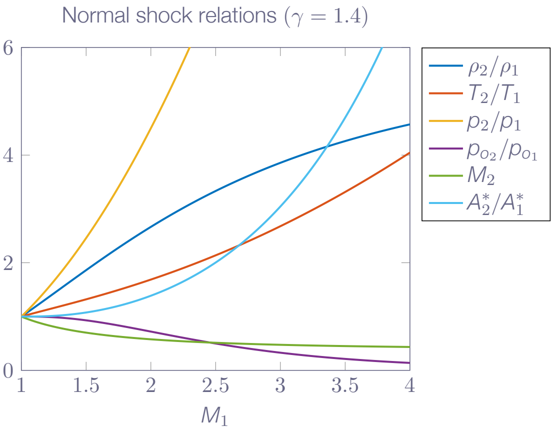


Note! for $\gamma > 1$ and $M_1 > 1$, the downstream Mach number **must** be less than 1.0, i.e we will **always** have subsonic flow downstream of a normal shock

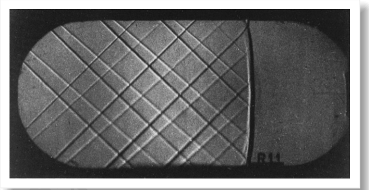
Normal Shocks - Summary

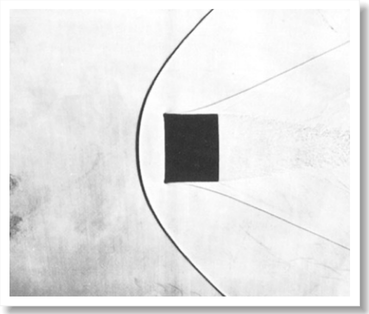
1. **Supersonic** flow **upstream** of normal shock
2. **Subsonic** flow **downstream** of normal shock
3. **Entropy increases** over the shock and consequently **total pressure decreases**
4. Sonic throat area increases
5. Very weak shock waves are nearly isentropic

Normal Shocks - Trends



Normal Shocks - Examples





Moving Normal Shocks

$W \rightarrow$

$p + \Delta p$	p	
$\rho + \Delta \rho$	ρ	frame of reference:
$T + \Delta T$	T	fixed to the fluid
$u = u_p$	$u = 0$	

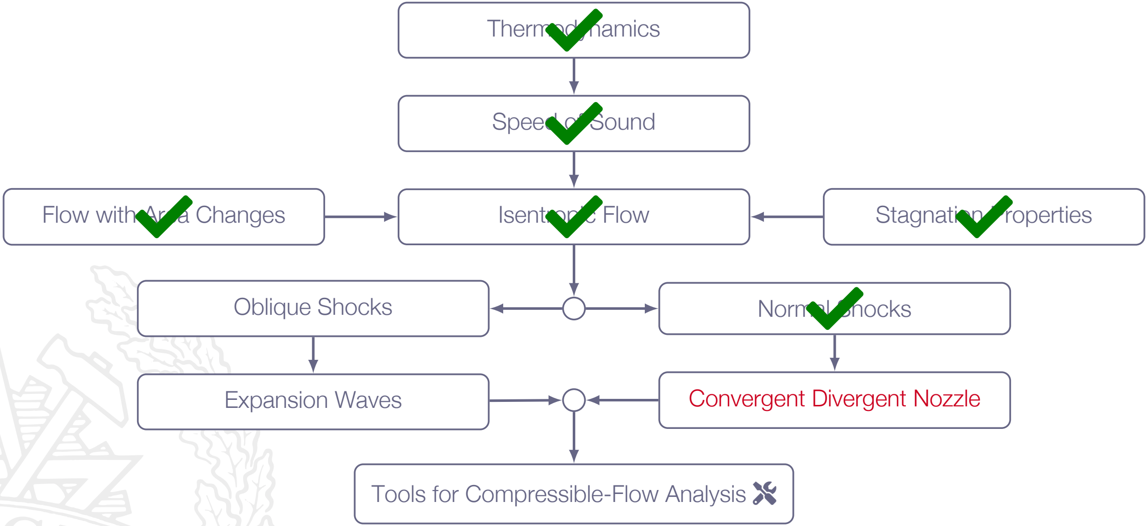
Change **frame of reference**:

1. coordinate system moving with the shock
2. thermodynamic properties does not change

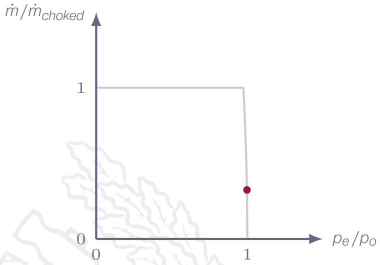
$p + \Delta p$	p	
$\rho + \Delta \rho$	ρ	frame of reference:
$T + \Delta T$	T	following the shock
$u = u_p - W$	$u = -W$	

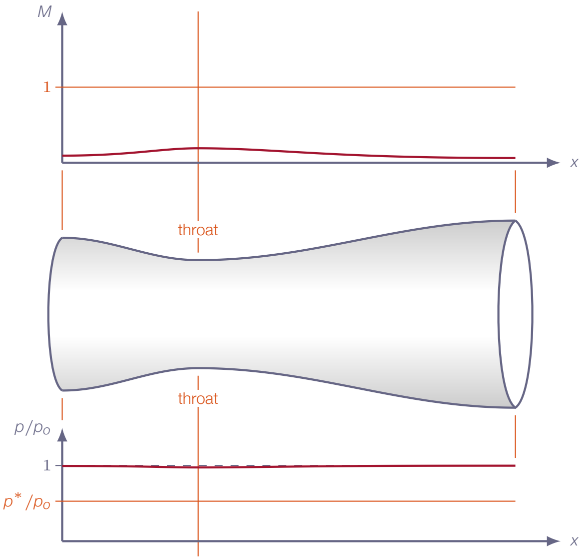
$$\boxed{h_2 - h_1 = \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)}$$

Roadmap - Compressible Flow

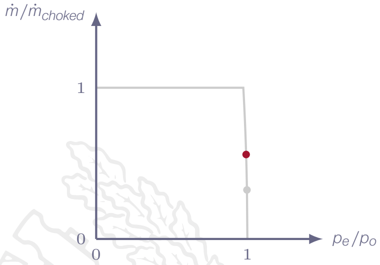


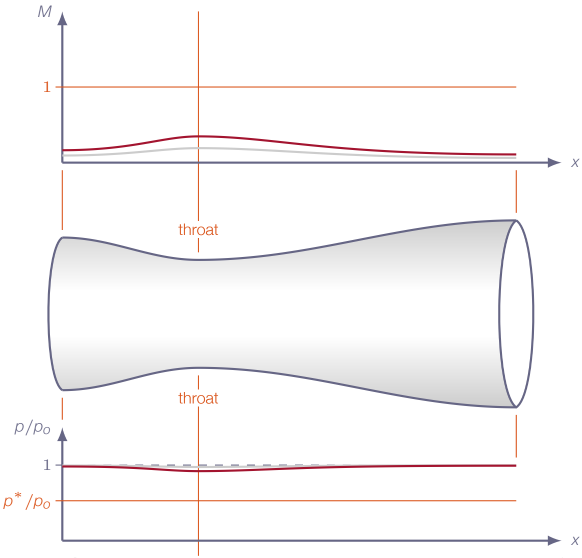
Convergent-Divergent Nozzle



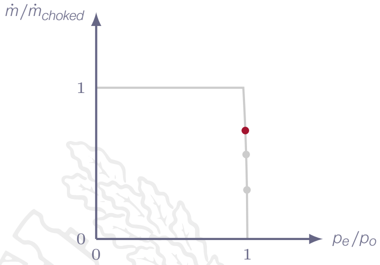


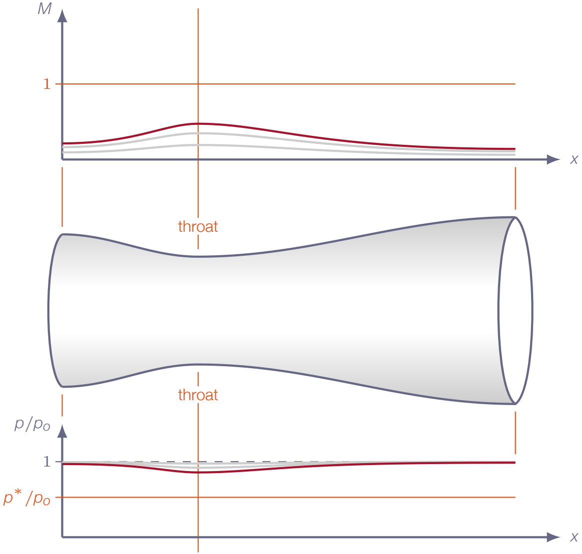
Convergent-Divergent Nozzle





Convergent-Divergent Nozzle





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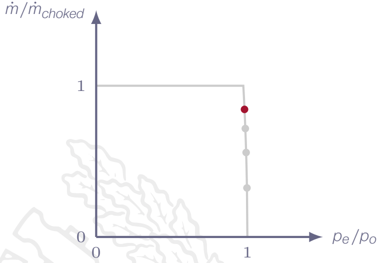

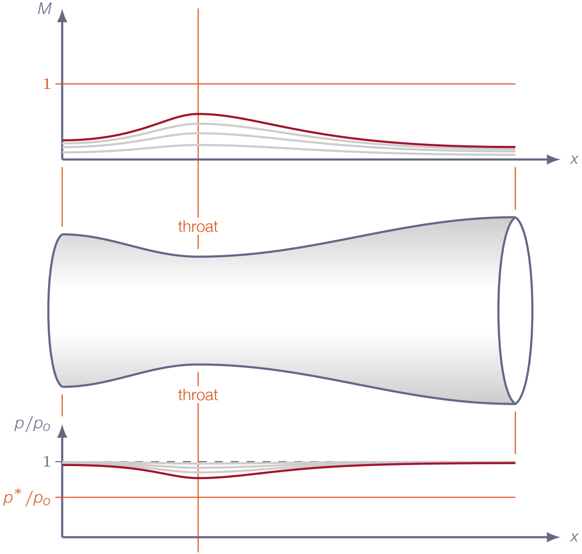

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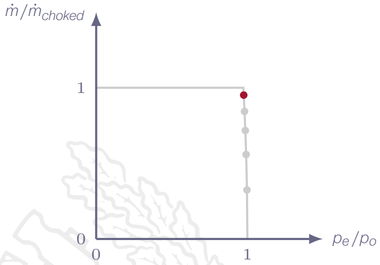

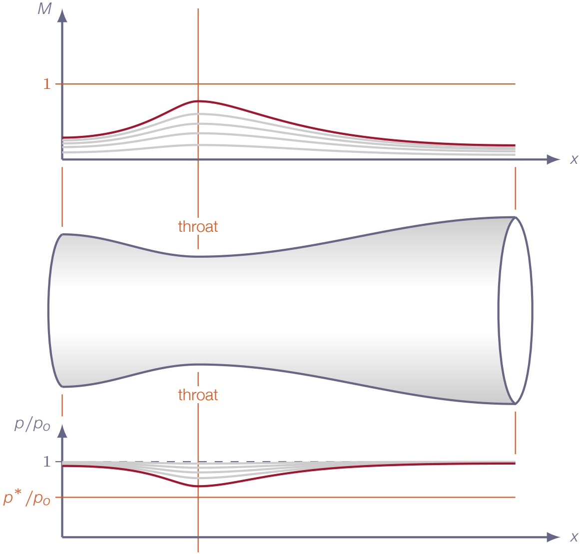

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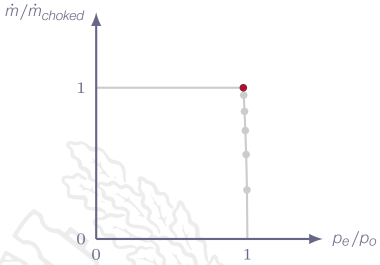

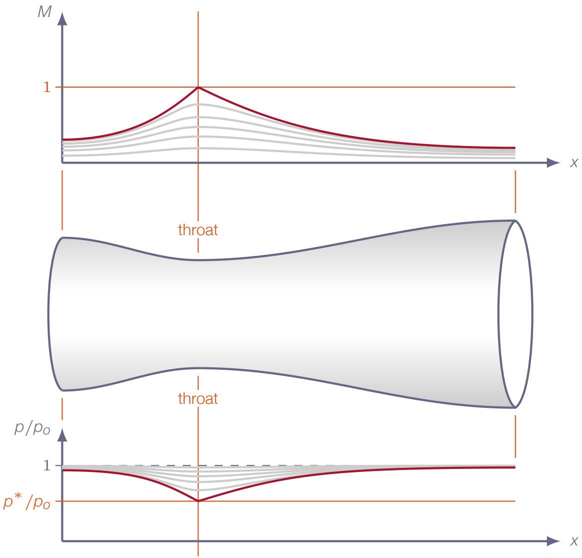

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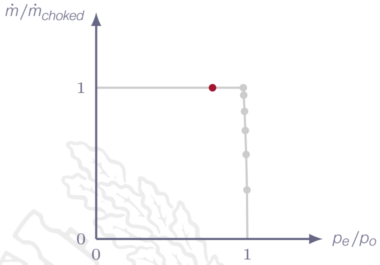

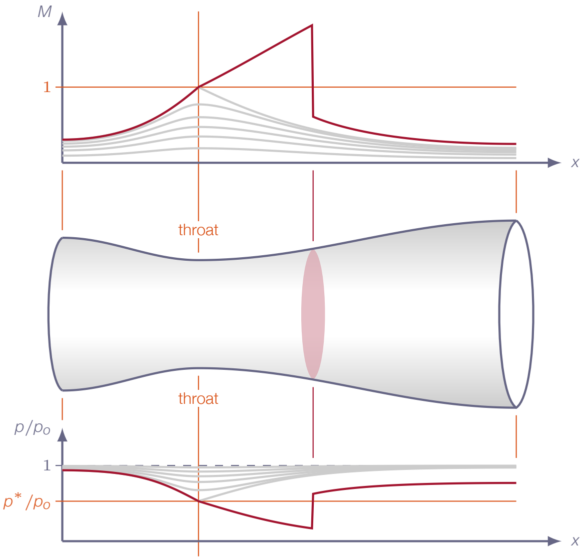

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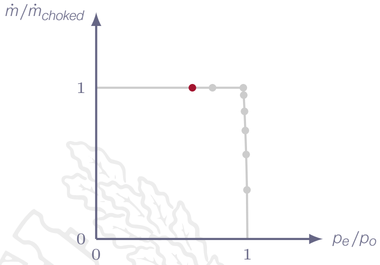
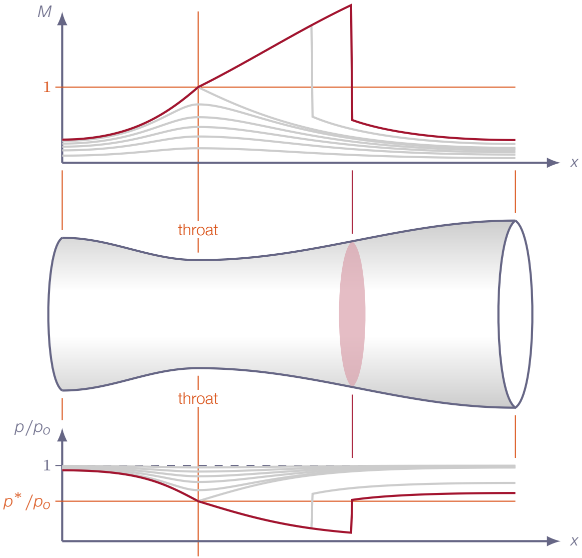
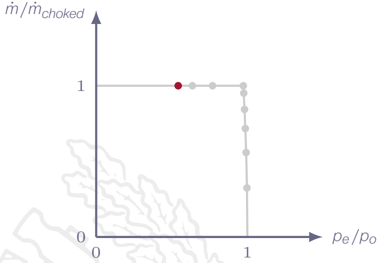
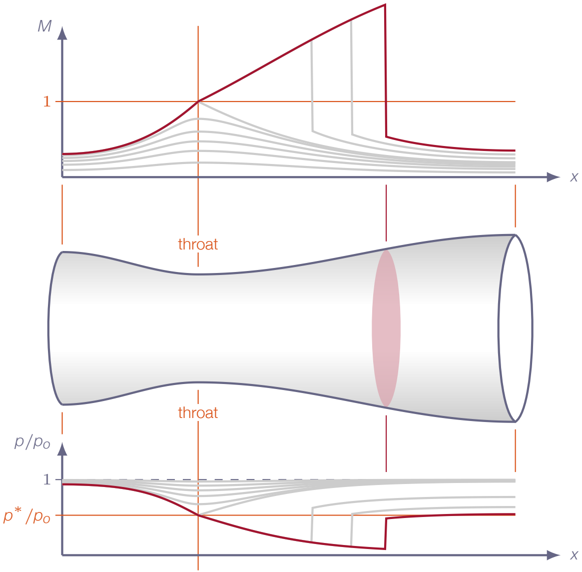


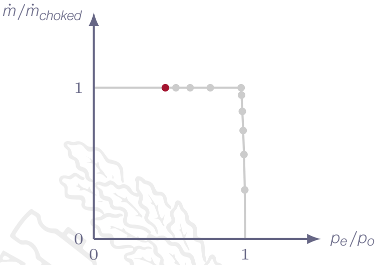


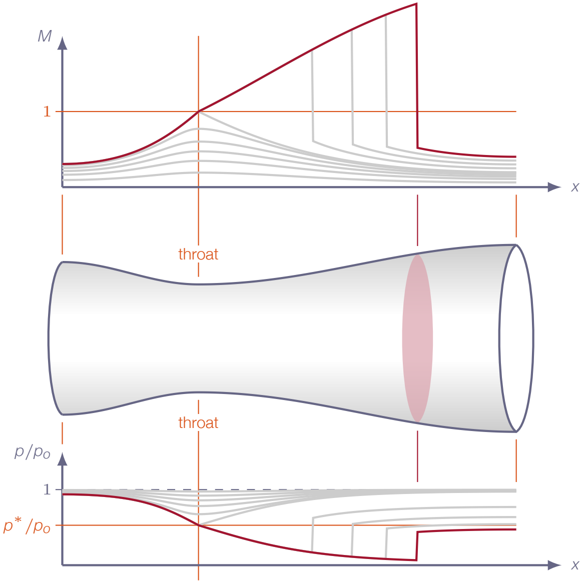
Convergent-Divergent Nozzle





Convergent-Divergent Nozzle





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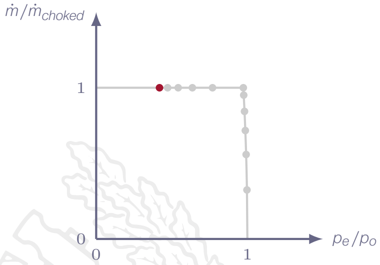
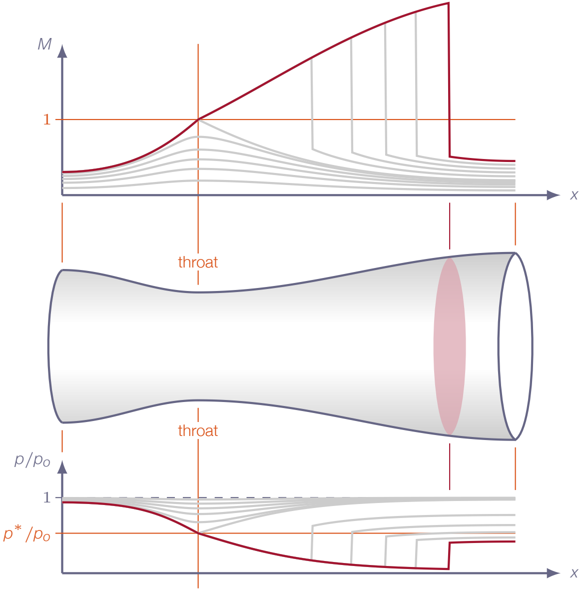
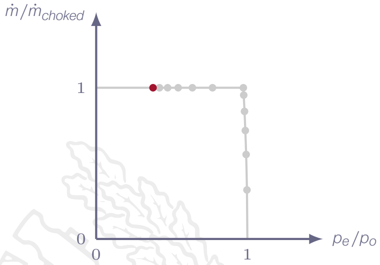
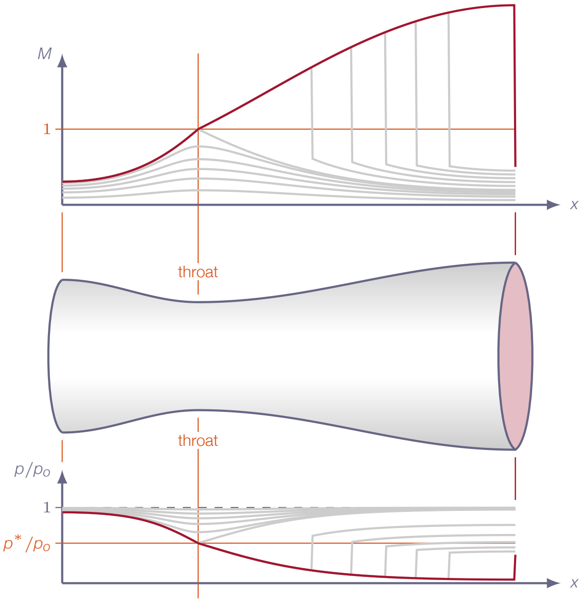


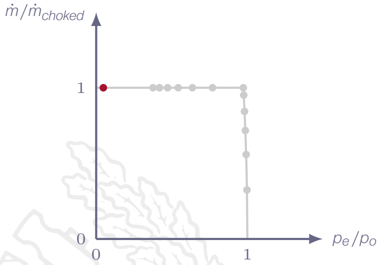


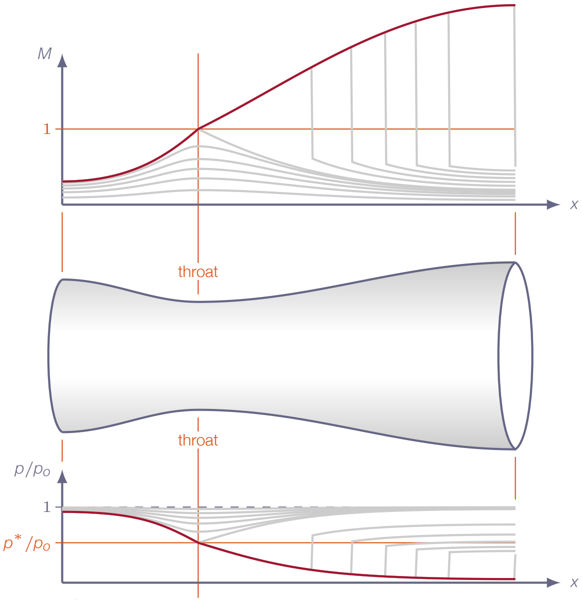
Convergent-Divergent Nozzle



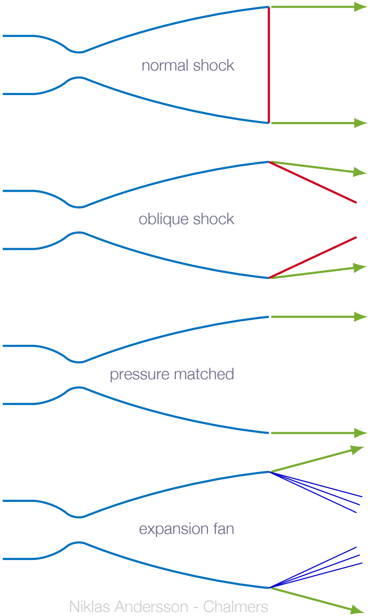


Convergent-Divergent Nozzle





Convergent-Divergent Nozzle



$p_o/p_e = (p_o/p_e)_{ne}$
normal shock at nozzle exit

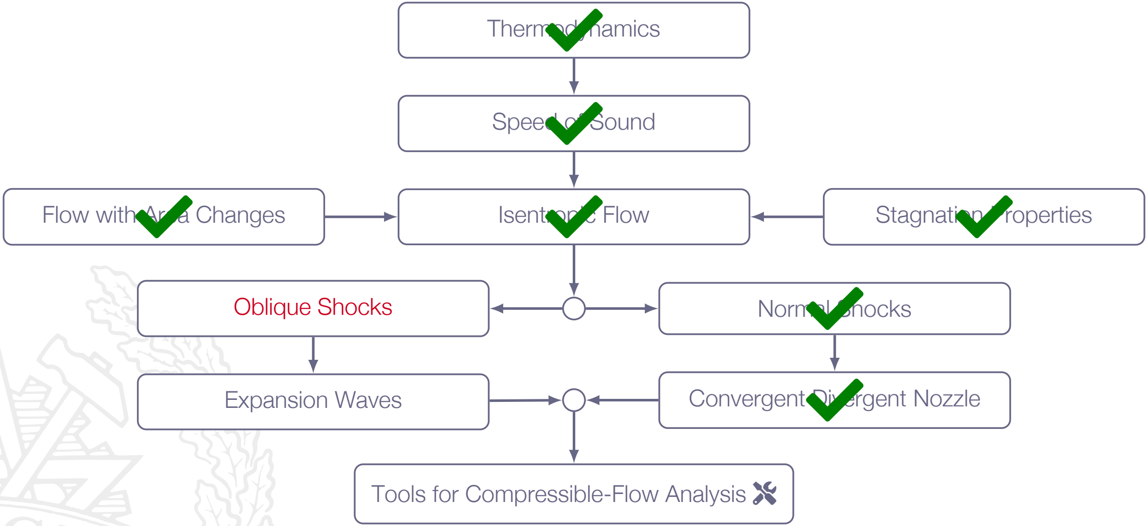
$(p_o/p_e)_{ne} < p_o/p_e < (p_o/p_e)_{sc}$
overexpanded nozzle flow

$p_o/p_e = (p_o/p_e)_{sc}$
pressure matched nozzle flow

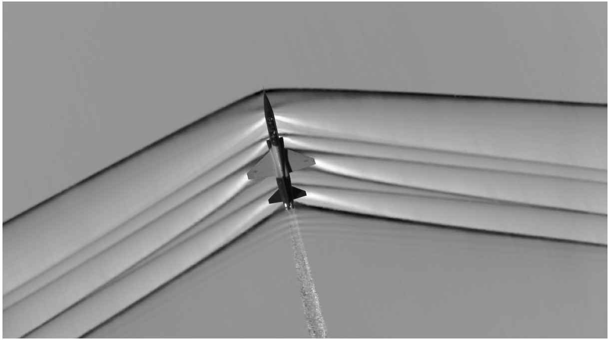
$p_o/p_e > (p_o/p_e)_{sc}$
underexpanded nozzle flow

Convergent-Divergent Nozzle

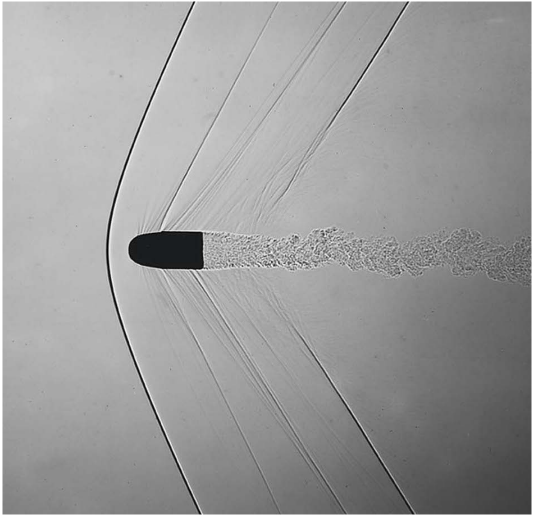
Roadmap - Compressible Flow



Oblique Shocks

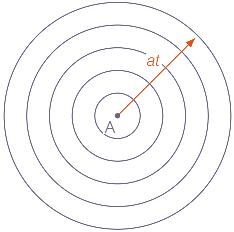


Oblique Shocks

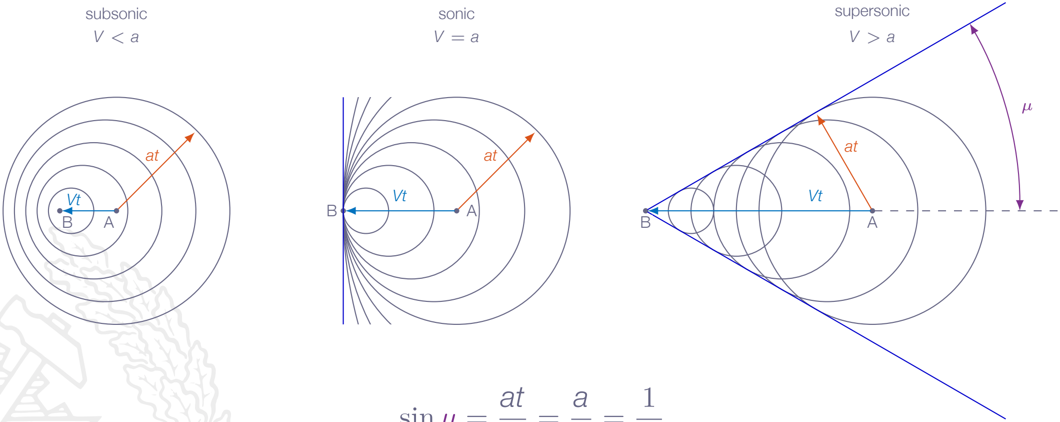


Mach Wave

Sound waves emitted from A (speed of sound a)



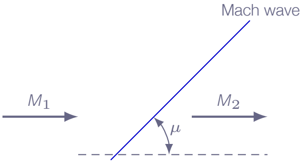
Mach Wave



$$\sin\mu = \frac{at}{Vt} = \frac{a}{V} = \frac{1}{M}$$

Mach Wave

A Mach wave is an infinitely weak oblique shock



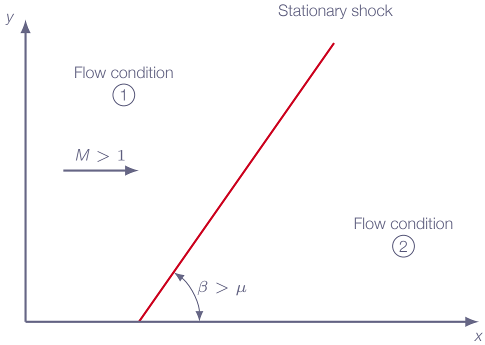
No substantial changes of flow properties over a single Mach wave

$M_1 > 1.0$ and $M_1 \approx M_2$

Isentropic

Oblique Shocks and Mach Waves

Two-dimensional steady-state flow



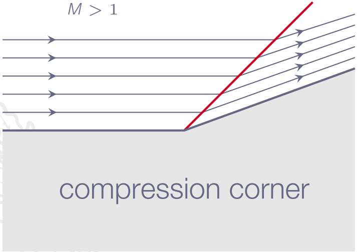
Significant changes of flow properties from 1 to 2

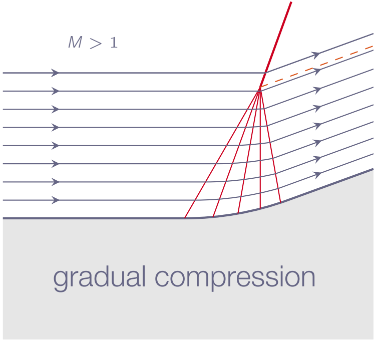
$M_1 > 1.0$, $\beta > \mu$, and $M_1 \neq M_2$

Not isentropic

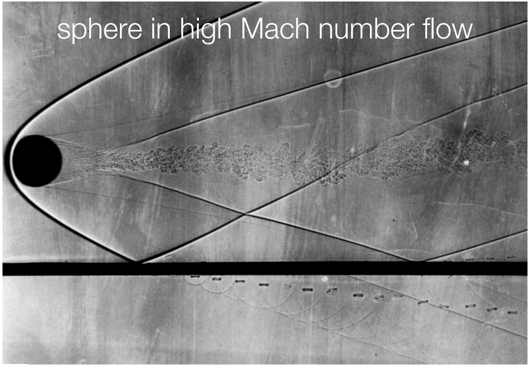
Oblique Shocks and Mach Waves

When does an oblique shock appear in a flow?

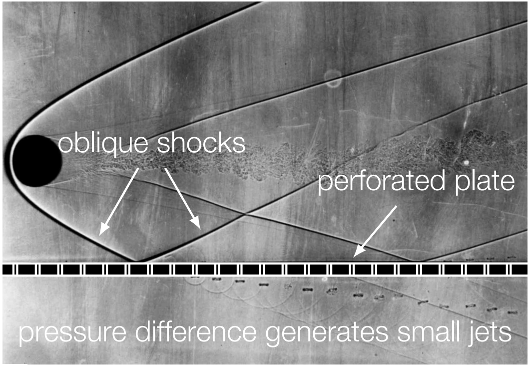




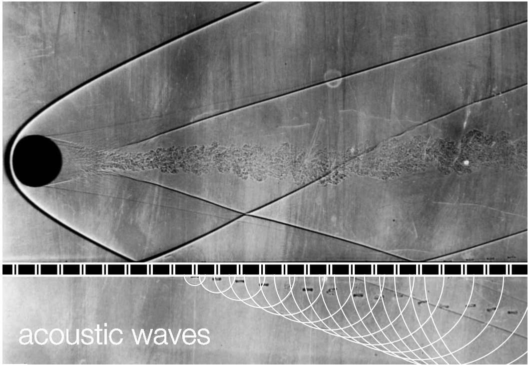
Oblique Shocks and Mach Waves



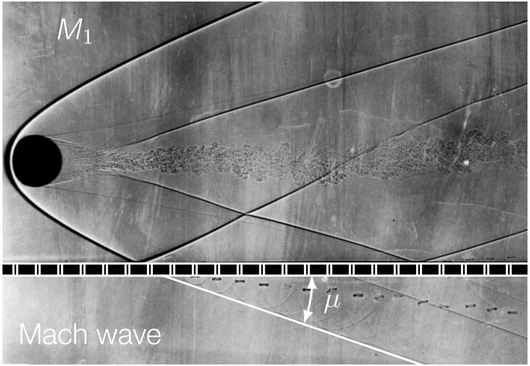
Oblique Shocks and Mach Waves



Oblique Shocks and Mach Waves

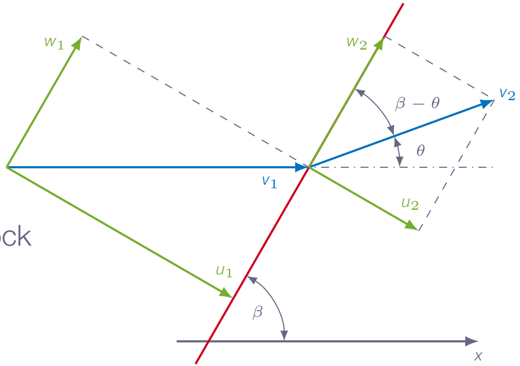


Oblique Shocks and Mach Waves



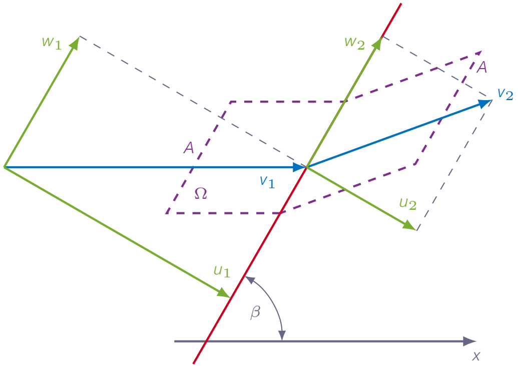
$$\mu = 19^\circ \Rightarrow M_1 = \frac{1}{\sin \mu} \approx 3.1$$

Oblique Shocks



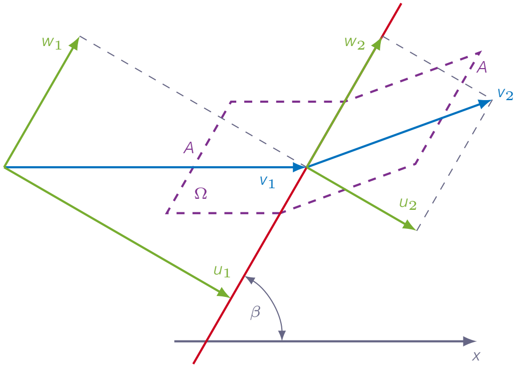
Stationary oblique shock

Oblique Shock Relations



1. Two-dimensional steady-state flow
2. Control volume aligned with flow stream lines

Oblique Shock Relations

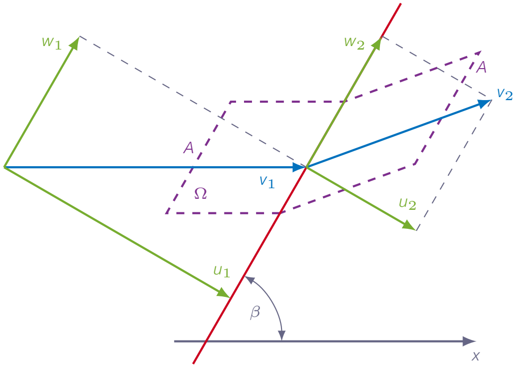


Velocity notations:

$$M_{n_1} = \frac{u_1}{a_1} = M_1 \sin(\beta) \qquad M_1 = \frac{v_1}{a_1}$$

$$M_{n_2} = \frac{u_2}{a_2} = M_2 \sin(\beta - \theta) \qquad M_2 = \frac{v_2}{a_2}$$

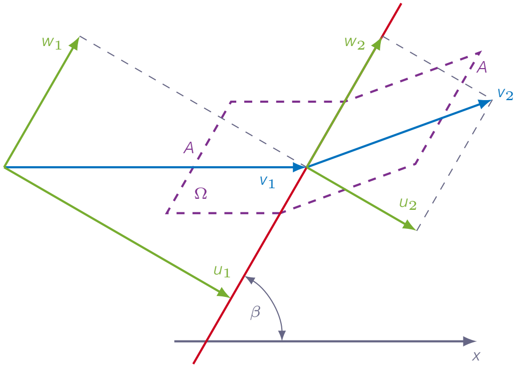
Oblique Shock Relations



Conservation of mass:

$$-\rho_1 u_1 A + \rho_2 u_2 A = 0 \Rightarrow \rho_1 u_1 = \rho_2 u_2$$

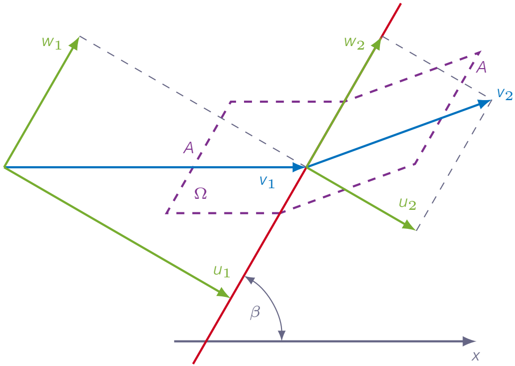
Oblique Shock Relations



Conservation of momentum (shock-normal direction):

$$-(\rho_1 u_1^2 + p_1)A + (\rho_2 u_2^2 + p_2)A = 0 \Rightarrow \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

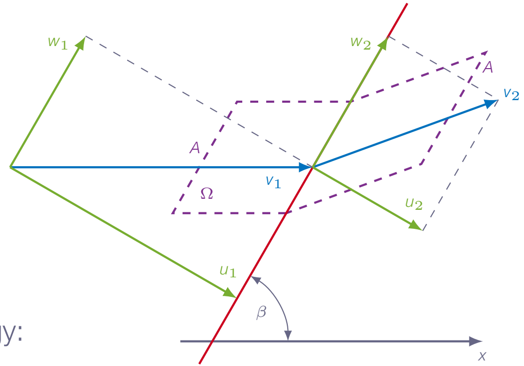
Oblique Shock Relations



Conservation of momentum (shock-tangential direction):

$$-\rho_1 u_1 w_1 A + \rho_2 u_2 w_2 A = 0 \Rightarrow w_1 = w_2$$

Oblique Shock Relations



Conservation of energy:

$$-\rho_1 u_1 [h_1 + \frac{1}{2}(u_1^2 + w_1^2)]A + \rho_2 u_2 [h_2 + \frac{1}{2}(u_2^2 + w_2^2)]A = 0 \Rightarrow h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

Oblique Shock Relations

We can use the same equations as for normal shocks if we replace M_1 with M_{n_1} and M_2 with M_{n_2}

$$\boxed{M_{n_2}^2 = \frac{M_{n_1}^2 + [2/(\gamma-1)]}{[2\gamma/(\gamma-1)]M_{n_1}^2 - 1}}$$

Ratios such as ρ_2/ρ_1, p_2/p_1, and T_2/T_1 can be calculated using the relations for normal shocks with M_1 replaced by M_{n_1}

Oblique Shock Relations

What about ratios involving stagnation flow properties, can we use the ones previously derived for normal shocks?

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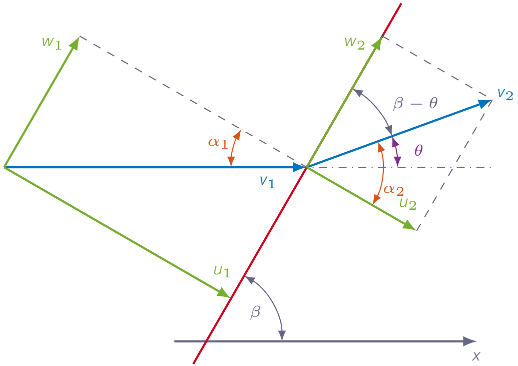
Note! Do not not use ratios involving total quantities, *e.g.* p_{o_2}/p_{o_1}, ρ_{o_2}/ρ_{o_1}, obtained from formulas or tables for normal shock

Deflection Angle (*for the interested*)



$$\theta = \alpha_2 - \alpha_1 = \tan^{-1}\left(\frac{w}{u_2}\right) - \tan^{-1}\left(\frac{w}{u_1}\right) \Rightarrow \frac{\partial \theta}{\partial w} = \frac{u_2}{w^2 + u_2^2} - \frac{u_1}{w^2 + u_1^2}$$

Deflection Angle (*for the interested*)

$$\frac{\partial\theta}{\partial w}=\frac{u_2}{w^2+u_2^2}-\frac{u_1}{w^2+u_1^2}=0\Rightarrow$$

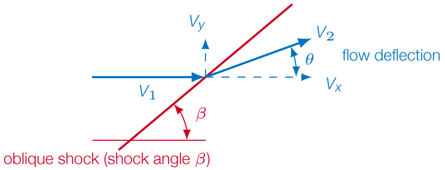
$$\frac{u_2(w^2+u_1^2)-u_1(w^2+u_2^2)}{(w^2+u_2^2)(w^2+u_1^2)}=0\Rightarrow\frac{(u_2-u_1)(w^2-u_1u_2)}{(w^2+u_2^2)(w^2+u_1^2)}=0$$

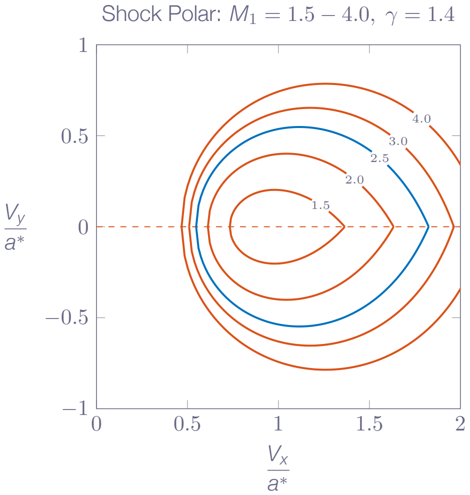
Two solutions:

1. $u_2=u_1$ (no deflection)
2. $w^2=u_1u_2$ (max deflection)

Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number





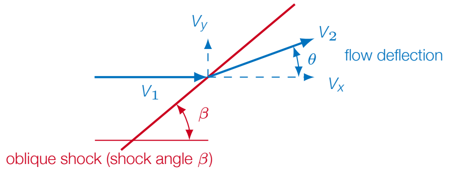
No-deflection cases:

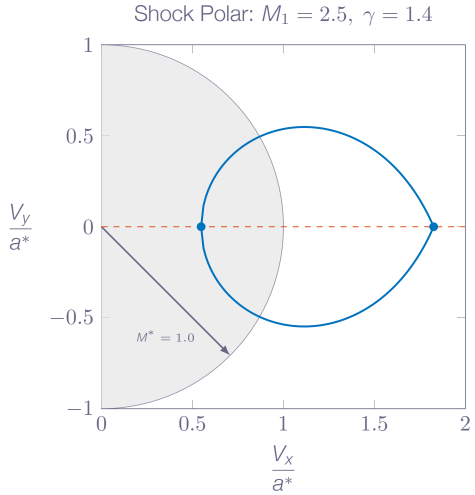
1. Normal shock
   (reduced shock-normal velocity)

2. Mach wave
   (unchanged wave-normal velocity)

Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number



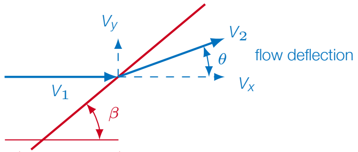


$$M^* = \frac{\sqrt{V_x^2 + V_y^2}}{a^*}$$

Solutions to the left of the sonic line are subsonic

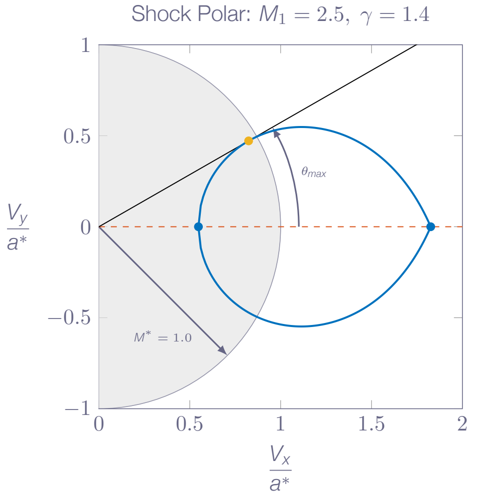
Shock Polar - Flow Deflection - θ_{max}

Graphical representation of all possible deflection angles for a specific Mach number



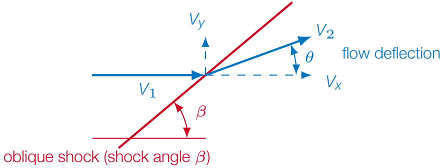
$$\tan\theta = \frac{V_y}{V_x}$$

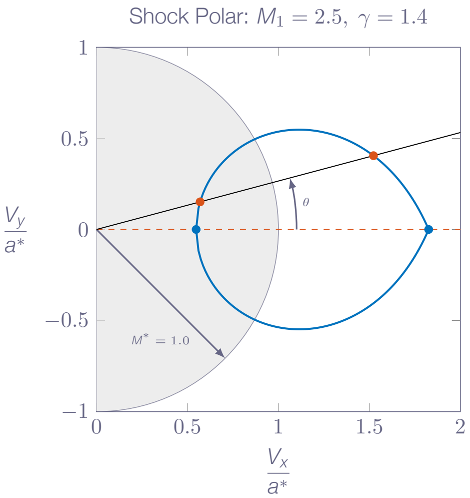
It is not possible to deflect the flow more than θ_{max}



Shock Polar - Flow Deflection

Graphical representation of all possible deflection angles for a specific Mach number





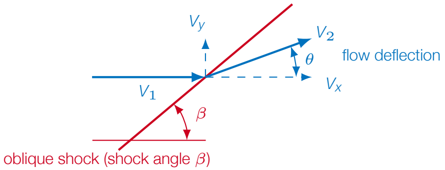
For each deflection angle $\theta < \theta_{max}$, there are two solutions

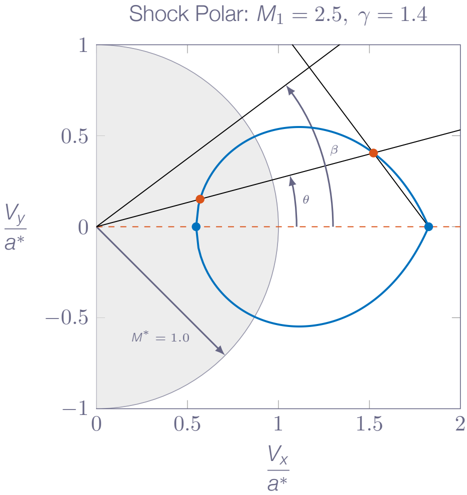
1. strong shock solution
2. weak shock solution

Weak shocks give lower losses and therefore the preferred solution

Shock Polar - Weak Solution

Graphical representation of all possible deflection angles for a specific Mach number

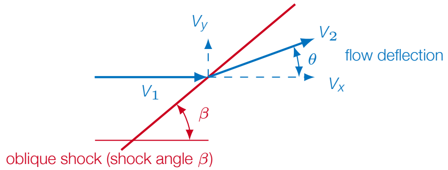


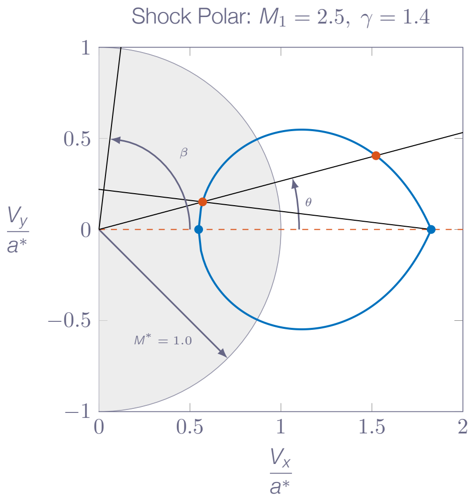


The shock polar can be used to calculate the shock angle β for a given deflection angle θ

Shock Polar - Strong Solution

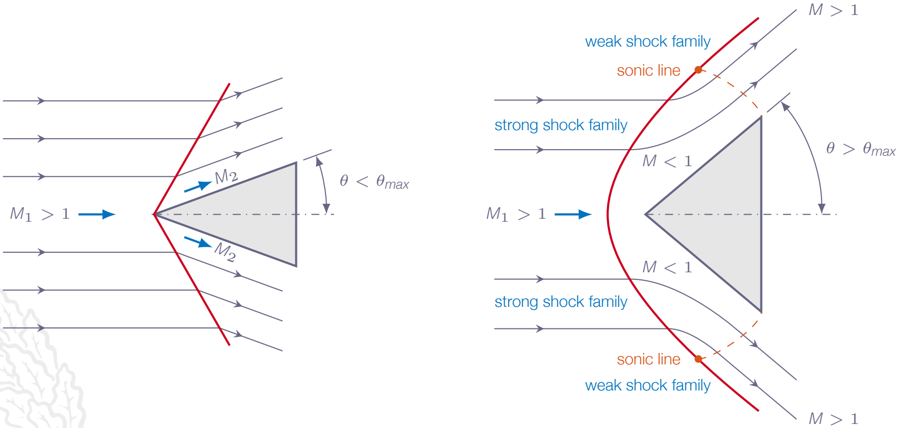
Graphical representation of all possible deflection angles for a specific Mach number





The shock polar can be used to calculate the shock angle β for a given deflection angle θ

Flow Deflection

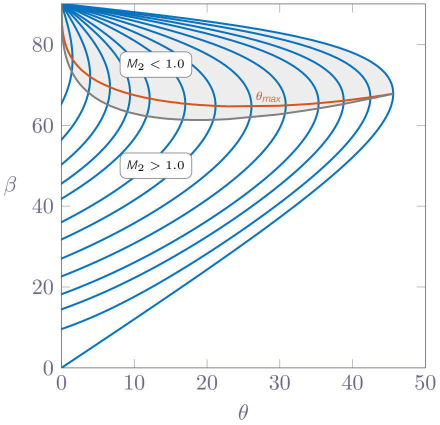


The θ-β-Mach Relation

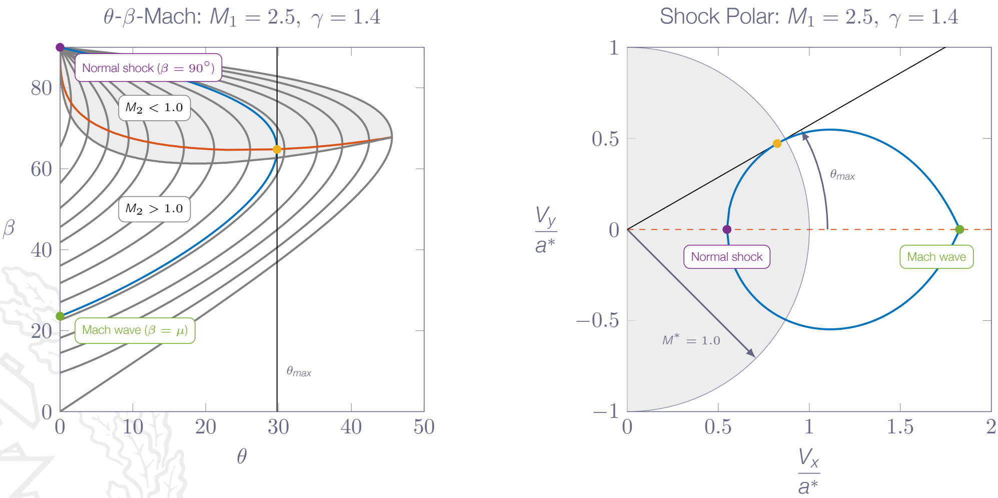
$$\tan(\theta) = \frac{2\cot(\beta)(M_1^2 \sin^2(\beta) - 1)}{M_1^2(\gamma + \cos(2\beta)) + 2}$$

A relation between:

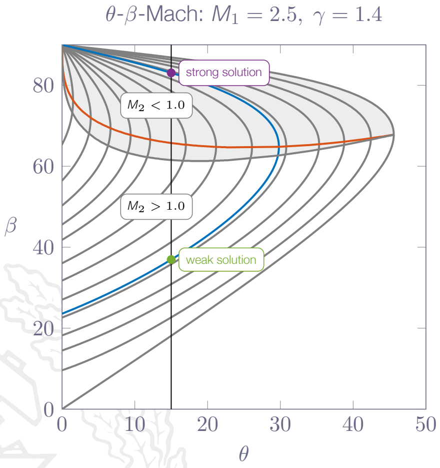
1. flow deflection angle θ
2. shock angle β
3. upstream flow Mach number M_1

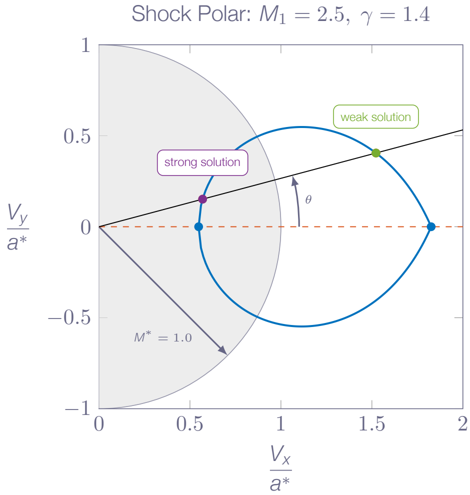


The θ-β-Mach Relation vs. Shock Polar



The θ-β-Mach Relation vs. Shock Polar

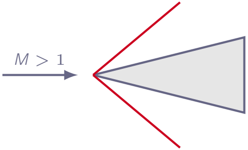




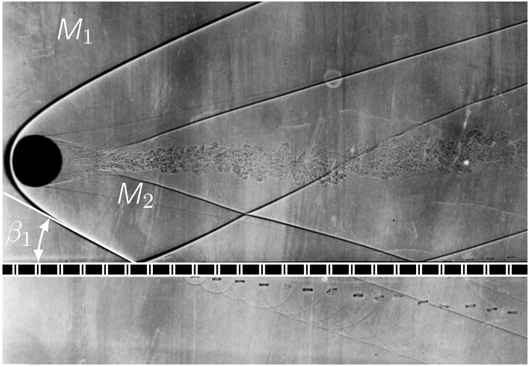
The θ-β-Mach Relation - Wedge Flow

Wedge flow oblique shock analysis:

1. θ-β-M relation $\Rightarrow$ β for given M_1 and θ
2. β gives M_{n_1} according to: $M_{n_1} = M_1 \sin(\beta)$
3. normal shock formula with M_{n_1} instead of M_1 $\Rightarrow$ M_{n_2} (instead of M_2)
4. M_2 given by $M_2 = M_{n_2} / \sin(\beta - \theta)$
5. normal shock formula with M_{n_1} instead of M_1 $\Rightarrow$ ρ_2/ρ_1, p_2/p_1, etc
6. upstream conditions + ρ_2/ρ_1, p_2/p_1, etc $\Rightarrow$ downstream conditions



Oblique Shocks and Mach Waves

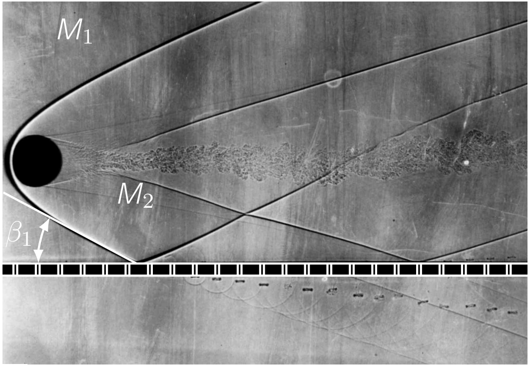


$M_1 > M_2$

$M_2 > 1.0$

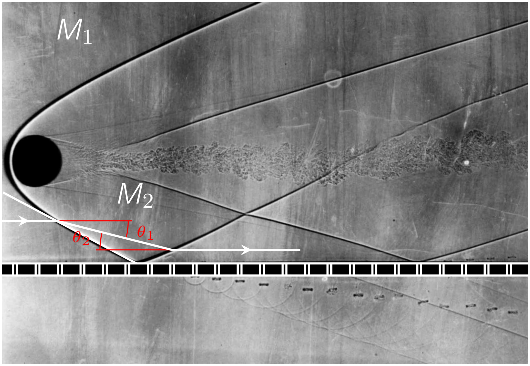
$$\theta_1 = f(M_1, \beta_1), \quad M_2 = f(M_1, \theta_1, \beta_1)$$

Oblique Shocks and Mach Waves



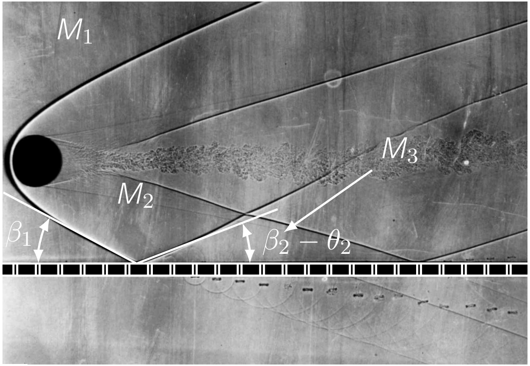
$$\left.\begin{array}{r}\beta_1 = 28^\circ \\ M_1 = 3.1\end{array}\right\} \Rightarrow \theta_1 \approx 11.2^\circ, \quad M_2 \approx 2.5$$

Oblique Shocks and Mach Waves



$$\theta_1 = \theta_2$$

Oblique Shocks and Mach Waves



$$M_1 > M_2 > M_3$$

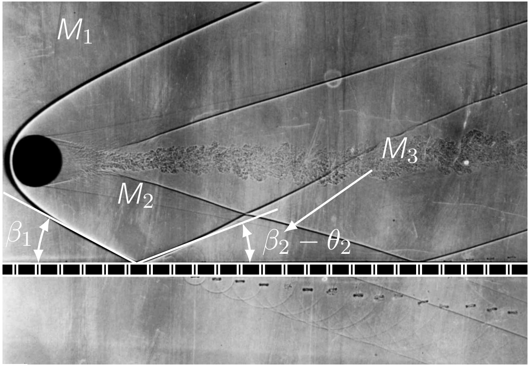
$$M_3 > 1.0$$

$$\beta_2 > \beta_1$$

$$\beta_2 = f(M_2, \theta_2), \quad M_3 = f(M_2, \theta_2, \beta_2)$$

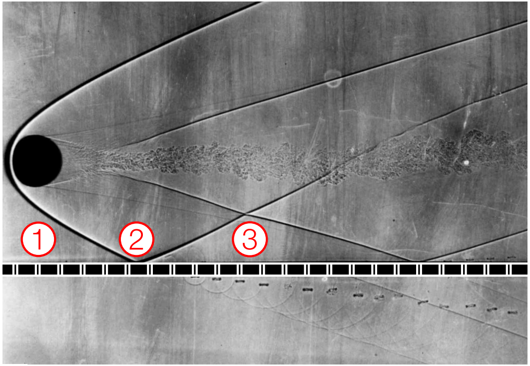
Note! Shock wave reflection at solid wall is **not** specular

Oblique Shocks and Mach Waves



$$\left.\begin{aligned} \theta_2 &= 11.2^\circ \\ M_2 &= 2.5 \end{aligned}\right\} \Rightarrow \beta_2 \approx 33^\circ, \quad M_3 \approx 2.0$$

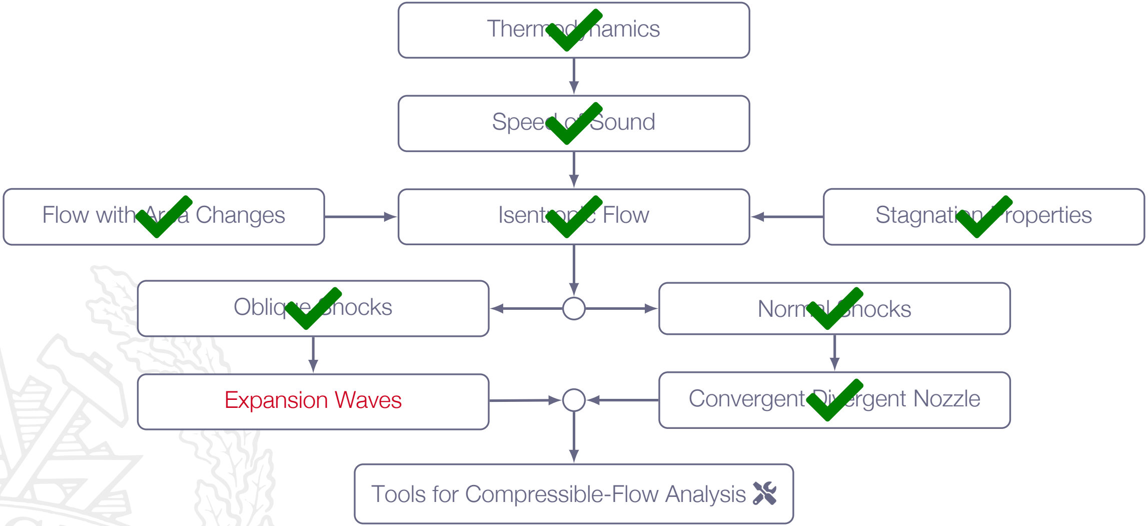
Oblique Shocks and Mach Waves



$$\frac{p_3}{p_1} = \frac{p_2}{p_1}\frac{p_3}{p_2} \approx 4.52$$

$$\frac{T_3}{T_1} = \frac{T_2}{T_1}\frac{T_3}{T_2} \approx 1.57$$

Roadmap - Compressible Flow



Expansion Waves

Gradual change of flow angle

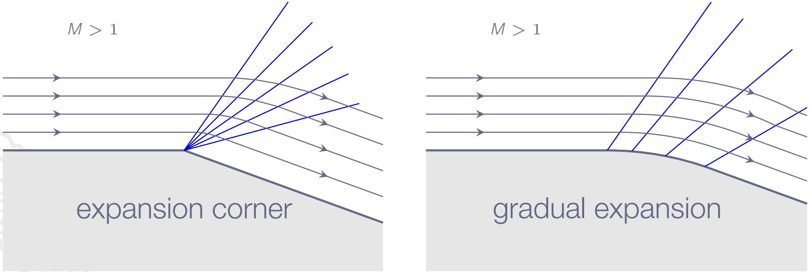
Increasing flow area

Increasing Mach number

Accumulation of infinitesimal flow deflections - isentropic

Expansion Waves

What is an expansion wave or expansion region?



The Prandtl-Meyer Function

The change of flow properties over an expansion region can be calculated using the Prandtl-Meyer function

The Prandtl-Meyer function derivation is based on the fact that each expansion wave gives an infinitesimal change in flow angle and flow properties

Prandtl-Meyer Function Derivation (*for the interested*)

For small deflection angles, linearization of the θ-β-Mach relation gives

$$\frac{dp}{p} \approx \frac{\gamma M^2}{(M^2-1)^{1/2}} d\theta$$

The momentum equation for inviscid flows gives

$$dp = -d(\rho V^2) = -\rho V dV - V \underbrace{d(\rho V)}_{=0} = -\rho V dV = -\rho V^2 \frac{dV}{V} = -\rho a^2 M^2 \frac{dV}{V} \Rightarrow$$

$$\Rightarrow \{\rho a^2 = \rho \gamma RT = \gamma p\} \Rightarrow \frac{dp}{p} = -\gamma M^2 \frac{dV}{V}$$

Prandtl-Meyer Function Derivation (*for the interested*)

Now, setting the two expressions for dp/p equal

$$-\gamma M^2 \frac{dV}{V} = \frac{\gamma M^2}{(M^2-1)^{1/2}} d\theta \Rightarrow d\theta = -(M^2-1)^{1/2} \frac{dV}{V}$$

$$V = Ma \Rightarrow dV = adM + Mda \Rightarrow \frac{dV}{V} = \frac{dM}{M} + \frac{da}{a}$$

$$d\theta = -(M^2-1)^{1/2} \left(\frac{dM}{M} + \frac{da}{a} \right)$$

Prandtl-Meyer Function Derivation (*for the interested*)

$$d\theta = -(M^2-1)^{1/2}\left(\frac{dM}{M}+\frac{da}{a}\right)$$

$$\frac{a_o}{a} = \left(1+\frac{\gamma-1}{2}M^2\right)^{1/2}$$

$$da = \left(1+\frac{\gamma-1}{2}M^2\right)^{-1/2} da_o + a_o d\left[\left(1+\frac{\gamma-1}{2}M^2\right)^{-1/2}\right]$$

isentropic $\Rightarrow$ $da_o = 0$

$$\frac{da}{a} = \frac{d\left[\left(1+\frac{\gamma-1}{2}M^2\right)^{-1/2}\right]}{\left(1+\frac{\gamma-1}{2}M^2\right)^{-1/2}} = \frac{-\frac{1}{2}\left(1+\frac{\gamma-1}{2}M^2\right)^{-3/2}(\gamma-1)MdM}{\left(1+\frac{\gamma-1}{2}M^2\right)^{-1/2}}$$

Prandtl-Meyer Function Derivation (*for the interested*)

$$d\theta = -(M^2-1)^{1/2}\left(\frac{dM}{M}+\frac{da}{a}\right)$$

$$\frac{da}{a} = \frac{-\frac{1}{2}(\gamma-1)MdM}{1+\frac{\gamma-1}{2}M^2} \Rightarrow d\theta = -\frac{2(M^2-1)^{1/2}}{2+(\gamma-1)M^2}\frac{dM}{M}$$

Introducing ω defined such that: $d\omega = -d\theta$, $\omega = 0$ when $M = 1$

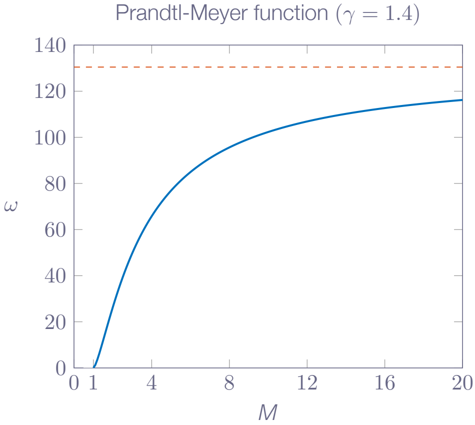
$$\int_0^{\omega} d\omega = \int_1^{M}\frac{2(M^2-1)^{1/2}}{2+(\gamma-1)M^2}\frac{dM}{M}$$

$$\omega(M) = \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}\tan^{-1}\left(\frac{M^2-1}{(\gamma+1)/(\gamma-1)}\right)^{1/2} - \tan^{-1}(M^2-1)^{1/2}$$

The Prandtl-Meyer Function

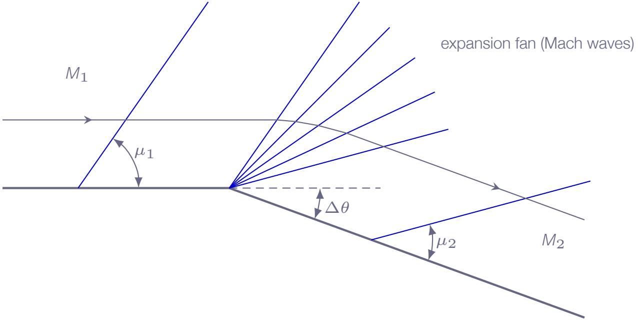
$$\omega(M) = \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2} \tan^{-1}\left(\frac{M^2-1}{(\gamma+1)/(\gamma-1)}\right)^{1/2} - \tan^{-1}(M^2-1)^{1/2}$$

$$\omega(M)|_{M\to\infty} = 130.45^\circ$$



Prandtl-Meyer Expansion Waves

Example:



1. $\theta_1 = 0$, $M_1 > 1$ is given
2. θ_2 is given
3. find M_2 such that $\Delta\theta = \theta_2 - \theta_1 = \omega(M_2) - \omega(M_1)$

Prandtl-Meyer Expansion Waves

Since the flow is isentropic, the isentropic relations apply:

(T_o and p_o are constant)

Calorically perfect gas:

$$\frac{T_o}{T} = \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]$$

$$\frac{p_o}{p} = \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{\frac{\gamma}{\gamma-1}}$$

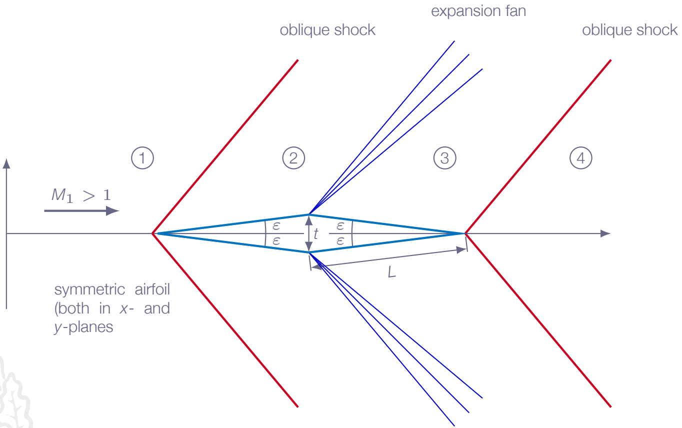
Prandtl-Meyer Expansion Waves

since $T_{o_1} = T_{o_2}$ and $p_{o_1} = p_{o_2}$

$$\frac{T_1}{T_2} = \frac{T_{o_2}}{T_{o_1}}\frac{T_1}{T_2} = \left(\frac{T_{o_2}}{T_2}\right) \Big/ \left(\frac{T_{o_1}}{T_1}\right) = \left[\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right]$$

$$\frac{p_1}{p_2} = \frac{p_{o_2}}{p_{o_1}}\frac{p_1}{p_2} = \left(\frac{p_{o_2}}{p_2}\right) \Big/ \left(\frac{p_{o_1}}{p_1}\right) = \left[\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right]^{\frac{\gamma}{\gamma-1}}$$

Diamond-Wedge Airfoil



Note! symmetric airfoil at zero incidence $\Rightarrow$ zero lift but what about drag?

Diamond-Wedge Airfoil

1-2 standard oblique shock calculation for flow deflection angle ε and upstream Mach number M_1

2-3 Prandtl-Meyer expansion for flow deflection angle 2ε and upstream Mach number M_2

3-4 standard oblique shock calculation for flow deflection angle ε and upstream Mach number M_3

Diamond-Wedge Airfoil - Wave Drag

Since conditions 2 and 3 are constant in their respective regions, we obtain:

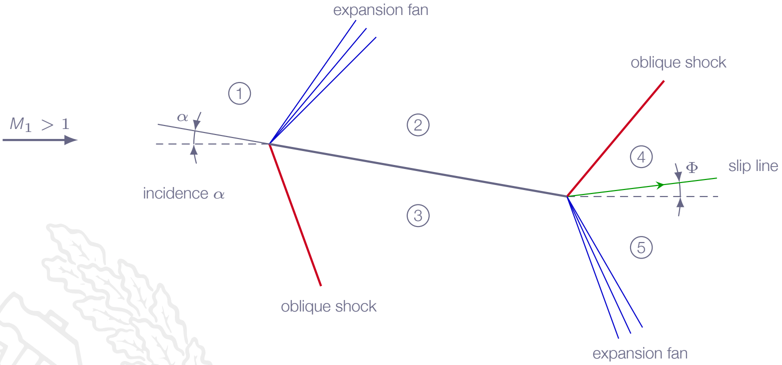
$$D = 2\left[p_2 L \sin(\varepsilon) - p_3 L \sin(\varepsilon)\right] = 2(p_2 - p_3)\frac{t}{2} = (p_2 - p_3)t$$

For supersonic free stream ($M_1 > 1$), with shocks and expansion fans according to figure we will always find that $p_2 > p_3$

which implies $D > 0$

Wave drag (drag due to flow loss at compression shocks)

Flat-Plate Airfoil



Flat-Plate Airfoil

It seems that the angle of the flow downstream of the flat plate would be different than the angle of the flow upstream of the plate. Can that really be correct?!

Flat-Plate Airfoil

It seems that the angle of the flow downstream of the flat plate would be different than the angle of the flow upstream of the plate. Can that really be correct?!

For the flow in the vicinity of the plate this is the correct picture. Further out from the plate, shock and expansion waves will interact and eventually sort the mismatch of flow angles out

Flat-Plate Airfoil

Flow states 4 and 5 must satisfy:

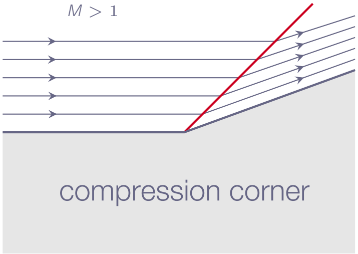
1. $p_4 = p_5$
2. flow direction 4 equals flow direction 5 (Φ)

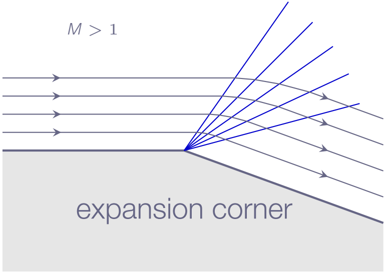
Shock between 2 and 4 as well as expansion fan between 3 and 5 will adjust themselves to comply with the requirements

For calculation of lift and drag only states 2 and 3 are needed

States 2 and 3 can be obtained using standard oblique shock formulas and Prandtl-Meyer expansion

Oblique Shocks and Expansion Waves

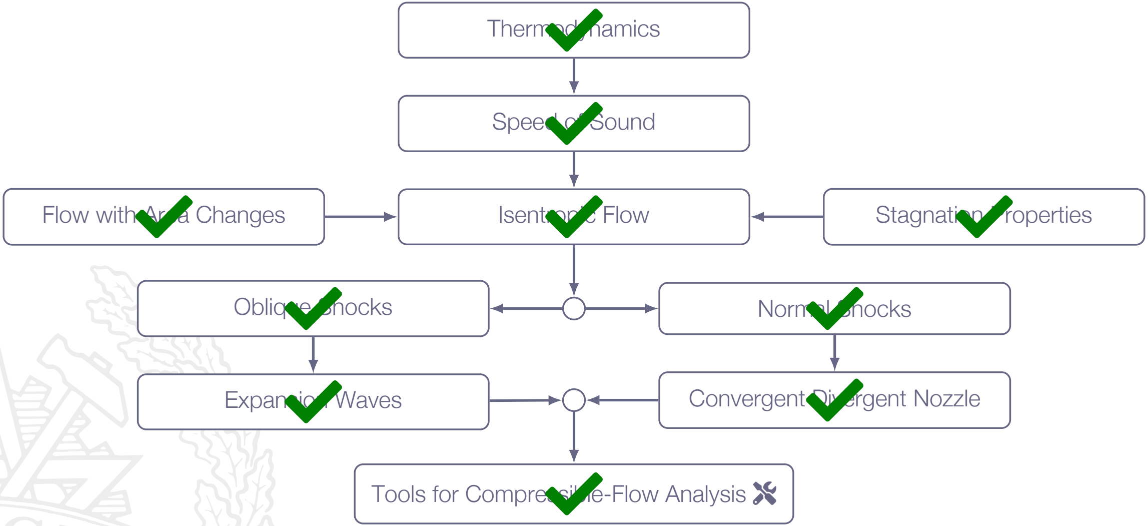




M	decrease
V	decrease
p	increase
ρ	increase
T	increase

M	increase
V	increase
p	decrease
ρ	decrease
T	decrease

Roadmap - Compressible Flow



Supersonic Stereo

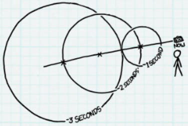
What if you somehow managed to make a stereo travel at twice the speed of sound, would it sound backwards to someone who was just casually sitting somewhere as it flies by?

—Tim Currie

Yes.

Technically, anyway. It would be pretty hard to hear.

The basic idea is pretty straightforward. The stereo is going faster than its own sound, so it will reach you first, followed by the sound it emitted one second ago, followed by the sound it emitted two seconds ago, and so forth.



The problem is that the stereo is moving at Mach 2, which means that two seconds ago, it was over a kilometer away. It's hard to hear music from that distance, particularly when your ears were just hit by (a) a sonic boom, and (b) pieces of a rapidly disintegrating stereo.

Wind speeds of Mach 2 would messily disassemble most consumer electronics. The force of the wind on the body of the stereo is roughly comparable to that of a dozen people standing on it: