Fluid Mechanics - MTF053

Chapter 7

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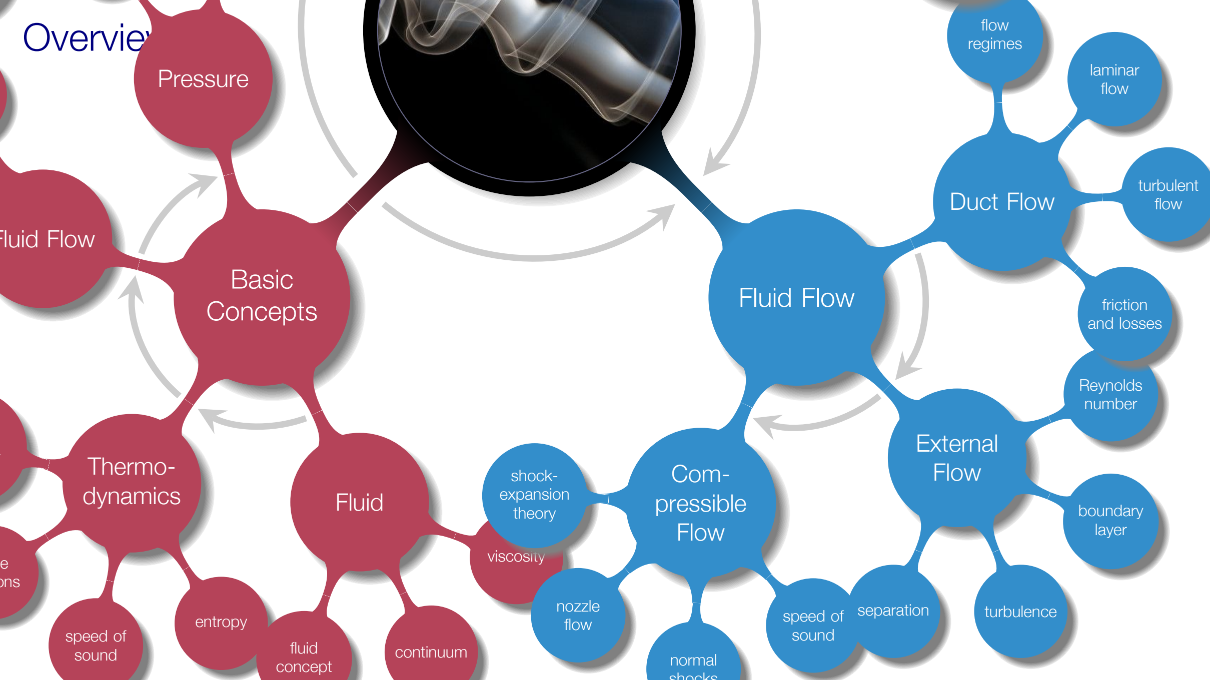
Chapter 7 - Flow Past Immersed Bodies

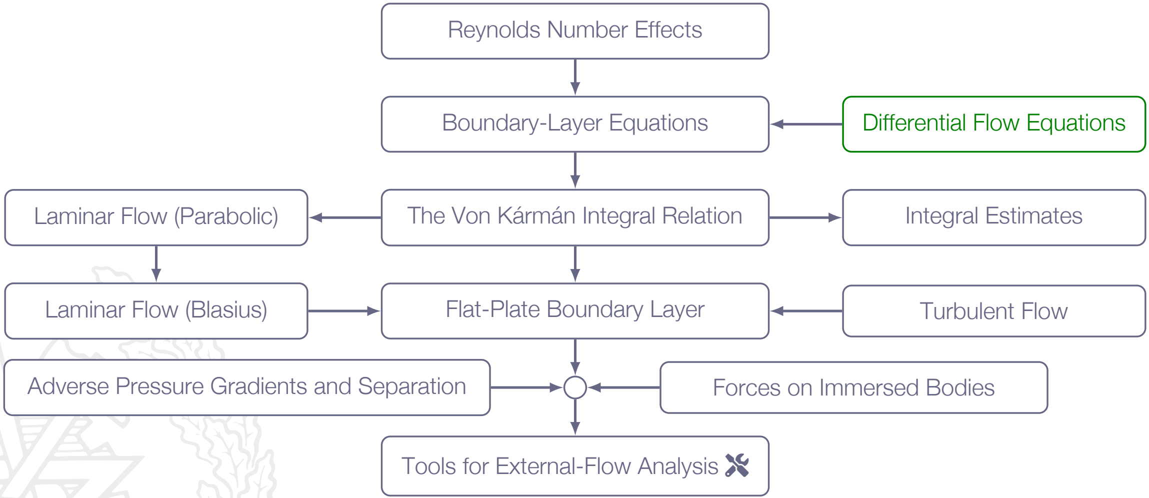

Overvie
Pressure
Fluid Flow
Basic Concepts
Thermo-dynamics
Fluid
viscosity
e
ons
speed of sound
entropy
fluid concept
continuum
flow regimes
laminar flow
Duct Flow
turbulent flow
Fluid Flow
friction and losses
Reynolds number
External Flow
boundary layer
turbulence
separation
speed of sound
Com-pressible Flow
shock-expansion theory
nozzle flow
normal shocks

Learning Outcomes

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category
6 **Explain** what a boundary layer is and when/where/why it appears
21 **Explain** how the flat plate boundary layer is developed (transition from laminar to turbulent flow)
22 **Explain** and use the Blasius equation
23 **Define** the Reynolds number for a flat plate boundary layer
24 **Explain** what is characteristic for a turbulent flow
29 **Explain** flow separation (separated cylinder flow)
30 **Explain** how to delay or avoid separation
31 **Derive** the boundary layer formulation of the Navier-Stokes equations
32 **Understand** and explain displacement thickness and momentum thickness
33 **Understand**, **explain** and **use** the concepts drag, friction drag, pressure drag, and lift

Let's take a deep dive into boundary-layer theory

Roadmap - Flow Past Immersed Bodies



Complementary Course Material

These lecture notes covers chapter 7 in the course book and additional course material that you can find in the following documents

- MTF053_Equation-for-Boundary-Layer-Flows.pdf
- MTF053_Turbulence.pdf


”Understanding the mechanisms behind flow-related forces is a key factor to success in many engineering applications”
S MB 7177

External Flow

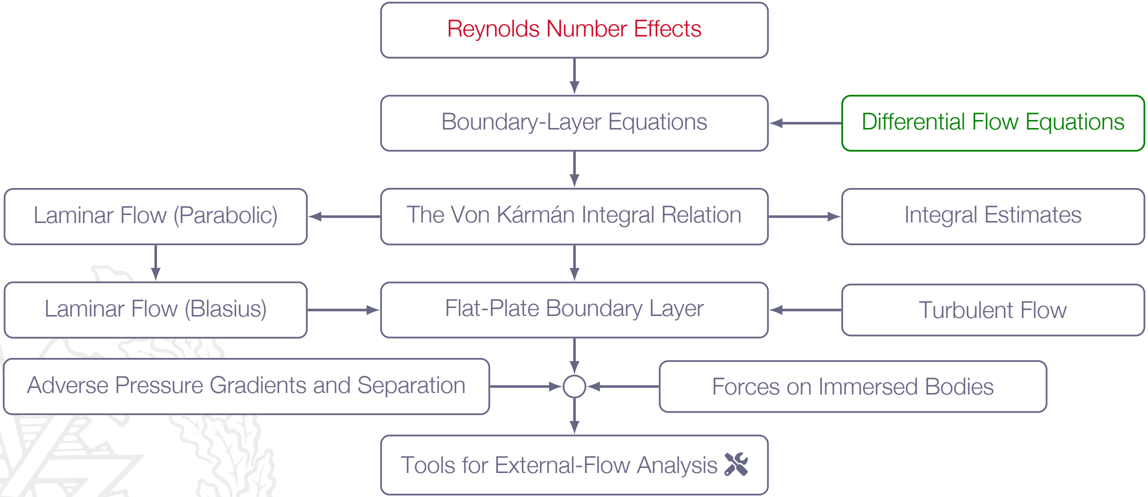
Significant viscous effects near the surface of an **immersed body**

Nearly inviscid far from the body

Unconfined - boundary layers are free to grow

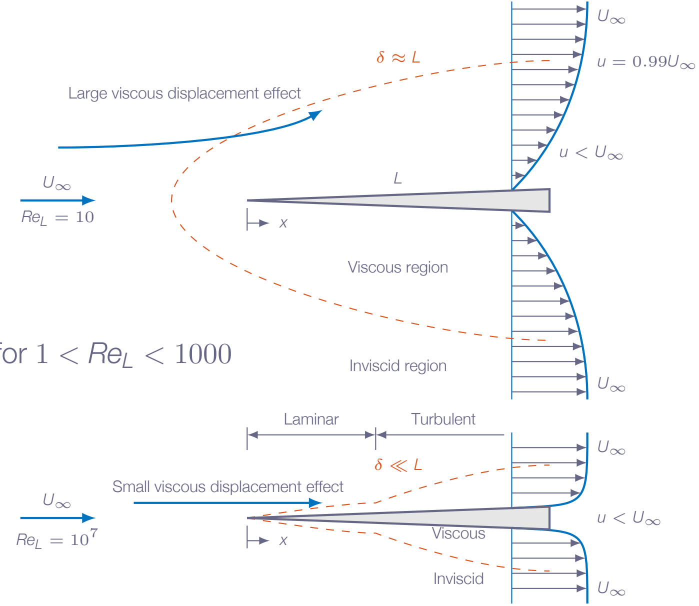
Most often CFD or experiments are needed to analyze an external flow unless the geometry is very simple

Roadmap - Flow Past Immersed Bodies



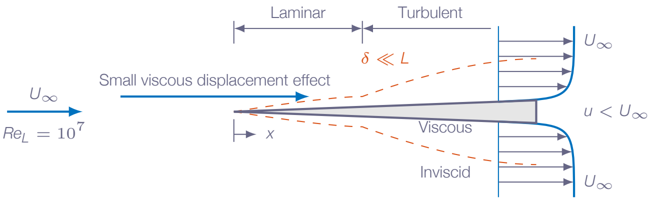
Reynolds Number Effects

$$Re_L = \frac{U_\infty L}{\nu}$$



Note: no simple theory exists for $1 < Re_L < 1000$

Reynolds Number Effects



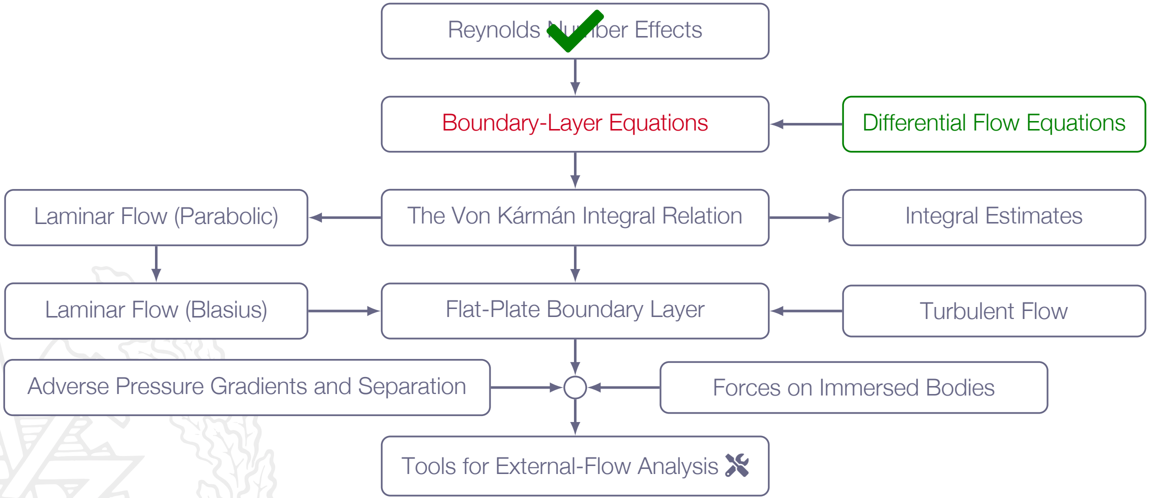
$$\frac{\delta}{x} \approx \begin{cases} \dfrac{5.0}{{Re_x}^{1/2}} & \textit{laminar} \quad 10^3 < Re_x < 10^6 \\ \dfrac{0.16}{{Re_x}^{1/7}} & \textit{turbulent} \quad 10^6 < Re_x \end{cases}$$

$$Re_L = \frac{U_\infty L}{\nu}$$

$$Re_x = \frac{U_\infty x}{\nu}$$

Note! Re_L and the **local Reynolds number** Re_x are not the same if $L \neq x$

Roadmap - Flow Past Immersed Bodies



Boundary Layer Equations

We will derive a set of equations suitable for **boundary-layer flow analysis**

Starting point: the **non-dimensional equations** derived in Chapter 5

We will assume two-dimensional, incompressible, steady-state flow

We will do an **order-of-magnitude** comparison of all the terms in the governing equations on non-dimensional form and identify terms that can be neglected in a **thin-boundary-layer** flow

Non-dimensional Flow Equations

The governing equations for two-dimensional, laminar, incompressible and steady-state flow with negligible body forces:

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

x-momentum:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

y-momentum:

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Non-dimensional Flow Equations

$$x^* = \frac{x}{L} \quad y^* = \frac{y}{L} \quad u^* = \frac{u}{U_\infty} \quad v^* = \frac{v}{U_\infty} \quad p^* = \frac{p}{\rho U_\infty^2}$$

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial (u^* U_\infty)}{\partial (x^* L)} + \frac{\partial (v^* U_\infty)}{\partial (y^* L)} = \frac{U_\infty}{L}\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*}\right) = 0$$

Non-dimensional Flow Equations

$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{L} \qquad u^* = \frac{u}{U_\infty} \qquad v^* = \frac{v}{U_\infty} \qquad p^* = \frac{p}{\rho U_\infty^2}$$

x-momentum:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(u^* U_\infty \frac{\partial(u^* U_\infty)}{\partial(x^* L)} + v^* U_\infty \frac{\partial(u^* U_\infty)}{\partial(y^* L)}\right) = -\frac{\partial(p^* \rho U_\infty^2)}{\partial(x^* L)} + \mu\left(\frac{\partial^2(u^* U_\infty)}{\partial(x^* L)^2} + \frac{\partial^2(u^* U_\infty)}{\partial(y^* L)^2}\right)$$

Non-dimensional Flow Equations

$$\frac{\rho {U_\infty}^2}{L}\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)=-\frac{\rho U_\infty^2}{L}\frac{\partial p^*}{\partial x^*}+\frac{\mu U_\infty}{L^2}\left(\frac{\partial^2 u^*}{\partial {x^*}^2}+\frac{\partial^2 u^*}{\partial {y^*}^2}\right)$$

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{\mu}{\rho U_\infty L}\left(\frac{\partial^2 u^*}{\partial {x^*}^2}+\frac{\partial^2 u^*}{\partial {y^*}^2}\right)$$

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{Re_L}\left(\frac{\partial^2 u^*}{\partial {x^*}^2}+\frac{\partial^2 u^*}{\partial {y^*}^2}\right)$$

Non-dimensional Flow Equations - Summary

continuity:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

x-momentum:

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re_L} \left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} \right)$$

y-momentum:

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re_L} \left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}} \right)$$

$$x^* = \frac{x}{L}$$

$$y^* = \frac{y}{L}$$

$$u^* = \frac{u}{U_\infty}$$

$$v^* = \frac{v}{U_\infty}$$

$$p^* = \frac{p}{\rho U_\infty^2}$$

$$Re_L = \frac{U_\infty L}{\nu}$$

Boundary Layer Equations

To be able to find the relative sizes of different terms in the equations, we will first have a look at the flow parameters and operators

$$u^* = u/U_\infty \sim 1$$

$$x^* = x/L \sim 1$$

$$y^* = y/L \sim \delta^*$$

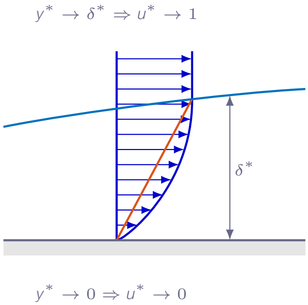
δ *denotes boundary layer thickness and* $\delta^* = \delta/L$

Note! here, u^* is **not** the friction velocity and δ^* is **not** the displacement thickness

Boundary Layer Equations

What about derivatives?

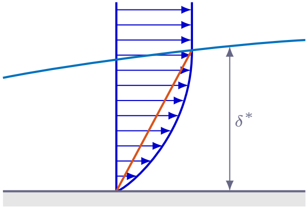
$$\frac{\partial u^*}{\partial y^*} \sim \frac{1-0}{\delta^*} = \frac{1}{\delta^*}$$



Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

$$y^* \rightarrow \delta^* \Rightarrow u^* \rightarrow 1, \; \frac{\partial u^*}{\partial y^*} \rightarrow 0$$



$$y^* \rightarrow 0 \Rightarrow u^* \rightarrow 0, \; \frac{\partial u^*}{\partial y^*} \rightarrow \frac{1}{\delta^*}$$

What about derivatives?

$$\frac{\partial u^*}{\partial y^*} \sim \frac{1-0}{\delta^*} = \frac{1}{\delta^*}$$

$$\frac{\partial^2 u^*}{\partial y^{*2}} = \frac{\partial}{\partial y^*}\frac{\partial u^*}{\partial y^*} \sim \frac{|0 - 1/\delta^*|}{\delta^*} = \frac{1}{\delta^{*2}}$$

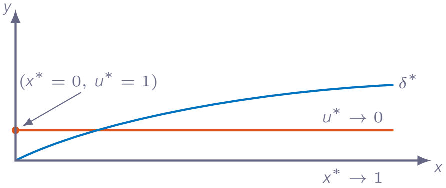
Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

$x^* \to 0 \Rightarrow u^* \to 0$

$x^* \to 1 \Rightarrow u^* \to 1$

$$\frac{\partial u^*}{\partial x^*} \sim \frac{|0-1|}{1-0} = 1$$



Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

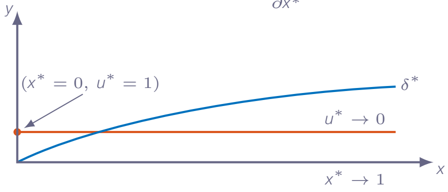
Boundary Layer Equations

$$\frac{\partial u^*}{\partial x^*} \sim \frac{|0-1|}{1-0} = 1$$

$$\frac{\partial^2 u^*}{\partial x^{*2}} = \frac{\partial}{\partial x^*}\frac{\partial u^*}{\partial x^*} \sim \frac{1-0}{1-0} = 1$$

$$x^* \to 0 \Rightarrow u^* \to 0,\ \frac{\partial u^*}{\partial x^*} \to 0$$

$$x^* \to 1 \Rightarrow u^* \to 1,\ \frac{\partial u^*}{\partial x^*} \to 1$$



Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

continuity:

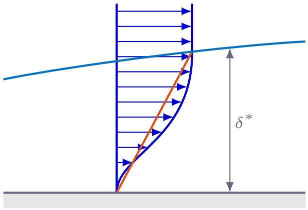
$$\underbrace{\frac{\partial u^*}{\partial x^*}}_{\sim\frac{1}{1}} + \underbrace{\frac{\partial v^*}{\partial y^*}}_{\sim\frac{?}{\delta^*}} = 0 \Rightarrow v^* \sim \delta^*$$

$\frac{\partial v^*}{\partial y^*}$ must be of the same order of magnitude as $\frac{\partial u^*}{\partial x^*}$ in order to fulfill the continuity equation

Boundary Layer Equations

$$\frac{\partial v^*}{\partial y^*} \sim \frac{\delta^*}{\delta^*} = 1$$

$y^* \rightarrow \delta^* \Rightarrow v^* \rightarrow \delta^*$

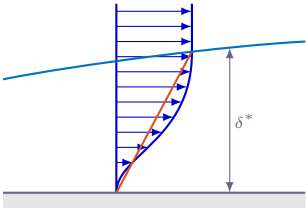


$y^* \rightarrow 0 \Rightarrow v^* \rightarrow 0$

Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

$$y^* \to \delta^* \Rightarrow v^* \to \delta^*, \ \frac{\partial v^*}{\partial y^*} \to 0$$



$$y^* \to 0 \Rightarrow v^* \to 0, \ \frac{\partial v^*}{\partial y^*} \to 1$$

$$\frac{\partial v^*}{\partial y^*} \sim \frac{\delta^*}{\delta^*} = 1$$

$$\frac{\partial^2 v^*}{\partial {y^*}^2} = \frac{\partial}{\partial y^*} \frac{\partial v^*}{\partial y^*} \sim \frac{|0-1|}{\delta^*} = \frac{1}{\delta^*}$$

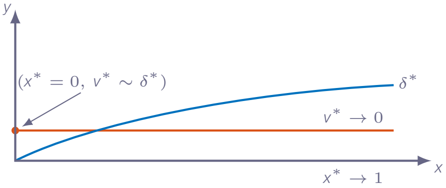
Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

$$\frac{\partial v^*}{\partial x^*} \sim \frac{|0-\delta^*|}{1-0} = \delta^*$$

$x^* \rightarrow 0 \Rightarrow v^* \rightarrow \delta^*$

$x^* \rightarrow 1 \Rightarrow v^* \rightarrow 0$



Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

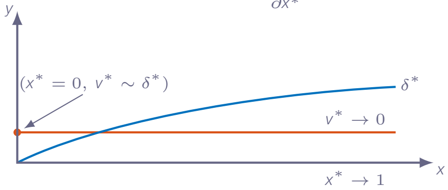
Boundary Layer Equations

$$x^* \to 0 \Rightarrow v^* \to \delta^*, \ \frac{\partial v^*}{\partial x^*} \to 0$$

$$x^* \to 1 \Rightarrow v^* \to 0, \ \frac{\partial v^*}{\partial x^*} \to \delta^*$$

$$\frac{\partial v^*}{\partial x^*} \sim \frac{|0 - \delta^*|}{1 - 0} = \delta^*$$

$$\frac{\partial^2 v^*}{\partial x^{*2}} = \frac{\partial}{\partial x^*}\frac{\partial v^*}{\partial x^*} \sim \frac{\delta^* - 0}{1 - 0} = \delta^*$$



Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

x-momentum:

$$\underbrace{u^*\frac{\partial u^*}{\partial x^*}}_{\sim 1\times 1=1} + \underbrace{v^*\frac{\partial u^*}{\partial y^*}}_{\sim \delta^*\frac{1}{\delta^*}=1} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re_L}\left(\underbrace{\frac{\partial^2 u^*}{\partial x^{*2}}}_{\sim 1} + \underbrace{\frac{\partial^2 u^*}{\partial y^{*2}}}_{\sim \frac{1}{\delta^{*2}}}\right)$$

the boundary layer is assumed to be very thin $\Rightarrow \delta^* \ll 1$ and thus

$$\frac{\partial^2 u^*}{\partial x^{*2}} \ll \frac{\partial^2 u^*}{\partial y^{*2}}$$

assuming the inertial forces to be of the same size as the friction forces in the boundary layer we get: $1/Re_L \sim \delta^{*2}$

Boundary Layer Equations

y-momentum:

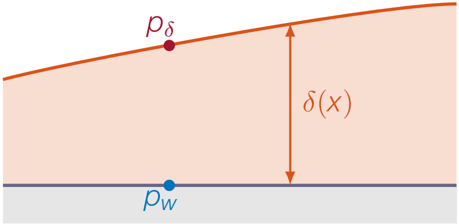
$$\underbrace{u^*\frac{\partial v^*}{\partial x^*}}_{\sim 1\times\delta^*=\delta^*} + \underbrace{v^*\frac{\partial v^*}{\partial y^*}}_{\sim\delta^*\frac{\delta^*}{\delta^*}=\delta^*} = -\frac{\partial p^*}{\partial y^*} + \underbrace{\frac{1}{Re_L}}_{\sim{\delta^*}^2}\left(\underbrace{\frac{\partial^2 v^*}{\partial {x^*}^2}}_{\sim\delta^*} + \underbrace{\frac{\partial^2 v^*}{\partial {y^*}^2}}_{\sim\frac{1}{\delta^*}}\right)$$

examining the equation we see that all terms are at most of size $\delta^* \Rightarrow \dfrac{\partial p^*}{\partial y^*} \sim \delta^*$

δ^* is small $\Rightarrow$ p is independent of y

Boundary Layer Equations

The pressure can be assumed to be constant in the vertical direction through the boundary layer and thus $p = p(x)$



$$|p_\delta^* - p_w^*| \approx \frac{\partial p^*}{\partial y^*}\delta^* \sim {\delta^*}^2$$

Boundary Layer Equations

With the knowledge gained, we now move back to the dimensional equations

laminar

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

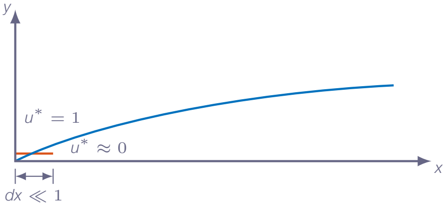
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

turbulent

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0$$

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{d\overline{p}}{dx} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial}{\partial y}\overline{u'v'}$$

Boundary Layer Equations



Limitations

1. The boundary layer equations **do not apply close to the start of the boundary layer** where $\dfrac{\partial u^*}{\partial x^*} \gg 1$
2. The equations are derived assuming a **thin boundary layer**

Boundary Layer Equations

The pressure derivative can be replaced with a velocity derivative

Outside of the boundary layer the flow is inviscid $\Rightarrow$ we can use the Bernoulli equation

$$p + \frac{1}{2}\rho U_\infty^2 = const \Rightarrow \frac{dp}{dx} + \rho U_\infty \frac{dU_\infty}{dx} = 0 \Rightarrow -\frac{1}{\rho}\frac{dp}{dx} = U_\infty \frac{dU_\infty}{dx}$$

Boundary Layer Equations

laminar boundary layer

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

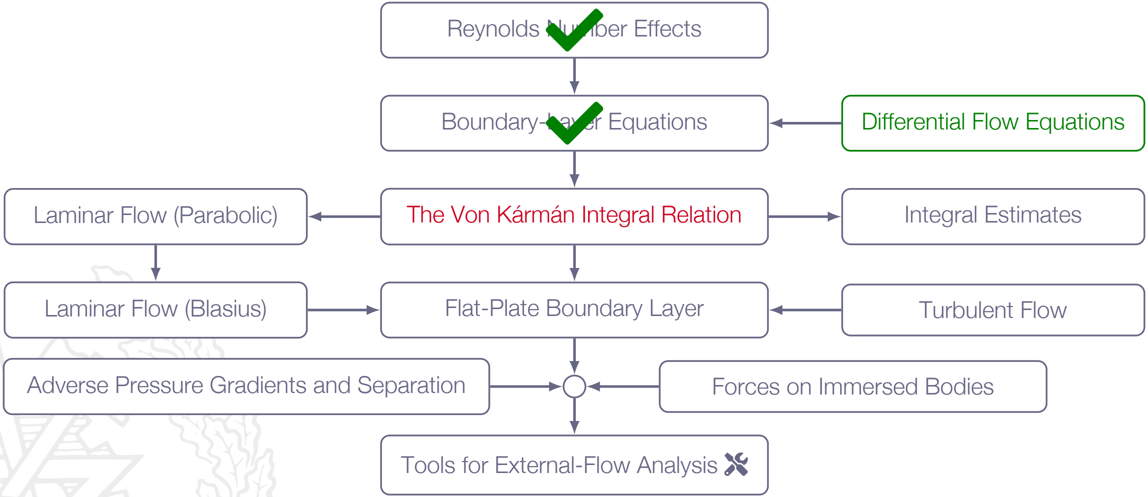
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + \nu \frac{\partial^2 u}{\partial y^2}$$

Two equations and two unknowns ⇒ possible to solve ☺

Boundary Layer Equations

Note! the boundary layer equations can be used for curved surfaces if the boundary layer thickness δ is small compared to the curvature radius r

Roadmap - Flow Past Immersed Bodies



The Von Kármán Integral Relation

Approximate solutions for $\delta(x)$ and $\tau_w(x)$

Control volume approach applied to a boundary layer

Assuming steady-state incompressible flow

$$\sum \mathbf{F} = \int_{cs} \mathbf{V}\rho(\mathbf{V} \cdot \mathbf{n})dA$$

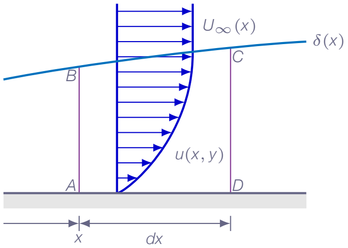
The Von Kármán Integral Relation

Massflow

$$\dot{m}_{AB} = \rho \int_0^{\delta} u dy$$

$$\dot{m}_{CD} = \rho \int_0^{\delta} u dy + \frac{d}{dx}\left[\rho \int_0^{\delta} u dy\right] dx$$

$$\dot{m}_{BC} = \rho \frac{d}{dx}\left[\int_0^{\delta} u dy\right] dx$$



The Von Kármán Integral Relation

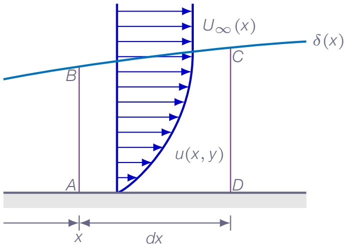
Momentum

$$I_{AB} = \rho \int_0^{\delta} u^2 dy$$

$$I_{CD} = \rho \int_0^{\delta} u^2 dy + \frac{d}{dx}\left[\rho \int_0^{\delta} u^2 dy\right] dx$$

$$I_{BC} = U\dot{m}_{BC} = \rho U_\infty \frac{d}{dx}\left[\int_0^{\delta} u dy\right] dx$$

$$I_{CD} - I_{AB} - I_{BC} = \rho \frac{d}{dx}\left[\int_0^{\delta} u^2 dy\right] dx - \rho U_\infty \frac{d}{dx}\left[\int_0^{\delta} u dy\right] dx$$



The Von Kármán Integral Relation

Pressure forces in the x-direction

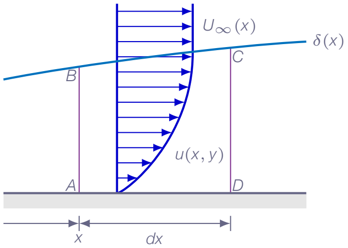
AB: $p\delta$

CD: $-\left(p + \frac{dp}{dx}dx\right)\left(\delta + \frac{d\delta}{dx}dx\right)$

BC: $\approx \left(p + \frac{1}{2}\frac{dp}{dx}dx\right)\frac{d\delta}{dx}dx$

Shear forces in the x-direction

AD: $-\tau_w dx$



The Von Kármán Integral Relation

Forces

$$dF_x =$$

$$-\tau_w dx + p\delta - \left[p\delta + p\frac{d\delta}{dx}dx + \delta\frac{dp}{dx}dx + \frac{dp}{dx}\frac{d\delta}{dx}dxdx\right] + p\frac{d\delta}{dx}dx + \frac{1}{2}\frac{dp}{dx}\frac{d\delta}{dx}dxdx$$

products of infinitesimal quantities can be regarded to be zero and thus

$$dF_x = -\tau_w dx - \delta\frac{dp}{dx}dx$$

The Von Kármán Integral Relation

Momentum equation

Now we have all components of the momentum equation defined

$$\rho \frac{d}{dx}\left[\int_0^\delta u^2 dy\right] - \rho U_\infty \frac{d}{dx}\left[\int_0^\delta u dy\right] = -\tau_w - \delta \frac{dp}{dx}$$

The momentum equation for boundary layers or **Von Kármán's integral relation**

Note! the relation is valid for laminar and turbulent flows (for turbulent flows use time-averaged quantities)

The Von Kármán Integral Relation

Outside of the boundary layer the flow is inviscid $\Rightarrow$ we can use Bernoulli

$$p + \frac{1}{2}\rho U_\infty^2 = const \Rightarrow \frac{dp}{dx} + \rho U_\infty \frac{dU_\infty}{dx} = 0 \Rightarrow -\frac{1}{\rho}\frac{dp}{dx} = U_\infty \frac{dU_\infty}{dx}$$

The Von Kármán Integral Relation

$$\frac{1}{\rho}\frac{dp}{dx} = -U_\infty \frac{dU_\infty}{dx} \Rightarrow \frac{\tau_w}{\rho} - \delta U_\infty \frac{dU_\infty}{dx} = U_\infty \frac{d}{dx}\left[\int_0^\delta u dy\right] - \frac{d}{dx}\left[\int_0^\delta u^2 dy\right]$$

$$\delta U_\infty \frac{dU_\infty}{dx} = U_\infty \frac{dU_\infty}{dx}\int_0^\delta dy$$

$$U_\infty \frac{d}{dx}\left[\int_0^\delta u dy\right] = \frac{d}{dx}\left[U_\infty \int_0^\delta u dy\right] - \frac{dU_\infty}{dx}\int_0^\delta u dy$$

$$\frac{\tau_w}{\rho} - U_\infty \frac{dU_\infty}{dx}\int_0^\delta dy = \frac{d}{dx}\left[U_\infty \int_0^\delta u dy\right] - \frac{dU_\infty}{dx}\int_0^\delta u dy - \frac{d}{dx}\left[\int_0^\delta u^2 dy\right]$$

The Von Kármán Integral Relation

$$\frac{\tau_w}{\rho} - U_\infty \frac{dU_\infty}{dx} \int_0^\delta dy = \frac{d}{dx}\left[U_\infty \int_0^\delta u dy\right] - \frac{dU_\infty}{dx} \int_0^\delta u dy - \frac{d}{dx}\left[\int_0^\delta u^2 dy\right]$$

$$U_\infty \frac{dU_\infty}{dx} \int_0^\delta dy - \frac{dU_\infty}{dx} \int_0^\delta u dy = \frac{dU_\infty}{dx} \int_0^\delta (U_\infty - u) dy$$

$$\frac{d}{dx}\left[U_\infty \int_0^\delta u dy\right] - \frac{d}{dx}\left[\int_0^\delta u^2 dy\right] = \frac{d}{dx} \int_0^\delta u(U_\infty - u) dy$$

The Von Kármán Integral Relation

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^{\delta} u(U_\infty - u)dy + \frac{dU_\infty}{dx}\int_0^{\delta}(U_\infty - u)dy$$

The Von Kármán Integral Relation

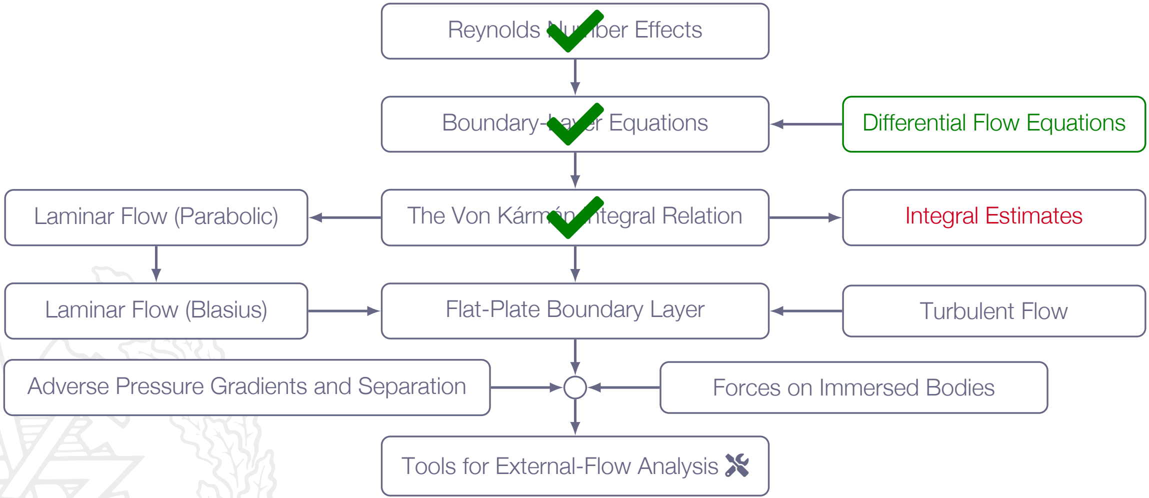
$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^{\delta} u(U_\infty - u)dy + \frac{dU_\infty}{dx}\int_0^{\delta}(U_\infty - u)dy$$

Constant freestream velocity gives

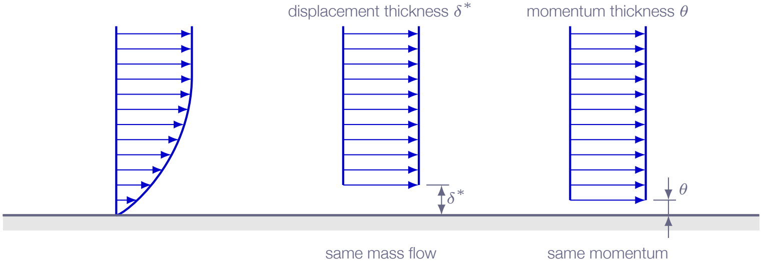
$$\frac{dU_\infty}{dx} = 0 \Rightarrow \frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^{\delta} u(U_\infty - u)dy$$

Ok, but what does this mean??

Roadmap - Flow Past Immersed Bodies

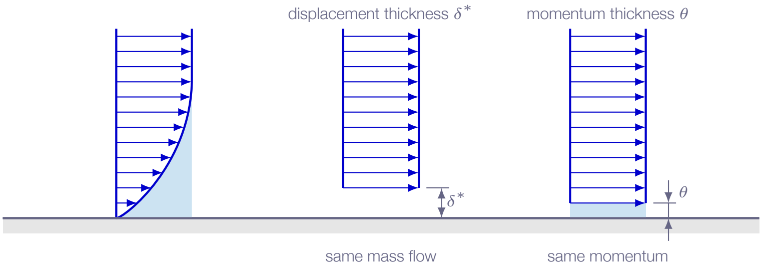


Momentum Integral Estimates



"The presence of a boundary layer will result in a small but finite displacement of the flow streamlines"

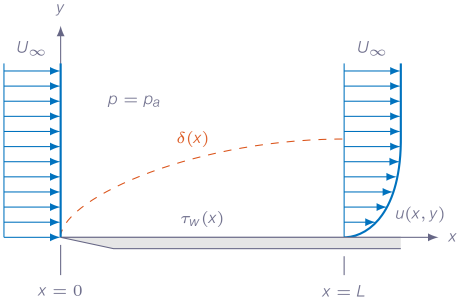
Momentum Thickness



$$\int_0^{\delta} \rho u(U_\infty - u)b\,dy = \rho U_\infty^2 b\theta \Rightarrow \theta = \int_0^{\delta} \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

Note! *b is the width of the flat plate*

Momentum Thickness



The drag D for a plate of width b

$$D(x) = b \int_0^x \tau_w(x) dx \Rightarrow \frac{dD}{dx} = b\tau_w$$

from before we have

$$\underbrace{\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^\delta u(U_\infty - u)dy}_{\text{the Von Kármán integral relation}} = \frac{d}{dx}U_\infty^2 \underbrace{\int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy}_{\text{the displacement thickness } \theta} = U_\infty^2 \frac{d\theta}{dx}$$

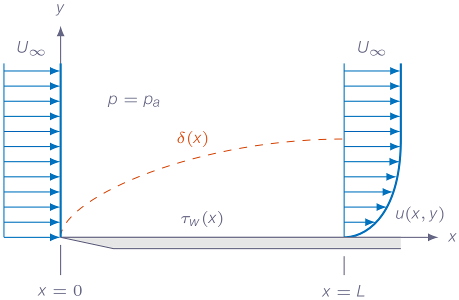
and thus

$$\frac{dD}{dx} = b\rho U_\infty^2 \frac{d\theta}{dx} \Rightarrow D(x) = \rho b U_\infty^2 \theta$$

Momentum Thickness

$$D(x) = \rho b U_\infty^2 \theta, \;\; \tau_w = \rho U_\infty^2 \frac{d\theta}{dx}$$

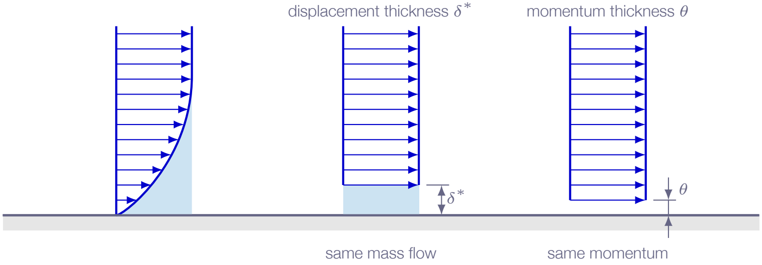
$$\theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$



Note!

1. the momentum thickness θ is a measure of the total drag
2. can be used both for laminar and turbulent flows
3. no assumption about velocity profile shape made

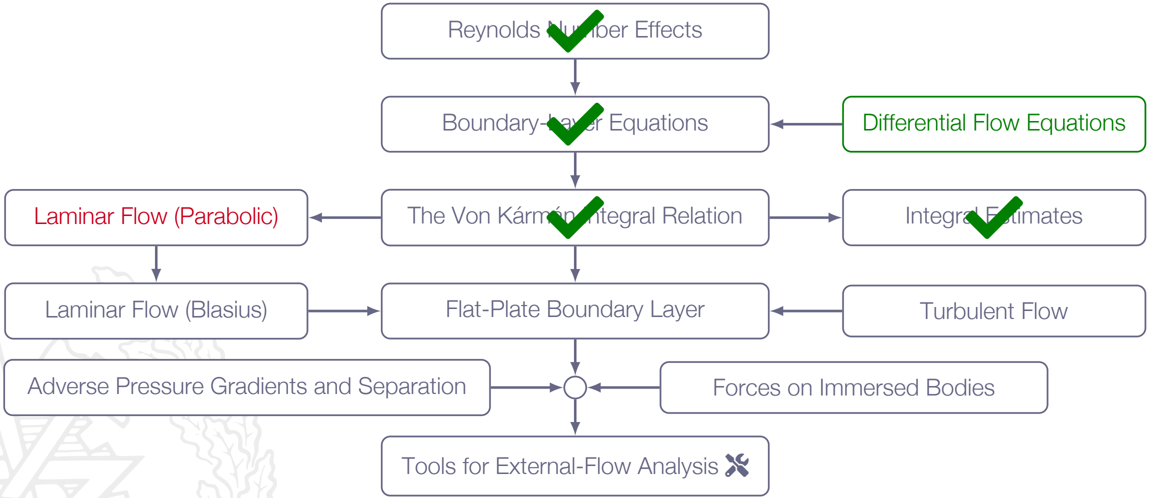
Displacement Thickness



$$\int_0^{\delta} \rho(U_\infty - u) b dy = \rho U_\infty b \delta^* \Rightarrow \delta^* = \int_0^{\delta} \left(1 - \frac{u}{U_\infty}\right) dy$$

δ^* is an estimate of the displacement in the wall-normal direction of streamlines in the outer part of the boundary layer due to the deficit of massflow caused by the no-slip condition at the wall - a measure of the boundary-layer thickness

Roadmap - Flow Past Immersed Bodies



Laminar Boundary Layer

The Von Kármán integral relation gives us the wall shear stress (τ_w) as a function of the velocity profile ($u(y)$) and the boundary-layer thickness (δ)

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^{\delta} u(U_\infty - u)dy$$

So now we need a velocity profile $u = u(y)$ to continue ...

Laminar Boundary Layer

Assumptions:

1. Boundary layer over a flat plate
2. Constant freestream velocity $U_\infty = const \Rightarrow \dfrac{dU_\infty}{dx} = 0$
3. Laminar flow
4. Parabolic velocity profile

Laminar Boundary Layer - Parabolic Velocity Profile

$$u(y) = A + By + Cy^2$$

The constants A, B, and C are defined using boundary conditions

1. no slip:

   $u(0) = 0 \Rightarrow A = 0$

2. constant velocity at $y = \delta$:

   $\left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0 \Rightarrow B + 2C\delta = 0 \Rightarrow B = -2\delta C$

3. freestream velocity:

   $u(\delta) = U_\infty \Rightarrow B\delta + C\delta^2 = U_\infty \Rightarrow \{B = -2\delta C\} \Rightarrow -C\delta^2 = U_\infty \Rightarrow C = -\frac{U_\infty}{\delta^2}$

Laminar Boundary Layer - Parabolic Velocity Profile

$$u(y) = A + By + Cy^2$$

$$A = 0, \quad B = \frac{2U_\infty}{\delta}, \quad C = -\frac{U_\infty}{\delta^2}$$

$$u(y) = U_\infty \left(\frac{2}{\delta} y - \frac{1}{\delta^2} y^2 \right)$$

Laminar Boundary Layer - Parabolic Velocity Profile

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^\delta u(U_\infty - u)dy$$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu\frac{2U_\infty}{\delta}$$

$$\int_0^\delta u(U_\infty - u)dy = \int_0^\delta U_\infty^2\left(\frac{2}{\delta}y - \frac{1}{\delta^2}y^2\right) - U_\infty^2\left(\frac{4}{\delta^2}y^2 - \frac{4}{\delta^3}y^3 + \frac{1}{\delta^4}y^4\right)dy = \frac{2}{15}U_\infty^2\delta$$

$$\frac{\mu}{\rho}\frac{2U_\infty}{\delta} = \frac{d}{dx}\left(\frac{2}{15}U_\infty^2\delta\right) \Rightarrow \frac{\nu}{\delta} = \frac{U_\infty}{15}\frac{d\delta}{dx} \Rightarrow \delta d\delta = 15\frac{\nu}{U_\infty}dx$$

Laminar Boundary Layer - Parabolic Velocity Profile

$$\delta d\delta = 15\frac{\nu}{U_\infty}dx$$

$$\frac{\delta^2}{2} = 15\frac{\nu}{U_\infty}x + C = \{x = 0 \Rightarrow \delta = 0 \Rightarrow C = 0\} = 15\frac{\nu}{U_\infty}x$$

$$\delta = \sqrt{\frac{30\nu x}{U_\infty}} \Rightarrow \frac{\delta}{x} = \sqrt{\frac{30\nu}{U_\infty x}} \approx \frac{5.5}{\sqrt{Re_x}}$$

Laminar Boundary Layer - Parabolic Velocity Profile

$$\tau_w = \mu \frac{2U_\infty}{\delta} = \frac{2\mu U_\infty}{\sqrt{\frac{30\nu x}{U_\infty}}} = \frac{2}{\sqrt{30}} \frac{\rho U_\infty^2}{\sqrt{\frac{U_\infty x}{\nu}}} \approx \frac{0.365}{\sqrt{Re_x}} \rho U_\infty^2$$

Introducing the **skin friction coefficient** c_f

$$c_f = \frac{2\tau_w}{\rho U_\infty^2} \approx \frac{0.73}{\sqrt{Re_x}}$$

Laminar Boundary Layer - Parabolic Velocity Profile

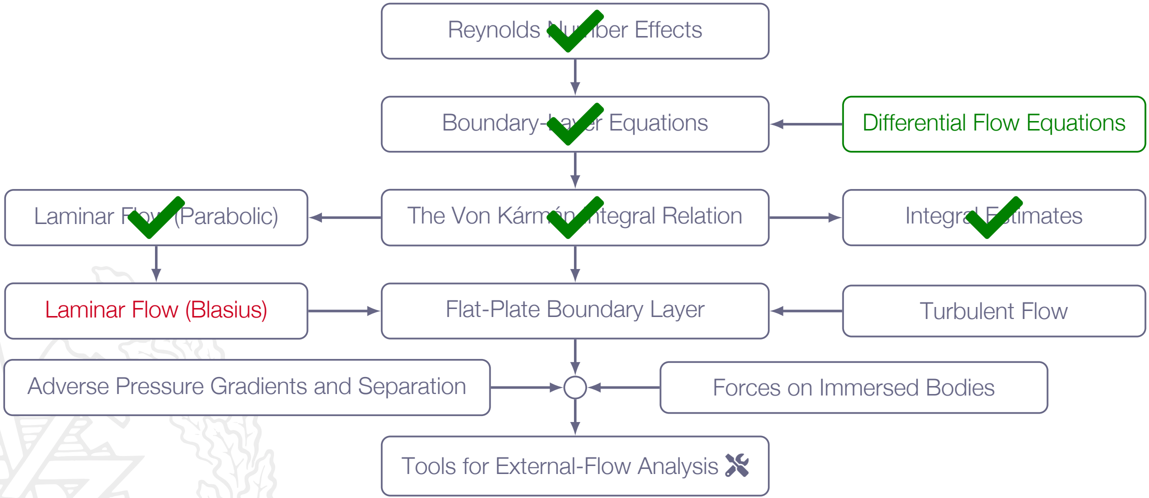
Note! *more accurate solutions for laminar flat plate boundary layers exists:*

$$c_f \approx \frac{0.664}{\sqrt{Re_x}}, \quad \frac{\delta}{x} \approx \frac{5.0}{\sqrt{Re_x}}$$

Ok, so where did we go wrong?

For external (unconfined) boundary layers, the velocity profile is not parabolic – but quite close to parabolic ...

Roadmap - Flow Past Immersed Bodies



The Blasius Velocity Profile

For laminar flow, the boundary layer equations can be solved for u and v

Blasius presented a solution 1908 where he had used a coordinate transformation and showed that $\frac{u}{U_\infty}$ is a function of a single dimensionless variable $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$

The coordinate transformation corresponds to a scaling of the y coordinate with the boundary layer thickness δ

$$\frac{\delta}{x} \propto \frac{1}{\sqrt{Re_x}} \Rightarrow \frac{y}{\delta} \propto \frac{y}{x/\sqrt{Re_x}} = \frac{y}{x}\sqrt{\frac{U_\infty x}{\nu}} = y\sqrt{\frac{U_\infty}{\nu x}} = \eta$$

The Blasius Velocity Profile

1. Rewrite the boundary layer equations using the stream function (Chapter 4)
2. Rewrite the equation again $\Psi = f(\eta)\sqrt{\nu U_\infty x}$ where η is the scaled wall-normal coordinate and $f(\eta)$ is a non-dimensional stream function
3. Lots of math ....

The Navier-Stokes equations are reduced to an ordinary differential equation (ODE)

$$f''' + \frac{1}{2}ff'' = 0$$

with the boundary conditions

$$\begin{cases} f(0) = f'(0) = 0 \\ f'_{\eta\to\infty} \to 1.0 \end{cases}$$

The Blasius Velocity Profile

$$\frac{u}{U_\infty} = f'(\eta)$$

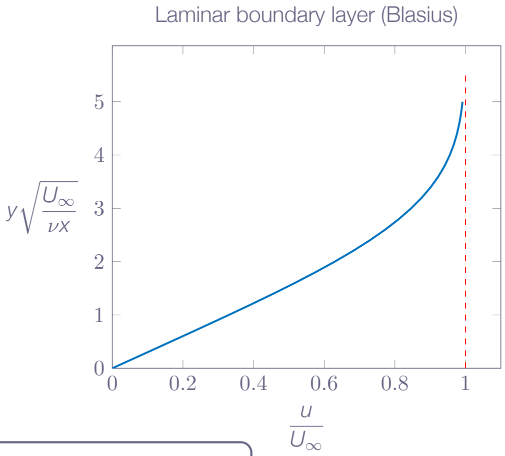
Note! $u/U_\infty \to 1$ as $y \to \infty$ and therefore δ is usually defined as the distance from the wall where $u/U_\infty = 0.99$

The Blasius Velocity Profile

$$\frac{u}{U_\infty} = f'(\eta)$$

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}}$$

$$\delta_{99\%}\sqrt{\frac{U_\infty}{\nu x}} \approx 5.0$$



$$\delta_{99\%}\sqrt{\frac{U_\infty}{\nu x}} \approx 5.0 \text{ or } \frac{\delta}{x} \approx \frac{5.0}{\sqrt{Re_x}}$$

The Blasius Velocity Profile

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu \left[\frac{du}{d\eta}\frac{\partial \eta}{\partial y}\right]_{\eta=0} = \mu U_\infty \left[\frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)\frac{\partial \eta}{\partial y}\right]_{\eta=0}$$

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \Rightarrow \frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{\nu x}} \Rightarrow \tau_w = \mu U_\infty \sqrt{\frac{U_\infty}{\nu x}} \frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)_{\eta=0} = \frac{\rho U_\infty^2}{\sqrt{Re_x}} \frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)_{\eta=0}$$

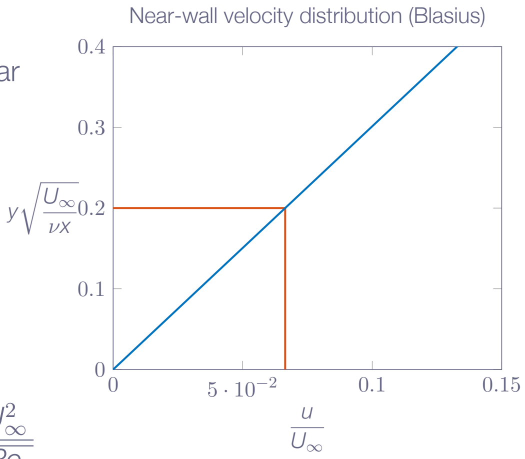
The Blasius Velocity Profile

close the the wall the velocity profile is linear

$$\eta = 0.2 \Rightarrow \frac{u}{U_\infty} \approx 0.0664$$

$$\frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)_{\eta=0} \approx \frac{0.0664}{0.2} = 0.332$$

$$\tau_w = \frac{\rho U_\infty^2}{\sqrt{Re_x}} \frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)_{\eta=0} \approx 0.332 \frac{\rho U_\infty^2}{\sqrt{Re_x}}$$



Laminar Boundary Layer - Blasius

$$\tau_w(x) \approx \frac{0.332\rho^{1/2}\mu^{1/2}U_\infty^{3/2}}{x^{1/2}}$$

Note! the wall shear stress drops off with increasing distance due to the boundary layer growth

Recall *for pipe flow, the wall shear stress is independent of x – pipe flow is confined and the boundary layer height is restricted*

Laminar Boundary Layer - Blasius

wall shear stress:

$$\tau_w(x) \approx \frac{0.332\rho^{1/2}\mu^{1/2}U_\infty^{3/2}}{x^{1/2}}$$

drag force:

$$D(x) = b\int_0^x \tau_w(x)dx \approx 0.664b\rho^{1/2}\mu^{1/2}U_\infty^{3/2}x^{1/2}$$

drag coefficient:

$$C_D = \frac{2D(L)}{\rho U_\infty^2 bL} \approx \frac{1.328}{\sqrt{Re_L}}$$

Laminar Boundary Layer - Blasius

From before we have $D(x) = \rho b \int_0^{\delta(x)} u(U_\infty - u)dy$

$$D(x) = \rho b U_\infty^2 \underbrace{\int_0^{\delta(x)} \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy}_{\theta(x)} = \rho b U_\infty^2 \theta(x)$$

$$b\int_0^x \tau_w(x)dx = \rho b U_\infty^2 \theta(x) \approx 0.664 b \rho^{1/2}\mu^{1/2}U_\infty^{3/2}x^{1/2} \Rightarrow \theta(x) \approx \frac{0.664\mu^{1/2}x^{1/2}}{\rho^{1/2}U_\infty^{1/2}}$$

$\Rightarrow \theta(x) \approx \dfrac{0.664\mu^{1/2}x}{\rho^{1/2}U_\infty^{1/2}x^{1/2}}$ and thus $\dfrac{\theta(x)}{x} \approx \dfrac{0.664}{\sqrt{Re_x}}$

Laminar Boundary Layer - Blasius

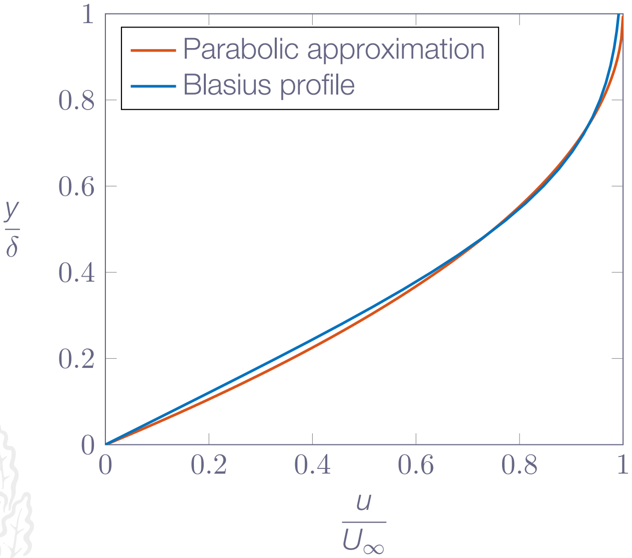
Displacement thickness:

$$\delta^* = \int_0^{\delta} \left(1 - \frac{u}{U_\infty}\right) dy$$

$$\frac{\delta^*}{x} \approx \frac{1.721}{Re_x^{1/2}}$$

Note! *since δ^* is much smaller than x for large values of Re_x, the velocity component in the wall-normal direction will be much smaller than the velocity parallel to the plate*

Laminar Boundary Layer



Laminar Boundary Layer

description	variable	laminar flow (Blasius)	turbulent flow (Prandtl)
boundary layer thickness	$\frac{\delta}{x}$	$\frac{5.0}{\sqrt{Re_x}}$	
displacement thickness	$\frac{\delta^*}{x}$	$\frac{1.721}{\sqrt{Re_x}}$	
momentum thickness	$\frac{\theta}{x}$	$\frac{0.664}{\sqrt{Re_x}}$	
shape factor	$H = \frac{\delta^*}{\theta}$	2.59	
wall shear stress	τ_w	$0.332 \frac{\rho U_\infty^2}{\sqrt{Re_x}}$	
local skin friction coefficient	$c_f = \frac{2\tau_w}{\rho U_\infty^2}$	$\frac{0.664}{\sqrt{Re_x}}$	
drag coefficient	C_D	$\frac{1.328}{\sqrt{Re_L}}$	

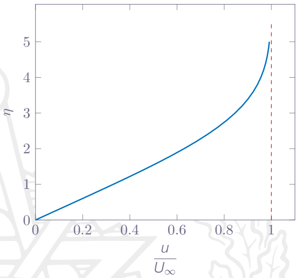
The Blasius Velocity Profile - Self Similarity

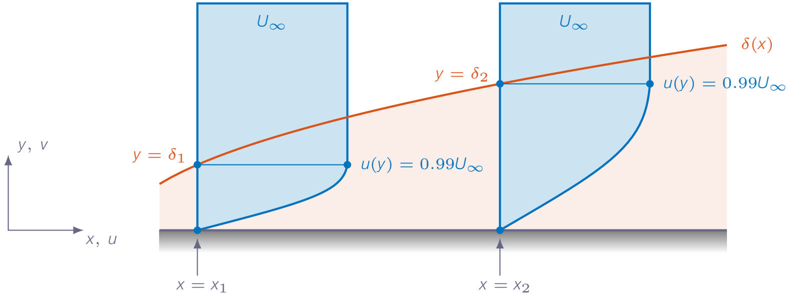
From before:

$$\eta(x,y) = y\sqrt{\frac{U_\infty}{\nu x}}$$

$$\frac{u}{U_\infty} = 0.99 \Rightarrow \eta \approx 5.0$$

Laminar boundary layer (Blasius)

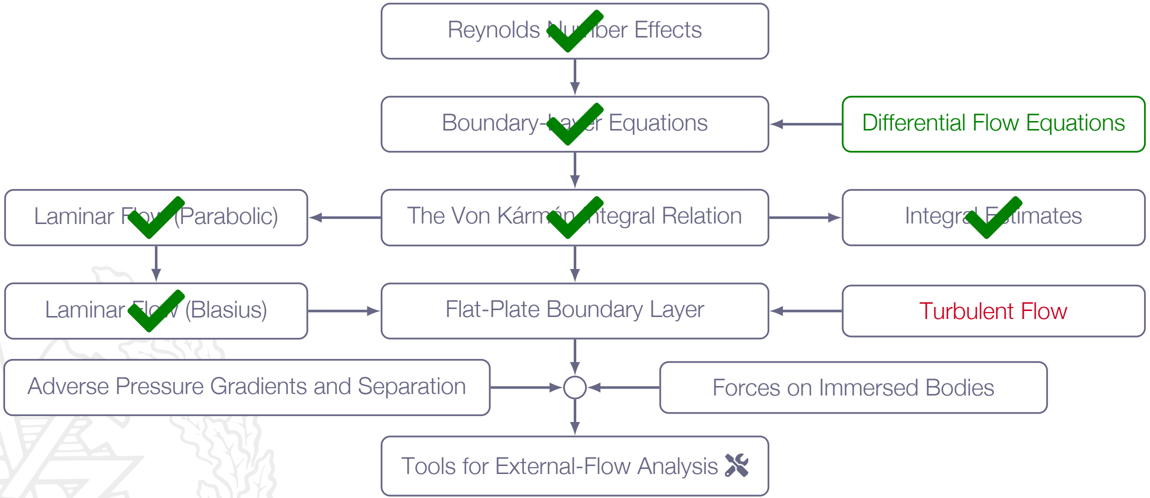




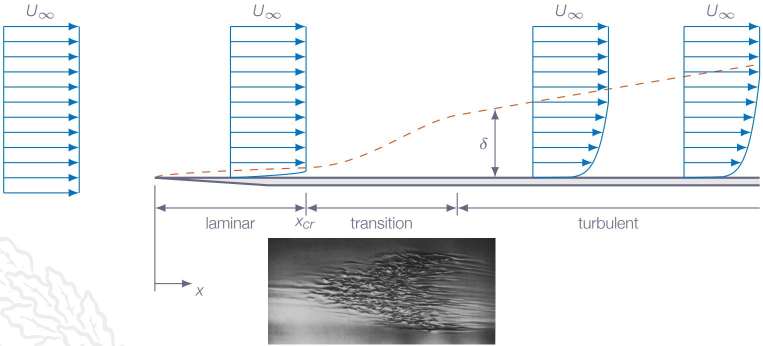
$$\eta(x_1,\delta_1) = \eta(x_2,\delta_2) \Rightarrow \delta_1\sqrt{\frac{U_\infty}{\nu x_1}} = \delta_2\sqrt{\frac{U_\infty}{\nu x_2}}$$

$$x_1 < x_2 \Rightarrow \sqrt{\frac{U_\infty}{\nu x_1}} > \sqrt{\frac{U_\infty}{\nu x_2}} \Rightarrow \delta_1 < \delta_2$$

Roadmap - Flow Past Immersed Bodies



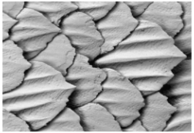
Boundary Layer Transition



$$u = 0.99U_\infty$$

$$Re_x = \frac{U_\infty x}{\nu}$$

$$Re_{x_{cr}} = \frac{U_\infty x_{cr}}{\nu} \approx 5.0 \times 10^5$$



Boundary Layer Transition

For low Re_x, disturbances in the flow are damped out by viscous forces

For somewhat higher Reynolds numbers, friction forces are less important and the flow becomes unstable

The transition region is short - can be treated as a point (the transition point)

Boundary Layer Transition

The onset of transition from laminar to turbulent is affected by a number of factors such as:

Turbulence in the freestream

Surface roughness

Pressure gradient

With a smooth surface, no turbulence in the freestream, and zero pressure gradient, the onset of transition can be pushed up to $Re_x \approx 3.0 \times 10^6$

As a **rule of thumb**, we can assume $Re_{x_{cr}} \approx 5.0 \times 10^5$

Boundary Layer Transition

Freestream turbulence:

frestream turbulence reduces the critical Reynolds number

with high turbulence intensity in the freestream, the transition can start already at $Re_x \approx 3.0 \times 10^5$ or lower

Boundary Layer Transition

Surface roughness:

surface roughness does not affect transition significantly if $Re_\epsilon = \frac{U_\infty \epsilon}{\nu} < 680$

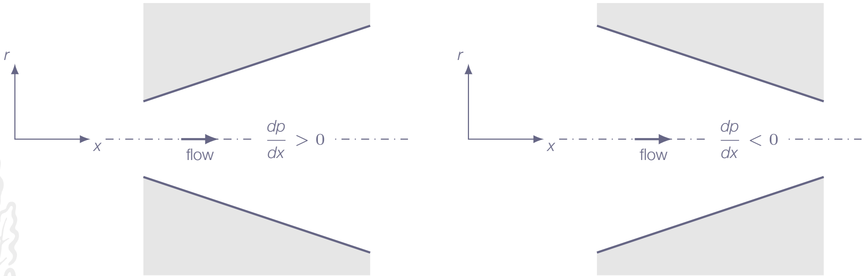
if $Re_\epsilon > 680$, the extent of the laminar region can be shortened significantly ($Re_x \approx 3.0 \times 10^5$)

Note! rule of thumb

Boundary Layer Transition

Negative pressure gradient:

decreasing pressure in the flow direction has a **stabilizing** effect on the flow and can delay transition from laminar to turbulent flow



Boundary Layer Transition

Forced transition:

a **trip wire** or **added surface roughness** can make the transition to turbulence really fast

the critical Reynolds number is not meaningful if the boundary layer is forced to transition

Flat Plate - Turbulent Boundary Layer

A turbulent boundary layer grows faster than a laminar boundary layer

- the velocity fluctuations (u', v', w') leads to **increased exchange of momentum**
- increased shear stress compared to the laminar case where we only have forces related to molecular viscosity
- larger portion of the fluid will be decelerated close to the wall

Flat Plate - Turbulent Boundary Layer

The Von Kármán integral relation and the integral estimates are valid for both laminar and turbulent boundary layers

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^\delta u(U_\infty - u)dy$$

$$\theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)d$$

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right)dy$$

We need a velocity profile $u(y)$ for turbulent boundary layers to be able to calculate τ_w, θ, and δ^*

Approach 1: the log law

Approach 2: Prandtl's power law approximation

Flat Plate - Turbulent Boundary Layer

Approach 1: the log law

$$\frac{u}{u^*} \approx \frac{1}{\kappa} \ln\left(\frac{yu^*}{\nu}\right) + B \quad \text{where } \kappa = 0.41 \text{ and } B = 5.0$$

u^* is the **friction velocity** defined as $u^* = \sqrt{\frac{\tau_w}{\rho}}$

at the edge of the boundary layer $u = U_\infty$ and $y = \delta$ and thus

$$\frac{U_\infty}{u^*} \approx \frac{1}{\kappa} \ln\left(\frac{\delta u^*}{\nu}\right) + B$$

Flat Plate - Turbulent Boundary Layer

Approach 1: the log law

The **skin friction coefficient** c_f is defined as $c_f = \frac{2\tau_w}{\rho U_\infty^2} \Rightarrow \tau_w = c_f \frac{1}{2}\rho U_\infty^2$

the **friction velocity** can be expressed as $u^* = \sqrt{\frac{\tau_w}{\rho}} = U_\infty \sqrt{\frac{c_f}{2}}$

insert in the **log-law** and we get

$$\sqrt{\frac{2}{c_f}} \approx \frac{1}{\kappa} \ln\left(Re_\delta \sqrt{\frac{c_f}{2}}\right) + B$$

rather difficult to work with ...

Flat Plate - Turbulent Boundary Layer

Approach 2: Prandtl's power law approximation

Prandtl suggested the following relations:

$$c_f \approx 0.02 Re_\delta^{-1/6}$$

$$\frac{u}{U_\infty} \approx \left(\frac{y}{\delta}\right)^{1/7}$$

from before we have the following relation: $\tau_w = \rho U_\infty^2 \dfrac{d\theta}{dx} \Rightarrow c_f = 2\dfrac{d\theta}{dx}$

calculate the **momentum thickness** $\theta = \displaystyle\int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy = \frac{7}{72}\delta$

Flat Plate - Turbulent Boundary Layer

Approach 2: Prandtl's power law approximation

Now, combining the two **skin friction coefficient** relations we see that

$$0.02Re_\delta^{-1/6} = 2\frac{d}{dx}\left(\frac{7}{72}\delta\right)$$

and thus $Re_\delta^{-1/6} \approx 9.72\frac{d\delta}{dx} = 9.72\frac{d(Re_\delta)}{d(Re_x)}$

integration gives $Re_\delta \approx 0.16Re_x^{6/7}$ or $\frac{\delta}{x} \approx \frac{0.16}{Re_x^{1/7}}$

Note! the turbulent boundary layer grows significantly faster than the laminar $\delta_{turb} \propto x^{6/7}$ vs $\delta_{lam} \propto x^{1/2}$

Flat Plate - Turbulent Boundary Layer

Approach 2: Prandtl's power law approximation

$$c_f \approx \frac{0.027}{Re_x^{1/7}}$$

$$\tau_{w_{turb}} \approx \frac{0.0135\mu^{1/7}\rho^{6/7}U_\infty^{13/7}}{x^{1/7}}$$

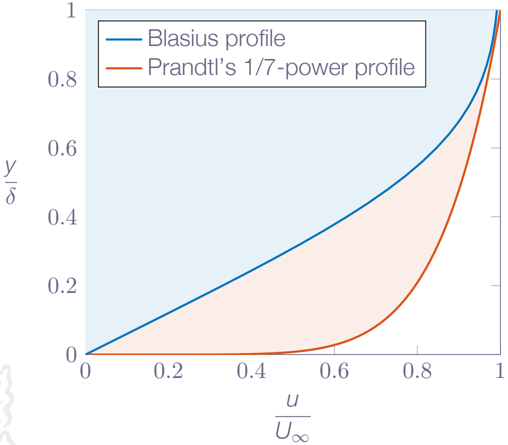
Note! friction drops slowly with x, increases nearly as ρ and U_∞^2, and is rather insensitive to viscosity

Flat Plate - Turbulent Boundary Layer

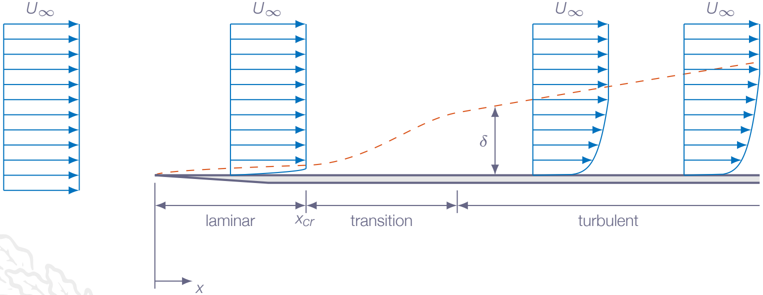
description	variable	laminar flow (Blasius)	turbulent flow (Prandtl)
boundary layer thickness	$\frac{\delta}{x}$	$\frac{5.0}{\sqrt{Re_x}}$	$\frac{0.16}{Re_x^{1/7}}$
displacement thickness	$\frac{\delta^*}{x}$	$\frac{1.721}{\sqrt{Re_x}}$	$\frac{0.02}{Re_x^{1/7}}$
momentum thickness	$\frac{\theta}{x}$	$\frac{0.664}{\sqrt{Re_x}}$	$\frac{0.016}{Re_x^{1/7}}$
shape factor	$H = \frac{\delta^*}{\theta}$	2.59	1.29
wall shear stress	τ_w	$0.332\frac{\rho U_\infty^2}{\sqrt{Re_x}}$	$0.0135\frac{\rho U_\infty^2}{Re_x^{1/7}}$
local skin friction coefficient	$c_f = \frac{2\tau_w}{\rho U_\infty^2}$	$\frac{0.664}{\sqrt{Re_x}}$	$\frac{0.027}{Re_x^{1/7}}$
drag coefficient	C_D	$\frac{1.328}{\sqrt{Re_L}}$	$\frac{0.031}{Re_L^{1/7}}$

Flat Plate - Turbulent Boundary Layer

The velocity profile in a turbulent boundary layer is quite far from the Blasius profile used for laminar boundary layers



Flat Plate Boundary Layer



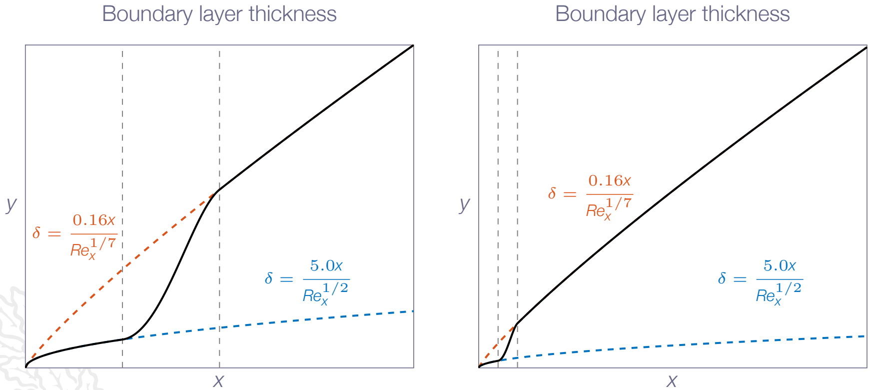
$u = 0.99U_\infty$

$$Re_x = \frac{U_\infty x}{\nu}$$

$$Re_{x_{cr}} = \frac{U_\infty x_{cr}}{\nu} \approx 5.0 \times 10^5$$

$$D = b\frac{1}{2}\rho U^2 \left[\int_{>0}^{x_{cr}} \frac{0.664}{Re_x^{1/2}} dx + \int_{x_{cr}}^{L} \frac{0.027}{Re_x^{1/7}} dx\right]$$

Flat Plate Boundary Layer



For a long boundary layer the length of the laminar region becomes relatively short in comparison with the length of the turbulent region

Wall Roughness

laminar:

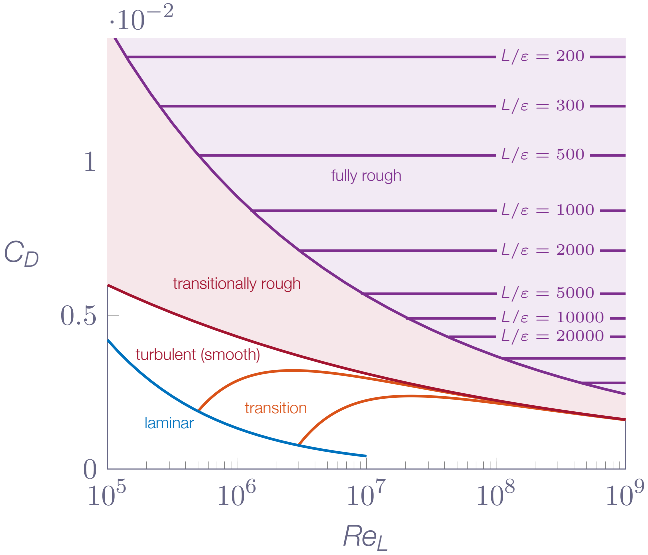
$$C_D = \frac{1.328}{Re_L^{1/2}}$$

turbulent (smooth):

$$C_D = \frac{0.031}{Re_L^{1/7}}$$

turbulent (fully rough):

$$C_D = (1.89 + 1.62 \log(L/\varepsilon))^{-2.5}$$



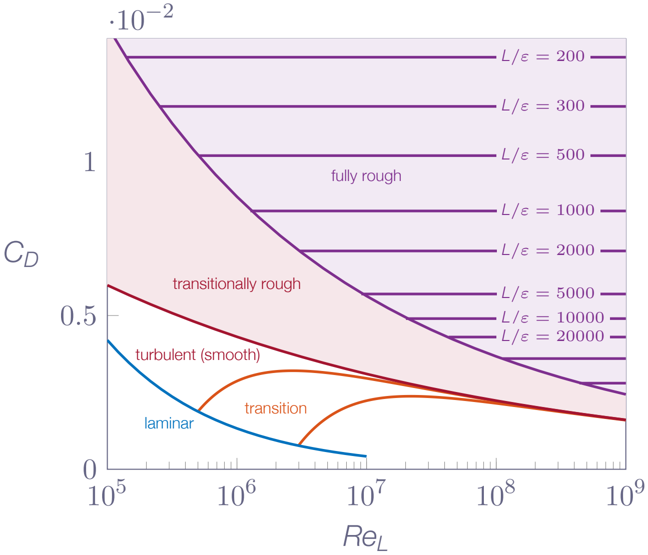
Wall Roughness

transition ($Re_{trans} = 5.0 \times 10^5$):

$$C_D = \frac{0.031}{Re_L^{1/7}} - \frac{1440}{Re_L}$$

transition ($Re_{trans} = 3.0 \times 10^6$):

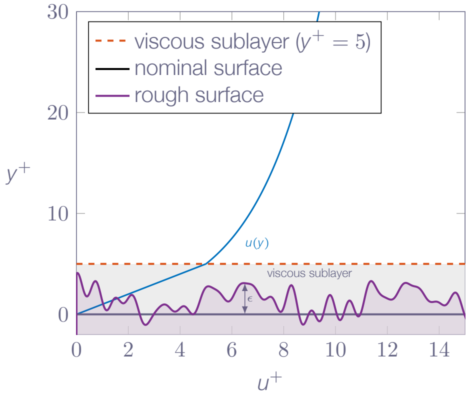
$$C_D = \frac{0.031}{Re_L^{1/7}} - \frac{8700}{Re_L}$$



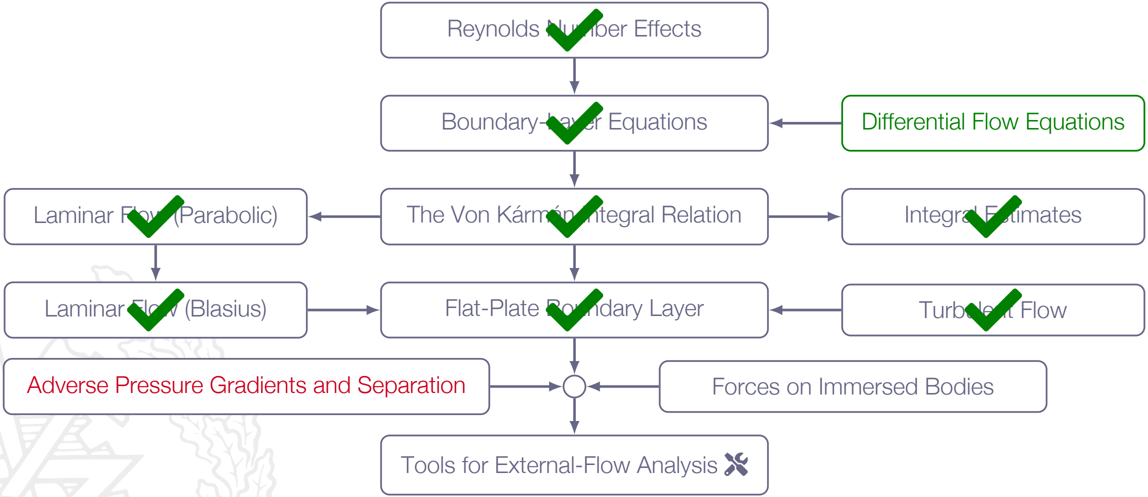
Wall Roughness

Recall: smooth surface:

Surface roughness (ϵ) within the viscous sublayer



Roadmap - Flow Past Immersed Bodies



Pressure Gradient

Adverse pressure gradient

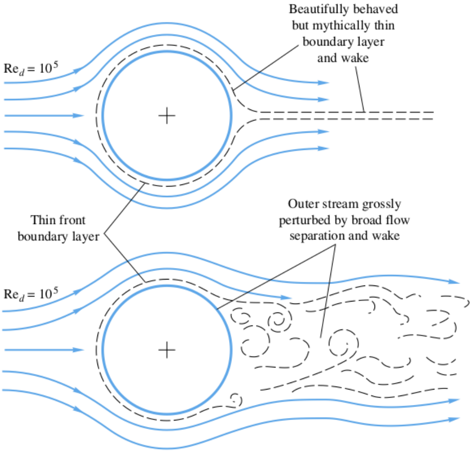
pressure increases in the flow direction
may lead to separation

Favorable pressure gradient

pressure decreases in the flow direction
the flow will not separate

Separation mechanism

loss of momentum near the wall
adverse pressure gradient
decelerated fluid will force flow to separate from the body



Pressure Gradient

Boundary layer formulation of the momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

with $u = v = 0$ close at the wall, we get

$$\left.\frac{\partial \tau}{\partial y}\right|_{wall} = \mu\left.\frac{\partial^2 u}{\partial y^2}\right|_{wall} \Rightarrow \left.\frac{\partial^2 u}{\partial y^2}\right|_{wall} = \frac{1}{\mu}\frac{dp}{dx}$$

Note! applies both for laminar and turbulent flow

Pressure Gradient

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{wall} = \frac{1}{\mu}\frac{dp}{dx}$$

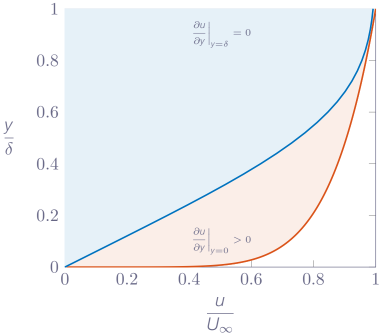
$$\frac{\partial u^2}{\partial y^2} \sim \frac{\left.\frac{\partial u}{\partial y}\right|_{y=\delta} - \left.\frac{\partial u}{\partial y}\right|_{y=0}}{\delta} < 0$$

Adverse pressure gradient ($\frac{dp}{dx} > 0$):

$\frac{\partial^2 u}{\partial y^2} > 0$ at the wall ($y = 0$)

$\frac{\partial^2 u}{\partial y^2} < 0$ in the outer layer ($y \to \delta$)

thus $\frac{\partial^2 u}{\partial y^2} = 0$ somewhere in the boundary layer



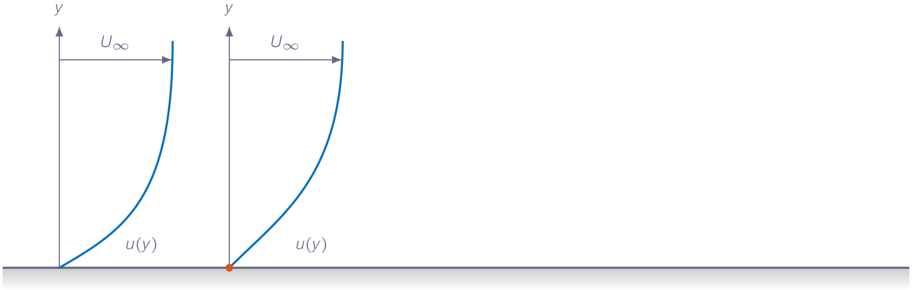
Pressure Gradient

Favorable gradient
$(dp/dx < 0)$

Point of inflection:
inside wall

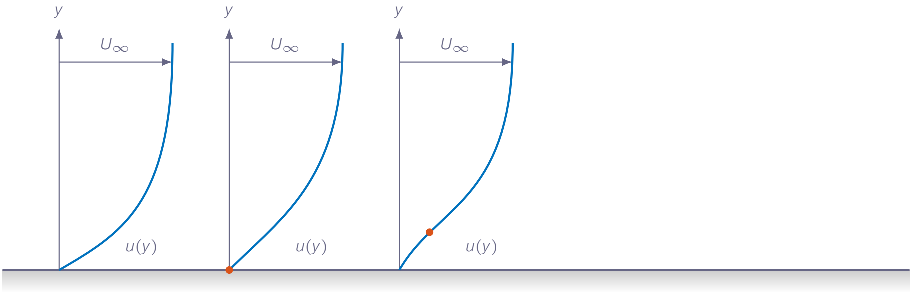
No separation

Pressure Gradient



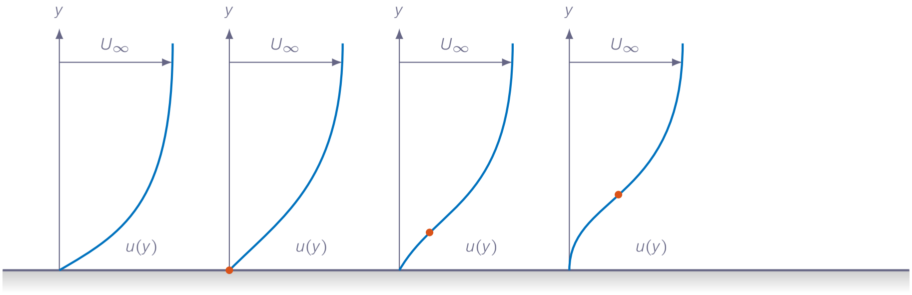
Favorable gradient ($dp/dx < 0$)	Zero gradient ($dp/dx = 0$)
Point of inflection: inside wall	Point of inflection: at the wall
No separation	No separation

Pressure Gradient



Favorable gradient $(dp/dx < 0)$	Zero gradient $(dp/dx = 0)$	Weak adverse gradient $(dp/dx > 0)$
Point of inflection: inside wall	Point of inflection: at the wall	Point of inflection: in the flow
No separation	No separation	No separation

Pressure Gradient



Favorable gradient $(dp/dx < 0)$

Point of inflection: inside wall

No separation

Zero gradient $(dp/dx = 0)$

Point of inflection: at the wall

No separation

Weak adverse gradient $(dp/dx > 0)$

Point of inflection: in the flow

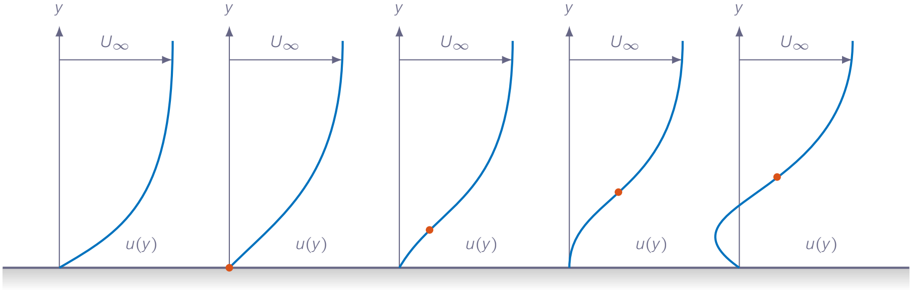
No separation

Critical adverse gradient $(dp/dx > 0)$

Point of inflection: in the flow

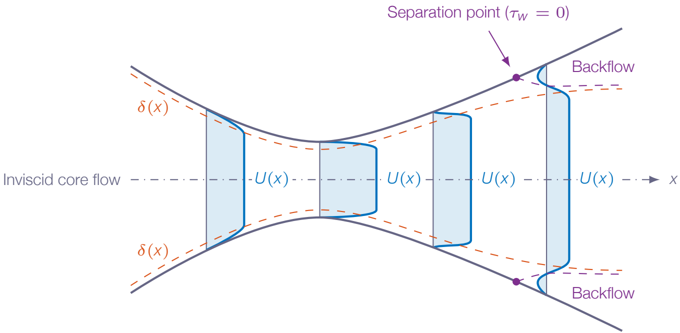
Separation
zero slope at wall
$\tau_w = 0$

Pressure Gradient



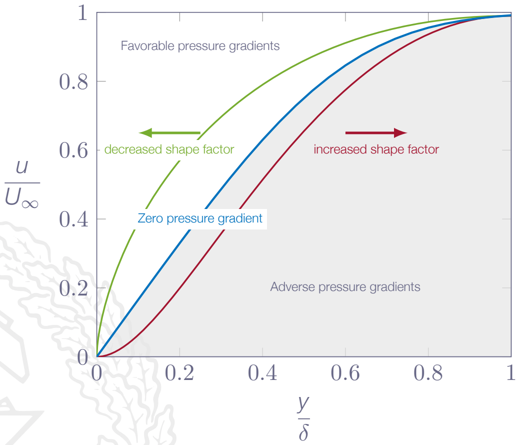
Favorable gradient $(dp/dx < 0)$	Zero gradient $(dp/dx = 0)$	Weak adverse gradient $(dp/dx > 0)$	Critical adverse gradient $(dp/dx > 0)$	Excessive adverse gradient $(dp/dx > 0)$
Point of inflection: inside wall	Point of inflection: at the wall	Point of inflection: in the flow	Point of inflection: in the flow	Point of inflection: in the flow
No separation	No separation	No separation	Separation zero slope at wall $\tau_w = 0$	Separated flow backflow at wall

Pressure Gradient



Nozzle	**Throat**	**Diffuser**
decreasing area	minimum area	increasing area
favorable pressure gradient	zero pressure gradient	adverse pressure gradient
$dp/dx < 0$	$dp/dx = 0$	$dp/dx > 0$
$dU/dx > 0$	$dU/dx = 0$	$dU/dx < 0$

Shape Factor



Shape factor: $H = \dfrac{\delta^*}{\theta}$

Laminar flow:

No pressure gradient: $H \approx 2.6$

Separation: $H \approx 3.5$

Turbulent flow:

No pressure gradient: $H \approx 1.3$

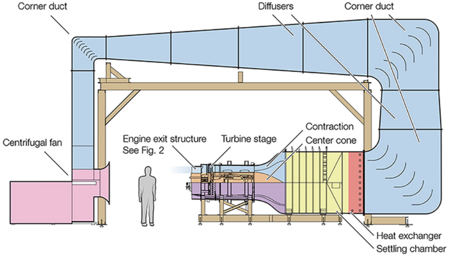
Separation: $H \approx 2.4$

Avoid or Delay Separation

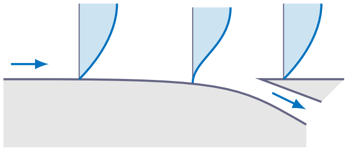
Decrease magnitude of adverse pressure gradient

Guide vanes

Streamlining



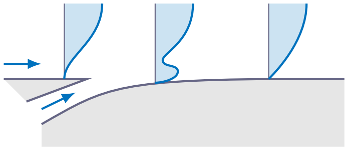
Avoid or Delay Separation



Remove decelerated fluid

Boundary layer suction

Avoid or Delay Separation



Increase near-wall momentum

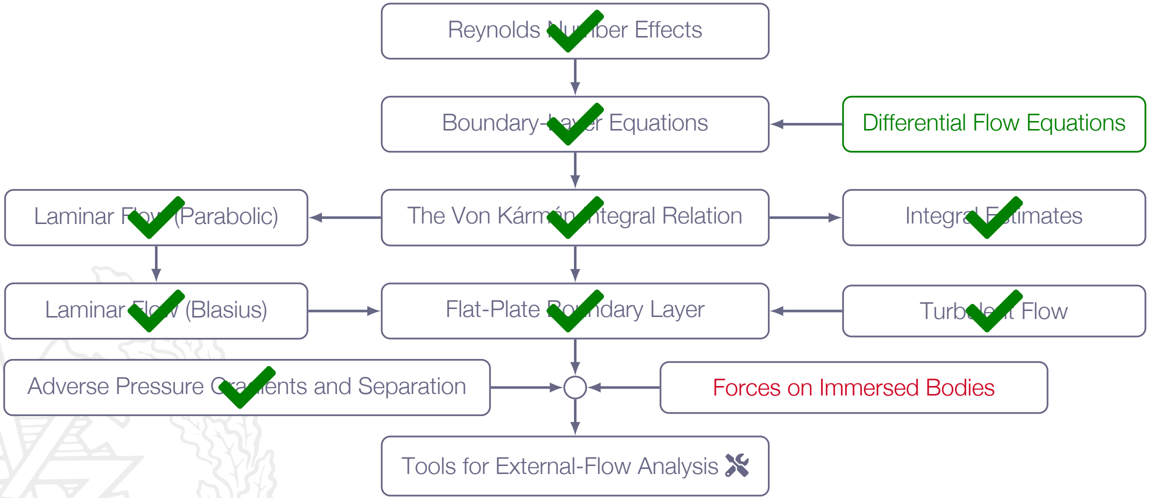
Injection of high-velocity fluid

Forced **transition to turbulence**

- surface roughness
- surface irregularities (dimples on the surface of a golf ball)
- trip wires

Negative consequence: comes with **increased friction**

Roadmap - Flow Past Immersed Bodies

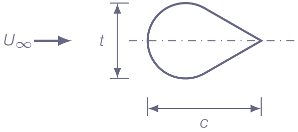


Drag of Immersed Bodies

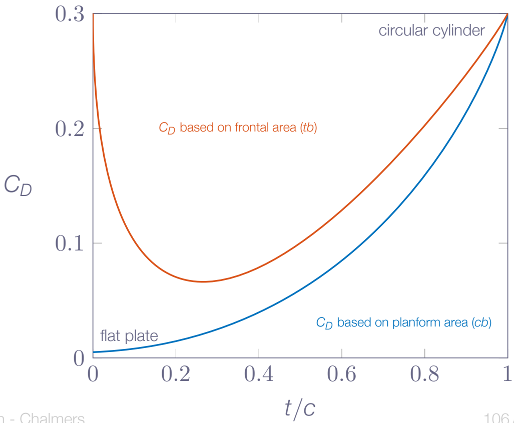
$$C_D = \frac{drag}{\frac{1}{2}\rho U_\infty^2 A} = f\left(\frac{U_\infty L}{\nu}\right)$$

Characteristic area A:

1. Frontal area
   blunt objects: *cylinders, cars*
2. Planform area
   wide flat bodies: *wings, hydrofoils*
3. Wetted area
   ships

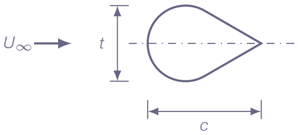


C_D based on frontal and planform area



Drag of Immersed Bodies

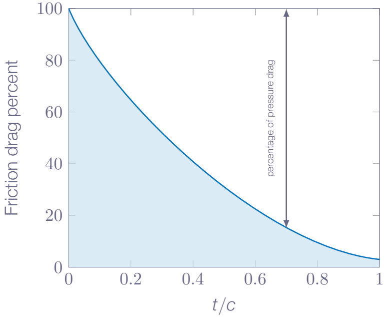
$$C_D = C_{D_{pressure}} + C_{D_{friction}}$$



Pressure drag:

"the difference between the high front stagnation pressure and the low wake pressure on the backside of the body"

"often larger than the friction drag"

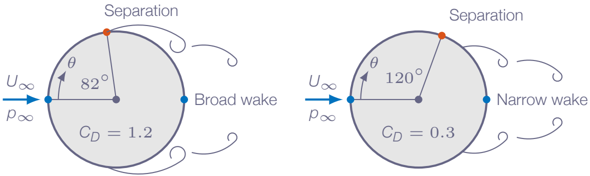


The relative importance of friction and pressure drag depends on:

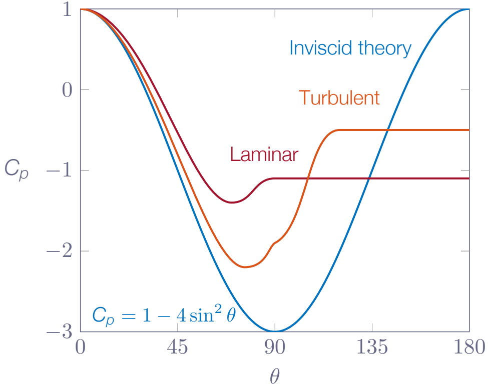
1. body shape
2. surface roughness

Note! for a cylinder, friction drag can be as low as a few percent of the total drag

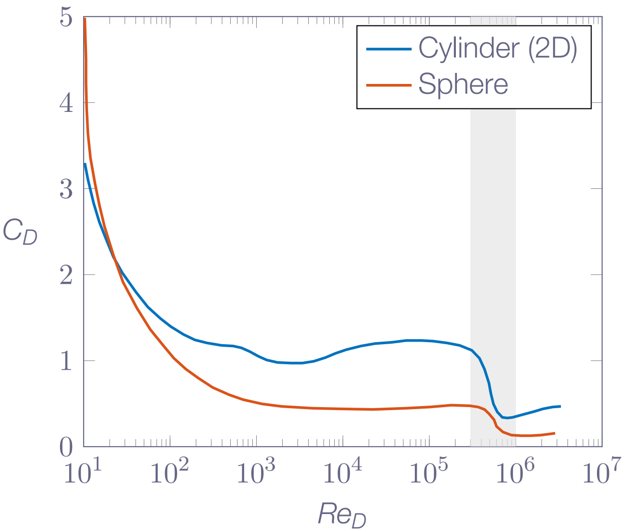
Cylinder Surface Pressure



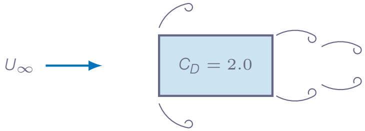
$$C_p = \frac{p - p_\infty}{\rho U_\infty^2/2}$$



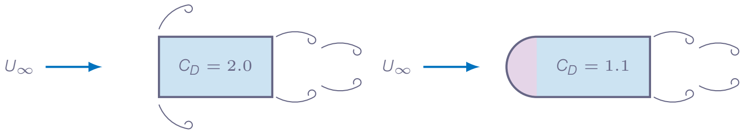
Cylinder Drag



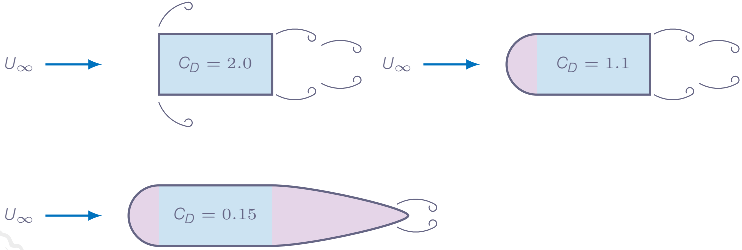
Streamlining



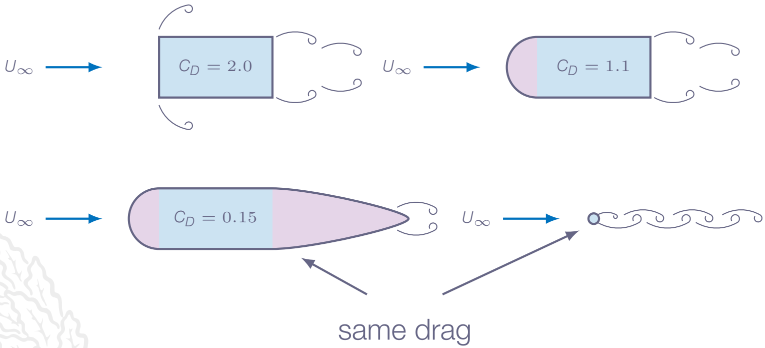
Streamlining



Streamlining



Streamlining



Drag Prediction

No reliable theory for drag prediction (with the exception of flat plates)

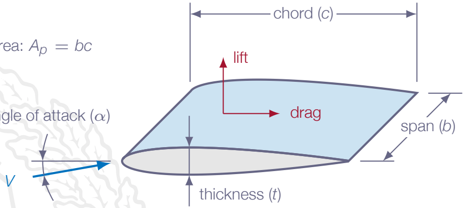
The separation point can be predicted with some accuracy but not the wake flow

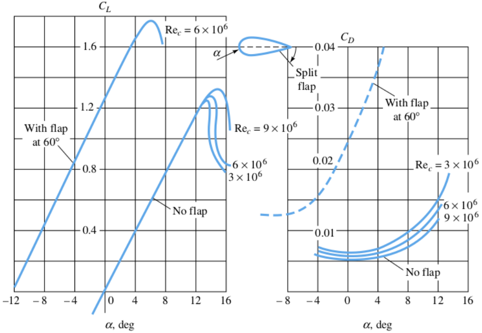
CFD or experiments needed

Wing Lift and Drag

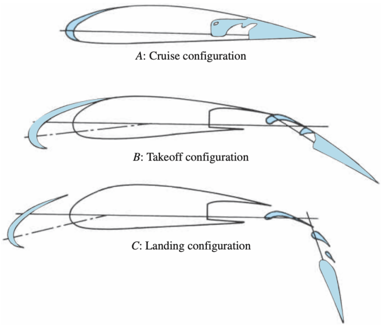
$$C_D = \frac{F_D}{\frac{1}{2}\rho U_\infty^2 A_p} \qquad C_L = \frac{F_L}{\frac{1}{2}\rho U_\infty^2 A_p}$$

planform area: $A_p = bc$

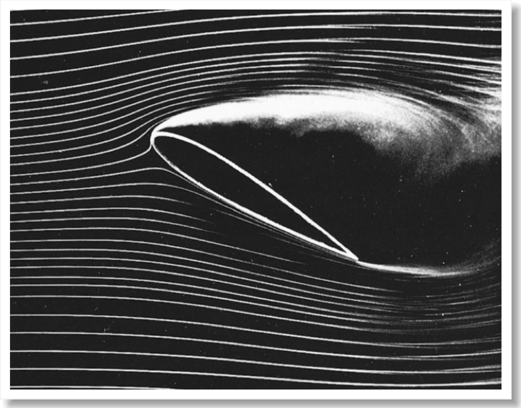




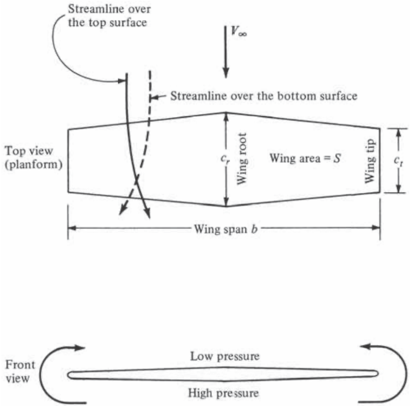
Wing Lift and Drag - High-Lift Devices

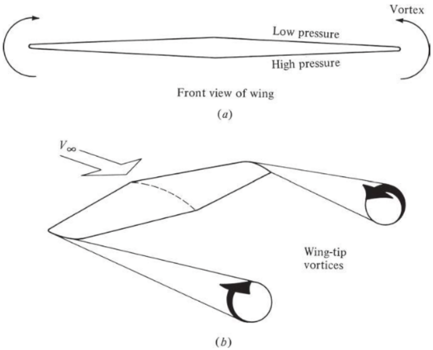


Wing Lift and Drag - Wing Stall

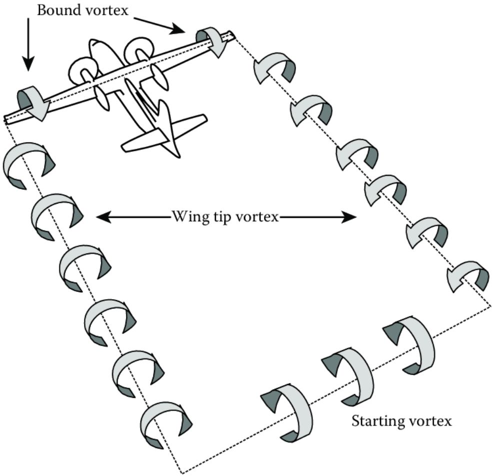


Wing Lift and Drag - Induced Drag



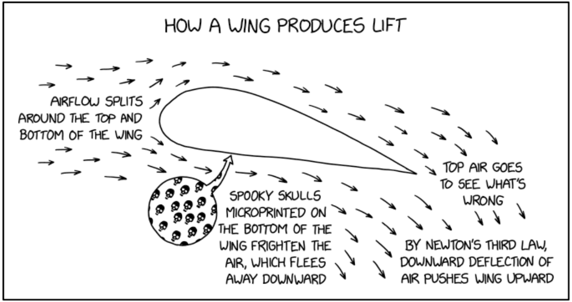


Wing Lift and Drag - Induced Drag

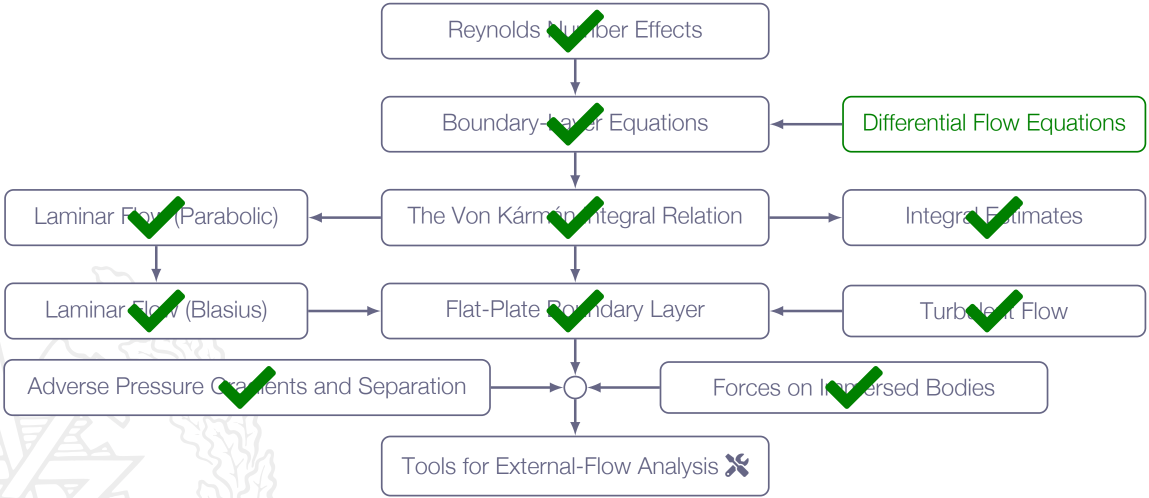


Wing Lift and Drag - Induced Drag

Wing Lift and Drag

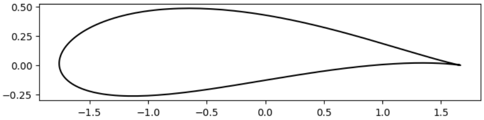


Roadmap - Flow Past Immersed Bodies



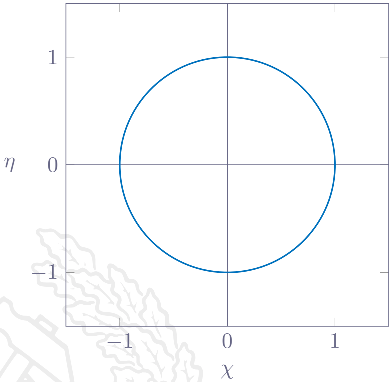
Joukowsky Transform

A Joukowsky wing is generated in the complex plane by applying the Joukowsky transform to a cylinder

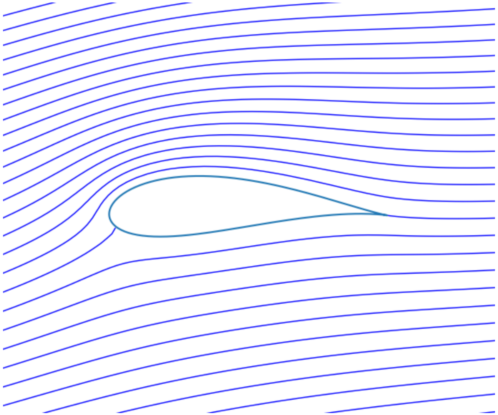


Since the potential flow around a cylinder is well known it is by using so-called conformal mapping possible to get the flow around the wing profile from the cylinder solution

Joukowsky Transform



$$\zeta = \chi + i\eta$$



$$z = \zeta + \frac{1}{\zeta} = x + iy$$

Complex Conjugate