Fluid Mechanics - MTF053

Chapter 7

Niklas Andersson

Chalmers University of Technology
Department of Mechanics and Maritime Sciences
Division of Fluid Mechanics
Gothenburg, Sweden

niklas.andersson@chalmers.se

Chapter 7 - Flow Past Immersed Bodies

Overvie

Pressure

Fluid Flow

Basic Concepts

Thermo-dynamics

Fluid

speed of sound

entropy

fluid concept

continuum

viscosity

Fluid Flow

Com-pressible Flow

shock-expansion theory

nozzle flow

normal shocks

speed of sound

separation

External Flow

turbulence

boundary layer

Reynolds number

friction and losses

Duct Flow

flow regimes

laminar flow

turbulent flow

Learning Outcomes

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

6 **Explain** what a boundary layer is and when/where/why it appears

21 **Explain** how the flat plate boundary layer is developed (transition from laminar to turbulent flow)

22 **Explain** and use the Blasius equation

23 **Define** the Reynolds number for a flat plate boundary layer

24 **Explain** what is characteristic for a turbulent flow

29 **Explain** flow separation (separated cylinder flow)

30 **Explain** how to delay or avoid separation

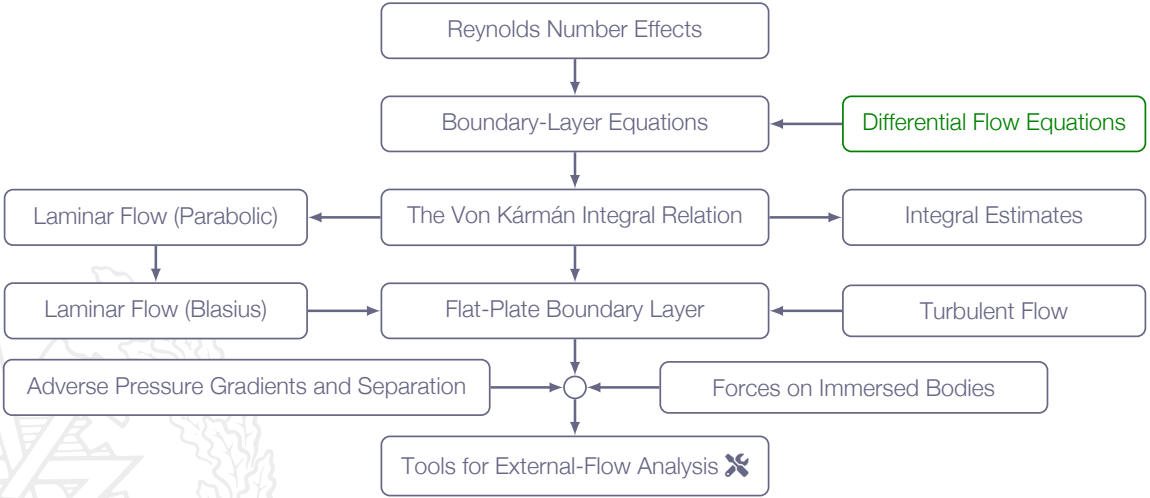
31 **Derive** the boundary layer formulation of the Navier-Stokes equations

32 **Understand** and explain displacement thickness and momentum thickness

33 **Understand**, **explain** and **use** the concepts drag, friction drag, pressure drag, and lift

Let's take a deep dive into boundary-layer theory

Roadmap - Flow Past Immersed Bodies



Complementary Course Material

These lecture notes covers chapter 7 in the course book and additional course material that you can find in the following documents

`MTF053_Equation-for-Boundary-Layer-Flows.pdf`

`MTF053_Turbulence.pdf`

”Understanding the mechanisms behind flow-related forces is a key factor to success in many engineering applications”

External Flow

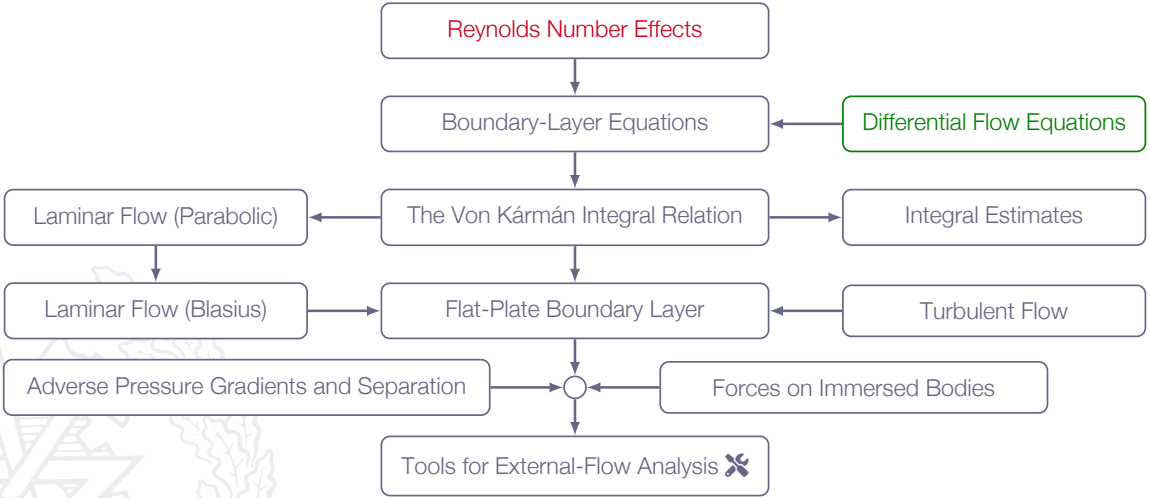
Significant viscous effects near the surface of an **immersed body**

Nearly inviscid far from the body

Unconfined - boundary layers are free to grow

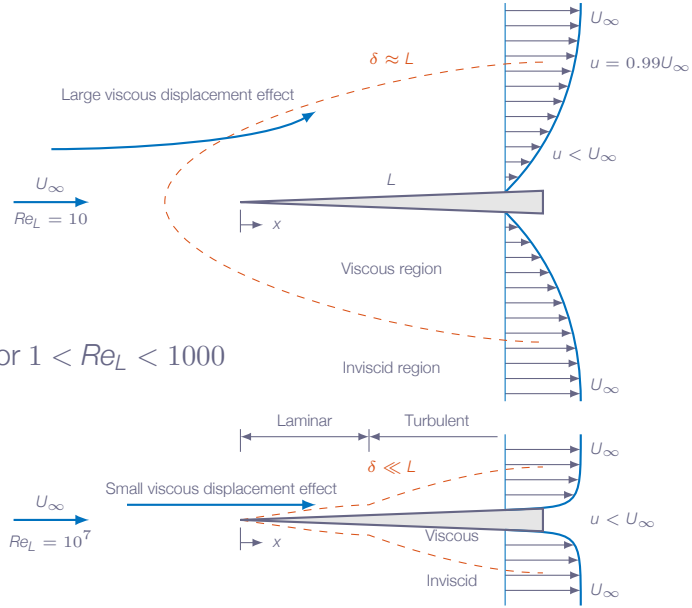
Most often CFD or experiments are needed to analyze an external flow unless the geometry is very simple

Roadmap - Flow Past Immersed Bodies



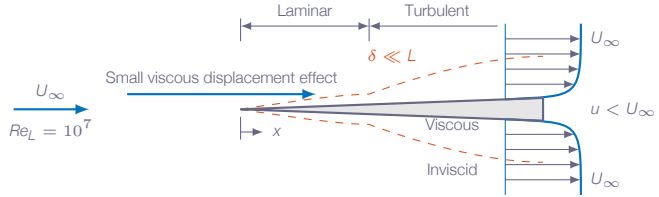
Reynolds Number Effects

$$Re_L = \frac{U_\infty L}{\nu}$$



Note: no simple theory exists for $1 < Re_L < 1000$

Reynolds Number Effects

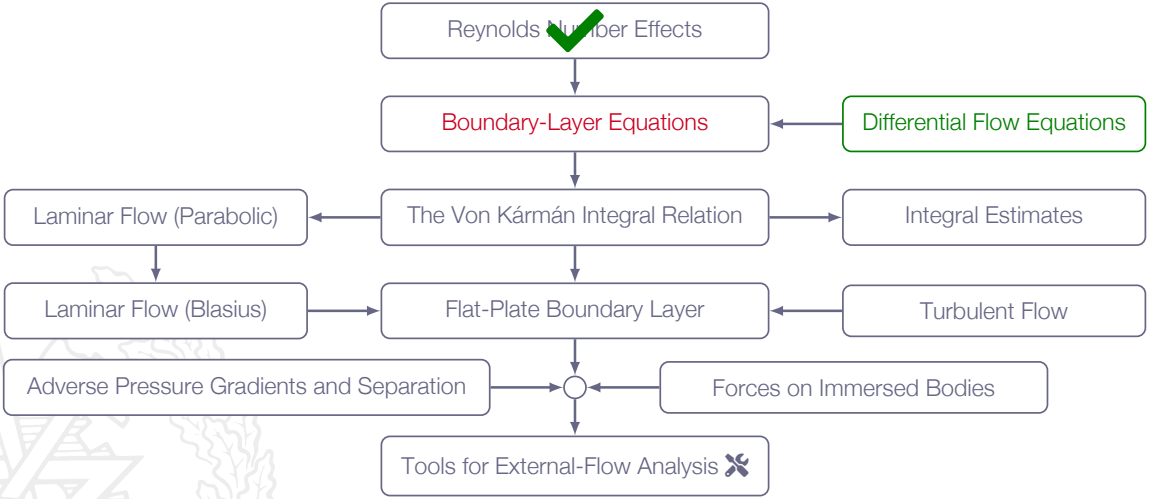


$$\frac{\delta}{x} \approx \begin{cases} \dfrac{5.0}{Re_x^{1/2}} & \textit{laminar} \quad 10^3 < Re_x < 10^6 \\ \dfrac{0.16}{Re_x^{1/7}} & \textit{turbulent} \quad 10^6 < Re_x \end{cases}$$

$$Re_L = \frac{U_\infty L}{\nu} \qquad Re_x = \frac{U_\infty x}{\nu}$$

Note! Re_L and the **local Reynolds number** Re_x are not the same

Roadmap - Flow Past Immersed Bodies



Boundary Layer Equations

We will derive a set of equations suitable for **boundary-layer flow analysis**

Starting point: the **non-dimensional equations** derived in Chapter 5

We will assume two-dimensional, incompressible, steady-state flow

We will do an **order-of-magnitude** comparison of all the terms in the governing equations on non-dimensional form and identify terms that can be neglected in a **thin-boundary-layer** flow

Boundary Layer Equations

continuity:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

x-momentum:

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re_L} \left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} \right)$$

y-momentum:

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re_L} \left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}} \right)$$

$$\begin{aligned} x^* &= \frac{x}{L} \\ y^* &= \frac{y}{L} \\ u^* &= \frac{u}{U_\infty} \\ v^* &= \frac{v}{U_\infty} \\ p^* &= \frac{p}{\rho U_\infty^2} \\ Re_L &= \frac{U_\infty L}{\nu} \end{aligned}$$

Boundary Layer Equations

To be able to find the relative sizes of different terms in the equations, we will first have a look at the flow parameters and operators

$$u^* = u/U_\infty \sim 1$$

$$x^* = x/L \sim 1$$

$$y^* = y/L \sim \delta^*$$

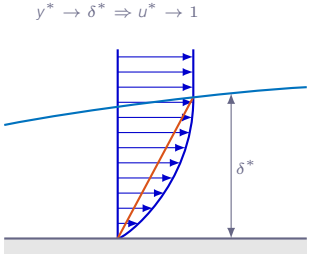
δ denotes boundary layer thickness and $\delta^* = \delta/L$

Note! here, u^* is **not** the friction velocity and δ^* is **not** the displacement thickness

Boundary Layer Equations

What about derivatives?

$$\frac{\partial u^*}{\partial y^*} \sim \frac{1-0}{\delta^*} = \frac{1}{\delta^*}$$



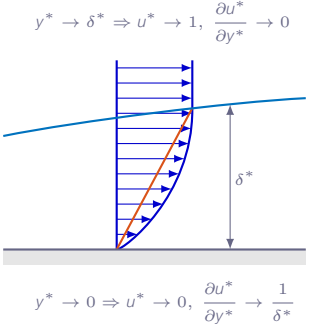
Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

What about derivatives?

$$\frac{\partial u^*}{\partial y^*} \sim \frac{1-0}{\delta^*} = \frac{1}{\delta^*}$$

$$\frac{\partial^2 u^*}{\partial y^{*2}} = \frac{\partial}{\partial y^*}\frac{\partial u^*}{\partial y^*} \sim \frac{|0 - 1/\delta^*|}{\delta^*} = \frac{1}{\delta^{*2}}$$



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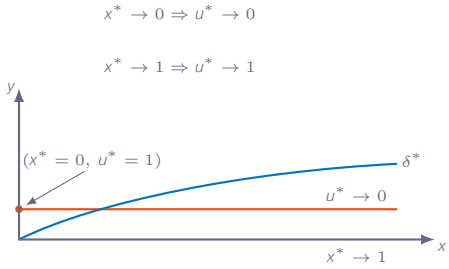
Boundary Layer Equations

$$\frac{\partial u^*}{\partial x^*} \sim \frac{|0-1|}{1-0} = 1$$



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Boundary Layer Equations

$$\frac{\partial u^*}{\partial x^*} \sim \frac{|0-1|}{1-0} = 1$$

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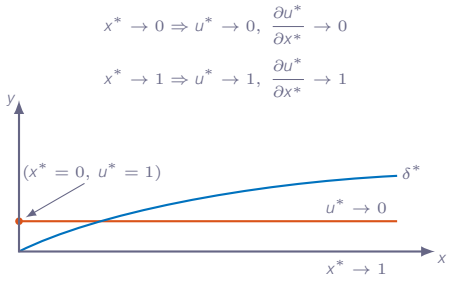

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Boundary Layer Equations

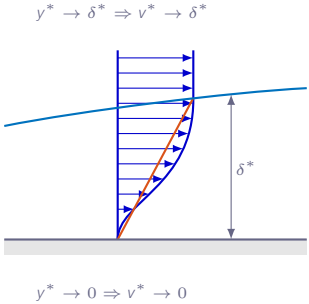
continuity:

$$\underbrace{\frac{\partial u^*}{\partial x^*}}_{\sim \frac{1}{1}} + \underbrace{\frac{\partial v^*}{\partial y^*}}_{\sim \frac{?}{\delta^*}} = 0 \Rightarrow v^* \sim \delta^*$$

$\frac{\partial v^*}{\partial y^*}$ must be of the same order of magnitude as $\frac{\partial u^*}{\partial x^*}$ in order to fulfill the continuity equation

Boundary Layer Equations

$$\frac{\partial v^*}{\partial y^*} \sim \frac{\delta^*}{\delta^*} = 1$$



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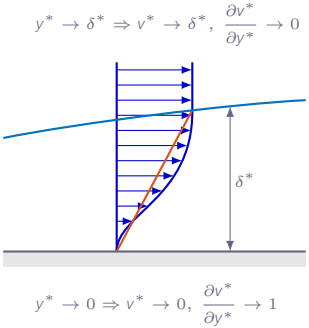
Boundary Layer Equations

$$\frac{\partial v^*}{\partial y^*} \sim \frac{\delta^*}{\delta^*} = 1$$

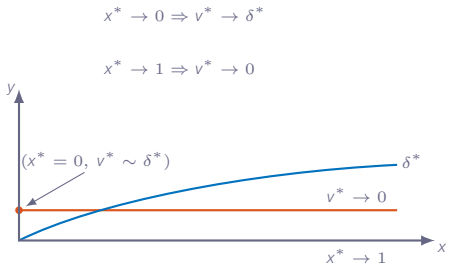
$$\frac{\partial^2 v^*}{\partial y^{*2}} = \frac{\partial}{\partial y^*}\frac{\partial v^*}{\partial y^*} \sim \frac{|0-1|}{\delta^*} = \frac{1}{\delta^*}$$



Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

$$\frac{\partial v^*}{\partial x^*} \sim \frac{|0 - \delta^*|}{1 - 0} = \delta^*$$



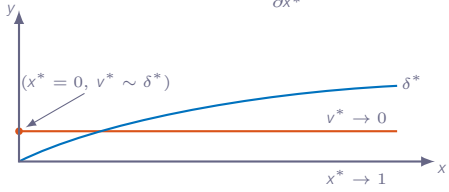
Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

$$x^* \to 0 \Rightarrow v^* \to \delta^*, \ \frac{\partial v^*}{\partial x^*} \to 0$$

$$x^* \to 1 \Rightarrow v^* \to 0, \ \frac{\partial v^*}{\partial x^*} \to \delta^*$$

$$\frac{\partial v^*}{\partial x^*} \sim \frac{|0 - \delta^*|}{1 - 0} = \delta^*$$



$$\frac{\partial^2 v^*}{\partial x^{*2}} = \frac{\partial}{\partial x^*}\frac{\partial v^*}{\partial x^*} \sim \frac{\delta^* - 0}{1 - 0} = \delta^*$$

Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

x-momentum:

$$\underbrace{u^*\frac{\partial u^*}{\partial x^*}}_{\sim 1\times 1=1} + \underbrace{v^*\frac{\partial u^*}{\partial y^*}}_{\sim \delta^* \frac{1}{\delta^*}=1} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re_L}\left(\underbrace{\frac{\partial^2 u^*}{\partial x^{*2}}}_{\sim 1} + \underbrace{\frac{\partial^2 u^*}{\partial y^{*2}}}_{\sim \frac{1}{\delta^{*2}}}\right)$$

the boundary layer is assumed to be very thin $\Rightarrow \delta^* \ll 1$ and thus

$$\frac{\partial^2 u^*}{\partial x^{*2}} \ll \frac{\partial^2 u^*}{\partial y^{*2}}$$

assuming the inertial forces to be of the same size as the friction forces in the boundary layer we get: $1/Re_L \sim \delta^{*2}$

Boundary Layer Equations

y-momentum:

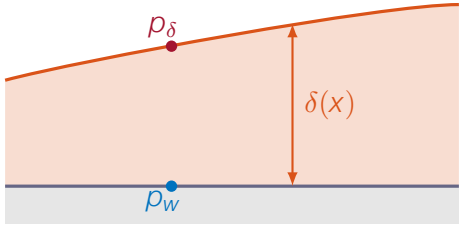
$$\underbrace{u^*\frac{\partial v^*}{\partial x^*}}_{\sim 1\times\delta^*=\delta^*} + \underbrace{v^*\frac{\partial v^*}{\partial y^*}}_{\sim\delta^*\frac{\delta^*}{\delta^*}=\delta^*} = -\frac{\partial p^*}{\partial y^*} + \underbrace{\frac{1}{Re_L}}_{\sim\delta^{*2}}\left(\underbrace{\frac{\partial^2 v^*}{\partial x^{*2}}}_{\sim\delta^*} + \underbrace{\frac{\partial^2 v^*}{\partial y^{*2}}}_{\sim\frac{1}{\delta^*}}\right)$$

examining the equation we see that all terms are at most of size $\delta^* \Rightarrow \dfrac{\partial p^*}{\partial y^*} \sim \delta^*$

δ^* is small $\Rightarrow$ p is independent of y

Boundary Layer Equations

The pressure can be assumed to be constant in the vertical direction through the boundary layer and thus $p = p(x)$



$$|p^*_\delta - p^*_w| \approx \frac{\partial p^*}{\partial y^*}\delta^* \sim {\delta^*}^2$$

Boundary Layer Equations

With the knowledge gained, we now move back to the dimensional equations

laminar

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

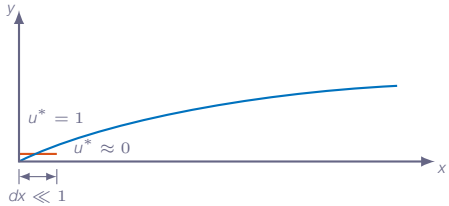
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

turbulent

$$\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}=0$$

$$\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{v}\frac{\partial \overline{u}}{\partial y}=-\frac{1}{\rho}\frac{d\overline{p}}{dx}+\nu\frac{\partial^2 \overline{u}}{\partial y^2}-\frac{\partial}{\partial y}\overline{u'v'}$$

Boundary Layer Equations



Limitations

1. The boundary layer equations **do not apply close to the start of the boundary layer** where $\dfrac{\partial u^*}{\partial x^*} \gg 1$
2. The equations are derived assuming a **thin boundary layer**

Boundary Layer Equations

The pressure derivative can be replaced with a velocity derivative

Outside of the boundary layer the flow is inviscid $\Rightarrow$ we can use the Bernoulli equation

$$p + \frac{1}{2}\rho U_\infty^2 = const \Rightarrow \frac{dp}{dx} + \rho U_\infty \frac{dU_\infty}{dx} = 0 \Rightarrow -\frac{1}{\rho}\frac{dp}{dx} = U_\infty \frac{dU}{dx}$$

Boundary Layer Equations

laminar boundary layer

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

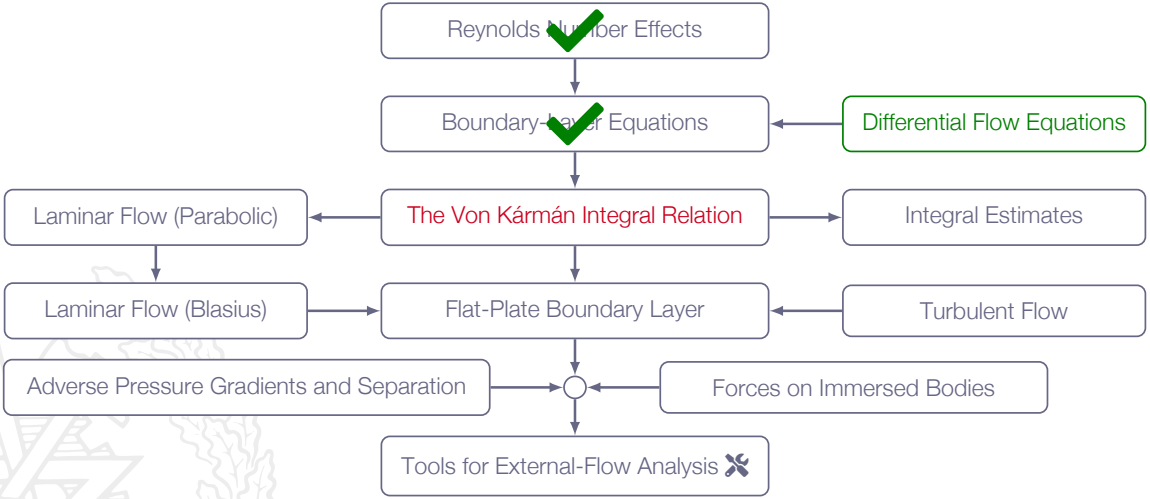
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

Two equations and two unknowns $\Rightarrow$ possible to solve ☺

Boundary Layer Equations

Note! the boundary layer equations can be used for curved surfaces if the boundary layer thickness δ is small compared to the curvature radius r

Roadmap - Flow Past Immersed Bodies



The Von Kármán Integral Relation

- Approximate solutions for $\delta(x)$ and $\tau_w(x)$
- Control volume approach applied to a boundary layer
- Assuming steady-state incompressible flow

$$\sum \mathbf{F} = \int_{CS} \mathbf{V}\rho(\mathbf{V} \cdot \mathbf{n})dA$$

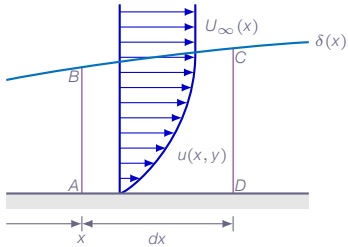
The Von Kármán Integral Relation

Massflow

$$\dot{m}_{AB} = \rho \int_0^{\delta} u dy$$

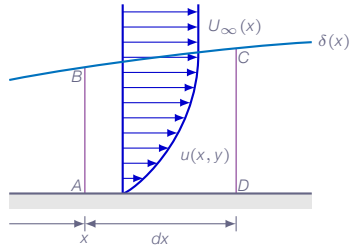
$$\dot{m}_{CD} = \rho \int_0^{\delta} u dy + \frac{d}{dx}\left[\rho \int_0^{\delta} u dy\right] dx$$

$$\dot{m}_{BC} = \rho \frac{d}{dx}\left[\int_0^{\delta} u dy\right] dx$$



The Von Kármán Integral Relation

Momentum



$$\dot{I}_{AB} = \rho \int_0^\delta u^2 dy$$

$$\dot{I}_{CD} = \rho \int_0^\delta u^2 dy + \frac{d}{dx}\left[\rho \int_0^\delta u^2 dy\right] dx$$

$$\dot{I}_{BC} = U\dot{m}_{BC} = \rho U_\infty \frac{d}{dx}\left[\int_0^\delta u dy\right] dx$$

$$\dot{I}_{CD} - \dot{I}_{AB} - \dot{I}_{BC} = \rho \frac{d}{dx}\left[\int_0^\delta u^2 dy\right] dx - \rho U_\infty \frac{d}{dx}\left[\int_0^\delta u dy\right] dx$$

The Von Kármán Integral Relation

Pressure forces in the x-direction

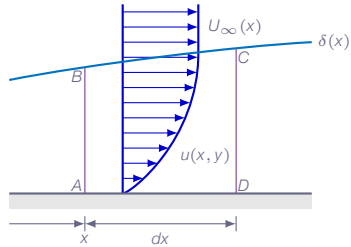
AB: $p\delta$

CD: $-\left(p+\frac{dp}{dx}dx\right)\left(\delta+\frac{d\delta}{dx}dx\right)$

BC: $\approx\left(p+\frac{1}{2}\frac{dp}{dx}dx\right)\frac{d\delta}{dx}dx$

Shear forces in the x-direction

AD: $-\tau_w dx$



The Von Kármán Integral Relation

Forces

$$dF_x = -\tau_w dx + p\delta - \left[p\delta + p\frac{d\delta}{dx}dx + \delta\frac{dp}{dx}dx + \frac{dp}{dx}\frac{d\delta}{dx}dxdx \right] + p\frac{d\delta}{dx}dx + \frac{1}{2}\frac{dp}{dx}\frac{d\delta}{dx}dxdx$$

products of infinitesimal quantities can be regarded to be zero and thus

$$dF_x = -\tau_w dx - \delta\frac{dp}{dx}dx$$

The Von Kármán Integral Relation

Momentum equation

Now we have all components of the momentum equation defined

$$\rho \frac{d}{dx}\left[\int_0^\delta u^2 dy\right] - \rho U_\infty \frac{d}{dx}\left[\int_0^\delta u dy\right] = -\tau_w - \delta \frac{dp}{dx}$$

The momentum equation for boundary layers or **Von Kármán's integral relation**

Note! the relation is valid for laminar and turbulent flows (for turbulent flows use time-averaged quantities)

The Von Kármán Integral Relation

Outside of the boundary layer the flow is inviscid $\Rightarrow$ we can use Bernoulli

$$p + \frac{1}{2}\rho U_\infty^2 = const \Rightarrow \frac{dp}{dx} + \rho U_\infty \frac{dU_\infty}{dx} = 0 \Rightarrow -\frac{1}{\rho}\frac{dp}{dx} = U_\infty \frac{dU_\infty}{dx}$$

The Von Kármán Integral Relation

$$\frac{1}{\rho}\frac{dp}{dx} = -U_\infty \frac{dU_\infty}{dx} \Rightarrow \frac{\tau_w}{\rho} - \delta U_\infty \frac{dU_\infty}{dx} = U_\infty \frac{d}{dx}\left[\int_0^\delta u dy\right] - \frac{d}{dx}\left[\int_0^\delta u^2 dy\right]$$

$$\delta U_\infty \frac{dU_\infty}{dx} = U_\infty \frac{dU_\infty}{dx}\int_0^\delta dy$$

$$U_\infty \frac{d}{dx}\left[\int_0^\delta u dy\right] = \frac{d}{dx}\left[U_\infty \int_0^\delta u dy\right] - \frac{dU_\infty}{dx}\int_0^\delta u dy$$

$$\frac{\tau_w}{\rho} - U_\infty \frac{dU_\infty}{dx}\int_0^\delta dy = \frac{d}{dx}\left[U_\infty \int_0^\delta u dy\right] - \frac{dU_\infty}{dx}\int_0^\delta u dy - \frac{d}{dx}\left[\int_0^\delta u^2 dy\right]$$

The Von Kármán Integral Relation

$$\frac{\tau_w}{\rho} - U_\infty \frac{dU_\infty}{dx} \int_0^\delta dy = \frac{d}{dx}\left[U_\infty \int_0^\delta u dy\right] - \frac{dU_\infty}{dx} \int_0^\delta u dy - \frac{d}{dx}\left[\int_0^\delta u^2 dy\right]$$

$$U_\infty \frac{dU_\infty}{dx} \int_0^\delta dy - \frac{dU_\infty}{dx} \int_0^\delta u dy = \frac{dU_\infty}{dx} \int_0^\delta (U_\infty - u) dy$$

$$\frac{d}{dx}\left[U_\infty \int_0^\delta u dy\right] - \frac{d}{dx}\left[\int_0^\delta u^2 dy\right] = \frac{d}{dx} \int_0^\delta u(U_\infty - u) dy$$

The Von Kármán Integral Relation

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^\delta u(U_\infty - u)dy + \frac{dU_\infty}{dx}\int_0^\delta (U_\infty - u)dy$$

The Von Kármán Integral Relation

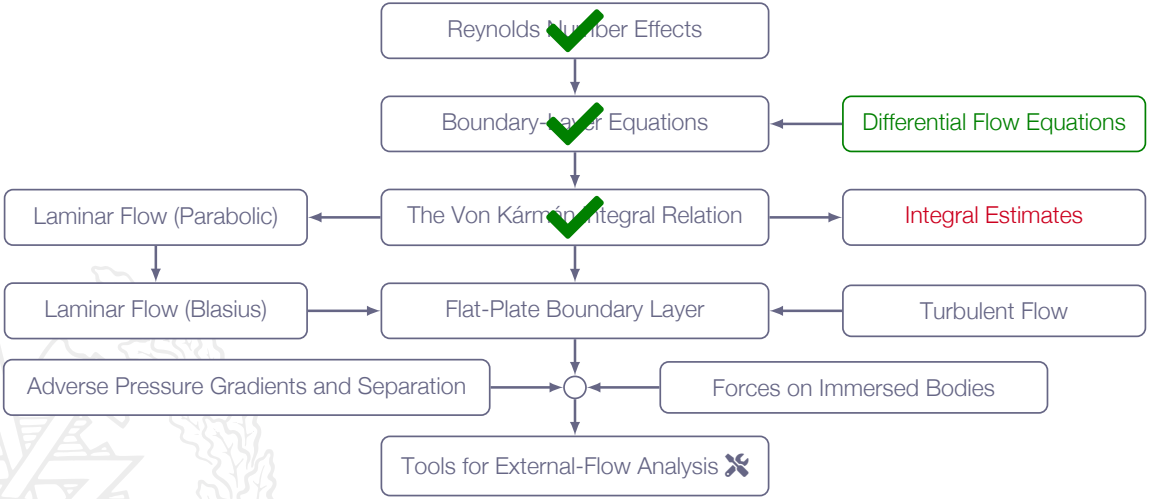
$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^\delta u(U_\infty - u)dy + \frac{dU_\infty}{dx}\int_0^\delta (U_\infty - u)dy$$

Constant freestream velocity gives

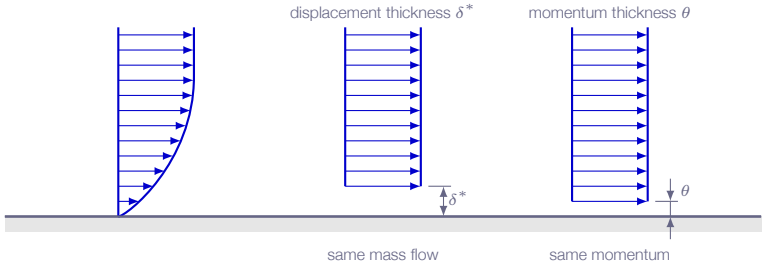
$$\frac{dU_\infty}{dx} = 0 \Rightarrow \frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^\delta u(U_\infty - u)dy$$

Ok, but what does this mean??

Roadmap - Flow Past Immersed Bodies

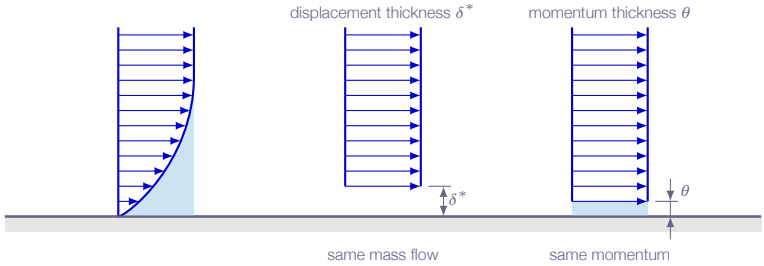


Momentum Integral Estimates



"The presence of a boundary layer will result in a small but finite displacement of the flow streamlines"

Momentum Thickness



$$\int_0^\delta \rho u(U_\infty - u) b dy = \rho U_\infty^2 b\theta \Rightarrow \theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

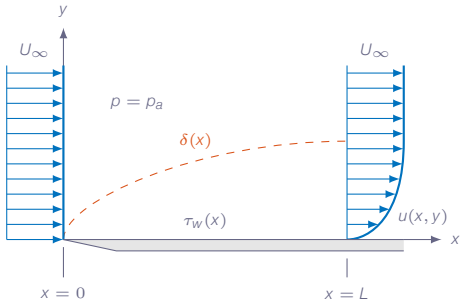
Note! *b is the width of the flat plate*

Momentum Thickness

The drag D for a plate of width b

$$D(x) = b\int_0^x \tau_w(x)dx \Rightarrow \frac{dD}{dx} = b\tau_w$$

from before we have



$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^\delta u(U_\infty - u)dy = \frac{d}{dx}U_\infty^2 \underbrace{\int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy}_{\theta} = U_\infty^2\frac{d\theta}{dx}$$

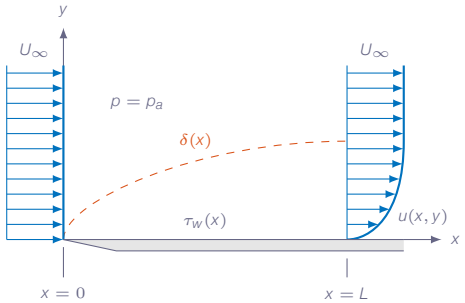
and thus

$$\frac{dD}{dx} = b\rho U_\infty^2\frac{d\theta}{dx} \Rightarrow D(x) = \rho b U_\infty^2\theta$$

Momentum Thickness

$$D(x) = \rho b U_\infty^2 \theta, \; \tau_w = \rho U_\infty^2 \frac{d\theta}{dx}$$

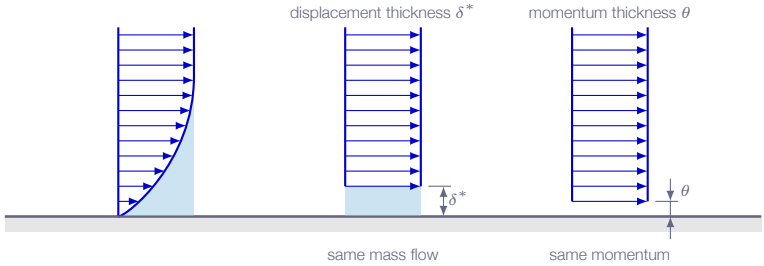
$$\theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$



Note!

1. the momentum thickness θ is a measure of the total drag
2. can be used both for laminar and turbulent flows
3. no assumption about velocity profile shape made

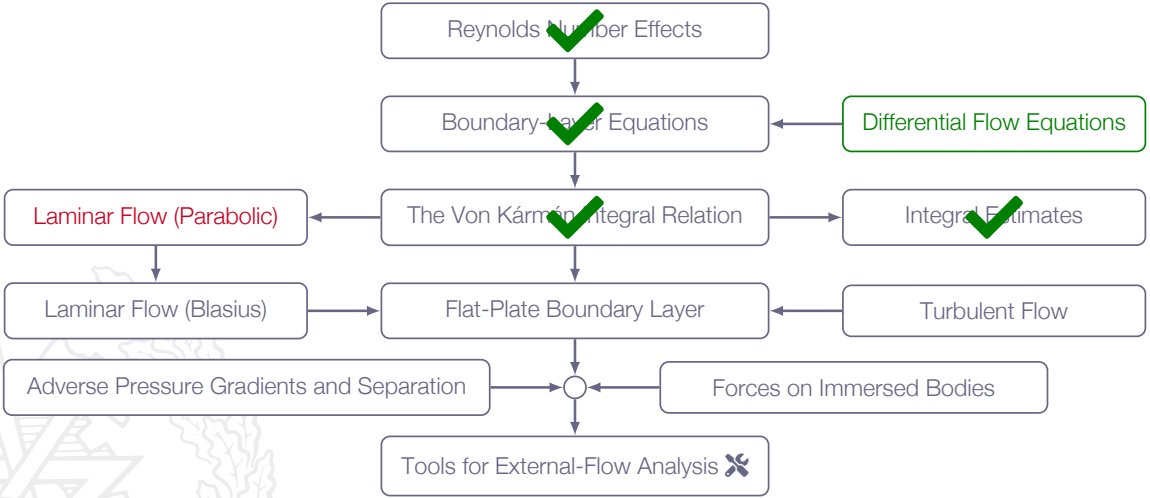
Displacement Thickness



$$\int_0^\delta \rho(U_\infty - u)bdy = \rho U_\infty b\delta^* \Rightarrow \delta^* = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) dy$$

δ^* is an estimate of the displacement in the wall-normal direction of streamlines in the outer part of the boundary layer due to the deficit of massflow caused by the no-slip condition at the wall - a measure of the boundary-layer thickness

Roadmap - Flow Past Immersed Bodies



Laminar Boundary Layer

The Von Kármán integral relation gives us the wall shear stress (τ_w) as a function of the velocity profile ($u(y)$) and the boundary-layer thickness (δ)

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^\delta u(U_\infty - u)dy$$

So now we need a velocity profile $u = u(y)$ to continue ...

Laminar Boundary Layer

Assumptions:

1. Boundary layer over a flat plate
2. Constant freestream velocity $U_\infty = const \Rightarrow \dfrac{dU_\infty}{dx} = 0$
3. Laminar flow
4. Parabolic velocity profile

Laminar Boundary Layer - Parabolic Velocity Profile

$$u(y) = A + By + Cy^2$$

The constants A, B, and C are defined using boundary conditions

1. no slip:
   $u(0) = 0 \Rightarrow A = 0$

2. constant velocity at $y = \delta$:
   $$\left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0 \Rightarrow B + 2C\delta = 0 \Rightarrow B = -2\delta C$$

3. freestream velocity:
   $$u(\delta) = U_\infty \Rightarrow B\delta + C\delta^2 = U_\infty \Rightarrow \{B = -2\delta C\} \Rightarrow -C\delta^2 = U_\infty \Rightarrow C = -\frac{U_\infty}{\delta^2}$$

Laminar Boundary Layer - Parabolic Velocity Profile

$$u(y) = A + By + Cy^2$$

$$A = 0, \quad B = \frac{2U_\infty}{\delta}, \quad C = -\frac{U_\infty}{\delta^2}$$

$$u(y) = U_\infty \left(\frac{2}{\delta} y - \frac{1}{\delta^2} y^2 \right)$$

Laminar Boundary Layer - Parabolic Velocity Profile

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^\delta u(U_\infty - u)dy$$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu\frac{2U_\infty}{\delta}$$

$$\int_0^\delta u(U_\infty - u)dy = \int_0^\delta U_\infty^2\left(\frac{2}{\delta}y - \frac{1}{\delta^2}y^2\right) - U_\infty^2\left(\frac{4}{\delta^2}y^2 - \frac{4}{\delta^3}y^3 + \frac{1}{\delta^4}y^4\right)dy = \frac{2}{15}U_\infty^2\delta$$

$$\frac{\mu}{\rho}\frac{2U_\infty}{\delta} = \frac{d}{dx}\left(\frac{2}{15}U_\infty^2\delta\right) \Rightarrow \frac{\nu}{\delta} = \frac{U_\infty}{15}\frac{d\delta}{dx} \Rightarrow \delta d\delta = 15\frac{\nu}{U_\infty}dx$$

Laminar Boundary Layer - Parabolic Velocity Profile

$$\delta d\delta = 15\frac{\nu}{U_\infty}dx$$

$$\frac{\delta^2}{2} = 15\frac{\nu}{U_\infty}x + C = \{x = 0 \Rightarrow \delta = 0 \Rightarrow C = 0\} = 15\frac{\nu}{U_\infty}x$$

$$\delta = \sqrt{\frac{30\nu x}{U_\infty}} \Rightarrow \frac{\delta}{x} = \sqrt{\frac{30\nu}{U_\infty x}} \approx \frac{5.5}{\sqrt{Re_x}}$$

Laminar Boundary Layer - Parabolic Velocity Profile

$$\tau_w = \mu \frac{2U_\infty}{\delta} = \frac{2\mu U_\infty}{\sqrt{\frac{30\nu x}{U_\infty}}} = \frac{2}{\sqrt{30}} \frac{\rho U_\infty^2}{\sqrt{\frac{U_\infty x}{\nu}}} \approx \frac{0.365}{\sqrt{Re_x}} \rho U_\infty^2$$

Introducing the **skin friction coefficient** c_f

$$c_f = \frac{2\tau_w}{\rho U_\infty^2} \approx \frac{0.73}{\sqrt{Re_x}}$$

Laminar Boundary Layer - Parabolic Velocity Profile

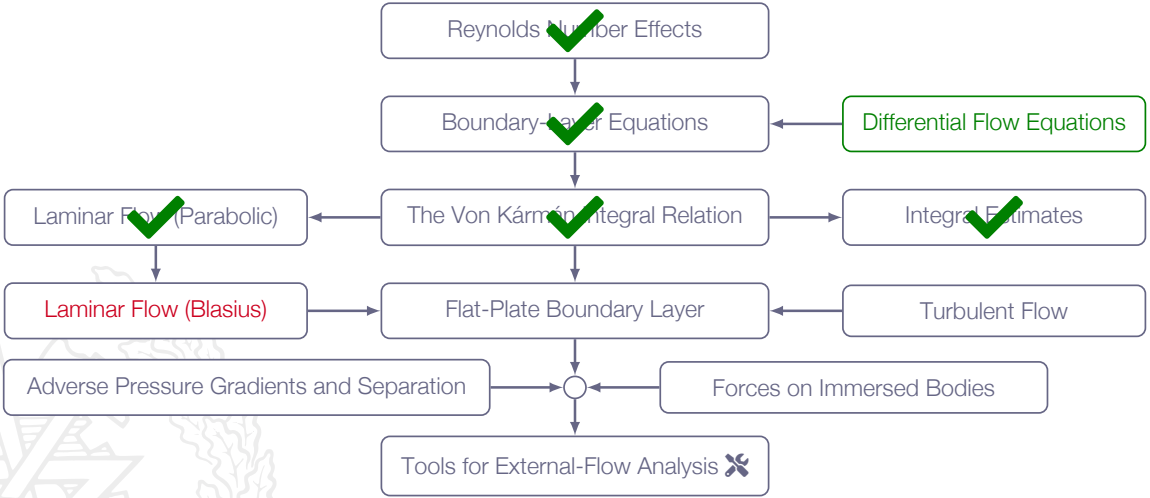
Note! *more accurate solutions for laminar flat plate boundary layers exists:*

$$c_f \approx \frac{0.664}{\sqrt{Re_x}}, \quad \frac{\delta}{x} \approx \frac{5.0}{\sqrt{Re_x}}$$

Ok, so where did we go wrong?

For external (unconfined) boundary layers, the velocity profile is not parabolic – but quite close to parabolic ...

Roadmap - Flow Past Immersed Bodies



The Blasius Velocity Profile

For laminar flow, the boundary layer equations can be solved for u and v

Blasius presented a solution 1908 where he had used a coordinate transformation and showed that $\frac{u}{U_\infty}$ is a function of a single dimensionless variable $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$

The coordinate transformation corresponds to a scaling of the y coordinate with the boundary layer thickness δ

$$\frac{\delta}{x} \propto \frac{1}{\sqrt{Re_x}} \Rightarrow \frac{y}{\delta} \propto \frac{y}{x/\sqrt{Re_x}} = \frac{y}{x}\sqrt{\frac{U_\infty x}{\nu}} = y\sqrt{\frac{U_\infty}{\nu x}} = \eta$$

The Blasius Velocity Profile

1. Rewrite the boundary layer equations using the stream function (Chapter 4)
2. Rewrite the equation again $\Psi = f(\eta)\sqrt{\nu U_\infty x}$ where η is the scaled wall-normal coordinate and $f(\eta)$ is a non-dimensional stream function
3. Lots of math ....

The Navier-Stokes equations are reduced to an ordinary differential equation (ODE)

$$f''' + \frac{1}{2} f f'' = 0$$

with the boundary conditions

$$\begin{cases} f(0) = f'(0) = 0 \\ f'_{\eta \to \infty} \to 1.0 \end{cases}$$

The Blasius Velocity Profile

$$\frac{u}{U_\infty} = f'(\eta)$$

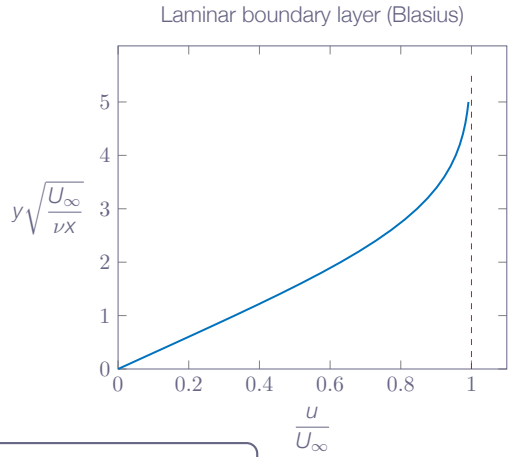
Note! $u/U_\infty \to 1$ as $y \to \infty$ and therefore δ is usually defined as the distance from the wall where $u/U_\infty = 0.99$

The Blasius Velocity Profile

$$\frac{u}{U_\infty} = f'(\eta)$$

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}}$$

$$\delta_{99\%}\sqrt{\frac{U_\infty}{\nu x}} \approx 5.0$$



$$\delta_{99\%}\sqrt{\frac{U_\infty}{\nu x}} \approx 5.0 \quad \text{or} \quad \frac{\delta}{x} \approx \frac{5.0}{\sqrt{Re_x}}$$

The Blasius Velocity Profile

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu U_\infty \left[\frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)\frac{d\eta}{dy}\right]_{\eta=0}$$

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \Rightarrow \frac{d\eta}{dy} = \sqrt{\frac{U_\infty}{\nu x}} \Rightarrow \tau_w = \mu U_\infty \sqrt{\frac{U_\infty}{\nu x}} \frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)_{\eta=0} = \frac{\rho U_\infty^2}{\sqrt{Re_x}} \frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)_{\eta=0}$$

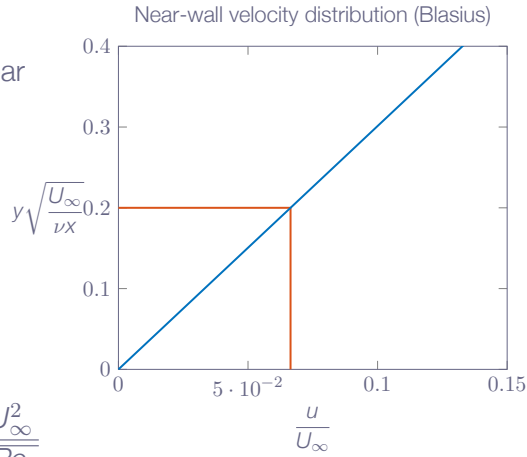
The Blasius Velocity Profile

close the the wall the velocity profile is linear

$$\eta = 0.2 \Rightarrow \frac{u}{U_\infty} \approx 0.0664$$

$$\frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)_{\eta=0} \approx \frac{0.0664}{0.2} = 0.332$$

$$\tau_w = \frac{\rho U_\infty^2}{\sqrt{Re_x}} \frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)_{\eta=0} \approx 0.332 \frac{\rho U_\infty^2}{\sqrt{Re_x}}$$



Laminar Boundary Layer - Blasius

$$\tau_w(x) \approx \frac{0.332\rho^{1/2}\mu^{1/2}U_\infty^{3/2}}{x^{1/2}}$$

Note! the wall shear stress drops off with increasing distance due to the boundary layer growth

Recall *for pipe flow, the wall shear stress is independent of x – pipe flow is confined and the boundary layer height is restricted*

Laminar Boundary Layer - Blasius

wall shear stress:

$$\tau_w(x) \approx \frac{0.332\rho^{1/2}\mu^{1/2}U_\infty^{3/2}}{x^{1/2}}$$

drag force:

$$D(x) = b\int_0^x \tau_w(x)dx \approx 0.664b\rho^{1/2}\mu^{1/2}U_\infty^{3/2}x^{1/2}$$

drag coefficient:

$$C_D = \frac{2D(L)}{\rho U_\infty^2 bL} \approx \frac{1.328}{\sqrt{Re_L}}$$

Laminar Boundary Layer - Blasius

From before we have $D(x) = \rho b \int_0^{\delta(x)} u(U_\infty - u)dy$

$$D(x) = \rho b U_\infty^2 \underbrace{\int_0^{\delta(x)} \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy}_{\theta(x)} = \rho b U_\infty^2 \theta(x)$$

$$b\int_0^x \tau_w(x)dx = \rho b U_\infty^2 \theta(x) \approx 0.664 b \rho^{1/2}\mu^{1/2}U_\infty^{3/2}x^{1/2} \Rightarrow \theta(x) \approx \frac{0.664\mu^{1/2}x^{1/2}}{\rho^{1/2}U_\infty^{1/2}}$$

$$\Rightarrow \theta(x) \approx \frac{0.664\mu^{1/2}x}{\rho^{1/2}U_\infty^{1/2}x^{1/2}} \text{ and thus } \frac{\theta(x)}{x} \approx \frac{0.664}{\sqrt{Re_x}}$$

Laminar Boundary Layer - Blasius

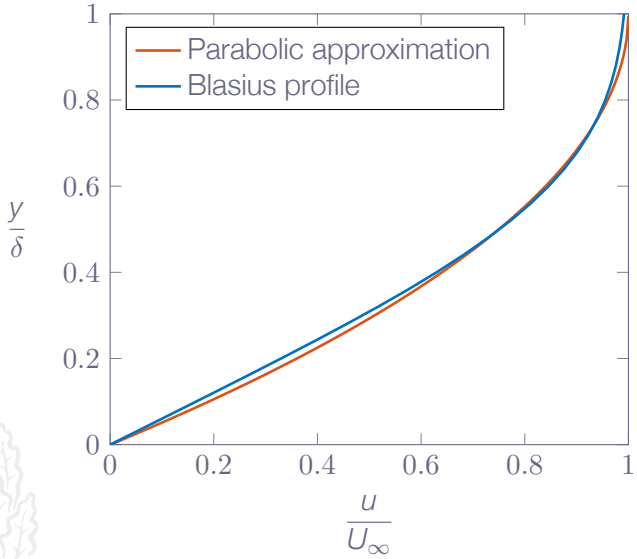
Displacement thickness:

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) dy$$

$$\frac{\delta^*}{x} \approx \frac{1.721}{Re_x^{1/2}}$$

Note! *since δ^* is much smaller than x for large values of Re_x, the velocity component in the wall-normal direction will be much smaller than the velocity parallel to the plate*

Laminar Boundary Layer



Laminar Boundary Layer

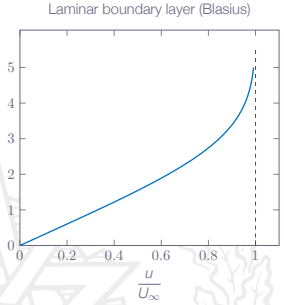
description	variable	laminar flow (Blasius)	turbulent flow (Prandtl)
boundary layer thickness	$\frac{\delta}{x}$	$\frac{5.0}{\sqrt{Re_x}}$	
displacement thickness	$\frac{\delta^*}{x}$	$\frac{1.721}{\sqrt{Re_x}}$	
momentum thickness	$\frac{\theta}{x}$	$\frac{0.664}{\sqrt{Re_x}}$	
shape factor	$H = \frac{\delta^*}{\theta}$	2.59	
wall shear stress	τ_w	$0.332 \frac{\rho U_\infty^2}{\sqrt{Re_x}}$	
local skin friction coefficient	$c_f = \frac{2\tau_w}{\rho U_\infty^2}$	$\frac{0.664}{\sqrt{Re_x}}$	
drag coefficient	C_D	$\frac{1.328}{\sqrt{Re_L}}$	

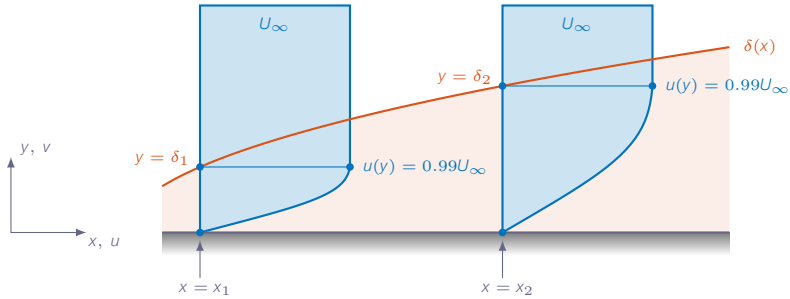
The Blasius Velocity Profile - Self Similarity

From before:

$$\eta(x,y) = y\sqrt{\frac{U_\infty}{\nu x}}$$

$$\frac{u}{U_\infty} = 0.99 \Rightarrow \eta \approx 5.0$$

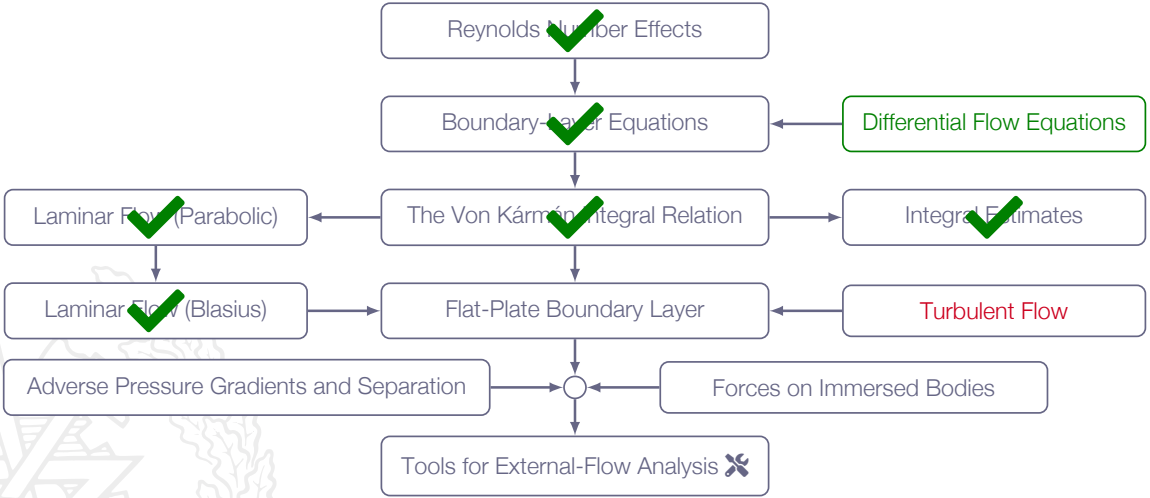




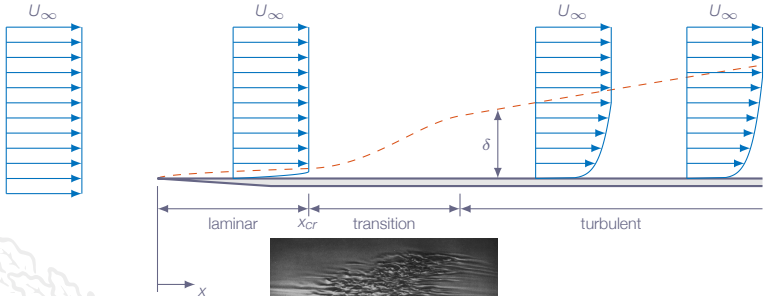
$$\eta(x_1,\delta_1) = \eta(x_2,\delta_2) \Rightarrow \delta_1\sqrt{\frac{U_\infty}{\nu x_1}} = \delta_2\sqrt{\frac{U_\infty}{\nu x_2}}$$

$$x_1 < x_2 \Rightarrow \sqrt{\frac{U_\infty}{\nu x_1}} > \sqrt{\frac{U_\infty}{\nu x_2}} \Rightarrow \delta_1 < \delta_2$$

Roadmap - Flow Past Immersed Bodies



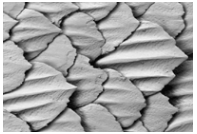
Boundary Layer Transition



$u = 0.99U_\infty$

$$Re_x = \frac{U_\infty x}{\nu}$$

$$Re_{x_{cr}} = \frac{U_\infty x_{cr}}{\nu} \approx 5.0 \times 10^5$$



Boundary Layer Transition

- For low Re_x, disturbances in the flow are damped out by viscous forces
- For somewhat higher Reynolds numbers, friction forces are less important and the flow becomes unstable
- The transition region is short - can be treated as a point (the transition point)

Boundary Layer Transition

The onset of transition from laminar to turbulent is affected by a number of factors such as:

- Turbulence in the freestream
- Surface roughness
- Pressure gradient

With a smooth surface, no turbulence in the freestream, and zero pressure gradient, the onset of transition can be pushed up to $Re_x \approx 3.0 \times 10^6$

As a rule of thumb, we can assume $Re_{x_{cr}} \approx 5.0 \times 10^5$

Boundary Layer Transition

Freestream turbulence:

- frestream turbulence reduces the critical Reynolds number
- with high turbulence intensity in the freestream, the transition can start already at $Re_x \approx 3.0 \times 10^5$ or lower

Boundary Layer Transition

Surface roughness:

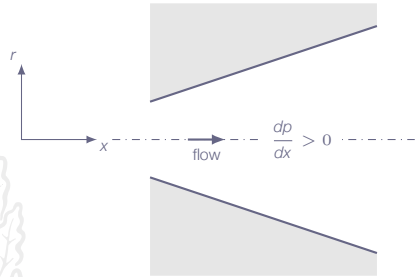
- surface roughness does not affect transition significantly if $Re_\epsilon = \frac{U_\infty \epsilon}{\nu} < 680$
- if $Re_\epsilon > 680$, the extent of the laminar region can be shortened significantly ($Re_x \approx 3.0 \times 10^5$)

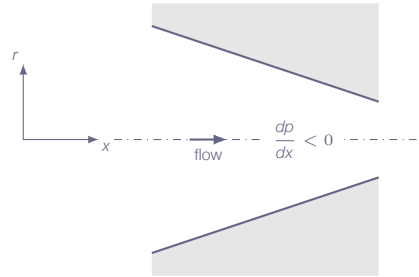
Note! rule of thumb

Boundary Layer Transition

Negative pressure gradient:

- decreasing pressure in the flow direction has a stabilizing effect on the flow and can delay transition from laminar to turbulent flow





Boundary Layer Transition

Forced transition:

- a trip wire or added surface roughness can make the transition to turbulence really fast
- the critical Reynolds number is not meaningful if the boundary layer is forced to transition

Flat Plate - Turbulent Boundary Layer

A turbulent boundary layer grows faster than a laminar boundary layer

- the velocity fluctuations (u', v', w') leads to increased exchange of momentum
- increased shear stress compared to the laminar case where we only have forces related to molecular viscosity
- larger portion of the fluid will be decelerated close to the wall

Flat Plate - Turbulent Boundary Layer

The Von Kármán integral relation and the integral estimates are valid for both laminar and turbulent boundary layers

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^\delta u(U_\infty - u)dy$$

$$\theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)d$$

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right)dy$$

We need a velocity profile $u(y)$ for turbulent boundary layers to be able to calculate τ_w, θ, and δ^*

- Approach 1: the log law
- Approach 2: Prandtl's power law approximation

Flat Plate - Turbulent Boundary Layer

Approach 1: the log law

$$\frac{u}{u^*} \approx \frac{1}{\kappa} \ln\left(\frac{y u^*}{\nu}\right) + B \quad \text{where } \kappa = 0.41 \text{ and } B = 5.0$$

u^* is the **friction velocity** defined as $u^* = \sqrt{\frac{\tau_w}{\rho}}$

at the edge of the boundary layer $u = U_\infty$ and $y = \delta$ and thus

$$\frac{U_\infty}{u^*} \approx \frac{1}{\kappa} \ln\left(\frac{\delta u^*}{\nu}\right) + B$$

Flat Plate - Turbulent Boundary Layer

Approach 1: the log law

The **skin friction coefficient** c_f is defined as $c_f = \dfrac{2\tau_w}{\rho U_\infty^2} \Rightarrow \tau_w = c_f \dfrac{1}{2} \rho U_\infty^2$

the **friction velocity** can be expressed as $u^* = \sqrt{\dfrac{\tau_w}{\rho}} = U_\infty \sqrt{\dfrac{c_f}{2}}$

insert in the **log-law** and we get

$$\sqrt{\frac{2}{c_f}} \approx \frac{1}{\kappa} \ln \left(Re_\delta \sqrt{\frac{c_f}{2}} \right) + B$$

rather difficult to work with ...

Flat Plate - Turbulent Boundary Layer

Approach 2: Prandtl's power law approximation

Prandtl suggested the following relations:

$$c_f \approx 0.02 Re_\delta^{-1/6}$$

$$\frac{u}{U_\infty} \approx \left(\frac{y}{\delta}\right)^{1/7}$$

from before we have the following relation: $\tau_w = \rho U_\infty^2 \frac{d\theta}{dx} \Rightarrow c_f = 2\frac{d\theta}{dx}$

calculate the **momentum thickness** $\theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy = \frac{7}{72}\delta$

Flat Plate - Turbulent Boundary Layer

Approach 2: Prandtl's power law approximation

Now, combining the two **skin friction coefficient** relations we see that

$$0.02 Re_\delta^{-1/6} = 2\frac{d}{dx}\left(\frac{7}{72}\delta\right)$$

and thus $Re_\delta^{-1/6} \approx 9.72\frac{d\delta}{dx} = 9.72\frac{d(Re_\delta)}{d(Re_x)}$

integration gives $Re_\delta \approx 0.16 Re_x^{6/7}$ or $\frac{\delta}{x} \approx \frac{0.16}{Re_x^{1/7}}$

Note! the turbulent boundary layer grows significantly faster than the laminar $\delta_{turb} \propto x^{6/7}$ vs $\delta_{lam} \propto x^{1/2}$

Flat Plate - Turbulent Boundary Layer

Approach 2: Prandtl's power law approximation

$$c_f \approx \frac{0.027}{Re_x^{1/7}}$$

$$\tau_{w_{turb}} \approx \frac{0.0135 \mu^{1/7} \rho^{6/7} U_\infty^{13/7}}{x^{1/7}}$$

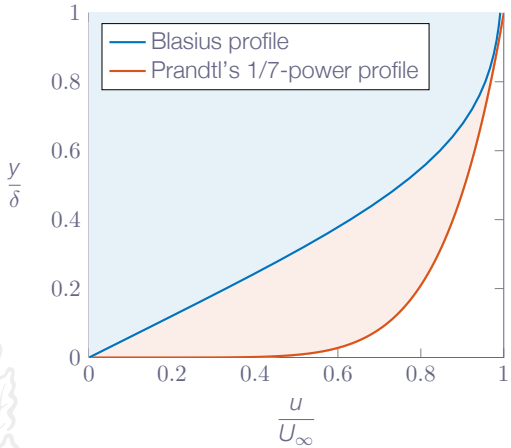
Note! friction drops slowly with x, increases nearly as ρ and U_∞^2, and is rather insensitive to viscosity

Flat Plate - Turbulent Boundary Layer

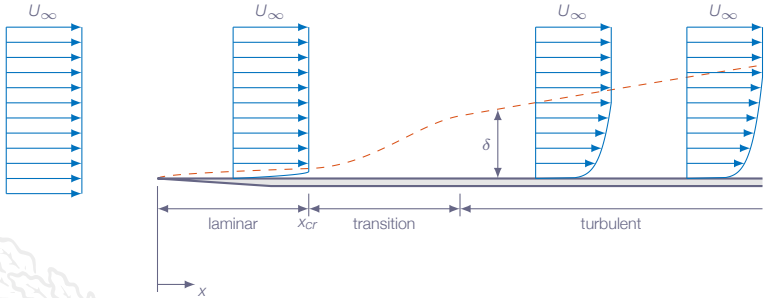
description	variable	laminar flow (Blasius)	turbulent flow (Prandtl)
boundary layer thickness	$\frac{\delta}{x}$	$\frac{5.0}{\sqrt{Re_x}}$	$\frac{0.16}{Re_x^{1/7}}$
displacement thickness	$\frac{\delta^*}{x}$	$\frac{1.721}{\sqrt{Re_x}}$	$\frac{0.02}{Re_x^{1/7}}$
momentum thickness	$\frac{\theta}{x}$	$\frac{0.664}{\sqrt{Re_x}}$	$\frac{0.016}{Re_x^{1/7}}$
shape factor	$H = \frac{\delta^*}{\theta}$	2.59	1.29
wall shear stress	τ_w	$0.332\frac{\rho U_\infty^2}{\sqrt{Re_x}}$	$0.0135\frac{\rho U_\infty^2}{Re_x^{1/7}}$
local skin friction coefficient	$c_f = \frac{2\tau_w}{\rho U_\infty^2}$	$\frac{0.664}{\sqrt{Re_x}}$	$\frac{0.027}{Re_x^{1/7}}$
drag coefficient	C_D	$\frac{1.328}{\sqrt{Re_L}}$	$\frac{0.031}{Re_L^{1/7}}$

Flat Plate - Turbulent Boundary Layer

The velocity profile in a turbulent boundary layer is quite far from the Blasius profile used for laminar boundary layers



Flat Plate Boundary Layer



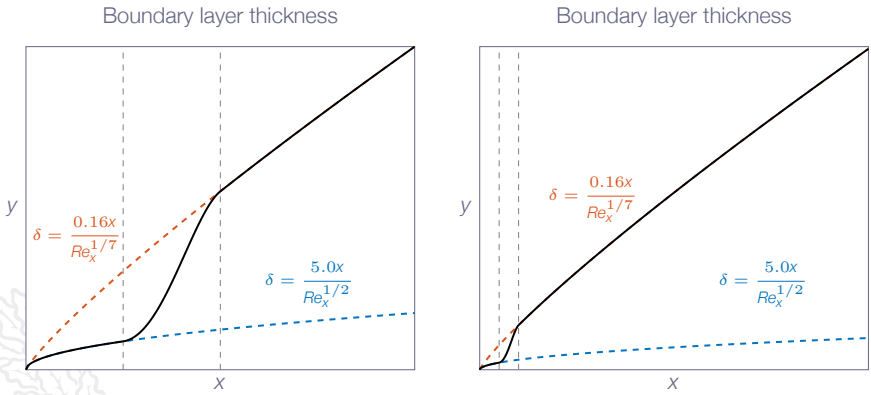
$u = 0.99U_\infty$

$Re_x = \dfrac{U_\infty x}{\nu}$

$Re_{x_{cr}} = \dfrac{U_\infty x_{cr}}{\nu} \approx 5.0 \times 10^5$

$$D = b\frac{1}{2}\rho U^2 \left[\int_{>0}^{x_{cr}} \frac{0.664}{Re_x^{1/2}} dx + \int_{x_{cr}}^{L} \frac{0.027}{Re_x^{1/7}} dx \right]$$

Flat Plate Boundary Layer



For a long boundary layer the length of the laminar region becomes relatively short in comparison with the length of the turbulent region

Wall Roughness

laminar:

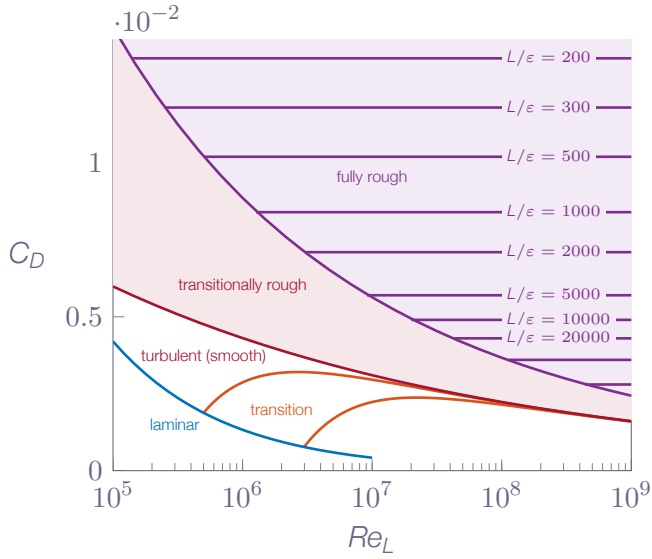
$$C_D = \frac{1.328}{Re_L^{1/2}}$$

turbulent (smooth):

$$C_D = \frac{0.031}{Re_L^{1/7}}$$

turbulent (fully rough):

$$C_D = (1.89 + 1.62\log(L/\varepsilon))^{-2.5}$$



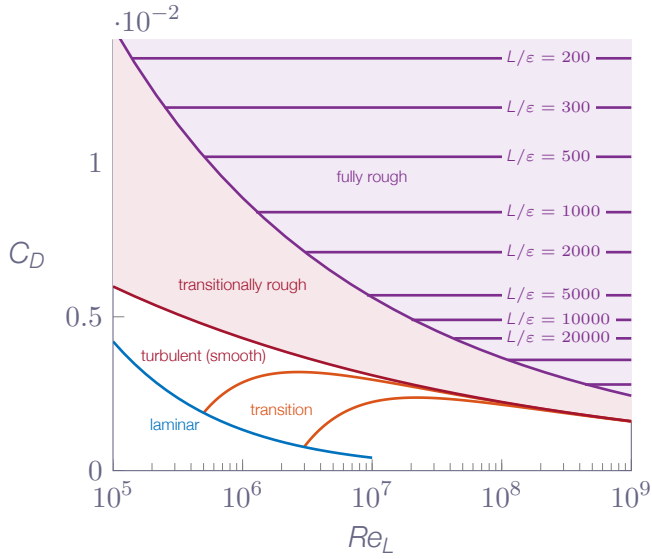
Wall Roughness

transition ($Re_{trans} = 5.0 \times 10^5$):

$$C_D = \frac{0.031}{Re_L^{1/7}} - \frac{1440}{Re_L}$$

transition ($Re_{trans} = 3.0 \times 10^6$):

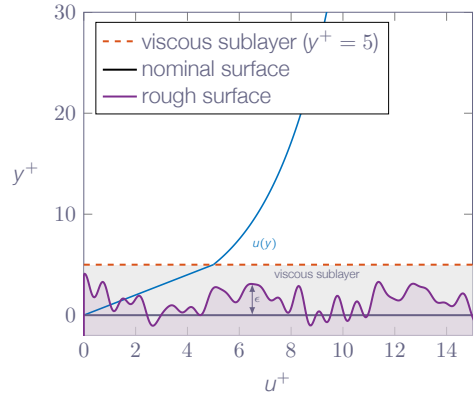
$$C_D = \frac{0.031}{Re_L^{1/7}} - \frac{8700}{Re_L}$$



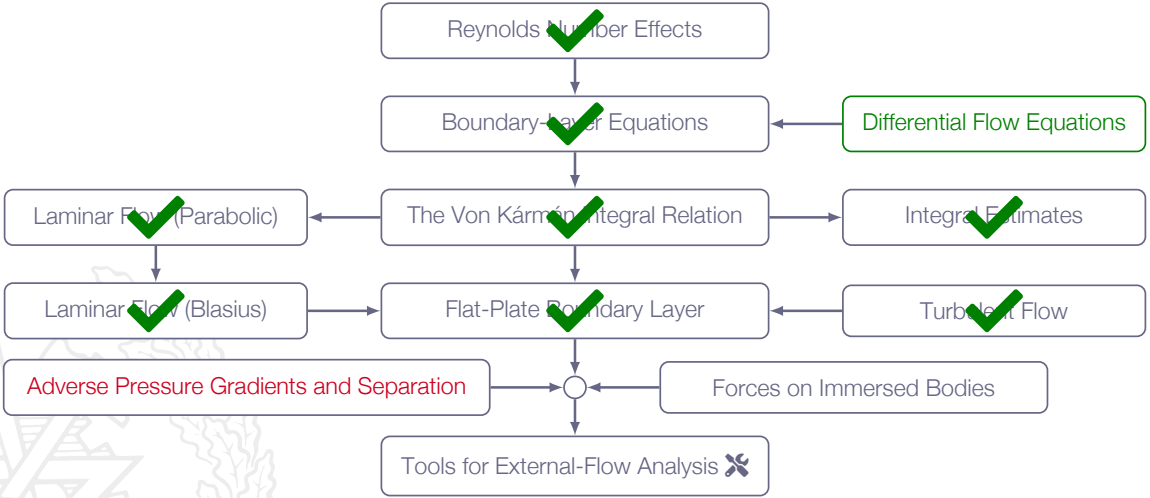
Wall Roughness

Recall: smooth surface:

Surface roughness (ϵ) within the viscous sublayer



Roadmap - Flow Past Immersed Bodies



Pressure Gradient

Adverse pressure gradient

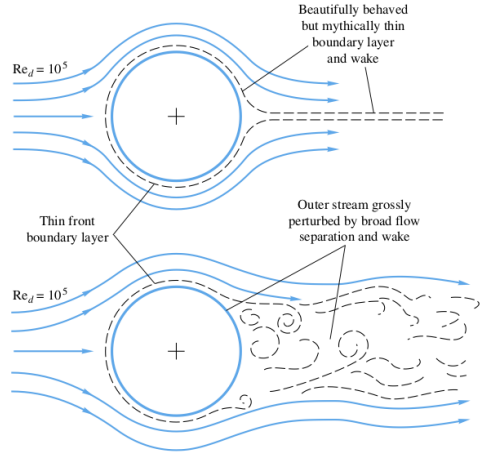
- pressure increases in the flow direction
- may lead to separation

Favorable pressure gradient

- pressure decreases in the flow direction
- the flow will not separate

Separation mechanism

- loss of momentum near the wall
- adverse pressure gradient
- decelerated fluid will force flow to separate from the body



Pressure Gradient

Boundary layer formulation of the momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

with $u = v = 0$ close at the wall, we get

$$\left.\frac{\partial \tau}{\partial y}\right|_{wall} = \mu \left.\frac{\partial^2 u}{\partial y^2}\right|_{wall} \Rightarrow \left.\frac{\partial^2 u}{\partial y^2}\right|_{wall} = \frac{1}{\mu}\frac{dp}{dx}$$

Note! applies both for laminar and turbulent flow

Pressure Gradient

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{wall} = \frac{1}{\mu}\frac{dp}{dx}$$

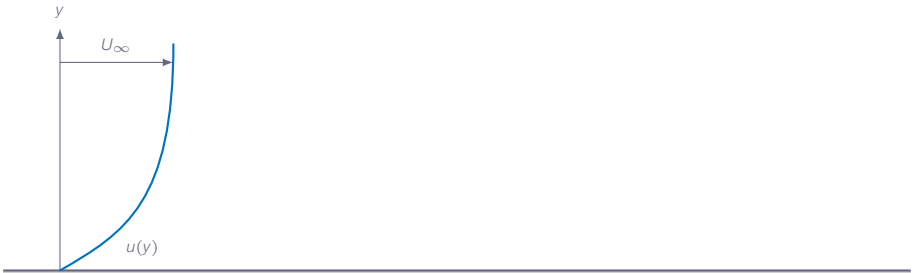
Adverse pressure gradient ($\frac{dp}{dx} > 0$):

$\frac{\partial^2 u}{\partial y^2} > 0$ at the wall

$\frac{\partial^2 u}{\partial y^2} < 0$ at the outer layer $y = \delta$

thus $\frac{\partial^2 u}{\partial y^2} = 0$ somewhere in the boundary layer

Pressure Gradient

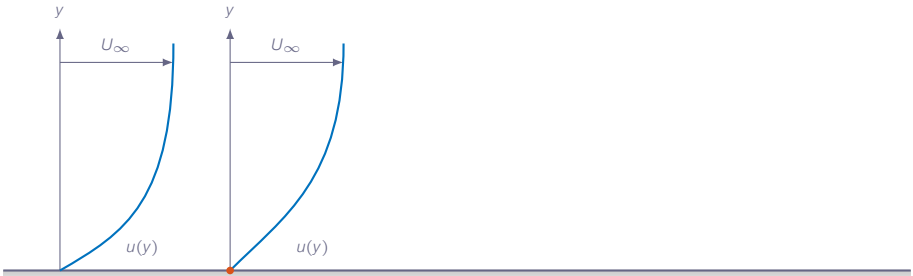


Favorable gradient
$(dp/dx < 0)$

Point of inflection:
inside wall

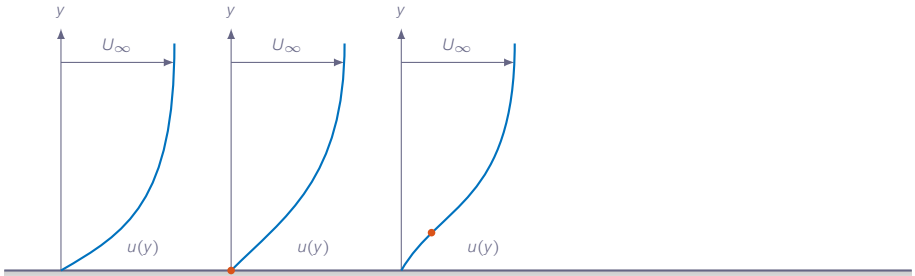
No separation

Pressure Gradient



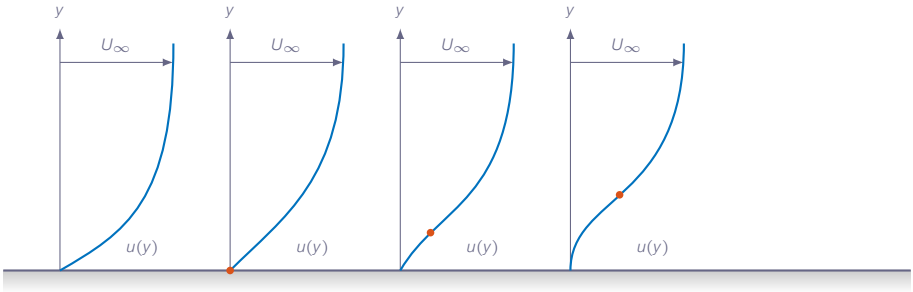
Favorable gradient ($dp/dx < 0$)	Zero gradient ($dp/dx = 0$)
Point of inflection: inside wall	Point of inflection: at the wall
No separation	No separation

Pressure Gradient



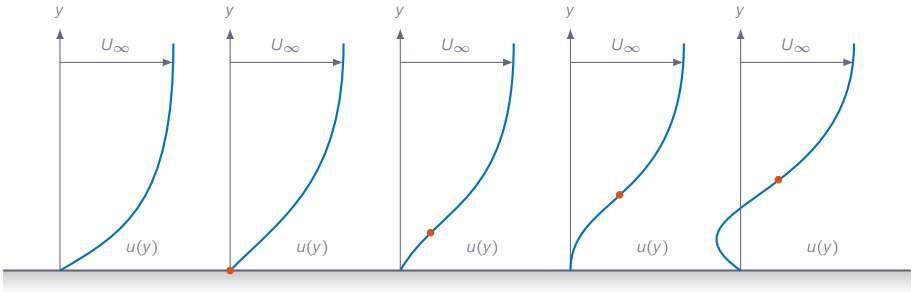
Favorable gradient ($dp/dx < 0$)	Zero gradient ($dp/dx = 0$)	Weak adverse gradient ($dp/dx > 0$)
Point of inflection: inside wall	Point of inflection: at the wall	Point of inflection: in the flow
No separation	No separation	No separation

Pressure Gradient



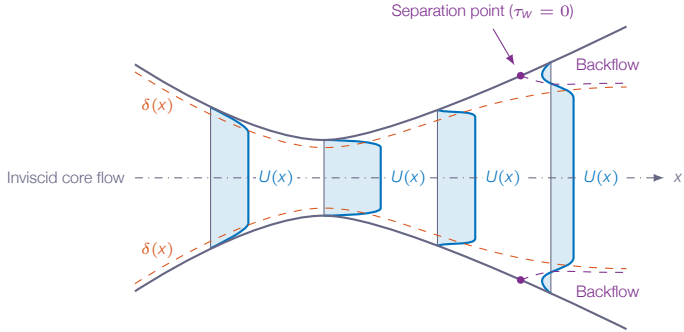
Favorable gradient ($dp/dx < 0$)	Zero gradient ($dp/dx = 0$)	Weak adverse gradient ($dp/dx > 0$)	Critical adverse gradient ($dp/dx > 0$)
Point of inflection: inside wall	Point of inflection: at the wall	Point of inflection: in the flow	Point of inflection: in the flow
No separation	No separation	No separation	Separation zero slope at wall $\tau_w = 0$

Pressure Gradient



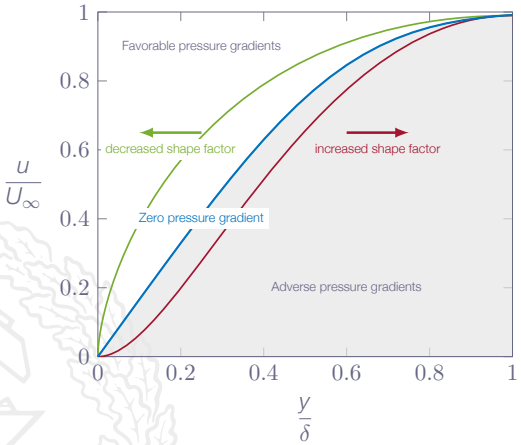
Favorable gradient ($dp/dx < 0$)	Zero gradient ($dp/dx = 0$)	Weak adverse gradient ($dp/dx > 0$)	Critical adverse gradient ($dp/dx > 0$)	Excessive adverse gradient ($dp/dx > 0$)
Point of inflection: inside wall	Point of inflection: at the wall	Point of inflection: in the flow	Point of inflection: in the flow	Point of inflection: in the flow
No separation	No separation	No separation	Separation zero slope at wall $\tau_w = 0$	Separated flow backflow at wall

Pressure Gradient



Nozzle	**Throat**	**Diffuser**
decreasing area	minimum area	increasing area
favorable pressure gradient	zero pressure gradient	adverse pressure gradient
$dp/dx < 0$	$dp/dx = 0$	$dp/dx > 0$
$dU/dx > 0$	$dU/dx = 0$	$dU/dx < 0$

Shape Factor



Shape factor: $H = \frac{\delta^*}{\theta}$

Laminar flow:

No pressure gradient: $H \approx 2.6$

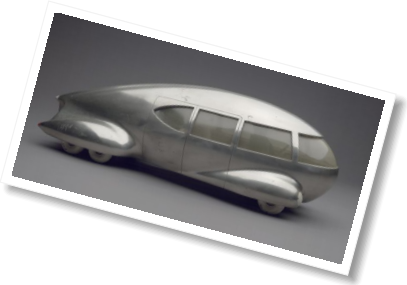
Separation: $H \approx 3.5$

Turbulent flow:

No pressure gradient: $H \approx 1.3$

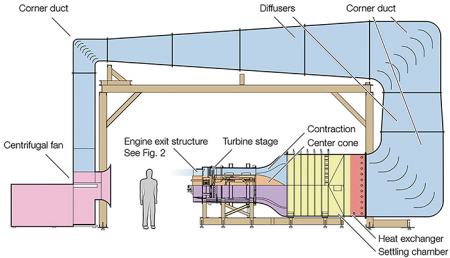
Separation: $H \approx 2.4$

Avoid or Delay Separation

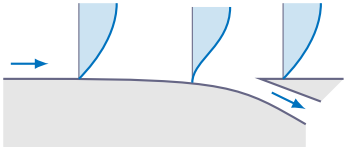


Decrease magnitude of adverse pressure gradient

- Guide vanes
- Streamlining



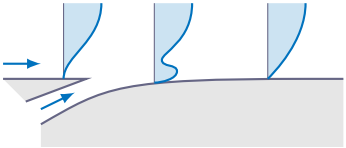
Avoid or Delay Separation



Remove decelerated fluid

Boundary layer suction

Avoid or Delay Separation



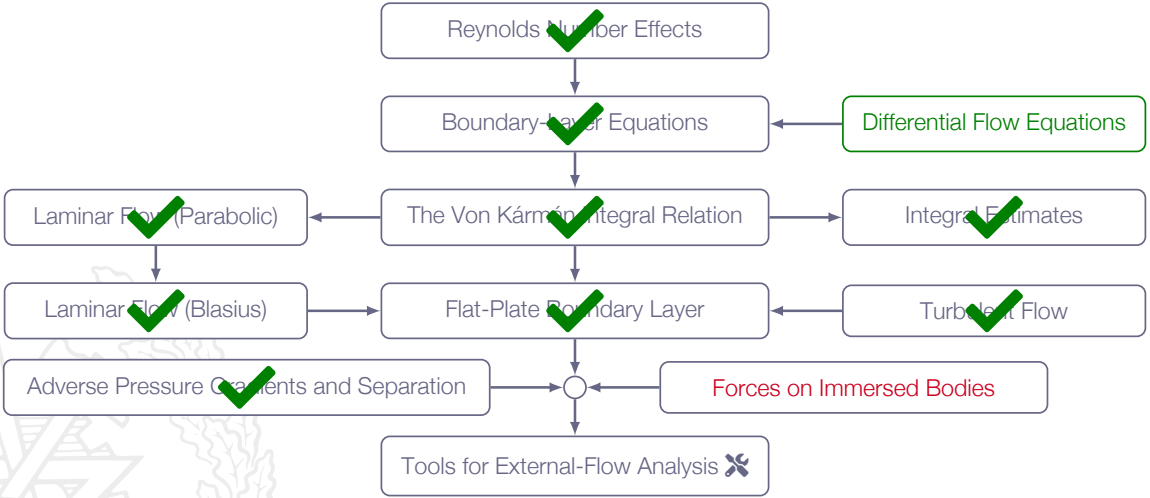
Increase near-wall momentum

Forced **transition to turbulence**

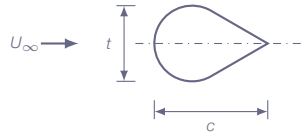
- surface roughness
- surface irregularities (dimples on the surface of a golf ball)
- trip wires

Negative consequence: comes with **increased friction**

Roadmap - Flow Past Immersed Bodies



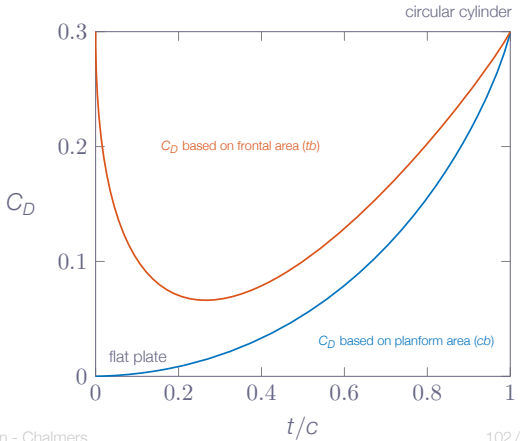
Drag of Immersed Bodies



$$C_D = \frac{drag}{\frac{1}{2}\rho U_\infty^2 A} = f\left(\frac{U_\infty L}{\nu}\right)$$

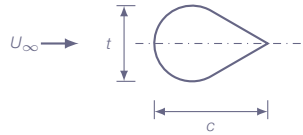
Characteristic area A:

1. Frontal area
   blunt objects: *cylinders, cars*
2. Planform area
   wide flat bodies: *wings, hydrofoils*
3. Wetted area
   ships



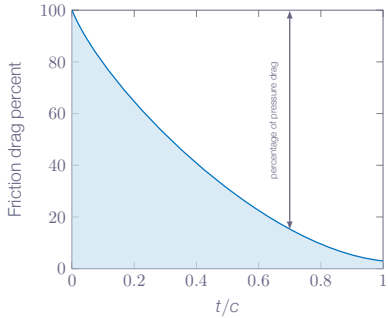
Drag of Immersed Bodies

$$C_D = C_{D_{pressure}} + C_{D_{friction}}$$



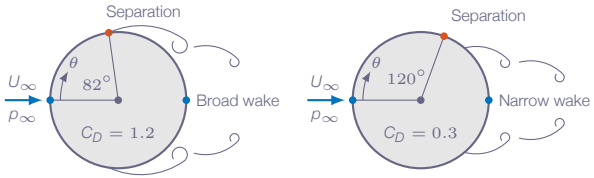
- Pressure drag
  - difference between the high front stagnation pressure and the low wake pressure on the backside of the body
  - often larger than the friction drag

- The relative importance of friction and pressure drag depends on
  - shape
  - surface roughness

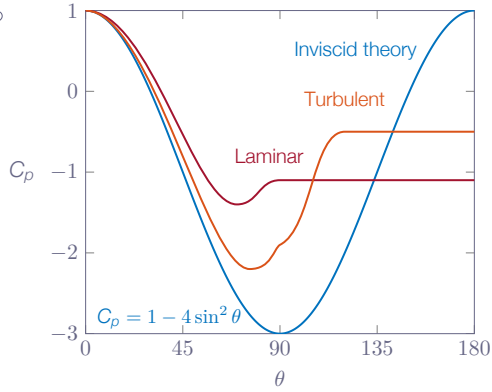


Note! for a cylinder, friction drag can be as low as a few percent of the total drag

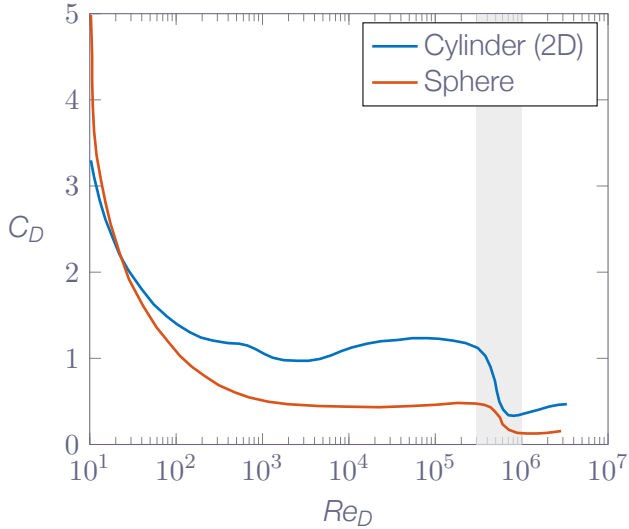
Cylinder Surface Pressure



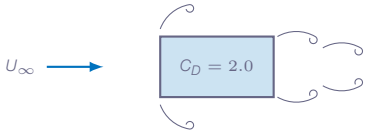
$$C_p = \frac{p - p_\infty}{\rho U_\infty^2 / 2}$$



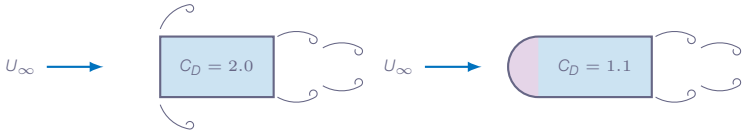
Cylinder Drag



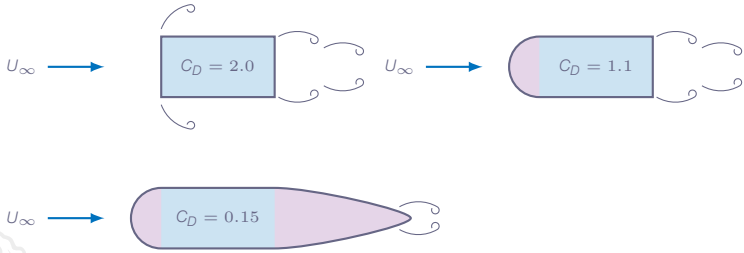
Streamlining



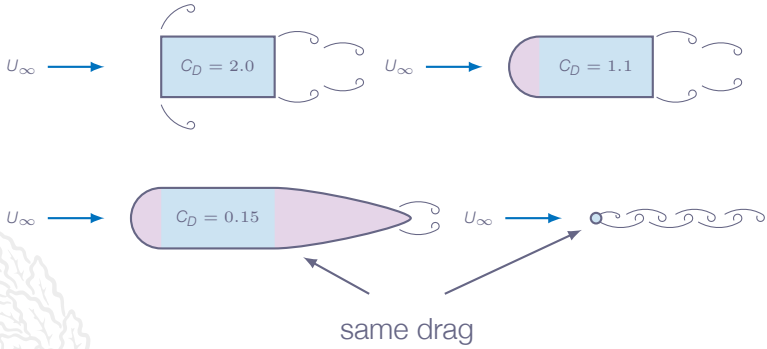
Streamlining



Streamlining



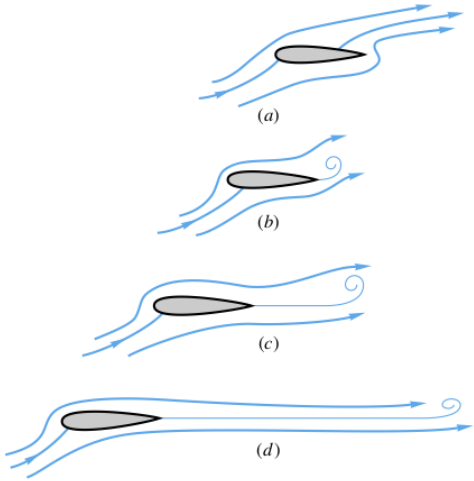
Streamlining



Drag Prediction

- No reliable theory for drag prediction (with the exception of flat plates)
- The separation point can be predicted with some accuracy but not the wake flow
- CFD or experiments needed

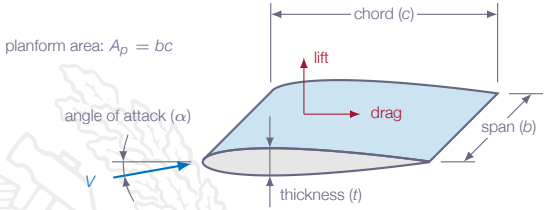
Wing Lift and Drag

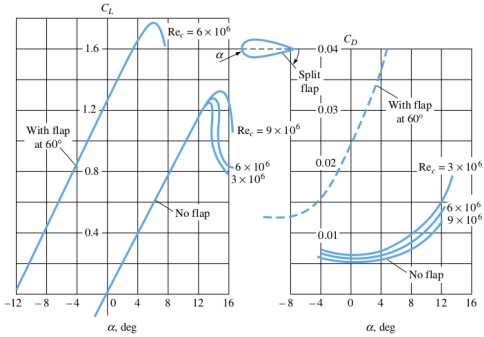


Wing Lift and Drag

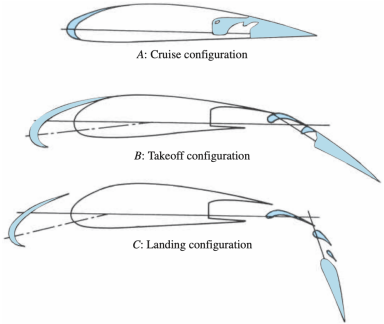
$$C_D = \frac{F_D}{\frac{1}{2}\rho U_\infty^2 A_p}$$

$$C_L = \frac{F_L}{\frac{1}{2}\rho U_\infty^2 A_p}$$

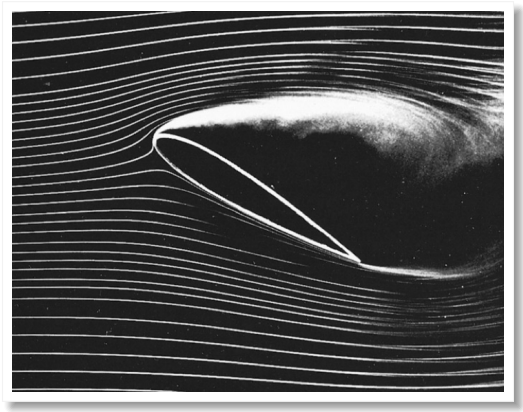




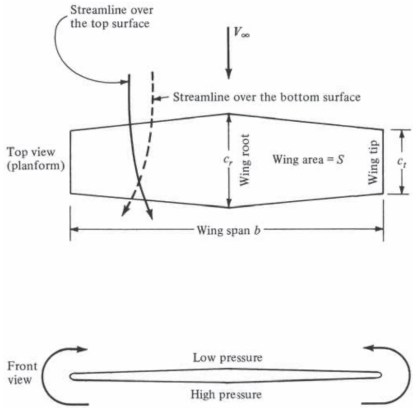
Wing Lift and Drag - High-Lift Devices

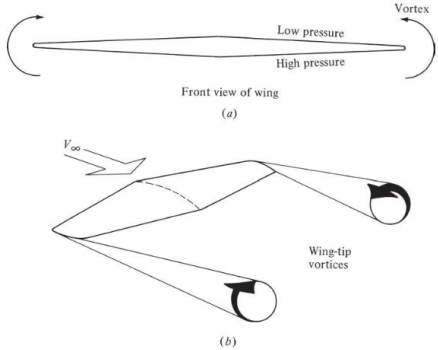


Wing Lift and Drag - Wing Stall

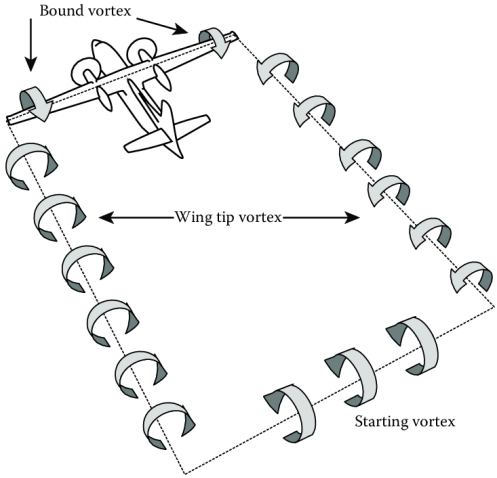


Wing Lift and Drag - Induced Drag



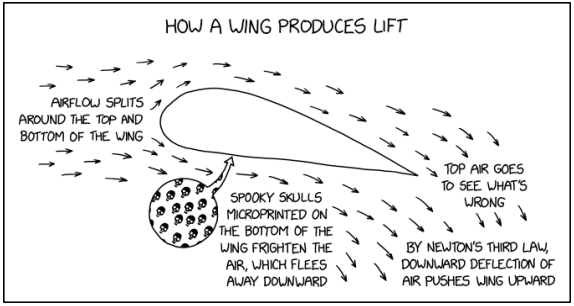


Wing Lift and Drag - Induced Drag

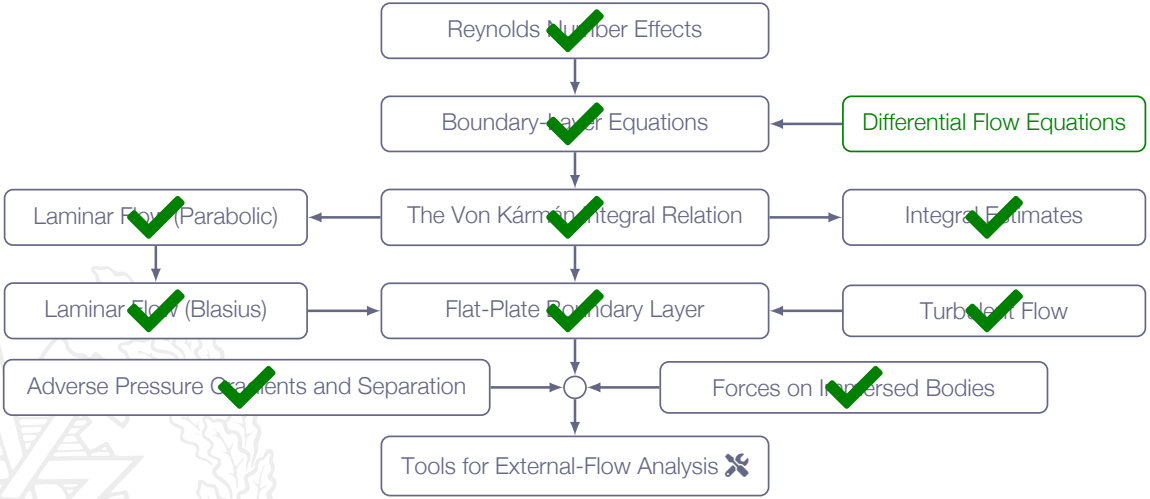


Wing Lift and Drag - Induced Drag

Wing Lift and Drag

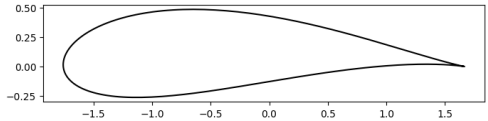


Roadmap - Flow Past Immersed Bodies



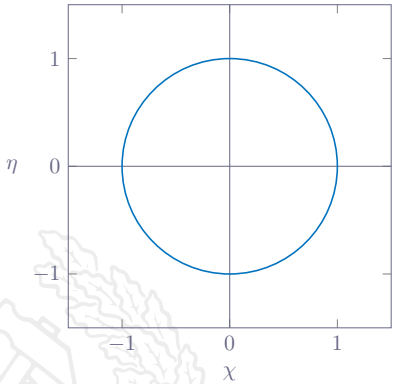
Joukowsky Transform

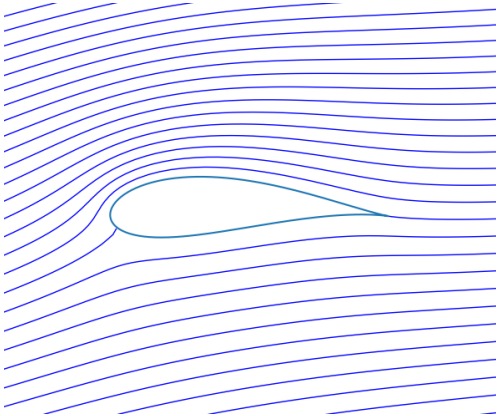
A Joukowsky wing is generated in the complex plane by applying the Joukowsky transform to a cylinder



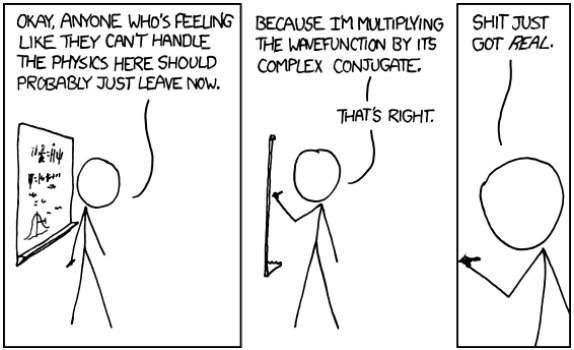
Since the potential flow around a cylinder is well known it is by using so-called conformal mapping possible to get the flow around the wing profile from the cylinder solution

Joukowsky Transform





$$\zeta = \chi + i\eta$$

$$z = \zeta + \frac{1}{\zeta} = x + iy$$

Complex Conjugate