Fluid Mechanics - MTF053

Chapter 6

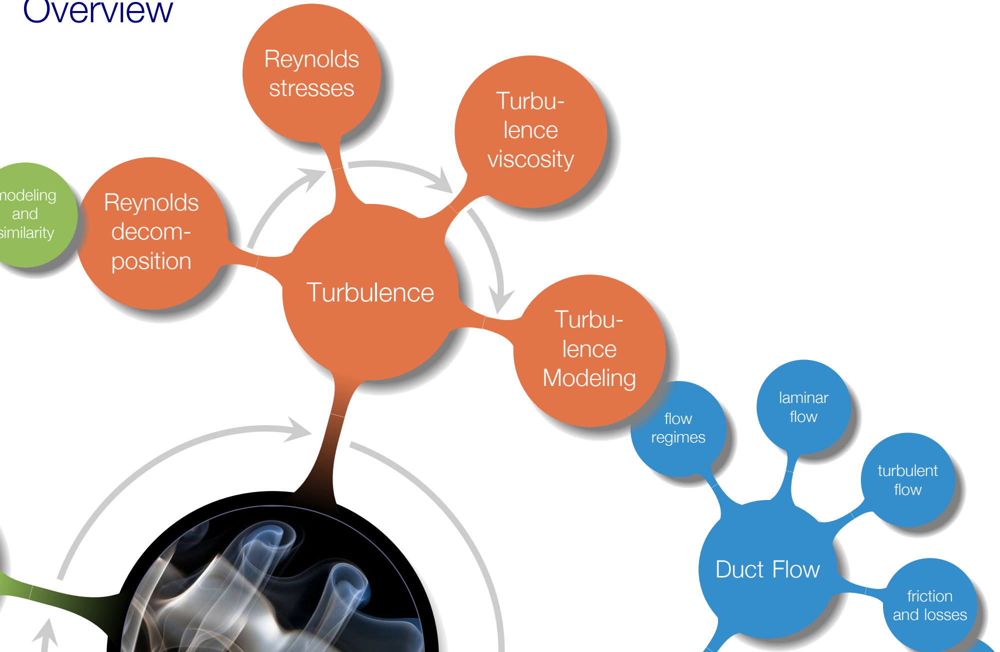
Niklas Andersson

Chalmers University of Technology
Department of Mechanics and Maritime Sciences
Division of Fluid Mechanics
Gothenburg, Sweden

`niklas.andersson@chalmers.se`

Chapter 6 - Viscous Flow in Ducts

Overview


Reynolds stresses
Turbu-lence viscosity
modeling and similarity
Reynolds decom-position
Turbulence
Turbu-lence Modeling
flow regimes
laminar flow
turbulent flow
Duct Flow
friction and losses

Learning Outcomes

3 **Define** the Reynolds number

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

6 **Explain** what a boundary layer is and when/where/why it appears

8 **Understand** and be able to **explain** the concept shear stress

18 **Explain** losses appearing in pipe flows

19 **Explain** the difference between laminar and turbulent pipe flow

20 **Solve** pipe flow problems using Moody charts

24 **Explain** what is characteristic for a turbulent flow

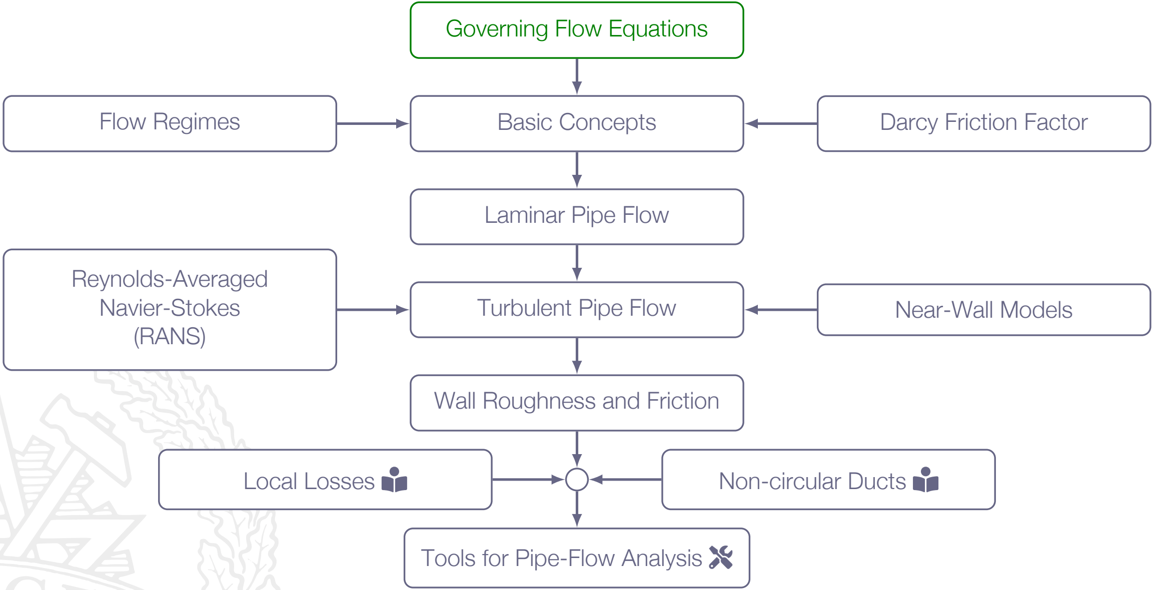
25 **Explain** Reynolds decomposition and derive the RANS equations

26 **Understand** and **explain** the Boussinesq assumption and turbulent viscosity

27 **Explain** the difference between the regions in a boundary layer and what is characteristic for each of the regions (viscous sub layer, buffer region, log region)

if you think about it, pipe flows are everywhere (a pipe flow is not a flow of pipes)

Roadmap - Viscous Flow in Ducts

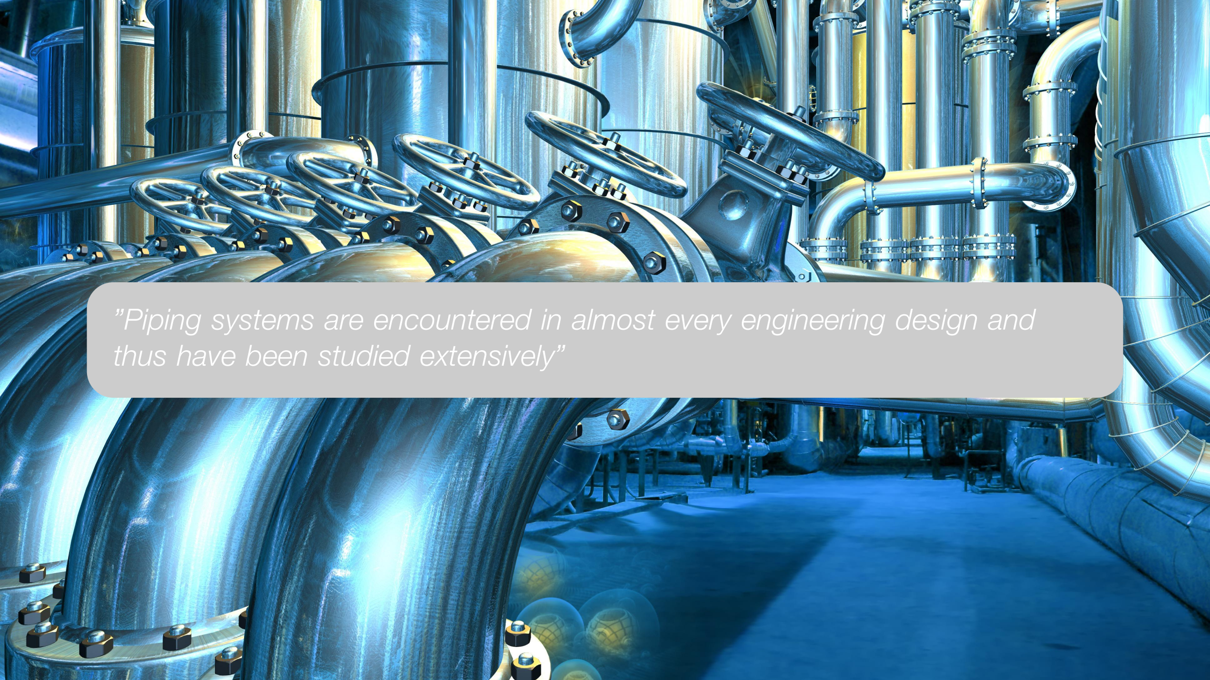


Complementary Course Material

These lecture notes covers chapter 6 in the course book and additional course material that you can find in the following documents

`MTF053_Equation-for-Boundary-Layer-Flows.pdf`

`MTF053_Turbulence.pdf`



”Piping systems are encountered in almost every engineering design and thus have been studied extensively”

Typical Pipe-Flow Problems

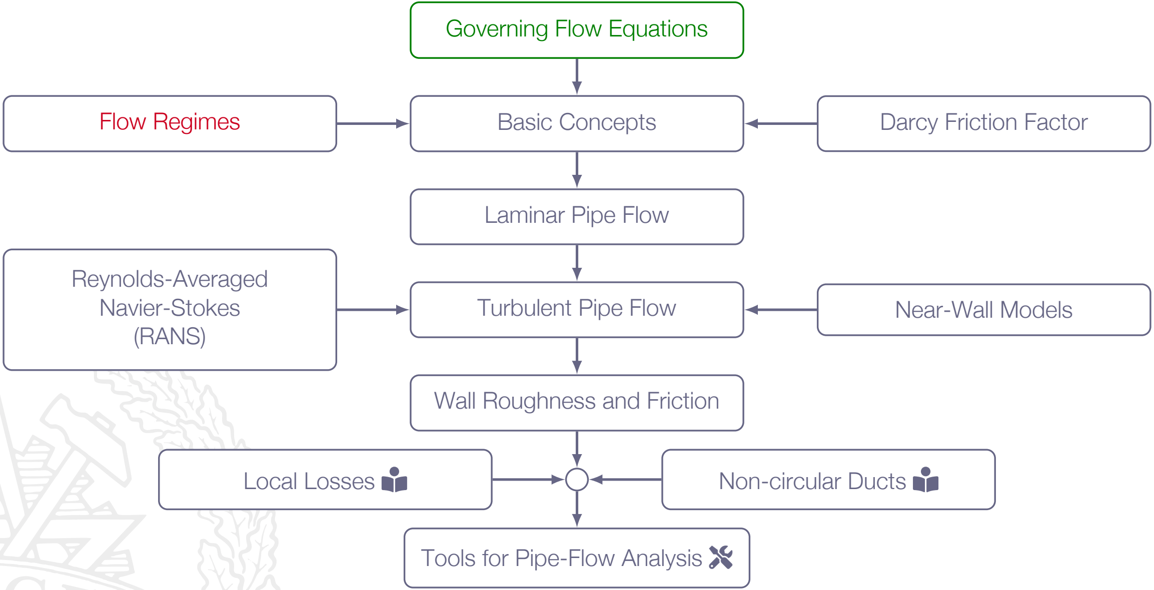
Example I:

Given pipe geometry, fluid properties, flow rate, and locations of valves, bends, diffusers etc - estimate the pressure drop needed to drive the flow

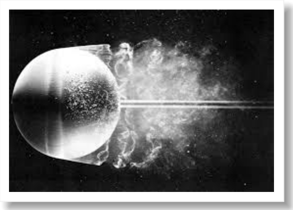
Example II:

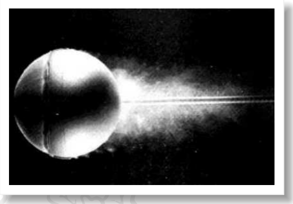
Given the pressure drop available from a pump - what flow rate can be expected

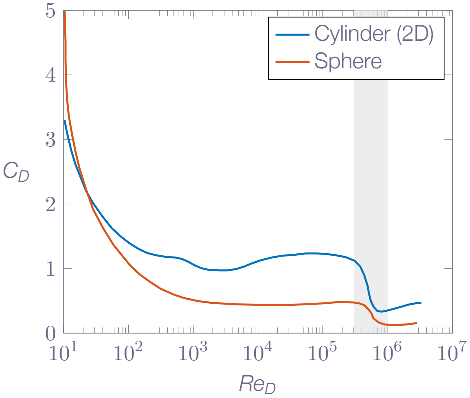
Roadmap - Viscous Flow in Ducts



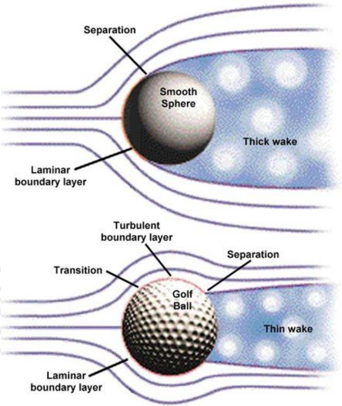
Transition to Turbulence

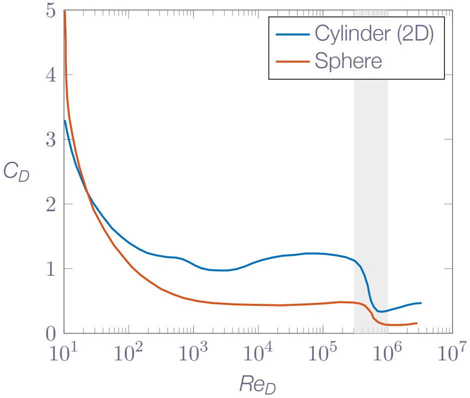






Transition to Turbulence



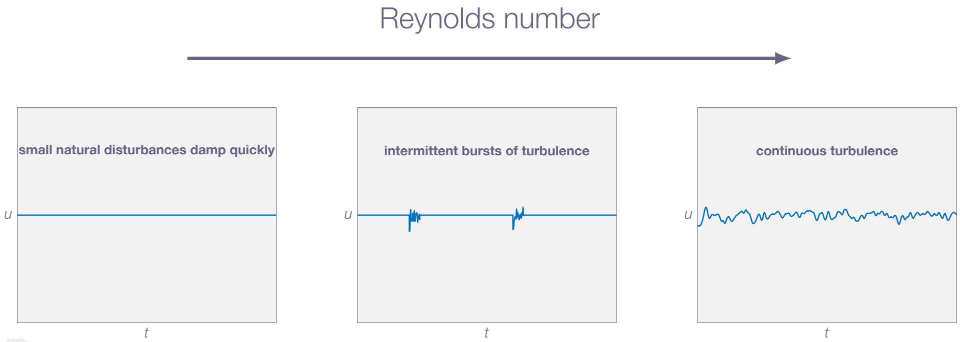


Transition to Turbulence

Factors that affects the transition to turbulent flow:

1. Wall roughness
2. Fluctuations in incoming flow
3. Reynolds number

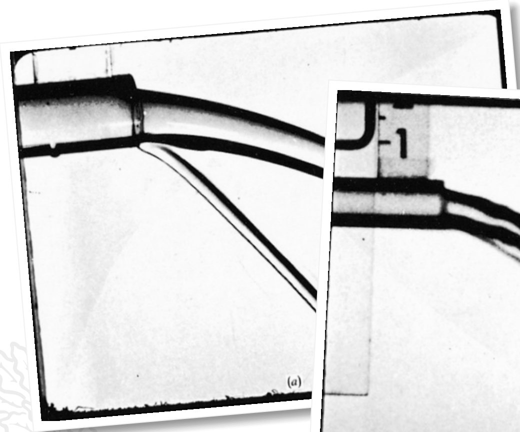
Transition to Turbulence

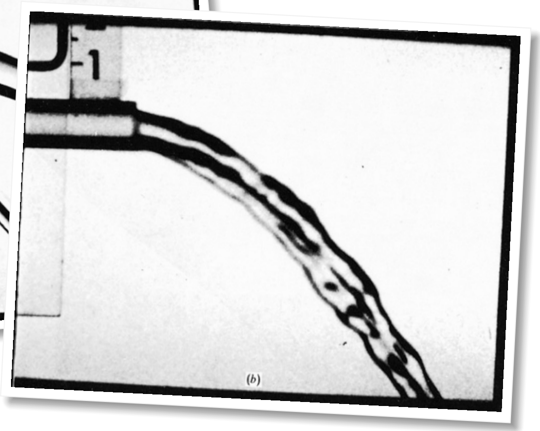


Fluctuations in the fully turbulent flow velocity signal:

- typically 1% to 20% of the average velocity
- not periodic
- random
- continuous range (spectrum) of frequencies

Transition to Turbulence





Transition to Turbulence - Viscous Flow in Ducts

0	$< Re <$	1	highly viscous laminar "creeping" motion
1	$< Re <$	100	laminar, strong Reynolds number dependence
100	$< Re <$	10^3	laminar, boundary layer theory useful
10^3	$< Re <$	10^4	transition to turbulence
10^4	$< Re <$	10^6	turbulent, moderate Reynolds number dependence
10^6	$< Re <$	∞	turbulent, slight Reynolds number dependence

Note! The ranges will vary somewhat with geometry and surface roughness

Transition to Turbulence - Viscous Flow in Ducts

An accepted design value for **pipe flow transition** is

$$Re_{d,crit} \approx 2300$$

Note!

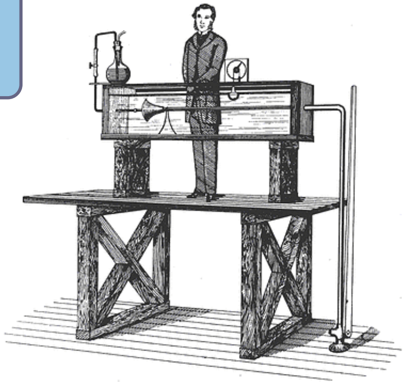
1. this value is **for pipe flows**, other applications have different transition Reynolds numbers
2. by careful design the Reynolds number can be pushed to higher values

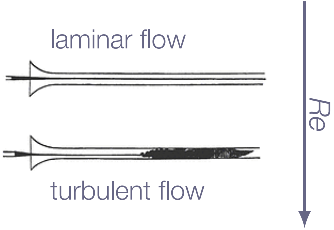
Transition to Turbulence - Viscous Flow in Ducts

The great majority of our analyses are concerned with laminar flow or with turbulent flow, and one should not normally design a flow operation in the transition region.

Transition to Turbulence - Osborne Reynolds (1842-1912)

$$Re = \frac{\rho U D}{\mu}$$



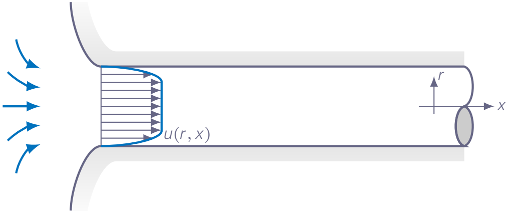


Internal Flows

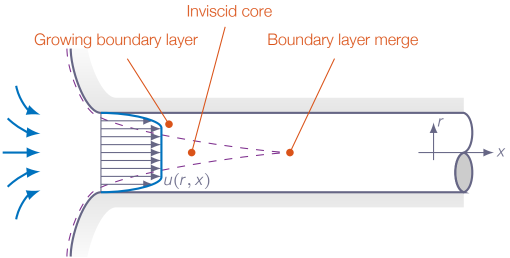
Wall-bounded flows - constrained by bounding walls

Boundary layers grows and meet at the center

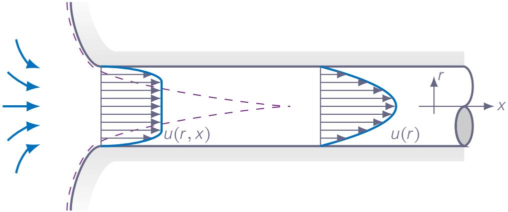
Velocity Profile Development



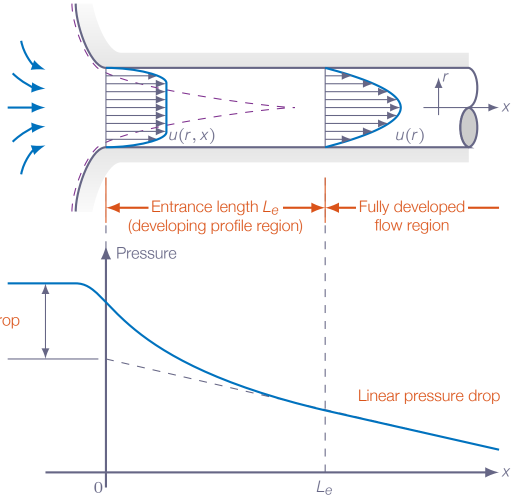
Velocity Profile Development



Velocity Profile Development



Velocity Profile Development



Velocity Profile Development

$$L_e = f(d, V, \rho, \mu)$$

where

$$V = \frac{Q}{A} = \frac{4Q}{\pi d^2}$$

and

$$Q = \int u dA = const$$

Dimensional analysis gives:

$$\frac{L_e}{d} = g\left(\frac{\rho V d}{\mu}\right) = g(Re_d)$$

Velocity Profile Development

Laminar flow:

$$\frac{L_e}{d} \approx 0.06 Re_d$$

The maximum laminar entrance length, at $Re_d = Re_{d,crit} = 2300$, is $L_e = 138d$, which is the longest development length possible

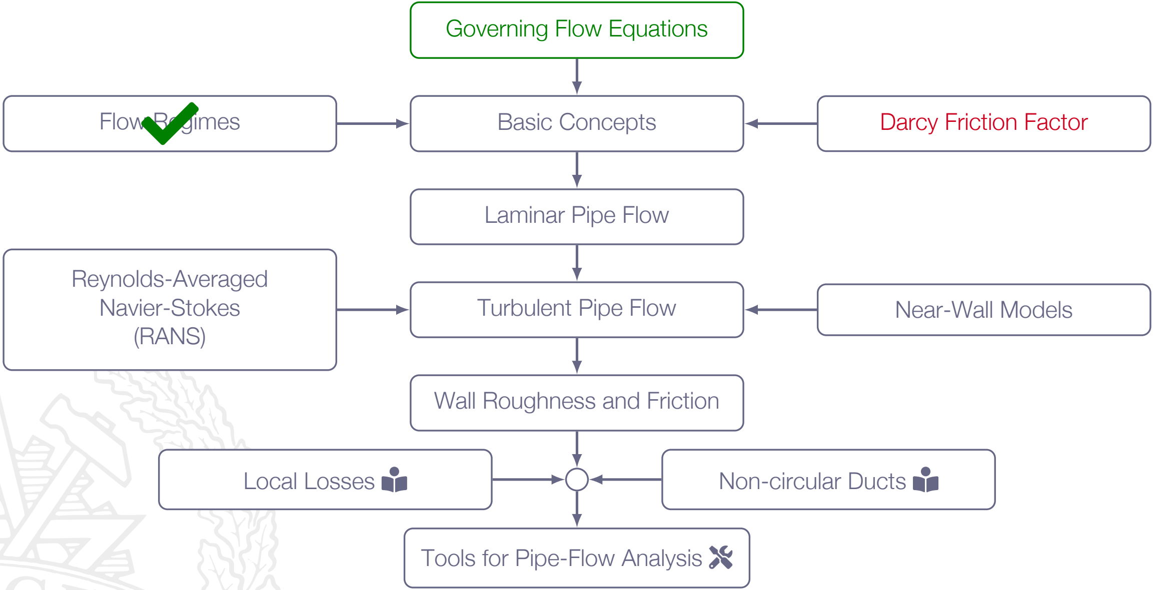
Velocity Profile Development

Turbulent flow ($Re_d \leq 10^7$):

$$\frac{L_e}{d} \approx 1.6 Re_d^{1/4}$$

Re_d	4.0×10^3	1.0×10^4	1.0×10^5	1.0×10^6	1.0×10^7
L_e/d	13	16	28	51	90

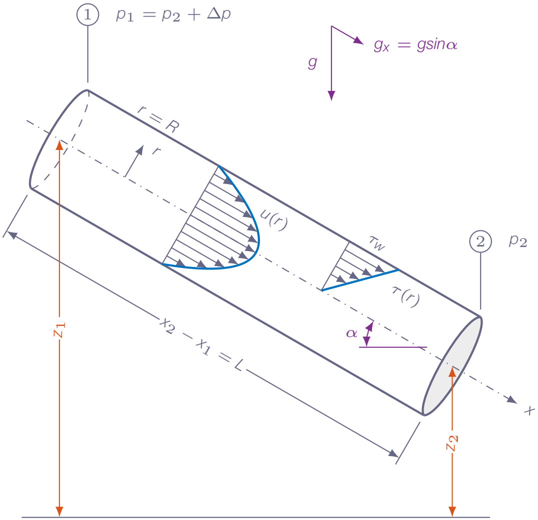
Roadmap - Viscous Flow in Ducts



Head Loss

Assumptions:

1. steady-state flow
2. incompressible
3. fully developed
4. no pumps or turbines



Head Loss

Continuity gives:

$$Q_1 = Q_2 = Q, \; V_1 = V_2 = V_{av}$$

Energy equation for steady flow without pumps or turbines:

$$\left(\frac{p}{\rho g} + \alpha \frac{V^2}{2g} + z\right)_1 = \left(\frac{p}{\rho g} + \alpha \frac{V^2}{2g} + z\right)_2 + h_f$$

Fully developed flow $\Rightarrow \alpha_1 = \alpha_2$

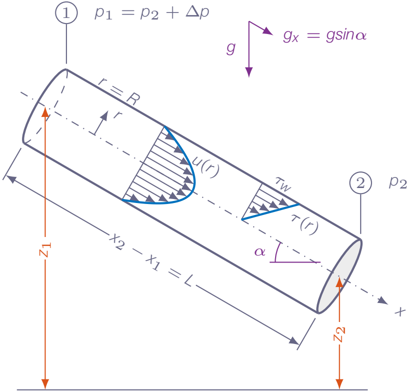
$$h_f = (z_1 - z_2) + \left(\frac{p_1 - p_2}{\rho g}\right) = \Delta z + \frac{\Delta p}{\rho g}$$

Head Loss

Apply the momentum equation along the pipe:

$$\sum F_x = \Delta p(\pi R^2) + \rho g(\pi R^2)L\sin\alpha - \tau_w(2\pi R)L$$

$$\sum F_x = \dot{m}(V_2 - V_1) = 0$$



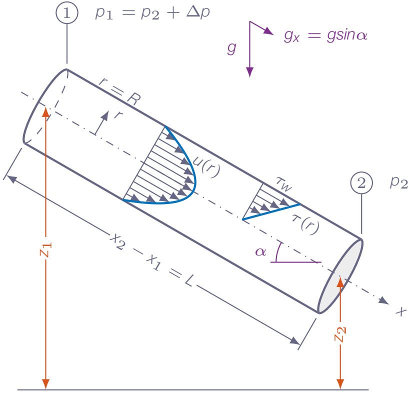
Head Loss

$$\Delta p(\pi R^2) + \rho g(\pi R^2)L\sin\alpha = \tau_w(2\pi R)L$$

$$\frac{\Delta p}{\rho g} + L\sin\alpha = \frac{2\tau_w}{\rho g}\frac{L}{R}$$

$$\frac{\Delta p}{\rho g} + \Delta z = \frac{4\tau_w}{\rho g}\frac{L}{d}$$

$$h_f = \frac{4\tau_w}{\rho g}\frac{L}{d}$$



Friction Factor

Henry Darcy 1803-1858

$$h_f = f_D \frac{L}{d} \frac{V_{av}^2}{2g}$$

where

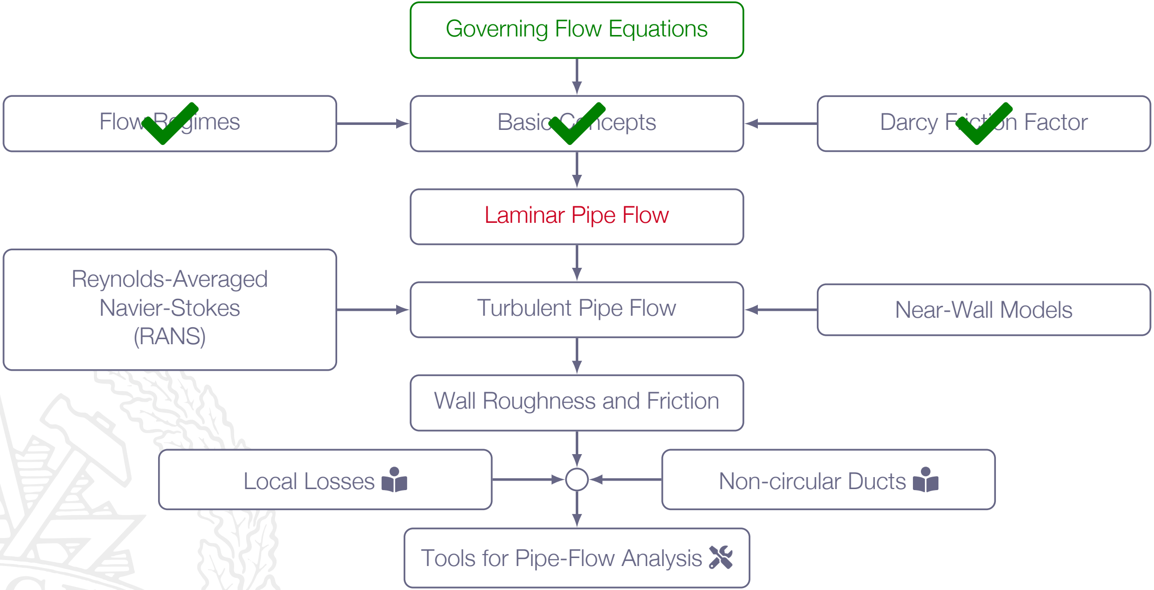
$$f_D = f(Re_d, \varepsilon/d, duct\ shape)$$

is the **Darcy friction factor**

$$\frac{4\tau_w}{\rho g} \frac{L}{d} = f_D \frac{L}{d} \frac{V_{av}^2}{2g} \Rightarrow f_D = \frac{8\tau_w}{\rho V_{av}^2}$$

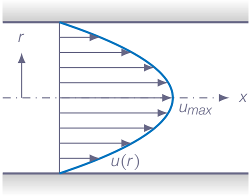
Note! for non-circular pipes, τ_w is an average value around the duct perimeter

Roadmap - Viscous Flow in Ducts



Fully-Developed Laminar Pipe Flow

Pressure driven (Poiseuille flow) in a circular pipe with the diameter D and radius R



Assumptions:

1. Steady state
2. Incompressible
3. Laminar
4. Fully developed

$$u(r) = u_{max}\left(1 - \left(\frac{r}{R}\right)^2\right) \Rightarrow \frac{du}{dr} = -2u_{max}\frac{r}{R^2} = \left\{V_{av} = \frac{u_{max}}{2}\right\} = -4V_{av}\frac{r}{R^2}$$

$$\tau_w = \mu\left|\frac{du}{dr}\right|_{r=R} = \frac{4\mu V_{av}}{R} = \frac{8\mu V_{av}}{D}$$

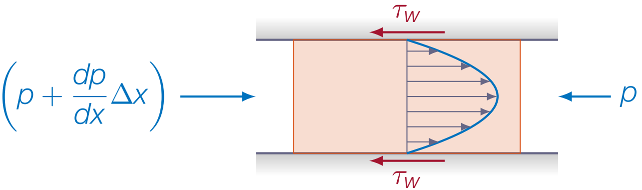
Fully-Developed Laminar Pipe Flow

For laminar flow:

$$f_D = \frac{8\tau_w}{\rho V_{av}^2} = \{\tau_w = \frac{8\mu V_{av}}{D}\} = \frac{64\mu}{\rho V_{av} D} = \frac{64}{Re_D}$$

Note! in laminar flow, the friction factor is inversely proportional to the Reynolds number

Fully-Developed Laminar Pipe Flow



$$\left(p + \frac{dp}{dx}\Delta x\right)\pi R^2 - p\pi R^2 - \tau_w 2\pi R \Delta x = 0 \Rightarrow$$

$$\frac{dp}{dx} = 2\frac{\tau_w}{R} = \left\{ \tau_w = \mu \left.\frac{du}{dr}\right|_{r=R} \right\} = \frac{2\mu}{R}\left.\frac{du}{dr}\right|_{r=R}$$

$$u(r) = u_{max}\left(1 - \left(\frac{r}{R}\right)^2\right) \Rightarrow \frac{du}{dr} = 2u_{max}\frac{r}{R^2}$$

$$\frac{dp}{dx} = \frac{4\mu}{R^2}u_{max} \Leftrightarrow u_{max} = -\frac{R^2}{4\mu}\frac{dp}{dx}$$

Fully-Developed Laminar Pipe Flow

$$-\frac{dp}{dx} = \left(\frac{\Delta p + \rho g \Delta z}{L}\right) \Rightarrow u_{max} = \left(\frac{\Delta p + \rho g \Delta z}{L}\right)\frac{R^2}{4\mu}$$

$$V_{av} = \frac{u_{max}}{2} = \left(\frac{\Delta p + \rho g \Delta z}{L}\right)\frac{R^2}{8\mu}$$

$$Q = \int u dA = V_{av}A = V_{av}\frac{\pi D^2}{4} = \left(\frac{\Delta p + \rho g \Delta z}{L}\right)\frac{\pi D^4}{128\mu}$$

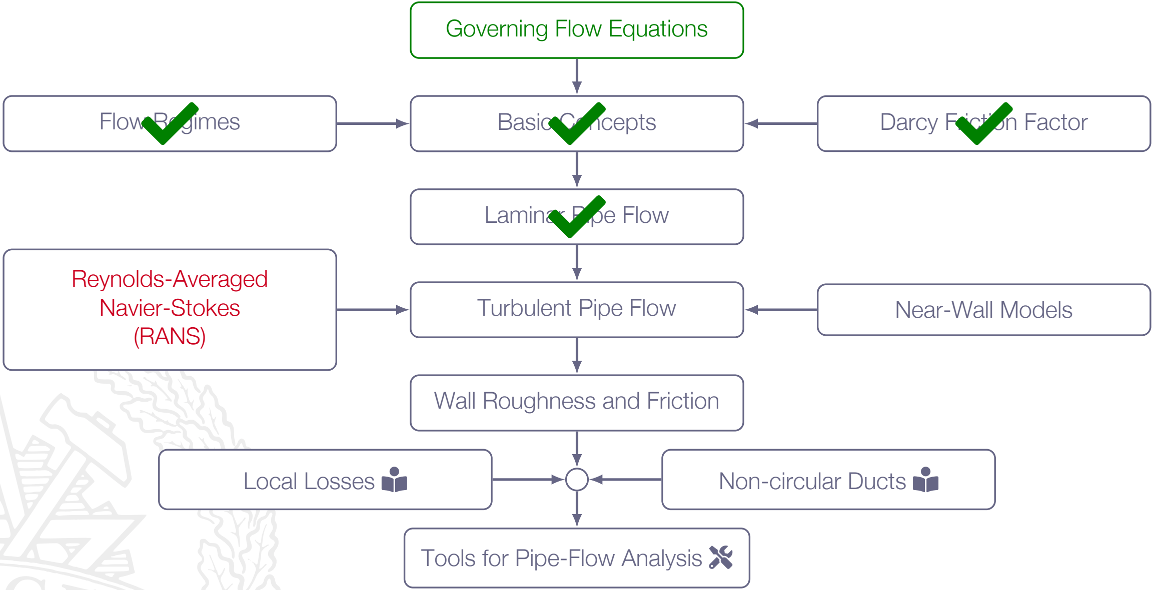
Fully-Developed Laminar Pipe Flow

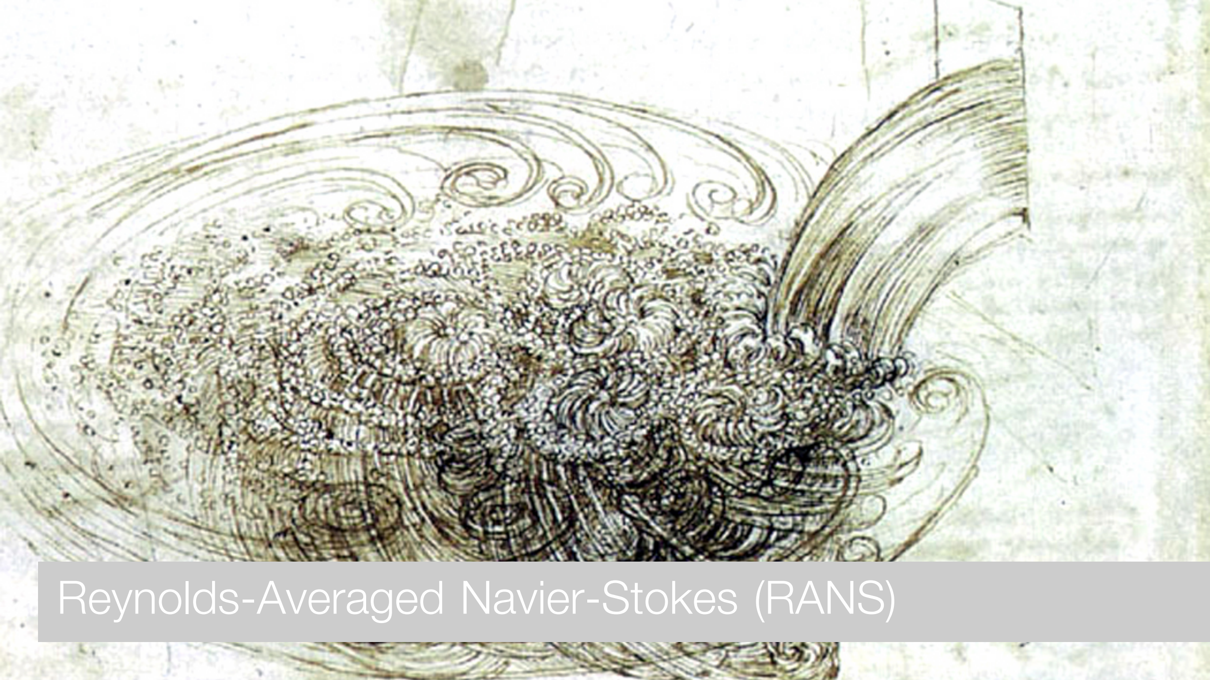
We can now calculate the head loss according to

$$h_f = f_D \frac{L}{D} \frac{V_{av}^2}{2g} \text{ where } f_D = \frac{8\tau_w}{\rho V_{av}^2}$$

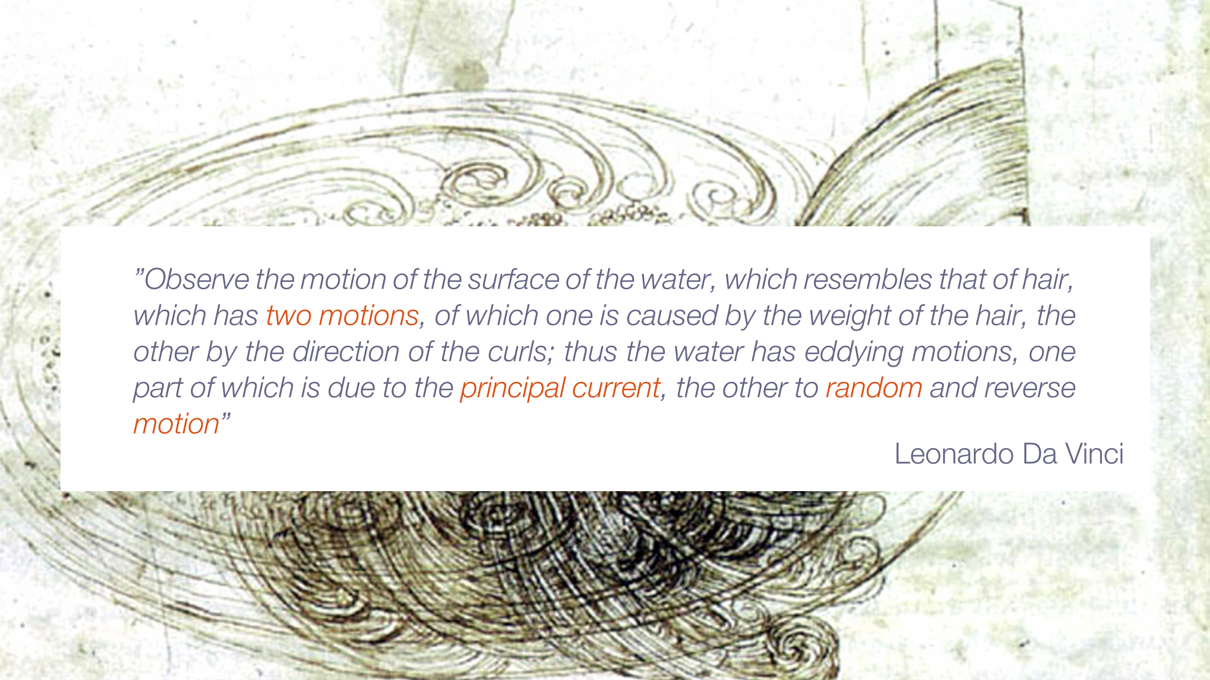
$$h_f = \frac{4\tau_w L}{\rho g D} = \left\{ \tau_w = \frac{8\mu V_{av}}{D} \right\} = \frac{16\mu V_{av} L}{\rho g D R} = \frac{32\mu V_{av} L}{\rho g D^2} = \left\{ V_{av} = \frac{4Q}{\pi D^2} \right\} = \frac{128\mu Q L}{\pi \rho g D^4}$$

Roadmap - Viscous Flow in Ducts





Reynolds-Averaged Navier-Stokes (RANS)



"Observe the motion of the surface of the water, which resembles that of hair, which has two motions, of which one is caused by the weight of the hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the principal current, the other to random and reverse motion"

Leonardo Da Vinci

Governing Equations

Assumptions:

1. constant density and viscosity
2. no thermal interaction

Flow equations:

continuity: $\nabla \cdot \mathbf{V} = 0$

momentum: $\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V}$

Governing Equations

The differential energy equation is not included here but let's have a look at it anyway

$$\rho \frac{D\hat{u}}{Dt} + p\nabla \cdot \mathbf{V} = \nabla \cdot (k\nabla T) + \phi$$

Pressure work:

pressure drives the flow through the duct

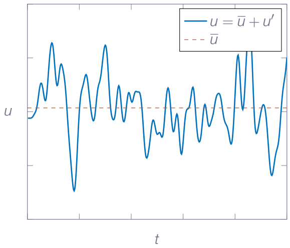
Viscous work:

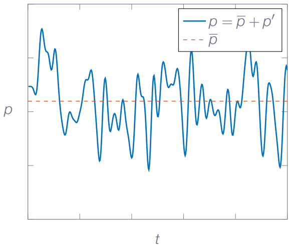
no-slip condition $\Rightarrow$ zero velocity at the walls $\Rightarrow$ no work done by wall shear stress

So, where does the energy go?

pressure work is balanced by **viscous dissipation** in the interior of the flow

Reynolds' Decomposition

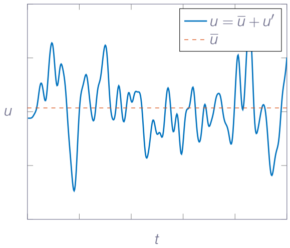


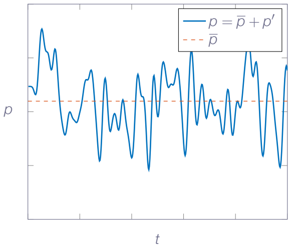


Not possible to solve analytically

Often, the time-averaged quantities are what we are looking for

Reynolds' Decomposition





$$\overline{u} = \frac{1}{T}\int_0^T u dt$$

$$u' = u - \overline{u}$$

$$\overline{u'} = \frac{1}{T}\int_0^T (u - \overline{u}) dt = \overline{u} - \overline{u} = 0$$

Reynolds' Decomposition

The mean square of the fluctuations are, however, not zero

$$\overline{u'^2} = \frac{1}{T}\int_0^T u'^2 dt \neq 0$$

measure of turbulence intensity

Mean of fluctuation products are generally not zero ($\overline{u'v'}$, $\overline{u'p'}$)

Reynolds' Decomposition

Reynolds' idea was to split all properties into mean and fluctuating parts:

$$u = \overline{u} + u',\ v = \overline{v} + v',\ w = \overline{w} + w',\ p = \overline{p} + p'$$

1. insert into the governing equations
2. time average the equations

Reynolds-Averaged Navier Stokes (RANS)

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Momentum (x-component):

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Reynolds-Averaged Navier Stokes (RANS)

Continuity:

$$\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}+\frac{\partial \overline{w}}{\partial z}+\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$$

time averaging the equation gives

$$\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}+\frac{\partial \overline{w}}{\partial z}=0$$

and as a consequence

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$$

Reynolds-Averaged Navier Stokes (RANS)

Momentum (x-component):

$$
\begin{aligned}
&\rho\left(\frac{\partial \overline{u}}{\partial t}+\frac{\partial u'}{\partial t}\right)+\\
&\rho\left(\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{u}\frac{\partial u'}{\partial x}+u'\frac{\partial \overline{u}}{\partial x}+u'\frac{\partial u'}{\partial x}\right)+\\
&\rho\left(\overline{v}\frac{\partial \overline{u}}{\partial y}+\overline{v}\frac{\partial u'}{\partial y}+v'\frac{\partial \overline{u}}{\partial y}+v'\frac{\partial u'}{\partial y}\right)+\\
&\rho\left(\overline{w}\frac{\partial \overline{u}}{\partial z}+\overline{w}\frac{\partial u'}{\partial z}+w'\frac{\partial \overline{u}}{\partial z}+w'\frac{\partial u'}{\partial z}\right)=\\
&-\frac{\partial \overline{p}}{\partial x}-\frac{\partial p'}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 \overline{u}}{\partial x^2}+\frac{\partial^2 \overline{u}}{\partial y^2}+\frac{\partial^2 \overline{u}}{\partial z^2}+\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 u'}{\partial y^2}+\frac{\partial^2 u'}{\partial z^2}\right)
\end{aligned}
$$

Reynolds-Averaged Navier Stokes (RANS)

Momentum (x-component):

time averaging the equation gives:

$$\rho\left(\frac{\partial\overline{u}}{\partial t}+\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}+\overline{w}\frac{\partial\overline{u}}{\partial z}+\overline{u'\frac{\partial u'}{\partial x}}+\overline{v'\frac{\partial u'}{\partial y}}+\overline{w'\frac{\partial u'}{\partial z}}\right)=$$
$$-\frac{\partial\overline{p}}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}+\frac{\partial^2\overline{u}}{\partial z^2}\right)$$

The highlighted terms can be rewritten as:

$$\overline{u'\frac{\partial u'}{\partial x}}+\overline{v'\frac{\partial u'}{\partial y}}+\overline{w'\frac{\partial u'}{\partial z}}=\frac{\partial\overline{u'u'}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}-\overline{u'\underbrace{\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)}_{=0}}$$

Reynolds-Averaged Navier Stokes (RANS)

the continuity equation reduces to

$$\boxed{\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0}$$

the axial component of the momentum equation:

$$\boxed{\begin{aligned}\rho \frac{D\bar{u}}{Dt} = -\frac{\partial \bar{p}}{\partial x} + \rho g_x + \frac{\partial}{\partial x}\left(\mu \frac{\partial \bar{u}}{\partial x} - \rho\overline{u'^2}\right) + \\ \frac{\partial}{\partial y}\left(\mu \frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial \bar{u}}{\partial z} - \rho\overline{u'w'}\right)\end{aligned}}$$

Reynolds-Averaged Navier Stokes (RANS)

By applying Reynolds' decomposition to our governing equations, we have introduced a number of new unknowns

The number of equations is the same as before, which means problems

Our new problem has a name

The closure problem

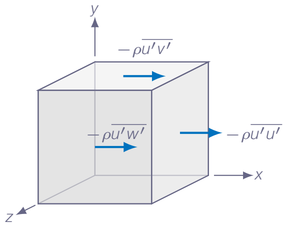
Reynolds Stresses

The three correlation terms $-\rho\overline{u'^2}$, $-\rho\overline{u'v'}$, and $-\rho\overline{u'w'}$ are called Reynolds stresses or turbulent stresses

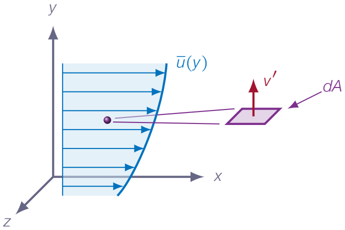
In duct and boundary layer flow, the stress $-\rho\overline{u'v'}$, associated with the direction normal to the wall, is dominant

$$\rho\frac{D\bar{u}}{Dt} \approx -\frac{\partial\bar{p}}{\partial x} + \rho g_x + \frac{\partial\tau}{\partial y}$$

$$\tau = \mu\frac{\partial\bar{u}}{\partial y} - \rho\overline{u'v'} = \tau_{lam} + \tau_{turb}$$



Reynolds Stresses



mass flow through surface element: $\dot{m}_y = \rho v' dA$

momentum balance in x-direction: $F_x = \dot{m}_y u = \rho v' \left(\overline{u} + u'\right) dA$

$$\tau_{dA} = -\frac{\overline{F}_x}{dA} = -\rho \overline{v' \left(\overline{u} + u'\right)} = -\rho \overline{v'\overline{u}} - \rho \overline{u'v'} = \left\{\overline{v'\overline{u}} = \overline{v'}\overline{u} = 0\right\} = -\rho \overline{u'v'}$$

$\Rightarrow -\rho\overline{u'v'}$ can be interpreted as a shear stress

Reynolds Stresses

Introducing **turbulent viscosity** μ_t defined such that

$$-\rho\overline{u'v'} = \mu_t \frac{\partial \overline{u}}{\partial y}$$

Boussinesq's assumption

With the turbulent viscosity, the total shear stress τ becomes:

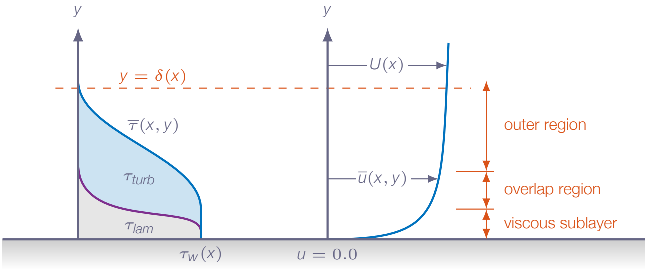
$$\tau = \mu \frac{\partial \overline{u}}{\partial y} - \rho\overline{u'v'} = (\mu + \mu_t)\frac{\partial \overline{u}}{\partial y}$$

Laminar vs Turbulent Shear Stress

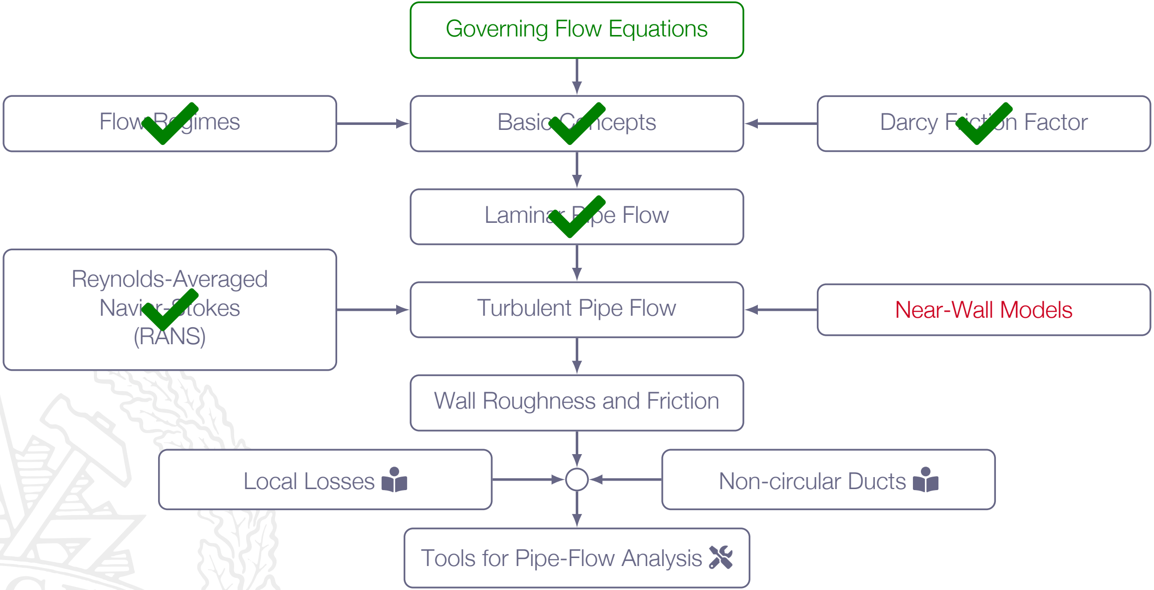
laminar shear (τ_{lam}) dominates in the near-wall region

turbulent shear (τ_{turb}) dominates in the outer region

both are important in the overlap layer



Roadmap - Viscous Flow in Ducts



Turbulent Pipe Flow - Boundary-Layer Equations

Momentum equation (x-component)

$$\rho \frac{D\bar{u}}{Dt} \approx -\frac{\partial \bar{p}}{\partial x} + \rho g_x + \frac{\partial \tau}{\partial y}$$

where

$$\tau = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'} = (\mu + \mu_t) \frac{\partial \bar{u}}{\partial y}$$

For boundary-layer flows

$$\rho \left(\bar{u} \frac{\partial \bar{u}}{dx} + \bar{v} \frac{\partial \bar{u}}{dy} \right) = -\frac{d\bar{p}}{dx} + \rho g_x + (\mu + \mu_t) \frac{\partial \bar{u}}{\partial y}$$

(will be discussed in more detail in later lectures)

Turbulent Pipe Flow - Boundary-Layer Equations

$$\rho\left(\bar{u}\frac{\partial\bar{u}}{dx}+\bar{v}\frac{\partial\bar{u}}{dy}\right)=-\frac{d\bar{p}}{dx}+\rho g_x+\frac{\partial\tau}{\partial y}$$

$$y\rightarrow 0\Rightarrow\begin{cases}\bar{u}\rightarrow 0\\ \bar{v}\rightarrow 0\end{cases}\Rightarrow$$

$$\frac{\partial\tau}{\partial y}=\frac{d\bar{p}}{dx}-\rho g_x$$

Turbulent Pipe Flow - Boundary-Layer Equations

$$\frac{\partial \tau}{\partial y} = \frac{d\bar{p}}{dx} - \rho g_x$$

$$\tau(y) = \left(\frac{d\bar{p}}{dx} - \rho g_x\right) y + C$$

$$\tau(0) = C = \tau_w \Rightarrow \tau(y) = \left(\frac{d\bar{p}}{dx} - \rho g_x\right) y + \tau_w$$

Note! with a negative pressure gradient, the shear stress will reduce with increasing distance from the wall

Turbulent Pipe Flow - Boundary-Layer Equations

$$\tau(y) = \left(\frac{d\bar{p}}{dx} - \rho g_x\right) y + \tau_w$$

At the wall, the shear stress is equal to the wall-shear stress

$$y \rightarrow 0 \Rightarrow \tau(y) \rightarrow \tau_w$$

In fact, assuming that the **shear stress** (τ) is **constant** and equal to the wall-shear stress (τ_w) is a valid assumption in the **near-wall region** (some distance from the wall but still close) as long as the pressure gradient is moderate.

Outside of the near-wall region, inertial effects has to be accounted for, i.e., $D\bar{u}/Dt$ will not be zero and thus the shear stress will not be equal to the wall-shear stress.

Turbulent Boundary Layers

A turbulent boundary layer may be divided into different regions where the physical processes leading to shear stress are clearly distinguishable

The viscous sublayer
: the shear stress is dominated by molecular viscosity (μ)

The buffer region
: molecular viscosity (μ) and turbulent viscosity (μ_t) are equally important

The log layer
: the shear stress is dominated by turbulent viscosity (μ_t)

The outer region
: inertial effects must be accounted for

Turbulent Boundary Layers

In the following we will discuss two turbulent boundary layer regions in detail:

The viscous sublayer - the region closest to the wall

The log region - outside of the viscous sublayer but still in the near-wall region

Viscous Sublayer

At the wall

$$\tau = \tau_w = \mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{u'v'}$$

$$y \to 0 \Rightarrow \begin{cases} u' \to 0 \\ v' \to 0 \end{cases} \Rightarrow$$

$$\tau = \mu \frac{\partial \overline{u}}{\partial y}$$

Viscous Sublayer

$$\tau = \mu \frac{\partial \bar{u}}{\partial y} \Rightarrow \bar{u}(y) = \frac{\tau_w}{\mu} y + C$$

$$\bar{u}(0) = 0 \Rightarrow C = 0 \Rightarrow$$

$$\bar{u}(y) = \frac{\tau_w}{\mu} y$$

Note! in the viscous sublayer, the average velocity increase linearly with the wall distance

Viscous Sublayer

Introducing friction velocity defined as

$$\boxed{u^* = \sqrt{\frac{\tau_w}{\rho}}}$$

and thus

$$\overline{u}(y) = \frac{\tau_w}{\mu}y = \frac{\rho u^{*2} y}{\mu} = \frac{u^{*2} y}{\nu}$$

which can be rewritten as:

$$\underbrace{\frac{\overline{u}}{u^*}}_{u^+} = \underbrace{\frac{u^* y}{\nu}}_{y^+} \text{ valid for } y^+ \leq 5 - 10$$

The Log Region

Now, let's move a bit further out from the wall

1. $\tau = const = \tau_w$ still (*we have not moved that far out from the wall*)
2. outside of the viscous sublayer $\mu_t \gg \mu$ and thus

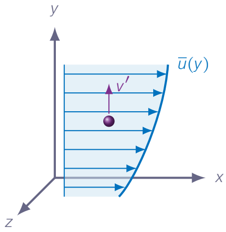
$$\tau = \tau_w = \mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{u'v'} \approx -\rho \overline{u'v'} = \mu_t \frac{\partial \overline{u}}{\partial y}$$

We need an estimate of μ_t to be able to solve this ...

The Log Region

Let's first examine the relation between u' and v' (the velocity fluctuations in the x and y directions)

The illustration below shows a fluid particle in a boundary-layer flow

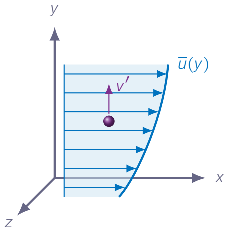


The Log Region

A positive v' fluctuation will lead to a vertical transport of the fluid particle

The fluid particle will end up in a position in the flow where the axial velocity is higher than where it came from, thus leading to a negative fluctuation in the axial velocity at that position $(u' < 0)$

In the same way, a negative v' fluctuation will lead to $u' > 0$

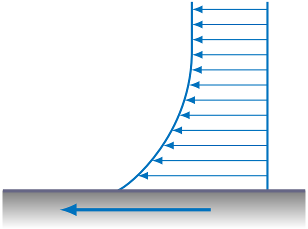


The product $u'v'$ will **always** be negative if $\partial\overline{u}/\partial y$ is positive in the wall-normal direction

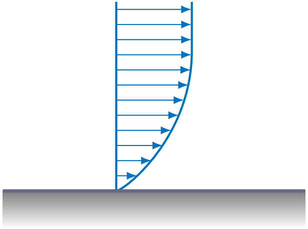
Thus $\tau_{turb} = -\rho\overline{u'v'} = \mu_t \dfrac{\partial\overline{u}}{\partial y}$ is positive

The Log Region

What about other type of boundary layers such as for example the flow over a moving surface

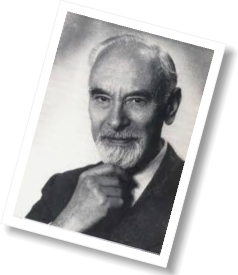


moving wall



frame of reference of the wall

The Log Region



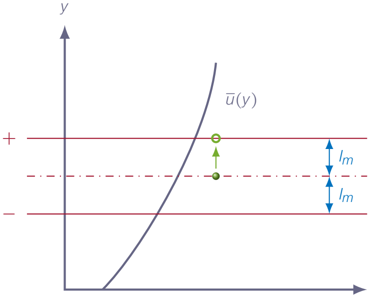
Ludwig Prandtl 1875-1953

Prandtl's mixing length concept

"the average distance that a small mass of fluid will travel before it exchanges its momentum with another mass of fluid"

$$\bar{u}(y + l_m) = \bar{u}(y) + l_m \frac{\partial \bar{u}}{\partial y}$$

$$\bar{u}(y - l_m) = \bar{u}(y) - l_m \frac{\partial \bar{u}}{\partial y}$$



Prandtl assumed $u' \approx l_m \dfrac{\partial \bar{u}}{\partial y}$

He further assumed v' to be of the same size as u'

The Log Region

Prandtl's mixing length concept

$$\tau_t = -\rho\overline{u'v'} \approx \rho l_m^2 \left(\frac{\partial \overline{u}}{\partial y}\right)^2$$

$$-\rho\overline{u'v'} \approx \mu_t \frac{\partial \overline{u}}{\partial y} \Rightarrow \mu_t \approx \rho l_m^2 \left|\frac{\partial \overline{u}}{\partial y}\right|$$

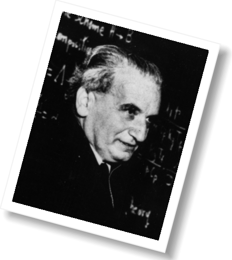
$$\nu_t = \frac{\mu_t}{\rho} \approx l_m^2 \left|\frac{\partial \overline{u}}{\partial y}\right|$$

The Log Region

Prandtl's mixing length concept

So, how do we estimate the mixing length l_m

$$l_m(y) = a_o + a_1 y + a_2 y^2 + \ldots$$



Theodore von Kármán 1881-1963

1. $y \rightarrow 0 \Rightarrow l_m \rightarrow 0 \Rightarrow a_o = 0$
2. small values of y (*we are still very close to the wall*) $\Rightarrow l_m = a_1 y$

$$l_m = \kappa y$$

where κ is Kármán's constant $\kappa \approx 0.41$

The Log Region

$$\mu_t \approx \rho l_m^2 \left|\frac{\partial \overline{u}}{\partial y}\right| = \rho \kappa^2 y^2 \left|\frac{\partial \overline{u}}{\partial y}\right|$$

$$\tau_w = \mu_t \frac{\partial \overline{u}}{\partial y} = \rho \kappa^2 y^2 \left(\frac{\partial \overline{u}}{\partial y}\right)^2 = \rho u^{*2}$$

$$\kappa^2 y^2 \left(\frac{\partial \overline{u}}{\partial y}\right)^2 = u^{*2} \Rightarrow$$

$$\frac{\partial \overline{u}}{\partial y} = \frac{u^*}{\kappa y}$$

The Log Region

$$\frac{\partial \overline{u}}{\partial y} = \frac{u^*}{\kappa y} \Rightarrow$$

$$\overline{u}(y) = \frac{u^*}{\kappa} \ln(y) + C$$

or in non-dimensional form

$$\underbrace{\frac{\overline{u}(y)}{u^*}}_{u^+} = \frac{1}{\kappa} \ln \underbrace{\left(\frac{yu^*}{\nu}\right)}_{y^+} + \underbrace{\frac{C}{u^*} - \ln\left(\frac{u^*}{\nu}\right)}_{B}$$

The Log Region

$$\boxed{u^+ = \frac{1}{\kappa}\ln\left(y^+\right) + B}$$

valid for $30 \lesssim y^+ \lesssim 1000$

From experiments we have:

$\kappa \approx 0.41$ and $4.9 < B < 5.5$

flow over a flat plate (external flow): $B \approx 4.9$
duct flow (internal flow): $B \approx 5.3$
White: $B \approx 5.0$

Outer region

In the outer region it has been found that

$$\frac{U - \overline{u}}{u^*} = f\left(\frac{y}{\delta}\right)$$

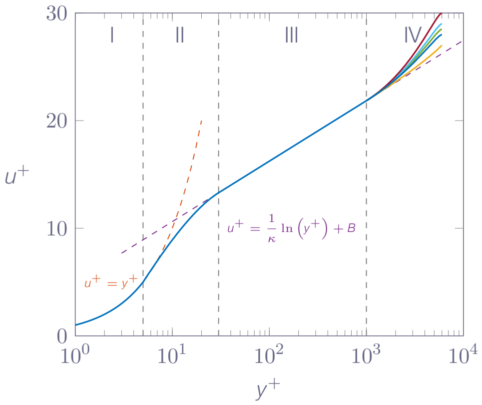
where δ is the thickness of the outer layer and U the velocity at the edge of the outer layer

Regions in a Turbulent Boundary Layer

between the viscous sublayer and the log region, none of the models works

in the outer region, inertial forces needs to be included

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) \neq 0$$



I: viscous sublayer
II: buffer layer
III: log-law region
IV: outer layer

Example - Pipe Flow Boundary Layer

Given data:
Air at 20°C flows through a 14-cm-diameter pipe. The flow is fully developed and the centerline velocity is 5.0 m/s

$$\text{Air @ } 20°\text{C} \Rightarrow \rho = 1.2\, kg/m^3,\ \mu = 1.8 \times 10^{-5}\, kg/(ms)$$
$$D = 0.14\, m$$
$$U_{max} = 5.0\, m/s$$

Assumptions:
steady-state, fully-developed, turbulent, incompressible pipe flow

Task:
From the provided data, estimate the friction velocity (u^*) and the wall-shear stress (τ_w)

Example - Pipe Flow Boundary Layer

Assume turbulent flow:

$$V_{av} = \frac{2U_{max}}{(1+m)(2+m)}$$

$m = 1/7$ gives $V_{av} = 4.08\ m/s$

$$Re_D = \frac{\rho V_{av} D}{\mu} \approx 38000 \gg Re_{D_{critical}} = 2300$$

The flow is turbulent

Example - Pipe Flow Boundary Layer

Assume that the log-law is valid all the way to the center of the pipe

$$u^+ = \frac{1}{\kappa}\ln(y^+) + B \Leftrightarrow 0 = \frac{1}{\kappa}\ln(y^+) + B - u^+$$

or (at the center of the pipe where $y = R$ and $u = U_{max}$)

$$0 = \frac{1}{\kappa}\ln\left(\frac{Ru^*}{\nu}\right) + B - \frac{U_{max}}{u^*}$$

where $\kappa = 0.41$ and $B = 5.0$

$$u^* = \sqrt{\frac{\tau_w}{\rho}}$$

Example - Pipe Flow Boundary Layer

Find estimates of u^* and τ_w using a Newton-Raphson solver

Using the definitions of y^+, u^+, and u^*, we can get a function $f(\tau_w)$

$$f(\tau_w) = \frac{1}{\kappa} \ln\left(\frac{R\sqrt{\tau_w}}{\sqrt{\rho}\nu}\right) + B - \frac{U_{max}\sqrt{\rho}}{\sqrt{\tau_w}}$$

The derivative of $f(\tau_w)$ is obtained as (*details on next slide*)

$$f'(\tau_w) = \frac{(1/\kappa)\sqrt{\tau_w} + U_{max}\sqrt{\rho}}{2\tau_w^{3/2}} = \frac{(1/\kappa) + u^+}{2\tau_w}$$

Example - Pipe Flow Boundary Layer

$$f(\tau_w) = \frac{1}{\kappa}\ln\left(\frac{R\sqrt{\tau_w}}{\sqrt{\rho}\nu}\right) + B - \frac{U_{max}\sqrt{\rho}}{\sqrt{\tau_w}}$$

$$f'(\tau_w) = \frac{\partial}{\partial \tau_w}\left(\frac{1}{\kappa}\ln\left(\frac{R\sqrt{\tau_w}}{\sqrt{\rho}\nu}\right)\right) - \frac{\partial}{\partial \tau_w}\left(\frac{U_{max}\sqrt{\rho}}{\sqrt{\tau_w}}\right) =$$

$$= \frac{\partial}{\partial \tau_w}\left(\frac{1}{\kappa}\left[\ln\left(\frac{R}{\sqrt{\rho}\nu}\right) + \ln\left(\sqrt{\tau_w}\right)\right]\right) - \left(-\frac{1}{2}\right)\frac{U_{max}\sqrt{\rho}}{\tau_w^{3/2}} =$$

$$= \frac{\partial}{\partial \tau_w}\left(\frac{1}{\kappa}\left[\ln\left(\frac{R}{\sqrt{\rho}\nu}\right) + \frac{1}{2}\ln\left(\tau_w\right)\right]\right) + \frac{U_{max}\sqrt{\rho}}{2\tau_w^{3/2}} =$$

$$= \left(\frac{1}{\kappa}\right)\frac{1}{2\tau_w} + \frac{U_{max}\sqrt{\rho}}{2\tau_w^{3/2}} = \frac{(1/\kappa)\sqrt{\tau_w} + U_{max}\sqrt{\rho}}{2\tau_w^{3/2}} = \frac{(1/\kappa) + u^+}{2\tau_w}$$

Example - Pipe Flow Boundary Layer

With the functions $f(\tau_w)$ and $f'(\tau_w)$ defined, we can set up an iterative Newton-Raphson solver to find τ_w using

$$\tau_{w_{n+1}} = \tau_{w_n} - \frac{f(\tau_{w_n})}{f'(\tau_{w_n})}$$

where $n+1$ and n are iteration numbers. Iterate until converged with the following convergence criterium:

$$\left|\frac{f(\tau_{w_n})}{f'(\tau_{w_n})}\right| \leq \tau_w \times 10^{-4}$$

Example - Pipe Flow Boundary Layer

```python
import numpy as np

def calc_yplus_uplus(rho,mu,tau_w,y,U):
    nu=mu/rho
    ustar=np.sqrt(tau_w/rho)
    yplus=y*ustar/nu
    uplus=U/ustar
    return yplus,uplus,ustar

mu    =  1.8e-5 # fluid viscosity (dynamic viscosity)
rho   =  1.2    # fluid density
u_max =  5.0    # centerline velocity
R     =  0.07   # pipe radius
kappa =  0.41   # von Kármán constant
B     =  5.0    # integration constant in the log-law
```

Example - Pipe Flow Boundary Layer

```
tau_w = mu*u_max/R # initial guess

yplus,uplus,ustar=calc_yplus_uplus(rho,mu,tau_w,R,u_max)

dtau_w = 10.*tau_w

while( abs(dtau_w) > 0.0001*tau_w ):
  f       = (1./kappa)*np.log(yplus)-uplus+B
  df      = 0.5*((1./kappa)+uplus)/tau_w
  dtau_w = -f/df
  tau_w   = tau_w+dtau_w
  yplus,uplus,ustar=calc_yplus_uplus(rho,mu,tau_w,R,u_max)
```

Example - Pipe Flow Boundary Layer

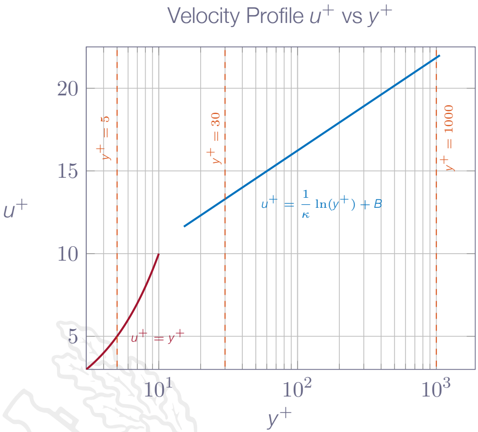
iteration	τ_w	f/f'
1	0.003531	2.244938e-03
2	0.009029	5.498838e-03
3	0.020451	1.142164e-02
4	0.038183	1.773146e-02
5	0.054798	1.661537e-02
6	0.061401	6.602591e-03
7	0.062021	6.204740e-04
8	0.062026	4.575602e-06

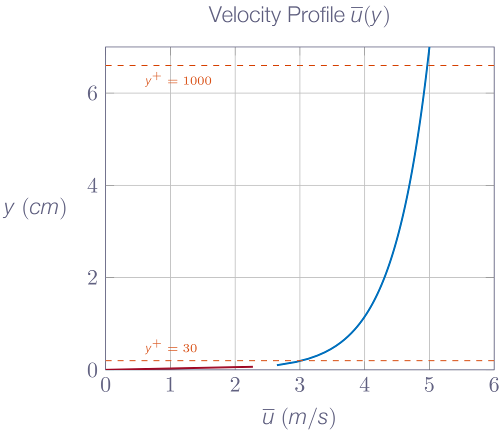
Example - Pipe Flow Boundary Layer

variable	dimension	value
y^+ (pipe center)		1061
u^*	m/s	0.227
τ_w	N/m^2	0.062

Note! $y^+ = 1061$ is actually outside the range of y^+ values for which the log-law is valid - but it is very close to the limit...

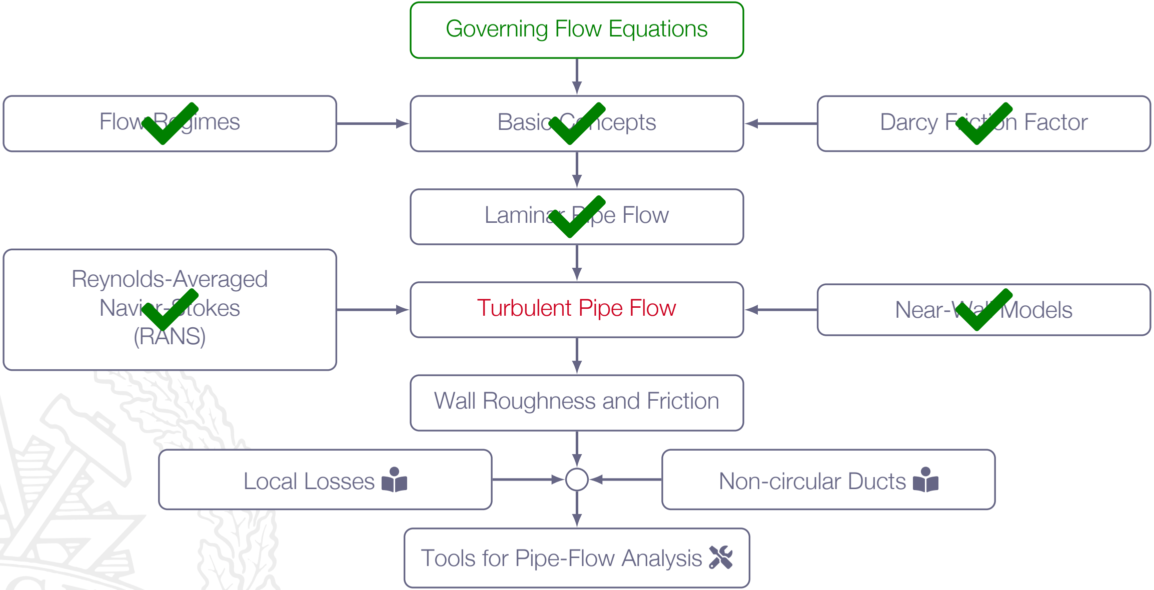
Example - Pipe Flow Boundary Layer



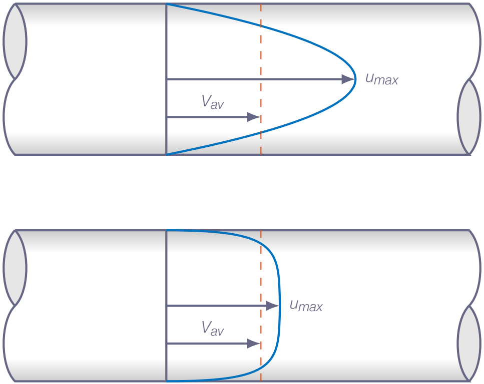


Note! The upper limit of the **viscous sublayer** $(y^+ = 5)$ corresponds to a distance from the wall of $y = 0.3\ mm$ or 0.2% of the pipe diameter and the lower bound for the **log region** $(y^+ = 30)$ corresponds to a wall distance of $y = 2.0\ mm$ or 1.4% of the pipe diameter.

Roadmap - Viscous Flow in Ducts



Turbulent Pipe Flow



Laminar flow

Turbulent flow

Turbulent Pipe Flow

As we did for laminar pipe flow, we will now obtain the friction factor for turbulent pipe flow

$$\left.\begin{aligned} \tau_w &= f_D \frac{\rho {V_{av}}^2}{8} \\ u^* &\equiv \sqrt{\frac{\tau_w}{\rho}} \end{aligned}\right\} \Rightarrow f_D = 8\left(\frac{V_{av}}{u^*}\right)^{-2}$$

So, what we need now is an estimate of the average flow velocity in the pipe (V_{av}/u^*)

There are different ways to do this and here is one example:

1. Assume that we can use the log-law all the way across the pipe
2. Integrate to get the average velocity
3. Insert the calculated average velocity into the relation above

Turbulent Pipe Flow

$$f_D = 8\left(\frac{V_{av}}{u^*}\right)^{-2}$$

$$\left.\begin{aligned}\frac{\overline{u}(r)}{u^*} &\approx \frac{1}{\kappa}\ln\frac{(R-r)u^*}{\nu}+B\\ \frac{V_{av}}{u^*} = \frac{Q}{Au^*} &= \frac{1}{\pi R^2}\int_0^R \frac{\overline{u}(r)}{u^*}2\pi r dr\end{aligned}\right\} \Rightarrow \frac{V_{av}}{u^*} \approx \frac{2}{R^2}\int_0^R\left[\frac{1}{\kappa}\ln\frac{(R-r)u^*}{\nu}+B\right]rdr$$

with $\kappa = 0.41$ and $B = 5.0$ we get

$$\frac{V_{av}}{u^*} \approx 2.44\ln\left(\frac{Ru^*}{\nu}\right)+1.34$$

details on next slide

Turbulent Pipe Flow

$$\frac{V_{av}}{u^*} = \frac{2}{R^2}\int_0^R \left[\frac{r}{\kappa}\ln\left(\frac{(R-r)u^*}{\nu}\right) + Br\right] dr = \frac{2}{\kappa R^2}\int_0^R \left[\ln(R-r) + \ln\left(\frac{u^*}{\nu}\right) + B\kappa\right] rdr =$$

$$= \frac{1}{\kappa}\left(\ln\left(\frac{u^*}{\nu}\right) + B\kappa\right) + \frac{2}{\kappa R^2}\int_0^R r\ln(R-r)dr =$$

$$= \frac{1}{\kappa}\ln\left(\frac{u^*}{\nu}\right) + B + \frac{2}{\kappa R^2}\left[\frac{1}{4}\left(-2(R^2-r^2)\ln(R-r) - r(2R+r)\right)\right]_0^R =$$

$$= \frac{1}{\kappa}\ln\left(\frac{Ru^*}{\nu}\right) + B - \frac{3}{2\kappa} = \{\kappa = 0.41,\ B = 5.0\} = 2.44\ln\left(\frac{Ru^*}{\nu}\right) + 1.34$$

Turbulent Pipe Flow

$$\frac{V_{av}}{u^*} \approx 2.44 \ln\left(\frac{Ru^*}{\nu}\right) + 1.34$$

The argument of the logarithm can be rewritten as

$$\frac{Ru^*}{\nu} = \frac{V_{av}D}{2\nu}\frac{u^*}{V_{av}} = \left\{Re_D = \frac{V_{av}D}{\nu},\ f_D = 8\left(\frac{u^*}{V_{av}}\right)^2\right\} = \frac{1}{2}Re_D\left(\frac{f_D}{8}\right)^{1/2}$$

and thus we get Prandtl's friction-factor function for smooth pipes:

$$\boxed{\frac{1}{\sqrt{f_D}} \approx 2.0\log_{10}(Re_D\sqrt{f_D}) - 0.8}$$

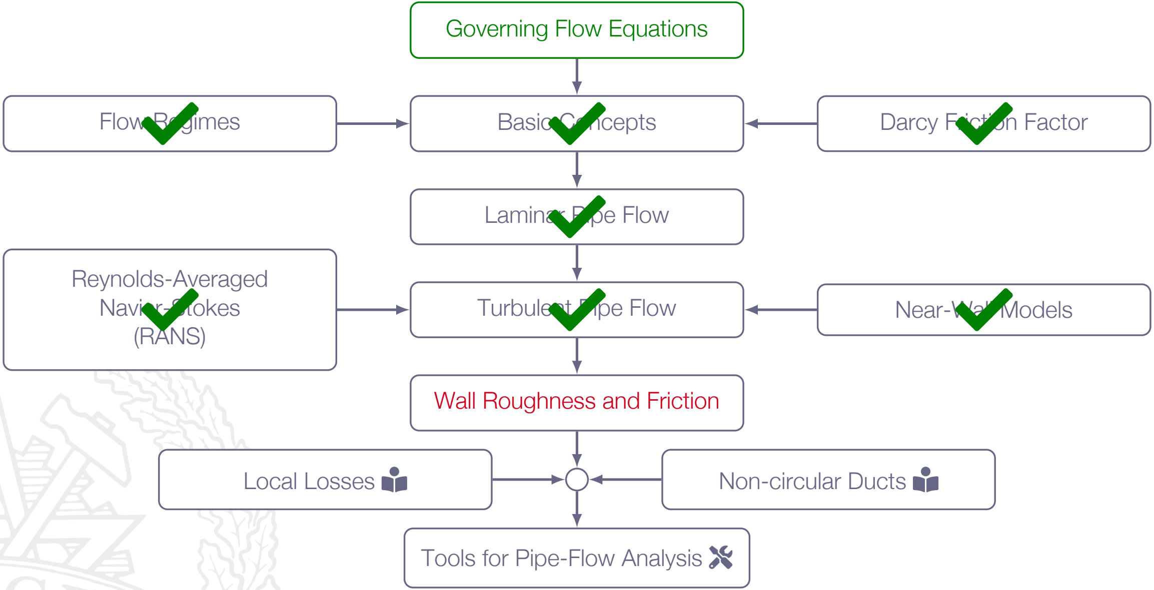
Turbulent Pipe Flow

Alternative 2:

If we assume that $\frac{\overline{u}(y)}{u^*} = 8.3\left(\frac{u^* y}{\nu}\right)^{1/7}$ applies all over the cross section we get

$$\boxed{f_D = \frac{0.3164}{Re_D^{1/4}}}$$

Roadmap - Viscous Flow in Ducts



Wall Roughness

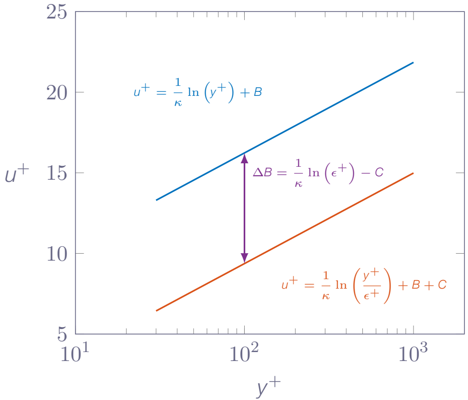
Effects of surface roughness on friction:

Negligible for **laminar** pipe flow

Significant for **turbulent** flow

breaks up the **viscous sublayer**

modified log law (changed the value of the integration constant B)



$$\Delta B \propto (1/\kappa)\ln\epsilon^+ \quad \text{where } \epsilon^+ = \frac{\epsilon u^*}{\nu}$$

ϵ is a representative measure of the surface roughness

Wall Roughness

$$\frac{\epsilon u^*}{\nu} < 5$$

hydraulically smooth

no effects of roughness

$$5 \leq \frac{\epsilon u^*}{\nu} \leq 70$$

transitional

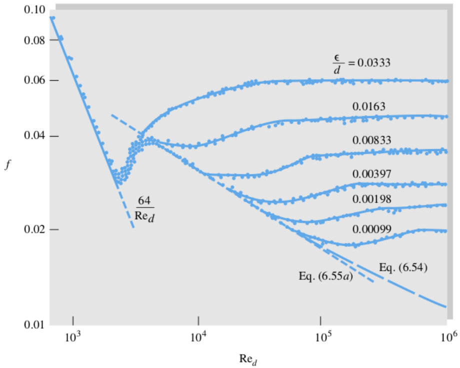
moderate Reynolds number effects

$$\frac{\epsilon u^*}{\nu} > 70$$

fully rough

sublayer totally broken up

independent of Reynolds number



Wall Roughness

$$\frac{\epsilon u^*}{\nu} < 5$$

hydraulically smooth

no effects of roughness

$$5 \leq \frac{\epsilon u^*}{\nu} \leq 70$$

transitional

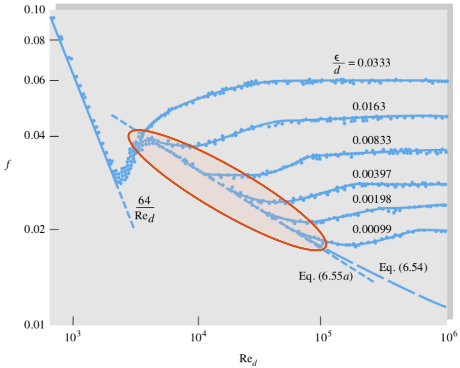
moderate Reynolds number effects

$$\frac{\epsilon u^*}{\nu} > 70$$

fully rough

sublayer totally broken up

independent of Reynolds number



Wall Roughness

$$\frac{\epsilon u^*}{\nu} < 5$$

hydraulically smooth

no effects of roughness

$$5 \leq \frac{\epsilon u^*}{\nu} \leq 70$$

transitional

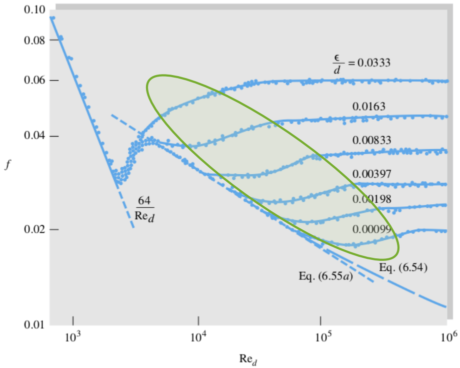
moderate Reynolds number effects

$$\frac{\epsilon u^*}{\nu} > 70$$

fully rough

sublayer totally broken up

independent of Reynolds number



Wall Roughness

$$\frac{\epsilon u^*}{\nu} < 5$$

hydraulically smooth

no effects of roughness

$$5 \leq \frac{\epsilon u^*}{\nu} \leq 70$$

transitional

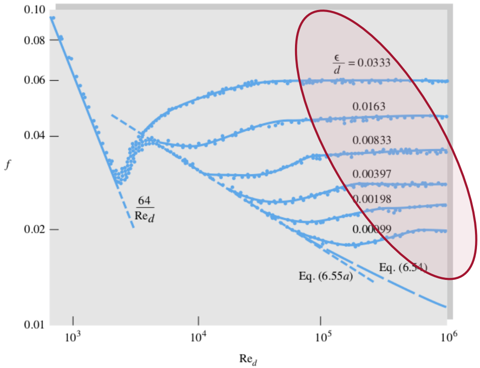
moderate Reynolds number effects

$$\frac{\epsilon u^*}{\nu} > 70$$

fully rough

sublayer totally broken up

independent of Reynolds number



Wall Roughness

$$\frac{\epsilon u^*}{\nu} < 5$$

hydraulically smooth

no effects of roughness

$$5 \leq \frac{\epsilon u^*}{\nu} \leq 70$$

transitional

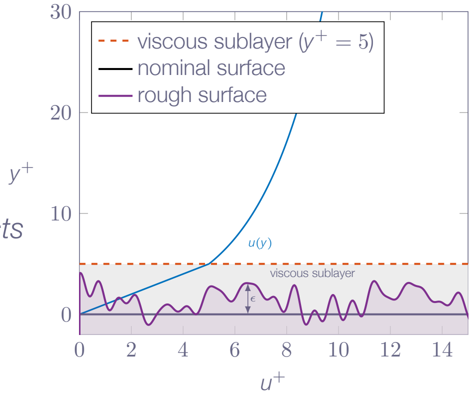
moderate Reynolds number effects

$$\frac{\epsilon u^*}{\nu} > 70$$

fully rough

sublayer totally broken up

independent of Reynolds number



Wall Roughness

Ra	Roughness Average	arithmetic average of the absolute values of the profile heights
Rq	RMS Roughness	root mean square average of the profile heights
Rp	Maximum Profile Peak Height	distance between the highest point of the profile and the mean line
Rpm	Average Maximum Profile Peak Height	average of the successive values of **Rp**
Rv	Maximum Profile Valley Depth	distance between the deepest valley of the profile and the mean line
Rt	Maximum Height of the Profile	vertical distance between the highest and lowest points of the profile
Rz	Average Maximum Height of the Profile	average of the successive values of **Rt**

Wall Roughness

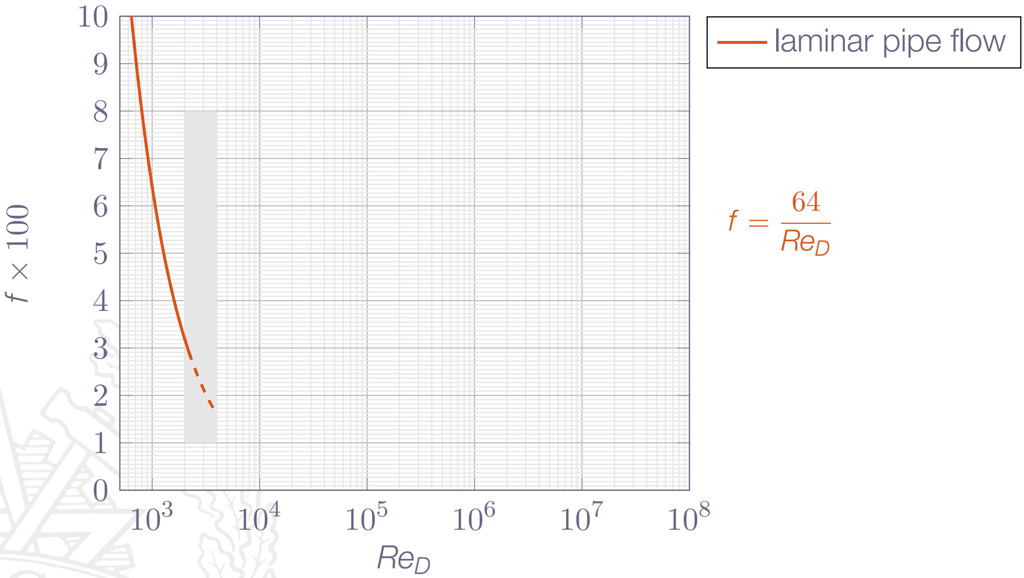
Colebrook (implicit):

$$\frac{1}{\sqrt{f_D}} = -2.0\log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.51}{Re_D\sqrt{f_D}}\right)$$

Haaland (explicit):

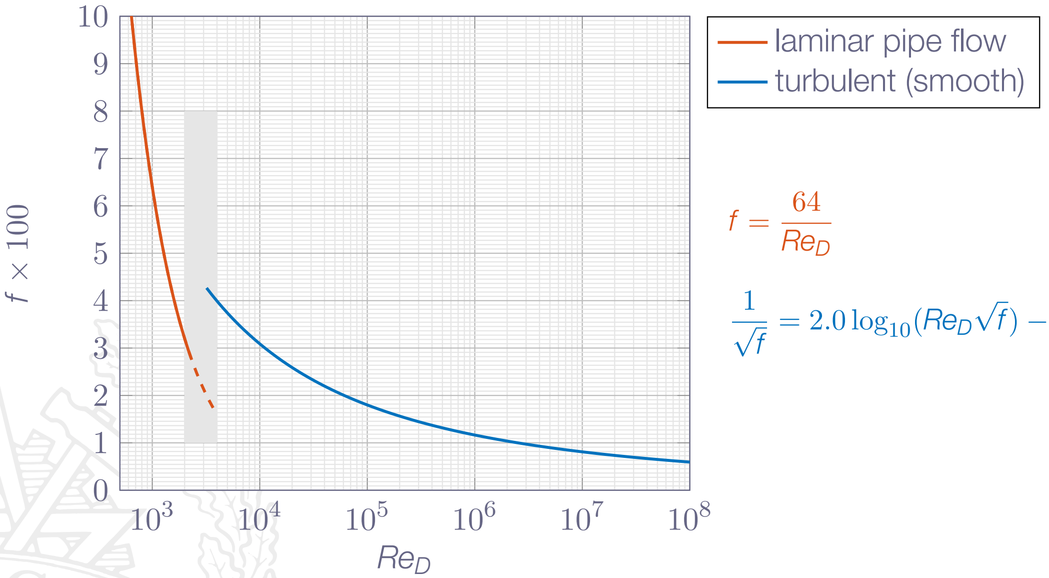
$$\frac{1}{\sqrt{f_D}} = -1.8\log_{10}\left(\frac{6.9}{Re_D} + \left(\frac{\epsilon/D}{3.7}\right)^{1.11}\right)$$

The Moody Chart



$$f = \frac{64}{Re_D}$$

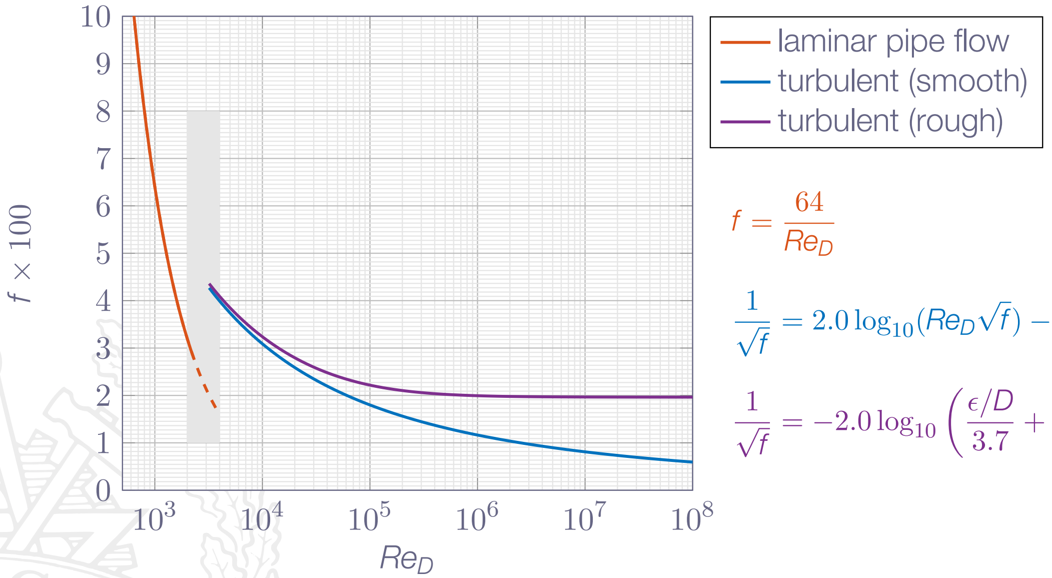
The Moody Chart



$$f = \frac{64}{Re_D}$$

$$\frac{1}{\sqrt{f}} = 2.0 \log_{10}(Re_D\sqrt{f}) - 0.8$$

The Moody Chart

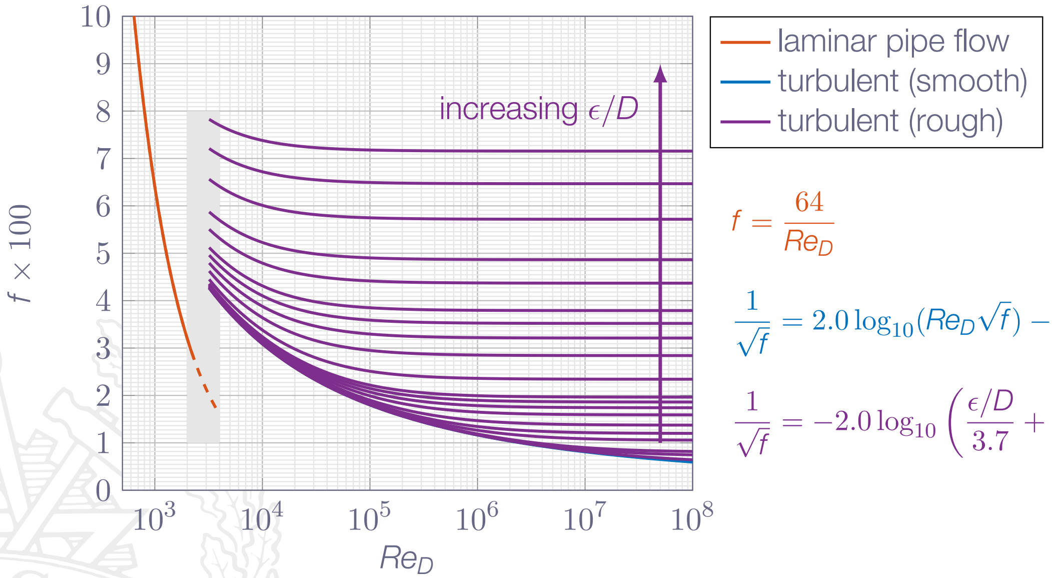


$$f = \frac{64}{Re_D}$$

$$\frac{1}{\sqrt{f}} = 2.0 \log_{10}(Re_D \sqrt{f}) - 0.8$$

$$\frac{1}{\sqrt{f}} = -2.0 \log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.5}{Re_D\sqrt{f}}\right)$$

The Moody Chart



$$f = \frac{64}{Re_D}$$

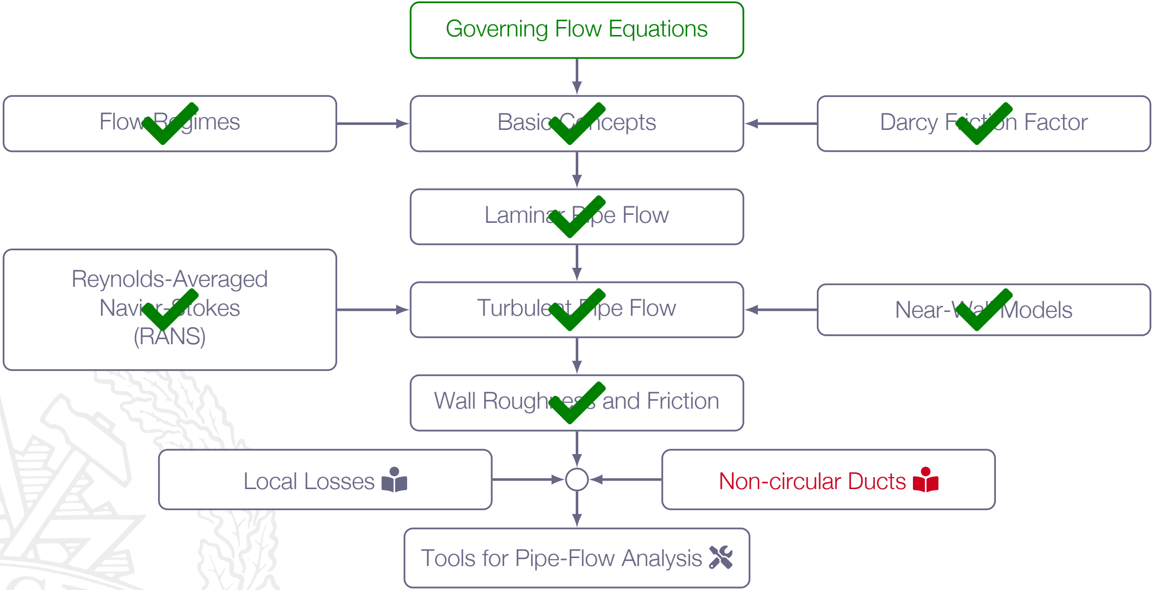
$$\frac{1}{\sqrt{f}} = 2.0\log_{10}(Re_D\sqrt{f}) - 0.8$$

$$\frac{1}{\sqrt{f}} = -2.0\log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.5}{Re_D\sqrt{f}}\right)$$

Wall Roughness

Material	**Condition**	ϵ $[mm]$	**Uncertainty** [%]
Steel	Sheet metal (new)	0.05	$\pm$ 60
	Stainless (new)	0.002	$\pm$ 50
	Commercial (new)	0.046	$\pm$ 30
	Riveted	3.0	$\pm$ 70
	Rusted	2.0	$\pm$ 50
Iron	Cast (new)	0.26	$\pm$ 50
	Wrought (new)	0.046	$\pm$ 20
	Galvanized (new)	0.15	$\pm$ 40
	Asphalted cast	0.12	$\pm$ 50
Brass	Drawn (new)	0.002	$\pm$ 50
Plastic	Drawn tubing	0.0015	$\pm$ 60
Glass	-	smooth	
Concrete	Smoothed	0.04	$\pm$ 60
	Rough	2.0	$\pm$ 50
Rubber	Smoothed	0.01	$\pm$ 60
Wood	Stave	0.5	$\pm$ 40

Roadmap - Viscous Flow in Ducts



Non-circular Ducts

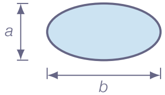
Use the same formulas of the Moody chart but replace the pipe diameter D with the hydraulic diameter D_h

$$D_h = \frac{4A}{\mathcal{P}}$$

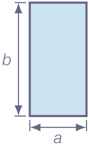
where A is the cross section area and $\mathcal{P}$ is the wetter perimeter

$$\Delta p_f = f_D \frac{L}{D_h} \frac{\rho V^2}{2}, \quad Re_{Dh} = \frac{V D_h}{\nu}, \quad \frac{\epsilon}{D_h}$$

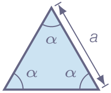
Non-circular Ducts



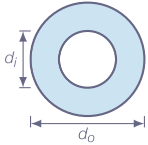
a/b	D_h	C
0.7	$1.17a$	65.0
0.5	$1.30a$	68.0
0.3	$1.44a$	73.0
0.2	$1.50a$	78.0
0.1	$1.55a$	79.0



b/a	D_h	C
1.0	$1.00a$	57.0
1.25	$1.11a$	57.6
2.0	$1.33a$	62.0
3.0	$1.50a$	69.0
4.0	$1.60a$	73.0
5.0	$1.67a$	78.0
8.0	$1.78a$	83.0
10.0	$1.82a$	85.0

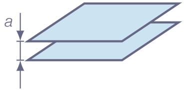


D_h	C
$0.58a$	53.0



d_i/d_o	C
$\frac{d_i}{d_o} = 0.10$	89.2
$\frac{d_i}{d_o} = 0.25$	94.0
$0.5 < \frac{d_i}{d_o} < 1.0$	96.0

$$D_h = d_o - d_i$$



D_h	C
$2.0a$	96.0

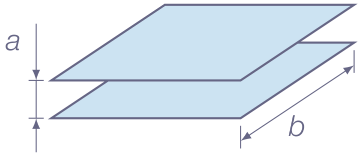
Non-circular Ducts

Laminar flow:

$$f_D = \frac{C}{Re_{D_h}}$$

(for circular pipes: $C = 64$ *and* $D_h = D$*)*

Non-circular Ducts



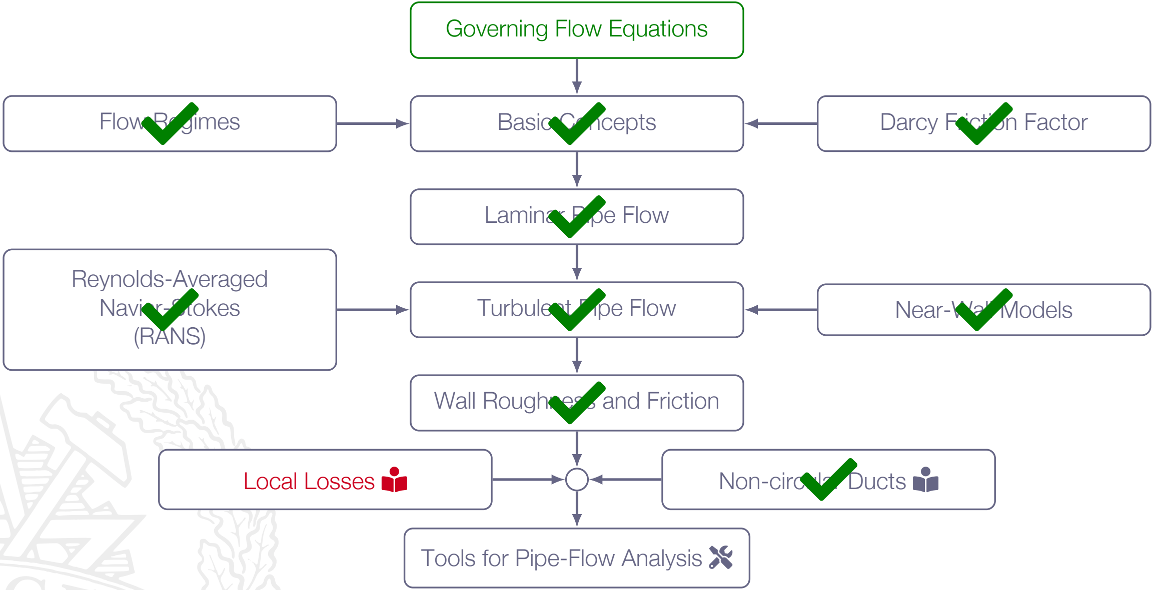
Flow between parallel plates:

vertical distance between plates: a

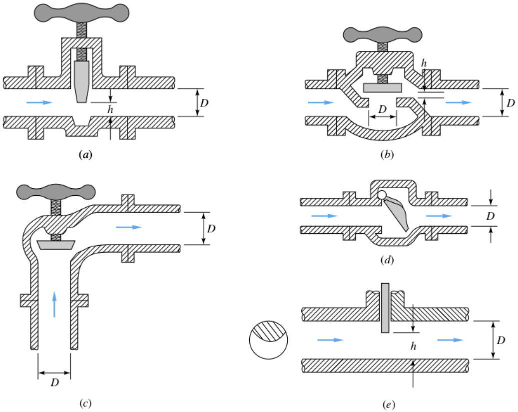
plate width: b

$$D_h = \frac{4A}{\mathcal{P}} = \left.\frac{4ab}{2a+2b}\right|_{b\to\infty} = \frac{4ab}{2b} = 2a$$

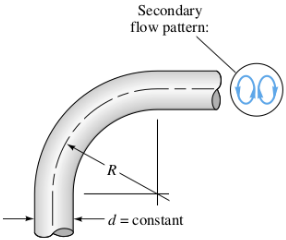
Roadmap - Viscous Flow in Ducts



Local Losses



Local Losses



Swirl generated by:

- Inlets or outlets
- Sudden area changes
- Bends
- Valves
- Gradual expansions or contractions

$$\Delta p_f = K\frac{\rho V^2}{2}$$

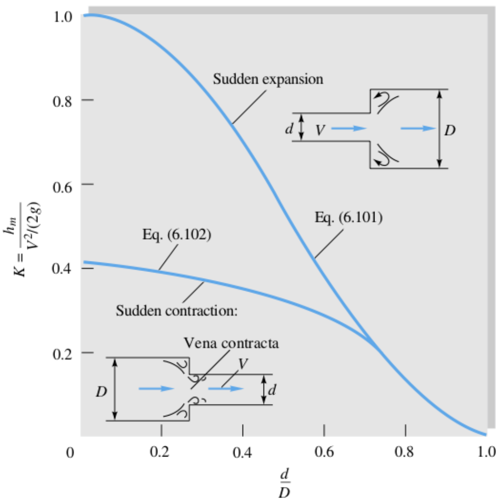
$$\Delta p_{f_{tot}} = \sum_i f_{D_i}\frac{L_i}{D_i}\frac{\rho V_i^2}{2} + \sum_j K_j\frac{\rho V_j^2}{2}$$

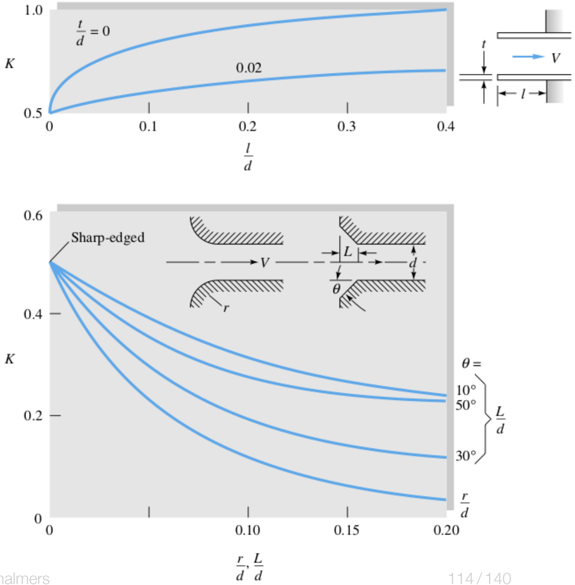
Local Losses

Generated swirl will be damped out by inner friction

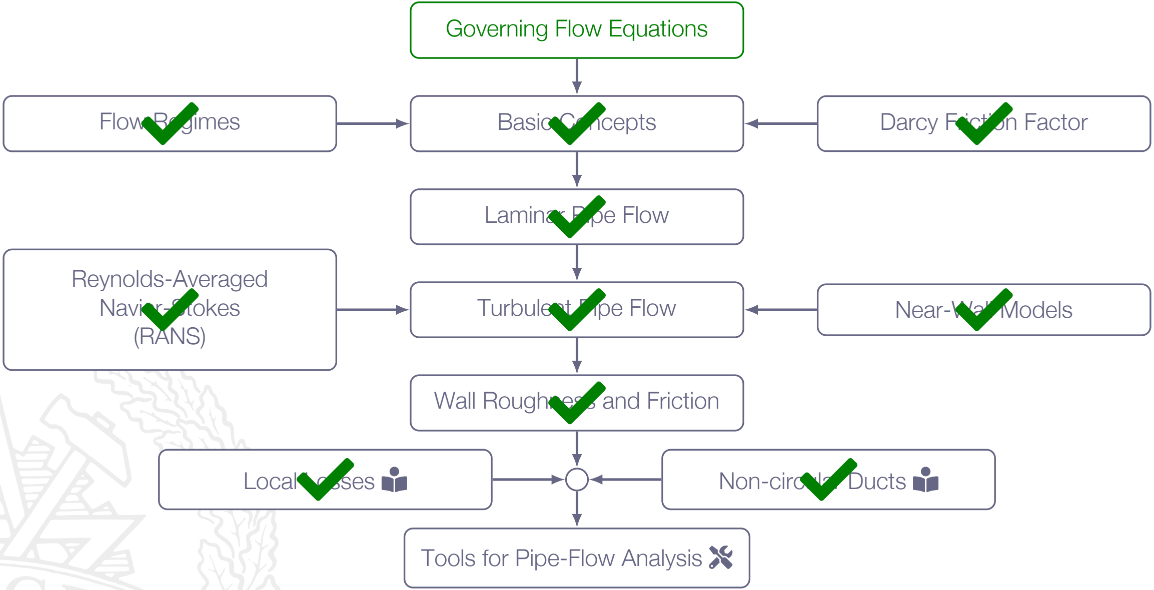
Kinetic energy is converted to internal energy, which results in a pressure loss

Local Losses





Roadmap - Viscous Flow in Ducts



Pipe Flow Example 1: Find Flow Rate (Rough Pipe)

Given data:
Oil with the density $\rho = 950.0 \ kg/m^3$ and viscosity $\nu = 2.0 \times 10^{-5} \ m^2/s$ flows through a $L = 100 \ m$ long pipe with the diameter $D = 0.3 \ m$. The roughness ratio is $\varepsilon/D = 2.0 \times 10^{-4}$ and the head loss is $h_f = 8.0 \ m$.

Assumptions:
steady-state, fully-developed, turbulent, incompressible pipe flow

Task:
Find the average flow velocity (V_{av}) and the flow rate (Q)

Pipe Flow Example 1: Find Flow Rate (Rough Pipe)

We are given a measure of the head loss (h_f) for the pipe

The definition of the **Darcy friction factor** gives a relation between head loss (h_f) and the average velocity (V_{av})

$$h_f = f \frac{V_{av}^2}{2g} \frac{L}{D}$$

To be able to calculate the average velocity (V_{av}), we need the friction factor (f)

Pipe Flow Example 1: Find Flow Rate (Rough Pipe)

The flow in the pipe is assumed to be turbulent and fully developed

For turbulent flows in rough pipes, **Colebrook's formula** gives a relation between friction factor (f) and average flow velocity (V_{av})

$$\frac{1}{\sqrt{f}} = -2.0 \log \left(\frac{\varepsilon / D}{3.7} + \frac{2.51}{Re_D \sqrt{f}} \right)$$

Use an iterative approach to find the friction factor (f) using Colebrook's relation and

$$Re_D = \frac{V_{av} D}{\nu}, \text{ where } V_{av} = \sqrt{\frac{2 h_f g D}{f L}}$$

Pipe Flow Example 1: Find Flow Rate (Rough Pipe)

```
import numpy as np

def GetVelocity(hf,f,D,L):
  return np.sqrt((2.*9.81*hf*D)/(f*L))

def GetReynoldsNumber(D,V,nu):
  return D*V/nu

def Colebrook(f,D,nu,eps,V):
  # Colebrook friction factor
  return -2.0*np.log10(((eps/D)/3.7)+(2.51/(GetReynoldsNumber(D,V,nu)*np.
    sqrt(f))))-1./np.sqrt(f)

def GetFlowRate(V,D):
  return (V*np.pi*D**2)/4.
```

Pipe Flow Example 1: Find Flow Rate (Rough Pipe)

```
nu    = 2.0e-5   # fluid viscosity [m^2/s]
D     = 3.0e-1   # pipe diameter [m]
L     = 1.0e2    # pipe length [m]
hf    = 8.0      # head loss [m]
eps   = 2.0e-4*D # surface roughness [m]
f     = 1.5e-2   # friction factor (inital guess)

# Newton-Raphson solver
f_old = 1.0e3
df    = 1.0e-6
while np.abs(f-f_old)>1.0e-6*f:
  f_old = f
  V   = GetVelocity(hf,f,D,L)
  ff  = Colebrook(f,D,nu,eps,V)
  dff = (Colebrook(f+df,D,nu,eps,V)-Colebrook(f-df,D,nu,eps,V))/(2.*df)
  f   = f_old-(ff/dff)
```

Pipe Flow Example 1: Find Flow Rate (Rough Pipe)

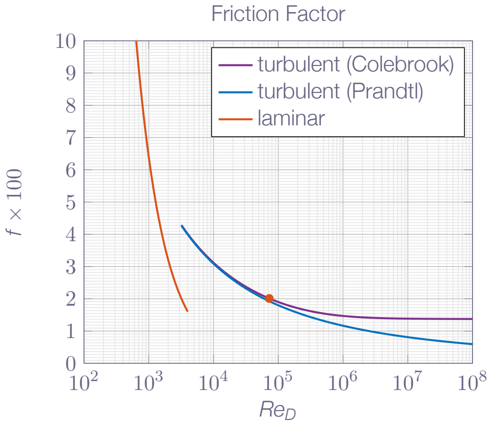
iteration	f	$\lvert f - f_{old} \rvert / f$
1	1.876228e-02	2.005234e-01
2	1.992732e-02	5.846445e-02
3	2.009150e-02	8.171902e-03
4	2.010758e-02	7.998039e-04
5	2.010907e-02	7.373985e-05
6	2.010920e-02	6.757868e-06
7	2.010921e-02	6.189787e-07

Pipe Flow Example 1: Find Flow Rate (Rough Pipe)

Result:

Average flow velocity	V_{av}	4.84	m/s
Flow rate	Q	0.342	m^3/s
Reynolds number	Re_D	72585	
Friction factor	f	0.0201	

IFLOW



Pipe Flow Example 1: Find Flow Rate (Rough Pipe)

Note! this specific case could actually have been solved without iterating since

$$\frac{1}{\sqrt{f}} = -2.0\log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{Re_D\sqrt{f}}\right)$$

$$Re_D = \frac{V_{av}D}{\nu}, \text{ where } V_{av} = \sqrt{\frac{2h_f gD}{fL}} \Rightarrow Re_D\sqrt{f} = \frac{\sqrt{h_f gD^3}}{\nu\sqrt{L}}$$

and thus

$$\frac{1}{\sqrt{f}} = -2.0\log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51\nu\sqrt{L}}{\sqrt{h_f gD^3}}\right) \Rightarrow f = 0.0201$$

Pipe Flow Example 2: Find Pipe Diameter (Rough Pipe)

Given data:
Oil with the density $\rho = 950.0\ kg/m^3$ and viscosity $\nu = 2.0 \times 10^{-5}\ m^2/s$ flows through a $L = 100\ m$ long pipe at a flow rate of $Q = 0.342\ m^3/s$. The surface roughness is $\varepsilon = 0.06\ mm$ and the head loss is $h_f = 8.0\ m$.

Assumptions:
steady-state, fully-developed, turbulent, incompressible pipe flow

Task:
Find the pipe diameter (D)

Pipe Flow Example 2: Find Pipe Diameter (Rough Pipe)

We are given a measure of the head loss (h_f) for the pipe

The definition of the **Darcy friction factor** gives a relation between head loss (h_f) and the pipe diameter (D)

$$h_f = f\frac{V_{av}^2}{2g}\frac{L}{D} = \left\{Q = V_{av}\frac{\pi D^2}{4}\right\} = f\frac{8Q^2L}{\pi^2 gD^5}$$

To be able to calculate the pipe diameter (D), we need the friction factor (f)

Pipe Flow Example 2: Find Pipe Diameter (Rough Pipe)

The flow in the pipe is assumed to be turbulent and fully developed

For turbulent flows in rough pipes, **Colebrook's formula** gives a relation between friction factor (f) and pipe diameter (D)

$$\frac{1}{\sqrt{f}} = -2.0 \log \left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{Re_D \sqrt{f}} \right)$$

Use an iterative approach to find the friction factor (f) using Colebrook's relation and

$$Re_D = \frac{V_{av} D}{\nu}, \text{ where } V_{av} = \frac{4Q}{\pi D^2}$$

Pipe Flow Example 2: Find Pipe Diameter (Rough Pipe)

```python
import numpy as np

def GetDiameter(hf,f,L,Q):
  return ((8.*f*Q**2*L)/(9.81*np.pi**2*hf))**(1./5.)

def GetReynoldsNumber(D,V,nu):
  return D*V/nu

def Colebrook(f,D,nu,eps,V):
  # Colebrook friction factor
  return -2.0*np.log10(((eps/D)/3.7)+(2.51/(GetReynoldsNumber(D,V,nu)*np.
    sqrt(f))))-1./np.sqrt(f)

def GetVelocity(Q,D):
  return 4.*Q/(np.pi*D**2)
```

Pipe Flow Example 2: Find Pipe Diameter (Rough Pipe)

```
nu    = 2.0e-5    # fluid viscosity [m^2/s]
L     = 1.0e2     # pipe length [m]
hf    = 8.0       # head loss [m]
eps   = 6.0e-5    # surface roughness [m]
Q     = 3.42e-1   # flow rate [m^3/s]
f     = 1.5e-2    # friction factor (inital guess)

# Newton-Raphson solver
f_old = 1.0e3
df    = 1.0e-6
while np.abs(f-f_old)>1.0e-6*f:
  f_old = f
  D   = GetDiameter(hf,f,L,Q)
  V   = GetVelocity(Q,D)
  ff  = Colebrook(f,D,nu,eps,V)
  dff = (Colebrook(f+df,D,nu,eps,V)-Colebrook(f-df,D,nu,eps,V))/(2.*df)
  f   = f_old-(ff/dff)
```

Pipe Flow Example 2: Find Pipe Diameter (Rough Pipe)

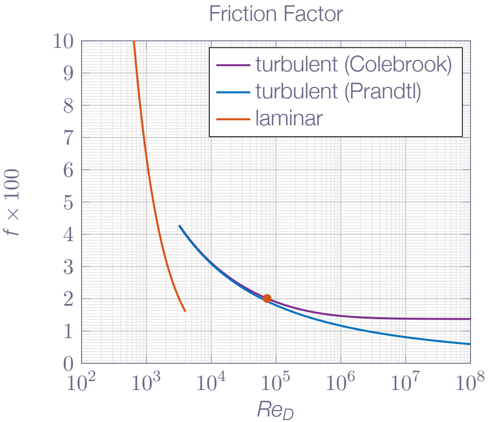
iteration	f	$\|f - f_{old}\|/f$
1	1.900357e-02	2.106745e-01
2	2.003493e-02	5.147839e-02
3	2.010728e-02	3.597900e-03
4	2.010953e-02	1.122501e-04
5	2.010960e-02	3.210179e-06
6	2.010960e-02	9.154651e-08

Pipe Flow Example 2: Find Pipe Diameter (Rough Pipe)

Result:

Pipe diameter	D	0.299	m
Average flow velocity	V_{av}	4.84	m/s
Reynolds number	Re_D	72579	
Friction factor	f	0.0201	

IFLOW



Pipe Flow Example 3: Find Pipe Diameter (Smooth Pipe)

Given data:

A smooth plastic pipe is to be designed to carry $Q = 0.25 \ m^3/s$ of water at $20°C$ through a $L = 300 \ m$ horizontal pipe with the exit at atmospheric pressure. The pressure drop is approximated to be $\Delta p = 1.7 \ MPa$.

Water @ $20°C$: $\rho = 998 \ kg/m^3$ and $\mu = 0.001 \ kg/(ms)$ ($\nu = 1.002 \times 10^{-6} \ m^2/s$)

Assumptions:

steady-state, fully-developed, turbulent, incompressible pipe flow

Task:

Find a suitable pipe diameter (D)

Pipe Flow Example 3: Find Pipe Diameter (Smooth Pipe)

The energy equation on integral form gives us a relation between the pressure drop Δp and the pipe head loss h_f

$$\left(\frac{p}{\rho g}+\frac{\alpha V^2}{2g}+z\right)_1=\left(\frac{p}{\rho g}+\frac{\alpha V^2}{2g}+z\right)_2+h_t-h_p+h_f$$

1. Steady-state, incompressible flow ($Q_1=Q_2=Q$) in a constant-diameter pipe ($D_1=D_2=D$) $\Rightarrow V_1=V_2=V_{av}$
2. Fully-developed turbulent pipe flow with constant average velocity $\Rightarrow$ $\alpha_1=\alpha_2=\alpha$
3. No information about elevation change is given so we will assume that $z_1=z_2=z$
4. There are no turbines or pumps in the pipe $\Rightarrow h_t=h_p=0.$

$$\frac{p_1-p_2}{\rho g}=\frac{\Delta p}{\rho g}=h_f$$

Pipe Flow Example 3: Find Pipe Diameter (Smooth Pipe)

Again, we will use the definition of the **Darcy friction factor** (f) to get a relation between the losses and the pipe diameter

$$h_f = f\frac{V^2}{2g}\frac{L}{D} \Rightarrow \left\{h_f = \frac{\Delta p}{\rho g}, Q = V_{av}\frac{\pi D^2}{4}\right\} \Rightarrow f = \frac{\pi^2 \Delta p}{8Q^2 L\rho}D^5$$

To be able to calculate the pipe diameter (D), we need the friction factor (f)

Pipe Flow Example 3: Find Pipe Diameter (Smooth Pipe)

The flow in the pipe is assumed to be turbulent and fully developed

For turbulent flows in smooth pipes, **Prandtl's formula** gives a relation between friction factor (f) and pipe diameter (D)

$$\frac{1}{\sqrt{f}} = 2.0 \log\left(Re_D \sqrt{f}\right) - 0.8$$

Use an iterative approach to find the friction factor (f) using Prandtl's relation and

$$Re_D = \frac{V_{av} D}{\nu}, \text{ where } V_{av} = \frac{4Q}{\pi D^2}$$

Pipe Flow Example 3: Find Pipe Diameter (Smooth Pipe)

```python
import numpy as np

def GetDiameter(Dp,rho,f,L,Q):
  return ((8.*f*Q**2*L*rho)/(np.pi**2*Dp))**(1./5.)

def GetReynoldsNumber(D,V,nu):
  return D*V/nu

def Prandtl(f,D,nu,V):
  # Prandtl friction factor
  return 2.0*np.log10(GetReynoldsNumber(D,V,nu)*np.sqrt(f))-0.8-(1./np.
    sqrt(f));

def GetVelocity(Q,D):
  return 4.*Q/(np.pi*D**2)
```

Pipe Flow Example 3: Find Pipe Diameter (Smooth Pipe)

```
rho    = 998.0      # fluid density [kg/m^3]
mu     = 1.0e-3     # fluid viscosity [kg/ms]
nu     = mu/rho     # fluid viscosity [m^2/s]
L      = 3.0e2      # pipe length [m]
Dp     = 1.7e6      # pressure drop [Pa]
Q      = 2.5e-1     # flow rate [m^3/s]
f      = 1.5e-2     # friction factor (inital guess)

# Newton-Raphson solver
f_old = 1.0e3
df    = 1.0e-6
while np.abs(f-f_old)>1.0e-6*f:
  f_old = f
  D   = GetDiameter(Dp,rho,f,L,Q)
  V   = GetVelocity(Q,D)
  ff  = Prandtl(f,D,nu,V)
  dff = (Prandtl(f+df,D,nu,V)-Prandtl(f-df,D,nu,V))/(2.*df)
  f   = f_old-(ff/dff)
```

Pipe Flow Example 3: Find Pipe Diameter (Smooth Pipe)

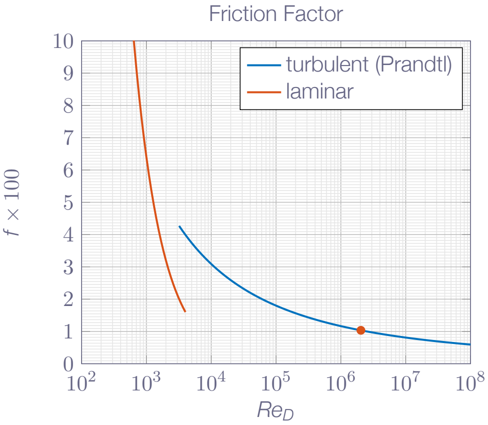
iteration	f	$\lvert f - f_{old} \rvert / f$
1	9.140310e-03	6.410822e-01
2	1.020433e-02	1.042711e-01
3	1.033763e-02	1.289507e-02
4	1.034327e-02	5.449435e-04
5	1.034345e-02	1.791398e-05
6	1.034346e-02	5.818538e-07

Pipe Flow Example 3: Find Pipe Diameter (Smooth Pipe)

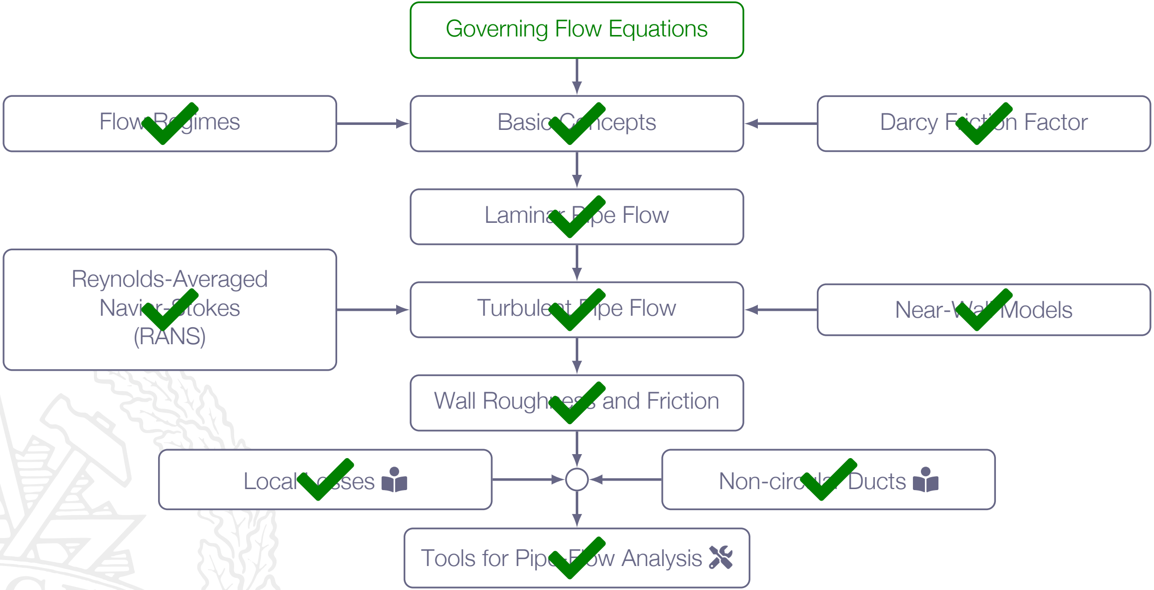
Result:

Pipe diameter	D	0.156	m
Average flow velocity	V_{av}	13.1	m/s
Reynolds number	Re_D	2036821	
Friction factor	f	0.01034	

IFLOW



Roadmap - Viscous Flow in Ducts



On-Demand Hyperloop-Style Water Delivery