Fluid Mechanics - MTF053

Lecture Notes

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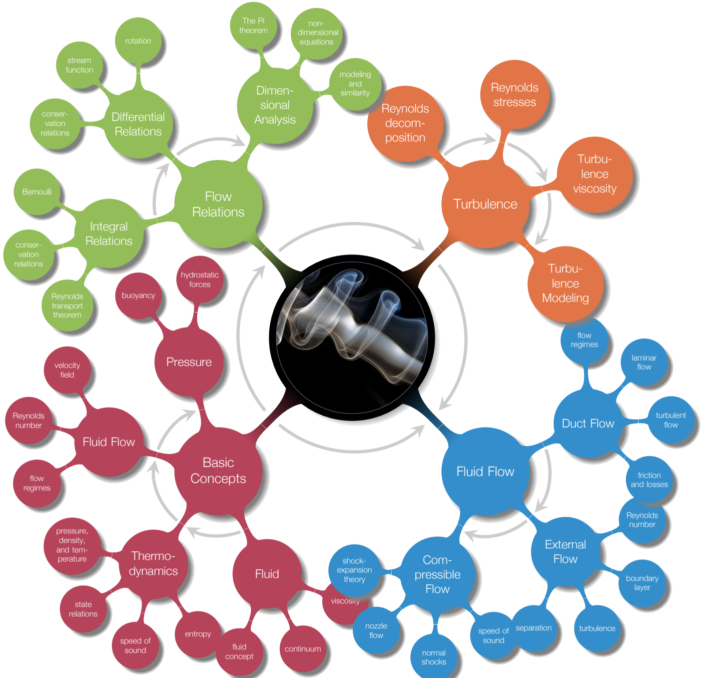
Overview



Flow Relations
Differential Relations
rotation
stream function
conser-vation relations
Integral Relations
Bernoulli
conser-vation relations
Reynolds transport theorem
Dimen-sional Analysis
The Pi theorem
non-dimensional equations
modeling and similarity
Turbulence
Reynolds decom-position
Reynolds stresses
Turbu-lence viscosity
Turbu-lence Modeling
Fluid Flow
Duct Flow
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laminar flow
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friction and losses
External Flow
Reynolds number
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nozzle flow
normal shocks
speed of sound
Basic Concepts
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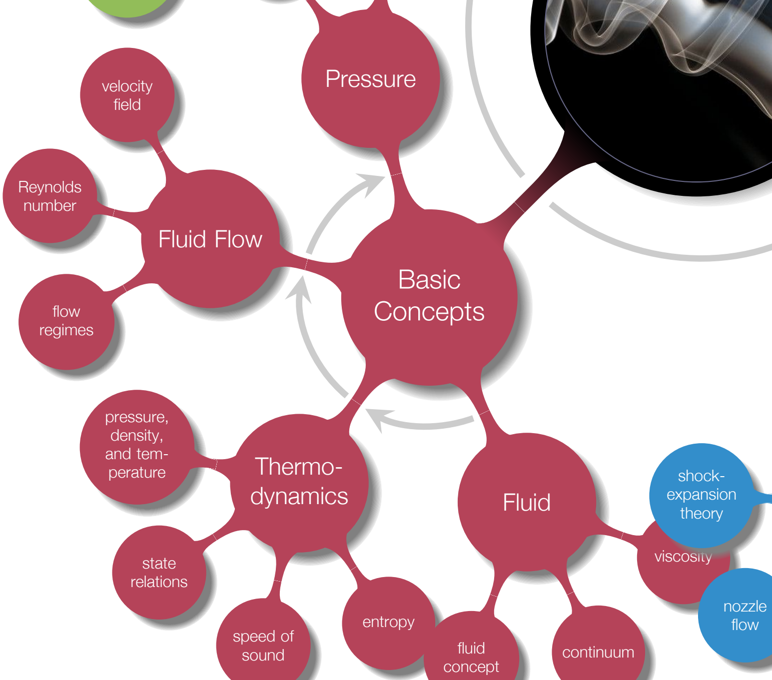
buoyancy
hydrostatic forces
Fluid Flow
velocity field
Reynolds number
flow regimes
Thermo-dynamics
pressure, density, and tem-perature
state relations
speed of sound
entropy
Fluid
fluid concept
continuum
viscosity

Chapter 1 - Introduction

Overview



Pressure
velocity field
Reynolds number
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flow regimes
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pressure, density, and tem-perature
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Fluid
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shock-expansion theory
nozzle flow

Learning Outcomes

1 **Explain** the difference between a fluid and a solid in terms of forces and deformation

2 **Understand** and be able to explain the viscosity concept

3 **Define** the Reynolds number

5 **Explain** the difference between Lagrangian and Eulerian frame of reference and know when to use which approach

7 **Explain** the concepts: streamline, pathline and streakline

8 **Understand** and be able to **explain** the concept shear stress

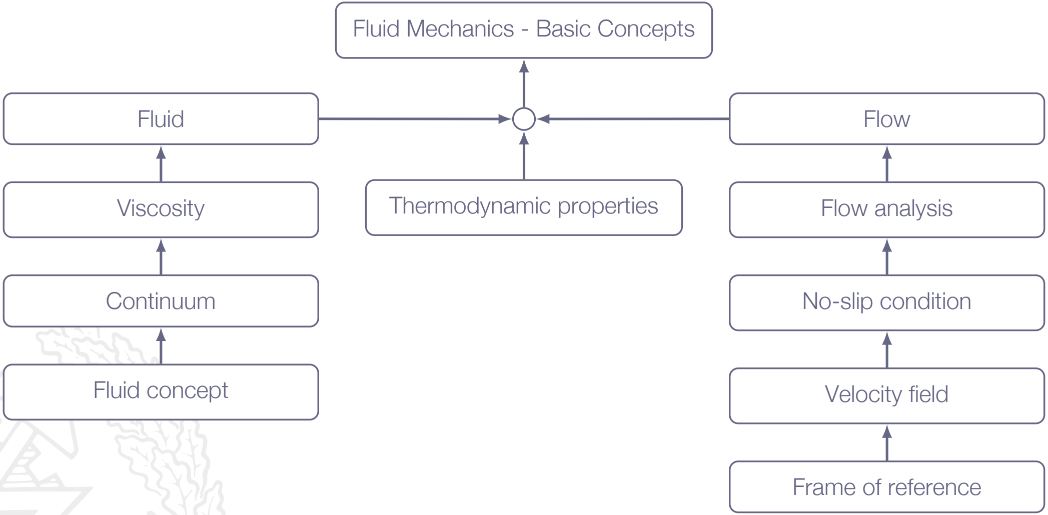
9 **Explain** how to do a force balance for fluid element (forces and pressure gradients)

10 **Understand and explain** buoyancy and cavitation

16 **Understand** and **explain** the concept Newtonian fluid

in this lecture we will find out what a fluid flow is

Roadmap - Introduction to Fluid Mechanics



Fluid Mechanics

"Fluid mechanics is the branch of physics concerned with the mechanics of fluids (liquids, gases, and plasmas) and the forces on them. It has applications in a wide range of disciplines, including mechanical, civil, chemical and biomedical engineering, geophysics, oceanography, meteorology, astrophysics, and biology."

Wikipedia

Fluid Flows in Your Daily Life

"When you think about it, almost everything on this planet either is a fluid or moves within or near a fluid"

Frank M. White

Fluid Flows in Your Daily Life

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Governing Equations

Continuity
Momentum
Energy

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) = \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) = \rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right) = \rho g_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

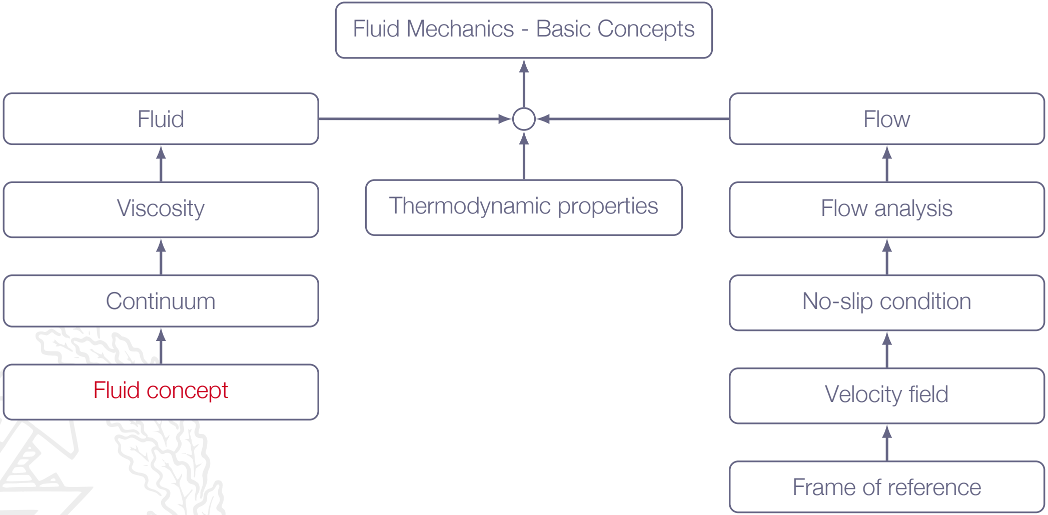
$$\rho C_v \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} \right) = p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \Phi$$

Fluid Flow Applications

- Analytical solutions limited to very specific simplified cases
- Complex geometries and flows leads to the need for experiments and Computational Fluid Dynamics (CFD)
- Chief obstacles to a general theory:
  - Geometry
  - Viscosity
  - Non-linearity
  - Turbulence

Understanding the basic principles is a key factor for a correct analysis

Roadmap - Introduction to Fluid Mechanics

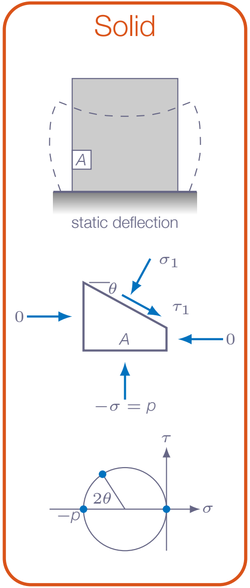


The Concept of a Fluid

"In physics, a fluid is a substance that continually deforms (flows) under an applied shear stress, or external force. Fluids are a phase of matter and include liquids, gases and plasmas. They are substances with zero shear modulus, or, in simpler terms, substances which cannot resist any shear force applied to them."

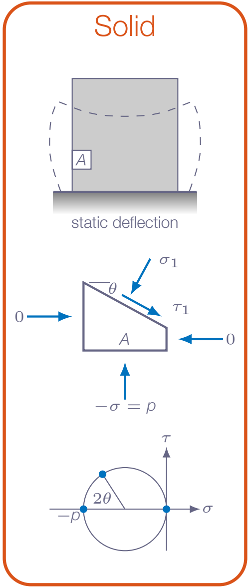
Wikipedia

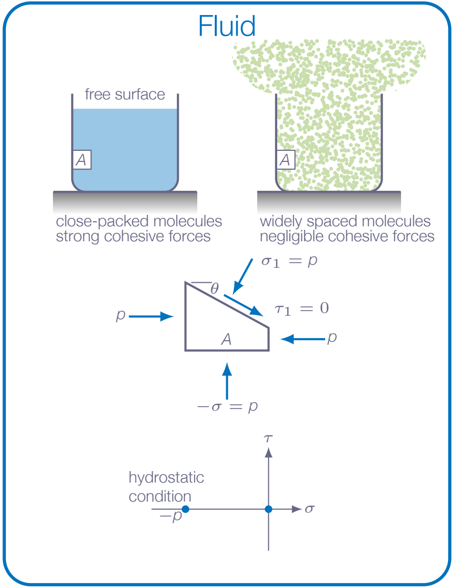
The Concept of a Fluid



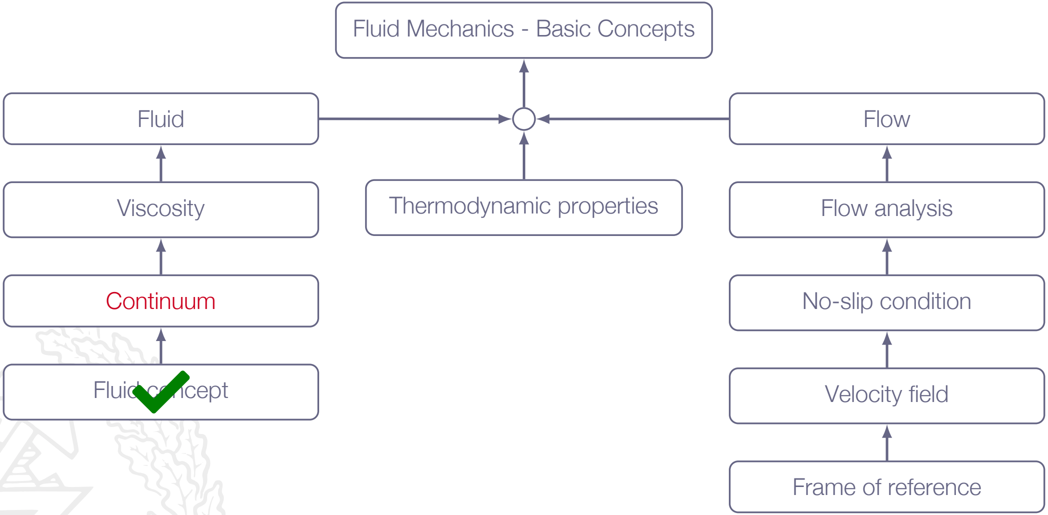
The Concept of a Fluid

"A solid can resist a shear stress by a static deflection; a fluid cannot"





Roadmap - Introduction to Fluid Mechanics

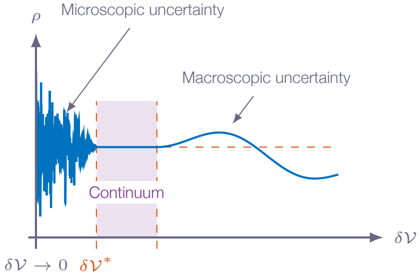


The Fluid as a Continuum

- Fluid density is essentially a point function
- fluid properties can be thought of as varying continually in space
- Volume large enough such that the number of molecules within the volume is constant
- Volume small enough not to introduce macroscopic fluctuations

$$\rho = \lim_{\delta\mathcal{V} \to \delta\mathcal{V}^*} \frac{\delta m}{\delta\mathcal{V}}$$

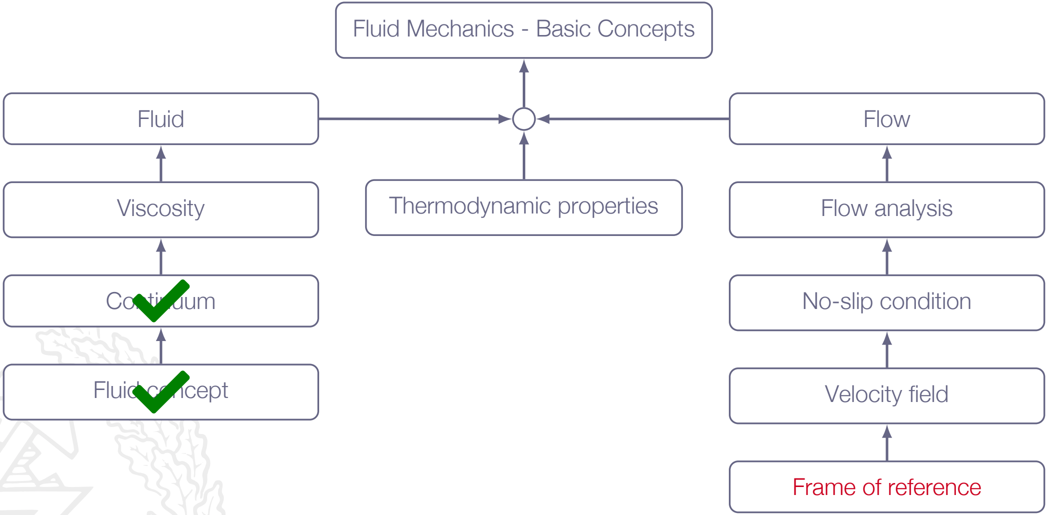
standard air: $\delta\mathcal{V}^* \approx 10^{-9} mm^3 \Rightarrow \sim 3 \times 10^7$ molecules



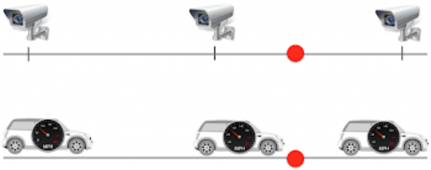
The Fluid as a Continuum

- Flow properties varies smoothly
- Differential calculus can be used

Roadmap - Introduction to Fluid Mechanics

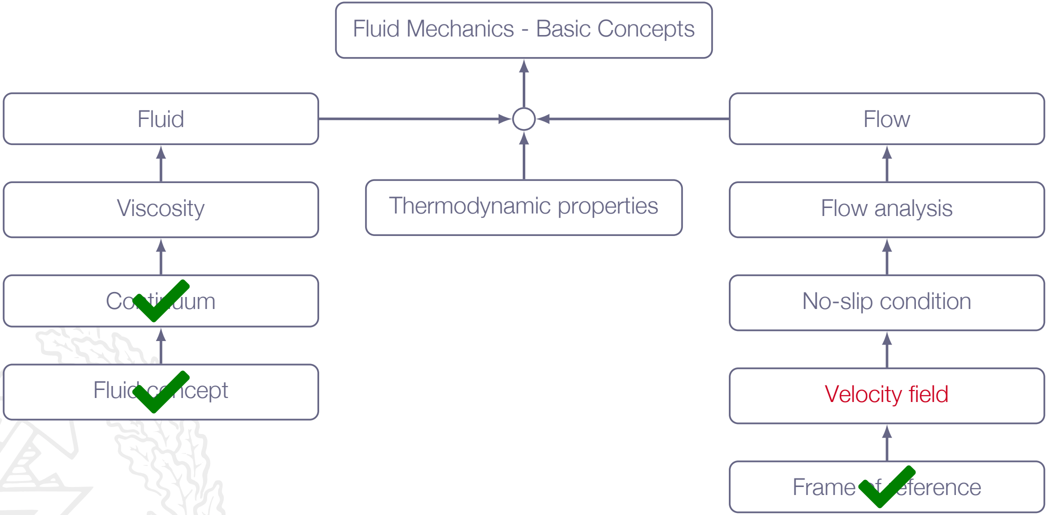


Frame of Reference



- Eulerian
  - fluid properties as function of position and time
  - most often used in fluid mechanics
- Lagrangian
  - follows a system in time and space
  - can be used in fluid mechanics
  - most often used in solid mechanics

Roadmap - Introduction to Fluid Mechanics



Properties of the Velocity Field

- The fluid velocity is a function of position and time
- Three components u, v, and w (one in each spatial direction)

$$\mathbf{V}(x,y,z,t) = u(x,y,z,t)\mathbf{e}_x + v(x,y,z,t)\mathbf{e}_y + w(x,y,z,t)\mathbf{e}_z$$

Properties of the Velocity Field

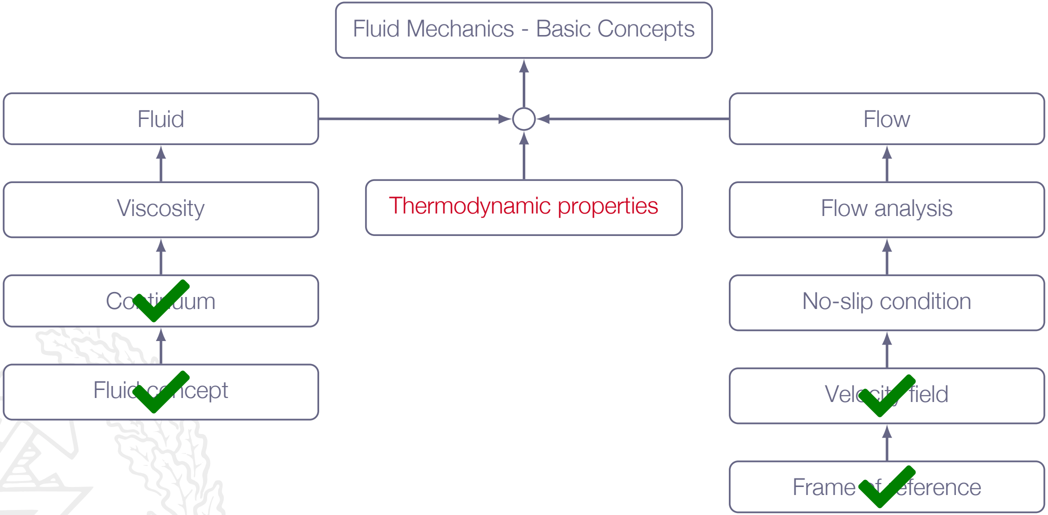
Acceleration:

$$\mathbf{V}(x,y,z,t) = u(x,y,z,t)\mathbf{e}_x + v(x,y,z,t)\mathbf{e}_y + w(x,y,z,t)\mathbf{e}_z$$

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + \left(\frac{\partial \mathbf{V}}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial \mathbf{V}}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right) + \left(\frac{\partial \mathbf{V}}{\partial z}\right)\left(\frac{\partial z}{\partial t}\right)$$

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$$

Roadmap - Introduction to Fluid Mechanics



Thermodynamic Properties

- Thermodynamic properties describe the state of a system, i.e., a collection of matter of fixed identity which interacts with its surroundings
- In this course, the system will be a small fluid element, and all properties will be assumed to be continuum properties of the flow field

Thermodynamic Properties

- Pressure p Pa
- Density ρ kg/m^3
- Temperature T K

most common properties

Thermodynamic Properties

- Pressure p Pa
- Density ρ kg/m^3
- Temperature T K

most common properties

- Internal energy $\hat{u}$
- Enthalpy $h = \hat{u} + p/\rho$
- Entropy s
- Specific heats C_p and C_v

work, heat, and energy balances

Thermodynamic Properties

- Pressure p Pa
- Density ρ kg/m^3
- Temperature T K

most common properties

- Internal energy $\hat{u}$
- Enthalpy $h = \hat{u} + p/\rho$
- Entropy s
- Specific heats C_p and C_v

work, heat, and energy balances

- Viscosity μ
- Thermal conductivity k

friction and heat conduction

Thermodynamic Properties

- For a single-phase substance, two basic properties are sufficient to get the values of all others

$$\rho = \rho(p, T),\ h = h(p, T),\ \mu = \mu(p, T)$$

- In the following it will be assumed that all thermodynamic properties exists as point functions in a flowing fluid
  - large enough number of molecules
  - any changes are slower than the flow time scale $\Rightarrow$ equilibrium

Thermodynamic Properties

Pressure: $p[Pa]$

- the compression stress at a point in a static fluid
- a fluid flow is often driven by pressure gradients
- if the pressure drops below the vapor pressure in a liquid, vapor bubbles will form

Temperature: $T[K]$

- related to internal energy
- large temperature differences $\Rightarrow$ heat transfer may be important

Density: $\rho[kg/m^3]$

- mass per unit volume
- nearly constant in liquids (incompressible) - for water, the density increases about one percent for a pressure increase by a factor of 220
- not constant for gases

$$\rho = \frac{p}{RT}$$

Potential and Kinetic Energies

The total stored energy per unit mass:

$$e = \hat{u} + \frac{1}{2}V^2 + gz$$

- the internal energy is a function of temperature
- the potential and kinetic energies are kinematic quantities

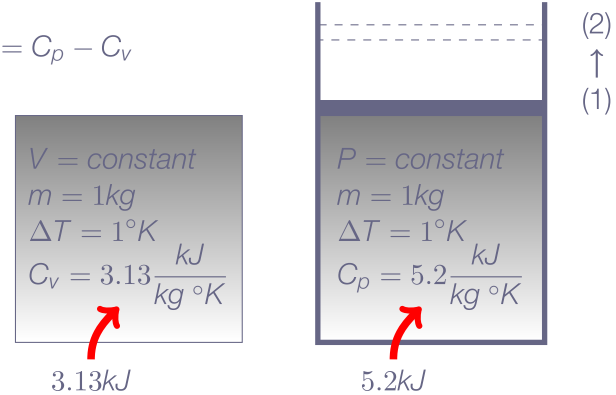
State Relations for Gases

The perfect gas law:

$$p = \rho R T$$

where R is the gas constant

$$R = C_p - C_v$$



State Relations for Gases

The ideal gas law requires: $\hat{u} = \hat{u}(T)$ and thus

specific heat (constant volume):

$$C_v = \left(\frac{\partial \hat{u}}{\partial T}\right)_\rho = \frac{d\hat{u}}{dT} = C_v(T)$$

State Relations for Gases

specific heat (constant pressure):

$$h = \hat{u} + \frac{p}{\rho} = \hat{u} + RT = h(T)$$

$$C_p = \left(\frac{\partial h}{\partial T}\right)_p = \frac{dh}{dT} = C_p(T)$$

ratio of specific heats:

$$\gamma = \frac{C_p}{C_v} \geq 1$$

State Relations for Gases

$$C_v = \frac{R}{\gamma - 1}$$

$$C_p = \frac{\gamma R}{\gamma - 1}$$

Speed of Sound

Speed of sound plays an important role when compressible effects are important (Chapter 9)

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

$$\tau_s = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_s \Rightarrow a = \sqrt{\frac{1}{\rho \tau_s}}$$

where τ_s is the fluid compressibility

for an ideal gas:

$$\boxed{a = \sqrt{\gamma R T}}$$

Vapor Pressure

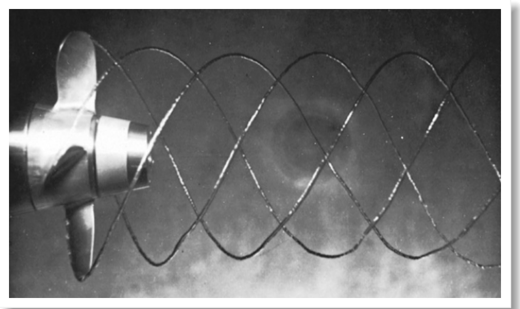
”the pressure at which a liquid boils and is in equilibrium with its own vapor”

Vapor pressure for water:

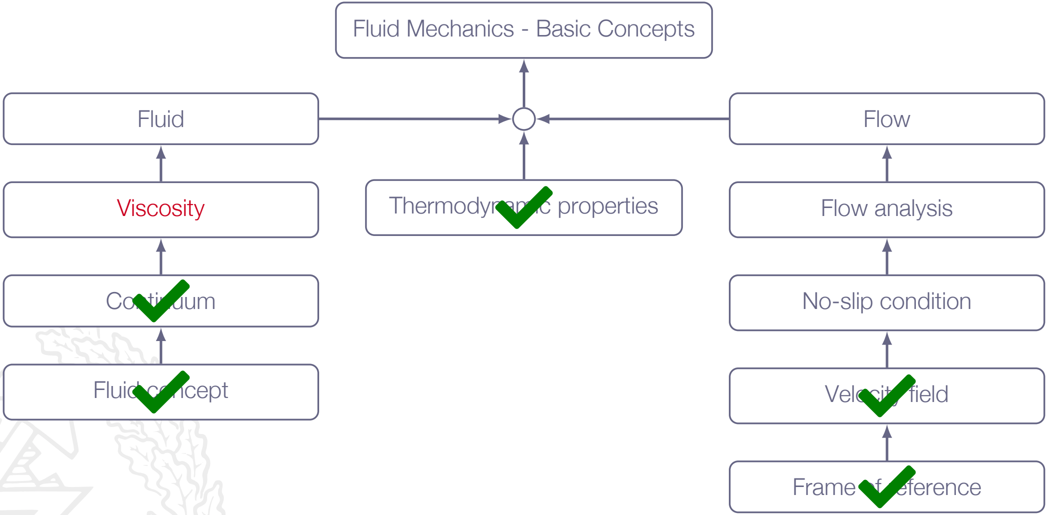
T [°C]	vapor pressure [Pa]
20	2340
100	101300

Vapor Pressure

- If the pressure in a liquid gets lower than the vapor pressure, vapor bubbles will appear in the liquid
- If the pressure drops below the vapor pressure due to the flow itself we get cavitation



Roadmap - Introduction to Fluid Mechanics



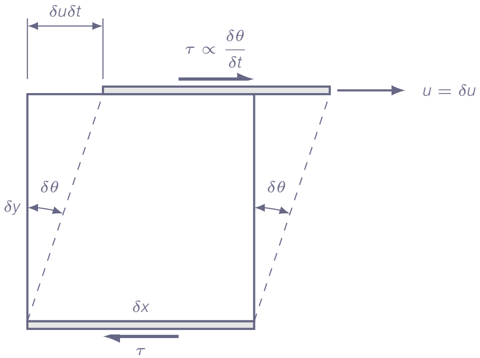
Viscosity

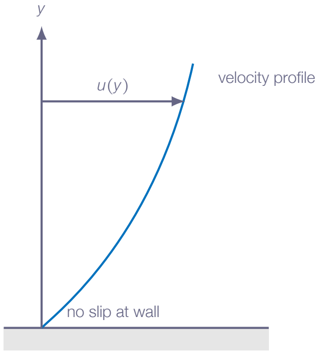
Viscosity

”relates the local stresses in a moving fluid to the strain rate of the fluid element”

”a quantitative measure of the fluid’s resistance to flow”

Viscosity





$$\tau \propto \frac{\delta\theta}{\delta t}, \ \tan\delta\theta = \frac{\delta u \delta t}{\delta y}$$

Viscosity

for infinitesimal changes:

$$\frac{d\theta}{dt} = \frac{du}{dy}$$

from before we know that $\tau \propto \dfrac{\delta\theta}{\delta t}$ and thus $\tau \propto \dfrac{d\theta}{dt}$

For newtonian fluids:

$$\boxed{\tau = \mu\frac{d\theta}{dt} = \mu\frac{du}{dy}}$$

where μ is the fluid viscosity

Viscosity

- Liquids have high viscosity that decreases with temperature
  - intermolecular forces decreases with temperature
- Gases have low viscosity that increases with temperature
  - increased temperature means increased molecular movement

Viscosity

Fluid	μ ($kg\,m^{-1}\,s^{-1}$)	ρ ($kg\,m^{-3}$)	ν ($m^2\,s^{-1}$)
Hydrogen	8.8 E-06	8.400 E-02	1.05 E-04
Air	1.8 E-05	1.200 E+00	1.51 E-05
Gasoline	2.9 E-04	6.800 E+02	4.22 E-07
Water	1.0 E-03	9.980 E+02	1.01 E-06
Mercury	1.5 E-03	1.358 E+04	1.16 E-07
SAE-30 Oil	2.9 E-02	8.910 E+02	3.25 E-04
Glycerin	1.5 E+00	1.264 E+03	1.18 E-03

Note! there are two different viscosities in the table (dynamic viscosity μ and kinematic viscosity $\nu = \mu/\rho$)

Viscosity

Inviscid flows: flows where viscous forces are negligible

Viscous flows: flows where viscous forces are important

Reynolds number

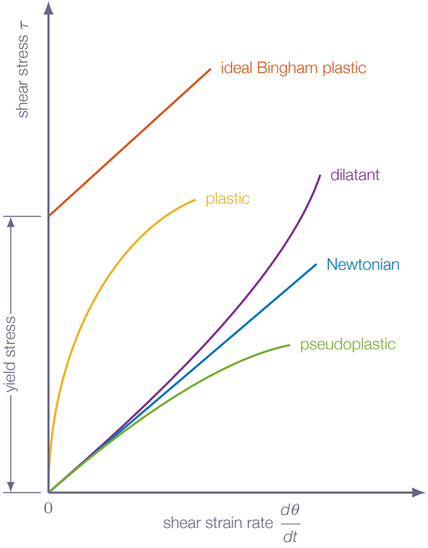
$$Re = \frac{\rho V L}{\mu}$$

- Non-dimensional number that relates viscous forces to inertial forces
- Very important parameter in fluid mechanics
- V and L are characteristic velocity and length scales of the flow

Reynolds number

Reynolds number	flow description
low	viscous, creeping motion (inertial forces negligible)
moderate	laminar flow
high	turbulent flow

Non-Newtonian Fluids



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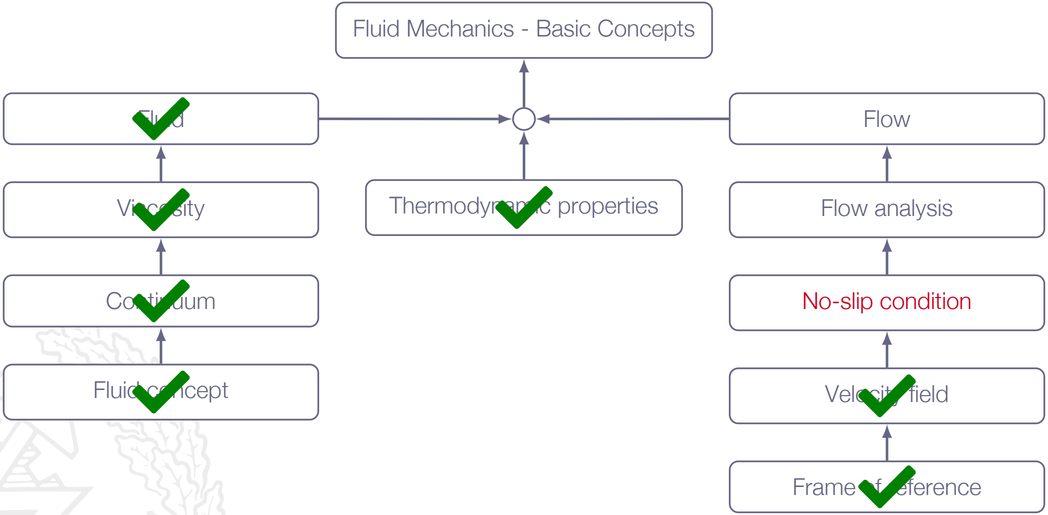
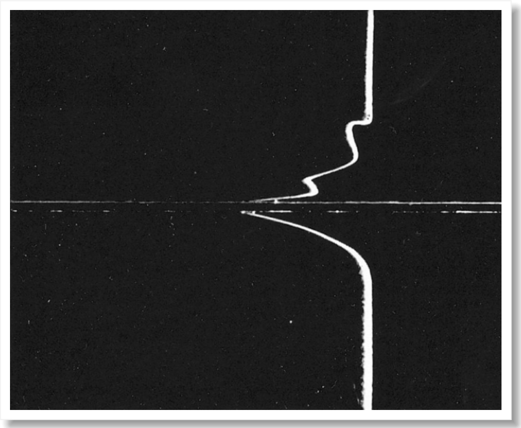


No Slip/No Temperature Jump

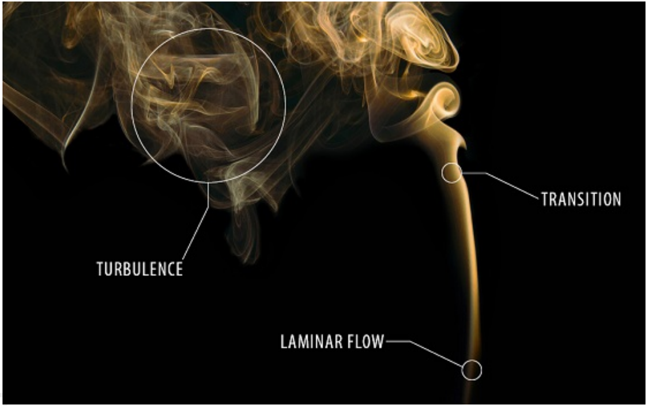
"When a fluid flow is bounded by a solid surface, molecular interactions cause the fluid in contact with the surface to seek momentum and energy equilibrium with that surface"

No Slip/No Temperature Jump

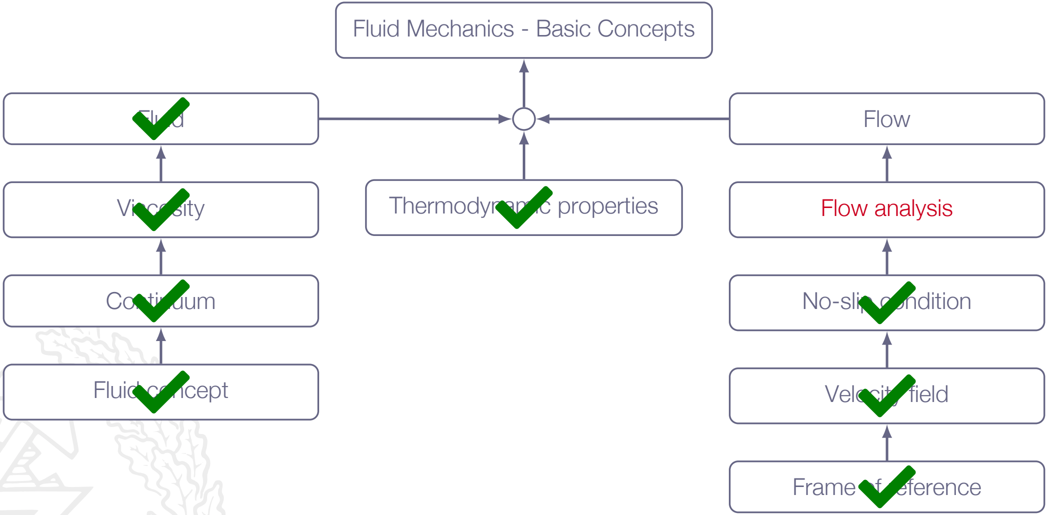
At a solid wall, the fluid will have the velocity and temperature of the wall



Laminar/Turbulent Flow



Roadmap - Introduction to Fluid Mechanics



Flow Analysis

Chapter 3 - Control-volume (integral) approach

Chapter 4 - Infinitesimal system (differential) approach

Chapter 5 - Dimensional analysis approach

Flow Analysis

- Conservation of mass (continuity)
- Conservation of momentum (Newton's second law)
- Conservation of energy (first law of thermodynamics)
- State relation (for example the ideal gas law)
- Second law of thermodynamics
- Boundary conditions

Flow Visualization

Streamline

- a line that is tangent to the velocity vector everywhere at an instant in time

Pathline

- the actual path traversed by a fluid particle

Streakline

- the locus of particles that have earlier passed through a prescribed point

Timeline

- a line formed by a set of particles at a given instant

Flow Visualization

Streamline

- a line that is tangent to the velocity vector everywhere at an instant in time

Pathline

- the actual path traversed by a fluid particle

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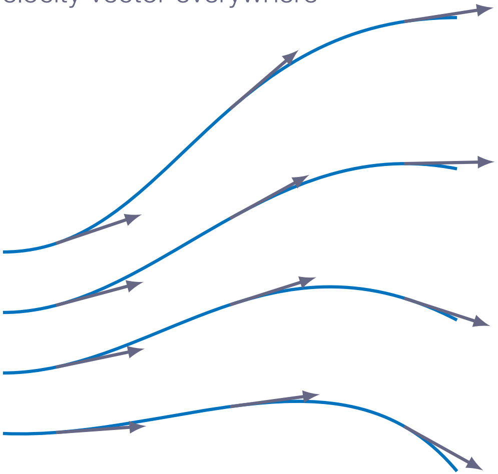
Timeline

- a line formed by a set of particles at a given instant

Note! In a steady-state flow, streamlines, pathlines and streaklines are identical

Streamline

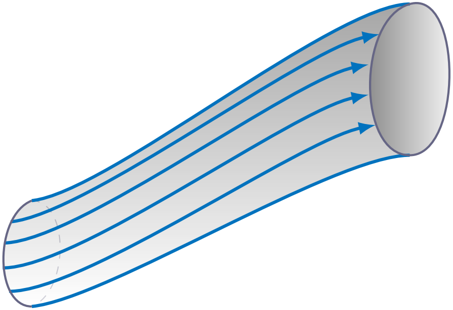
Tangent to flow velocity vector everywhere



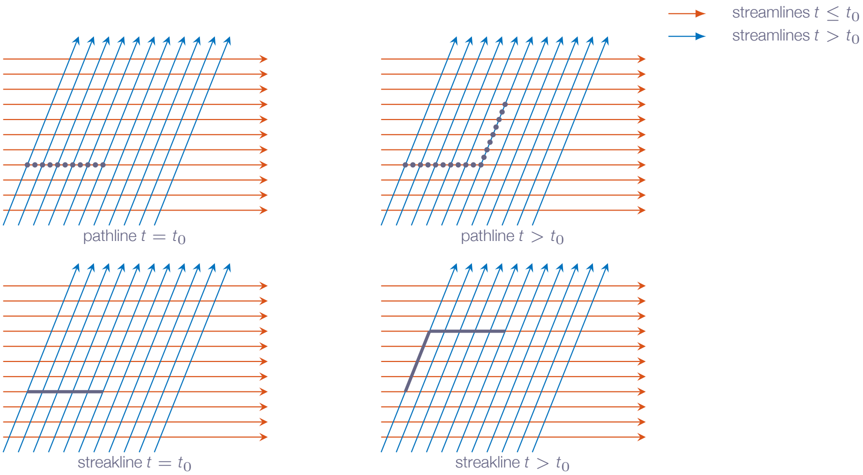
Streamtube

”Constructed” from individual streamlines

No flow across streamtube ”walls” (by definition)



Pathline vs Streakline



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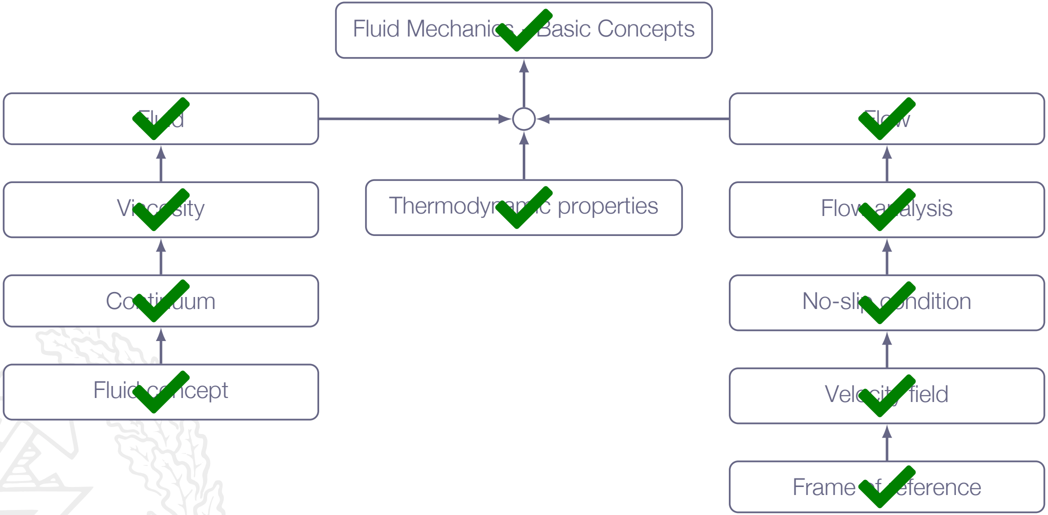
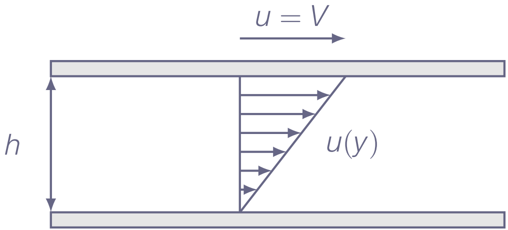


Example - Flow Between Plates

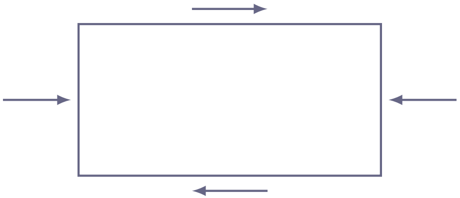


- No acceleration
- No pressure gradients
- two-dimensional flow

Example - Flow Between Plates

$$\left(\tau + \frac{\partial \tau}{\partial y}\Delta y\right)\Delta x \times 1$$

$$p\Delta y \times 1$$



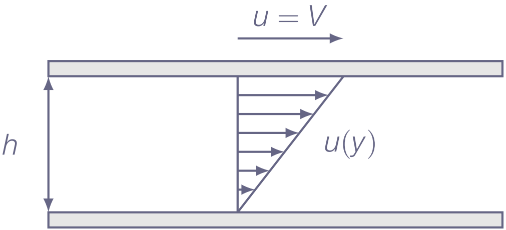
$$\left(p + \frac{\partial p}{\partial x}\Delta x\right)\Delta y \times 1$$

$$\tau \Delta x \times 1$$

$$\sum F_x = p\Delta y - \left(p + \frac{\partial p}{\partial x}\Delta x\right)\Delta y + \left(\tau + \frac{\partial \tau}{\partial y}\Delta y\right)\Delta x - \tau\Delta x = 0$$

$$\frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x} = 0 \Rightarrow \tau = const$$

Example - Flow Between Plates



$$\frac{du}{dy} = \frac{\tau}{\mu} = const$$

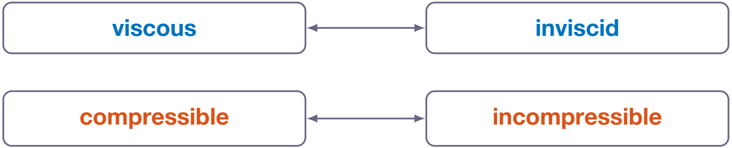
$$u = a + by$$

$$\begin{cases} y = 0 \Rightarrow u = 0 \\ y = h \Rightarrow u = V \end{cases}$$

$$u = \frac{V}{h} y$$

Flow Categories

Flow Categories



Flow Categories

viscous ↔ inviscid

compressible ↔ incompressible

turbulent ↔ laminar

Flow Categories

viscous ↔ inviscid

compressible ↔ incompressible

turbulent ↔ laminar

unsteady ↔ steady-state

Flow Categories

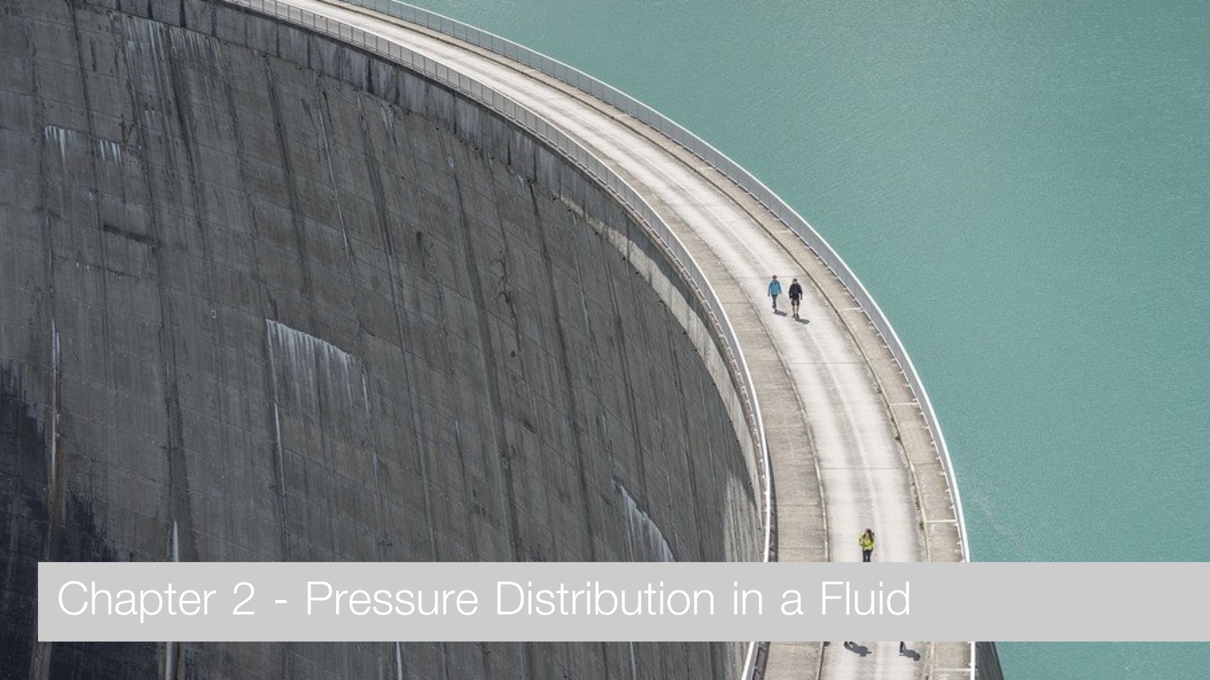
viscous ↔ inviscid

compressible ↔ incompressible

turbulent ↔ laminar

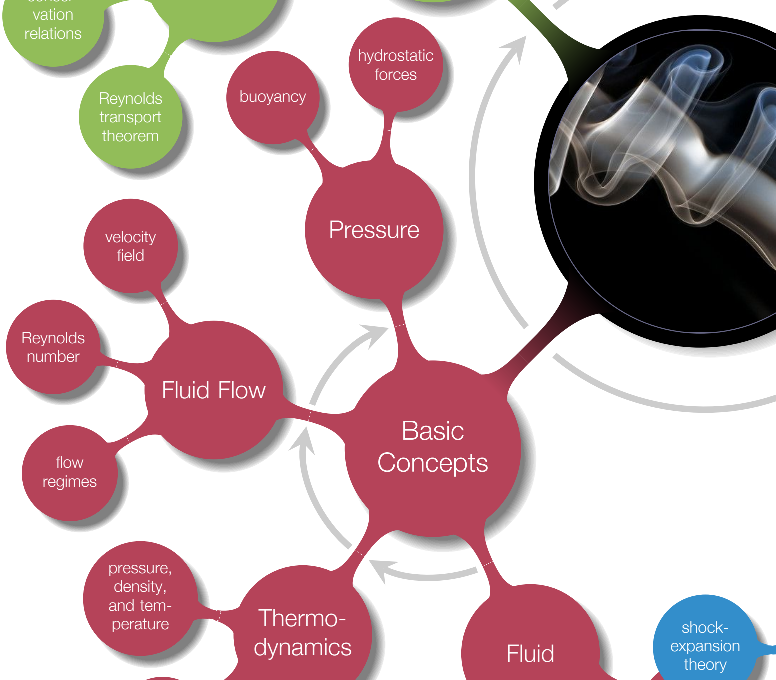
unsteady ↔ steady-state

3D ↔ 1D/2D



Chapter 2 - Pressure Distribution in a Fluid

Overview



vation relations
Reynolds transport theorem
buoyancy
hydrostatic forces
Pressure
velocity field
Reynolds number
Fluid Flow
flow regimes
Basic Concepts
pressure, density, and temperature
Thermo-dynamics
Fluid
shock-expansion theory

Learning Outcomes

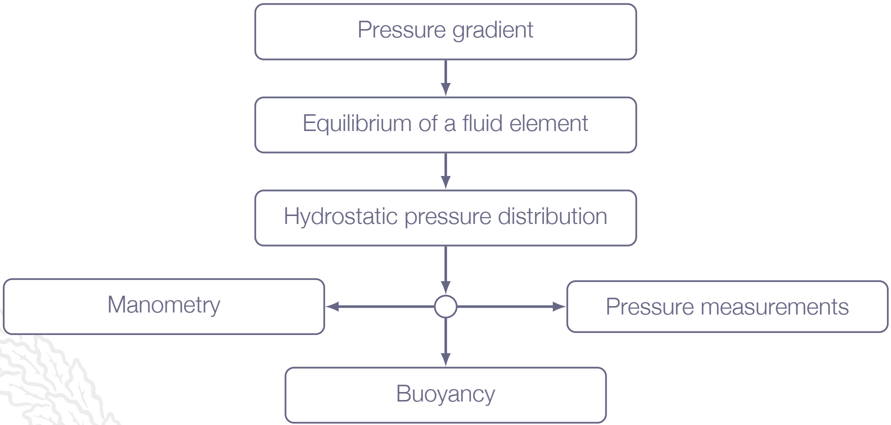
9 **Explain** how to do a force balance for fluid element (forces and pressure gradients)

10 **Understand and explain** buoyancy and cavitation

11 **Solve** problems involving hydrostatic pressure and buoyancy

we will have a look at the pressure distribution in a fluid at rest, i.e. no flow yet...

Roadmap - Pressure Distribution in a Fluid



Motivation

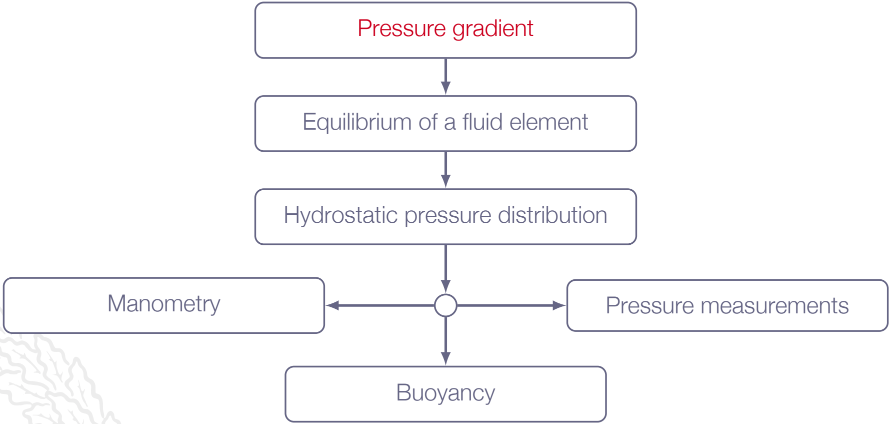
- Many problems does not include fluid motion
  - pressure distribution in a static fluid
  - pressure on solid surfaces due to presence of static fluid
  - floating and submerged bodies

Motivation

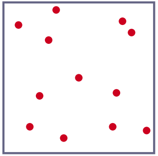
Examples:

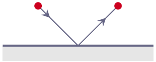
- pressure distribution in the atmosphere and in oceans
- design of pressure measurement devices
- buoyancy on a submerged body
- behavior of floating bodies

Roadmap - Pressure Distribution in a Fluid



Pressure

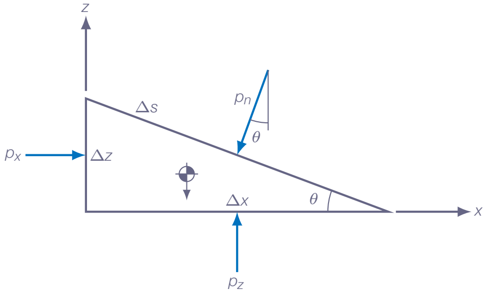




- Pressure is a thermodynamic property
- Pressure is not a force and has no direction
- Forces arise when the molecules of the fluid interacts with the surface of an immersed body
- A force in the surface-normal direction is generated due to the collision of fluid molecules and the surface

Pressure Variation in a Fluid at Rest

- Fluid at rest - no shear (by definition)
- Pressures p_x, p_z, and p_n may be different
- Small element $\Rightarrow$ constant pressure on each face

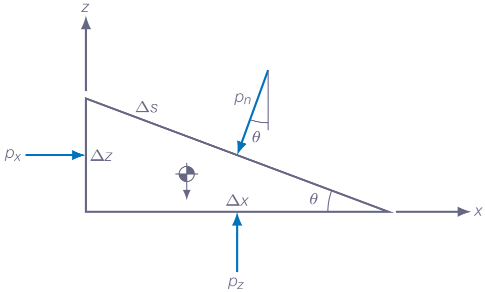


Pressure Variation in a Fluid at Rest

$$\sum F_x = 0 = p_x b \Delta z - p_n b \Delta s \sin\theta$$

$$\sum F_z = 0 = p_z b \Delta x - p_n b \Delta s \cos\theta - \frac{1}{2}\rho g b \Delta x \Delta z$$

$$\begin{cases} \Delta z = \Delta s \sin\theta \\ \Delta x = \Delta s \cos\theta \end{cases}$$

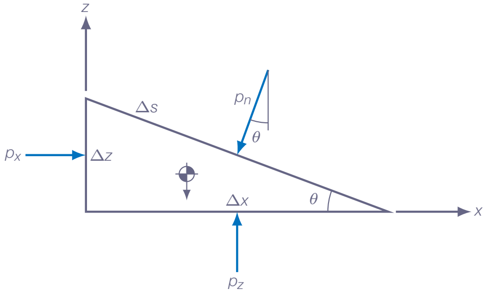


Pressure Variation in a Fluid at Rest

$$\sum F_x = 0 = p_x b\Delta z - p_n b\Delta z$$

$$\sum F_z = 0 = p_z b\Delta x - p_n b\Delta x - \frac{1}{2}\rho g b\Delta x\Delta z$$

$$\begin{cases} p_x = p_n \\ p_z = p_n + \frac{1}{2}\rho g\Delta z \end{cases}$$

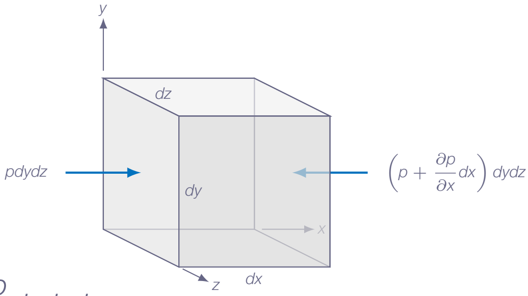


Pressure Variation in a Fluid at Rest

- Since θ is arbitrary, the result is general
- There is no pressure change in the horizontal direction
- The pressure change in the vertical direction is proportional to the depth

"The pressure in a static fluid is a point property, independent of orientation"

Pressure Forces on a Fluid Element



$$p = p(x, y, z, t)$$

$$dF_x = pdydz - \left(p + \frac{\partial p}{\partial x} dx\right) dydz = -\frac{\partial p}{\partial x} dxdydz$$

$$dF_p = -\left[\frac{\partial p}{\partial x}\mathbf{e}_x + \frac{\partial p}{\partial y}\mathbf{e}_y + \frac{\partial p}{\partial z}\mathbf{e}_z\right] dxdydz$$

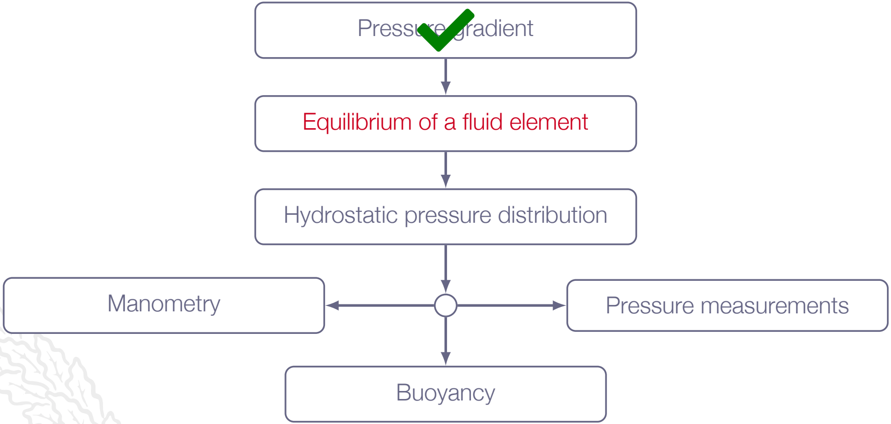
$$\mathbf{f}_p = -\nabla p$$

$\mathbf{f}$ *is the net force per unit volume*

Pressure Forces on a Fluid Element

”it is not the pressure but the pressure gradient causing a net force which must be balanced by gravity or acceleration”

Roadmap - Pressure Distribution in a Fluid



Equilibrium of a Fluid Element

- Force balance for a small element
  - pressure gradients gives surface forces
  - body forces (electromagnetic or gravitational potentials)
  - surface forces due to viscous stresses

Newton's second law:

$$\sum \mathbf{f} = \mathbf{f}_p + \mathbf{f}_g + \mathbf{f}_v = -\nabla p + \rho \mathbf{g} + \mathbf{f}_v = \rho \mathbf{a}$$

Equilibrium of a Fluid Element

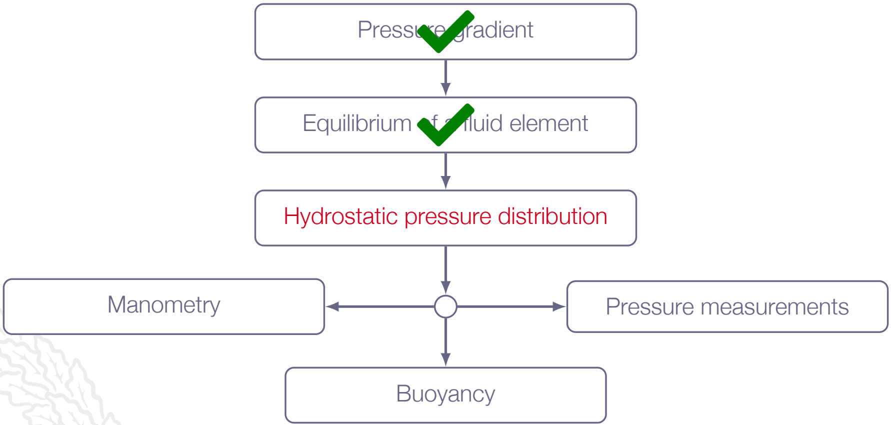
- Hydrostatic problems:
  - no viscous forces
  - no acceleration

Newton's second law reduces to:

$$\nabla p = \rho \mathbf{g}$$

(the general form of Newton's second law will be studied later)

Roadmap - Pressure Distribution in a Fluid



Hydrostatic Pressure in Liquids

$$\nabla p = \rho \mathbf{g}$$

- ∇p is perpendicular everywhere to surfaces of constant p
- The normal of constant-pressure surfaces will be aligned with $\mathbf{g}$

Hydrostatic Pressure in Liquids

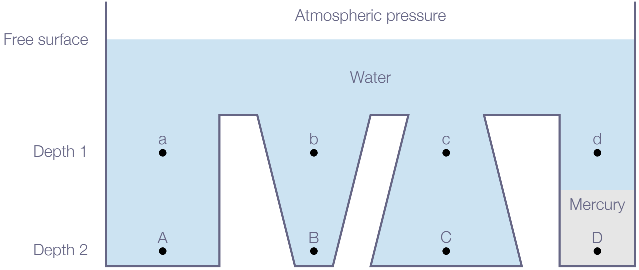
$$\mathbf{g} = -g\mathbf{e}_z$$

$$\frac{dp}{dz} = -\rho g$$

$$p_2 - p_1 = -\int_1^2 \rho g dz$$

Hydrostatic Pressure in Liquids

for liquids, we assume constant density $\Rightarrow p_2 - p_1 = -\int_1^2 \rho g dz = -\rho g(z_2 - z_1)$



$$p_a = p_b = p_c = p_d$$
$$p_A = p_B = p_C \neq p_D$$

Hydrostatic Pressure in Liquids

- Is the incompressible assumption for liquids a good assumption?

  the density is 4.6 percent higher at the deepest part of the ocean - so yes!

$$p_2 - p_1 = -\int_1^2 \rho g dz = -\rho g(z_2 - z_1)$$

Hydrostatic Pressure in Liquids

$$p_2 - p_1 = -\int_1^2 \rho g dz = -\rho g(z_2 - z_1)$$

Why is mercury used for pressure measurements?

Hydrostatic Pressure in Gases

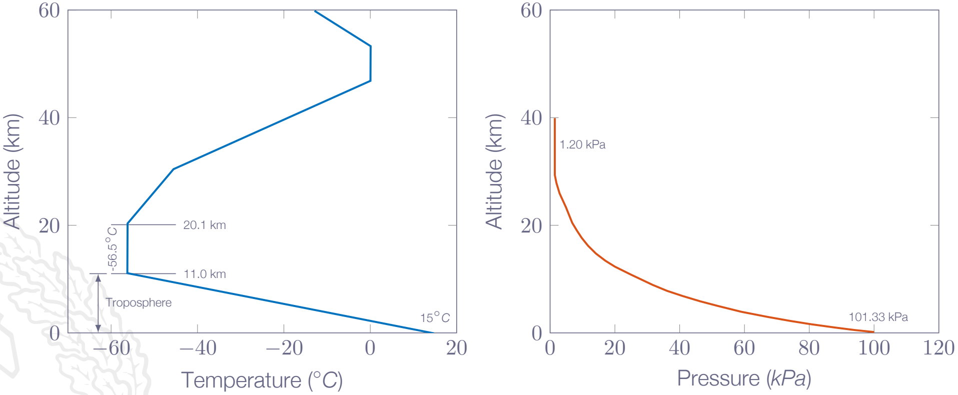
$$\frac{dp}{dz} = -\rho g = -\frac{p}{RT}g$$

both pressure and temperature varies with altitude

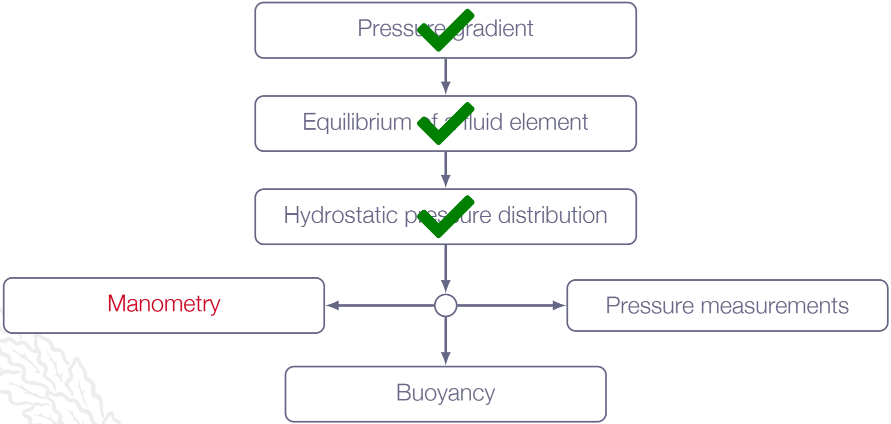
$$\int_1^2 \frac{dp}{p} = \ln\frac{p_2}{p_1} = -\frac{g}{R}\int_1^2 \frac{dz}{T}$$

Temperature variation $T(z)$ needed

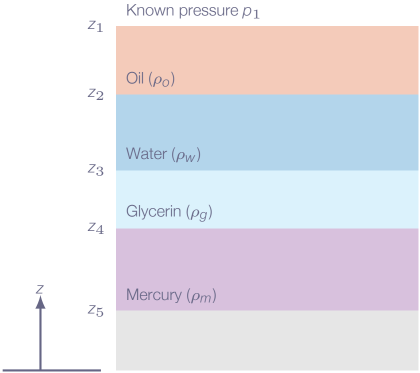
Hydrostatic Pressure in Gases



Roadmap - Pressure Distribution in a Fluid



Manometry



$$p_2 - p_1 = -\rho_o g(z_2 - z_1)$$

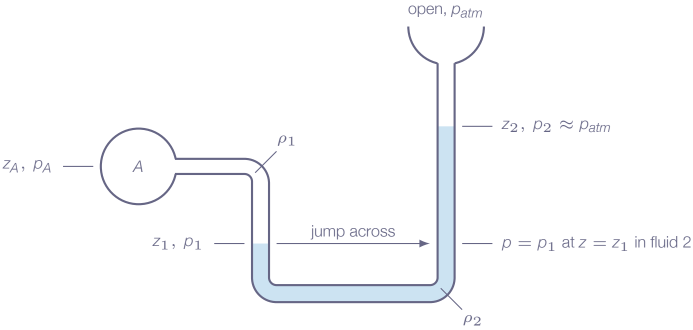
$$p_3 - p_2 = -\rho_w g(z_3 - z_2)$$

$$p_4 - p_3 = -\rho_g g(z_4 - z_3)$$

$$p_5 - p_4 = -\rho_m g(z_5 - z_4)$$

$$p_5 - p_1 = -\rho_m g(z_5 - z_4) - \rho_g g(z_4 - z_3) - \rho_w g(z_3 - z_2) - \rho_o g(z_2 - z_1)$$

Manometry

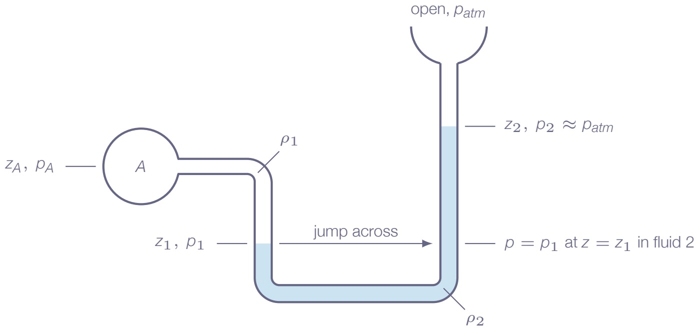


Pascal's law:

"Any two points at the same elevation in a continuous mass of the same static fluid will be at the same pressure"

Manometry

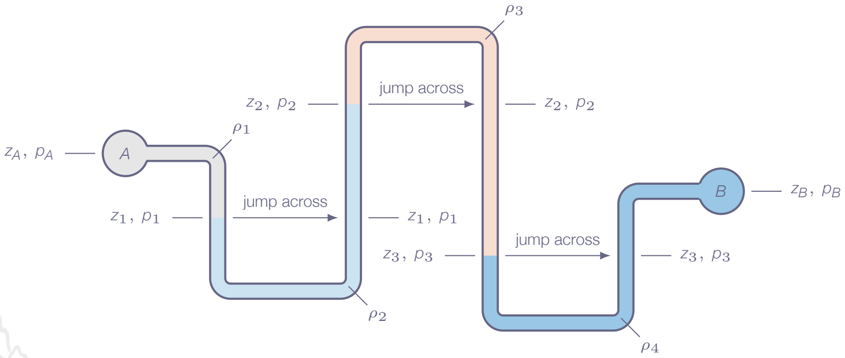
$$p_A + \rho_1 g(z_A - z_1) - \rho_2 g(z_2 - z_1) = p_2 \approx p_{atm}$$



Pascal's law:

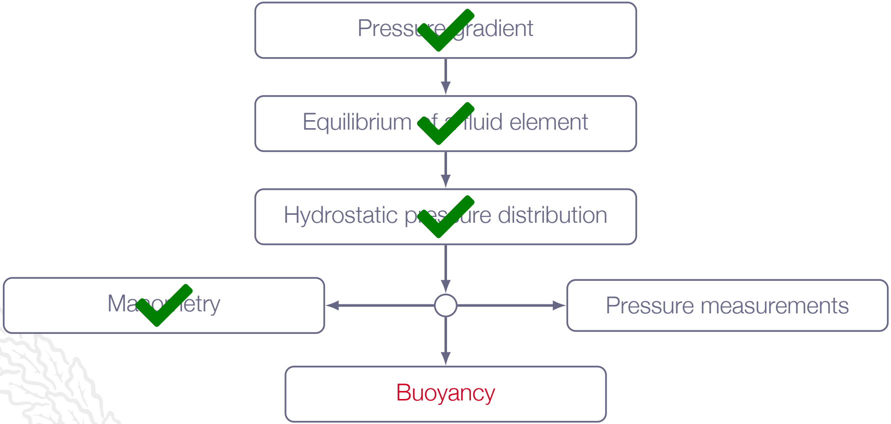
"Any two points at the same elevation in a continuous mass of the same static fluid will be at the same pressure"

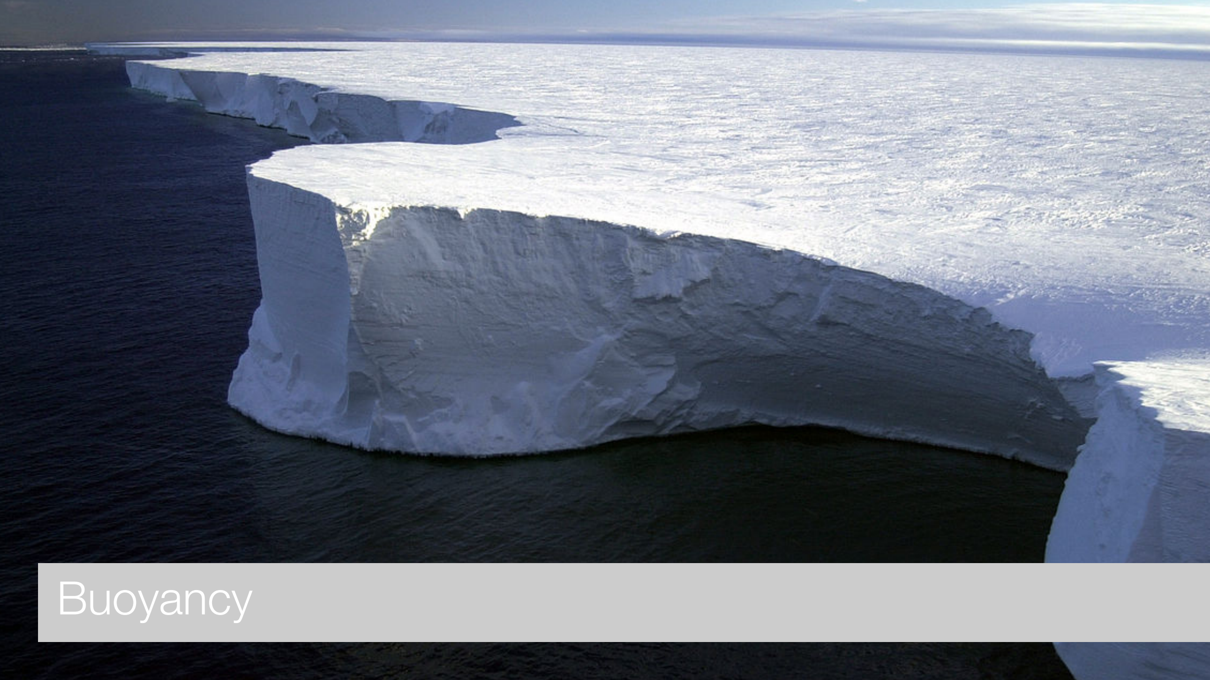
Manometry



ρ_3
jump across
z_2, p_2
z_2, p_2
ρ_1
z_A, p_A
A
jump across
z_1, p_1
z_1, p_1
B
z_B, p_B
jump across
z_3, p_3
z_3, p_3
ρ_2
ρ_4

Roadmap - Pressure Distribution in a Fluid





Buoyancy

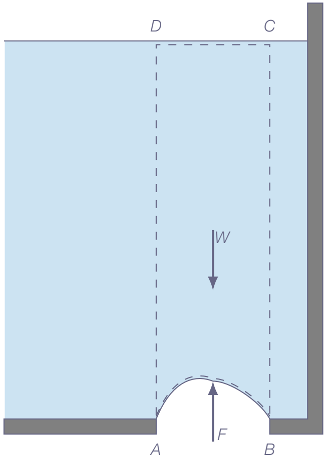
Buoyancy

Archimedes:

- A body immersed in a fluid experiences a vertical buoyant force equal to the weight of the fluid it displaces
- A floating body displaces its own weight in the fluid in which it floats

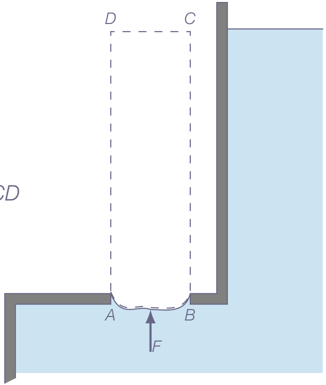
Buoyancy

$$F = \rho g \mathcal{V}_{ABCD}$$

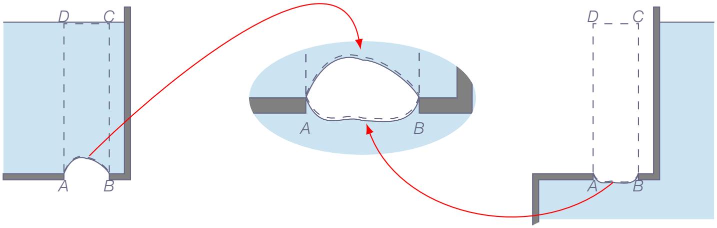


Buoyancy

$$F = \rho g \mathcal{V}_{ABCD}$$



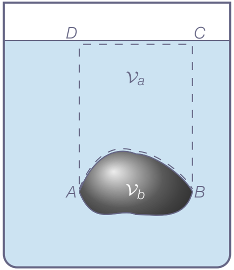
Buoyancy



Buoyancy

$$F_{up} = \rho g(\mathcal{V}_a + \mathcal{V}_b)$$

$$F_{down} = \rho g \mathcal{V}_a$$



Buoyancy

In general

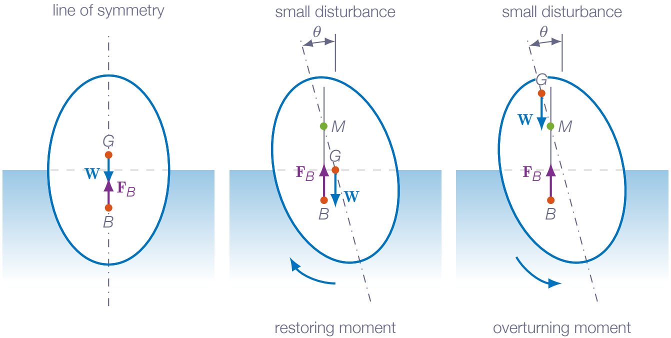
$$\mathbf{F}_B = \sum \rho_i g (displacement\ volume)_i$$

Floating bodies

$$\mathbf{F}_B = body\ weight$$

Buoyancy - Stability

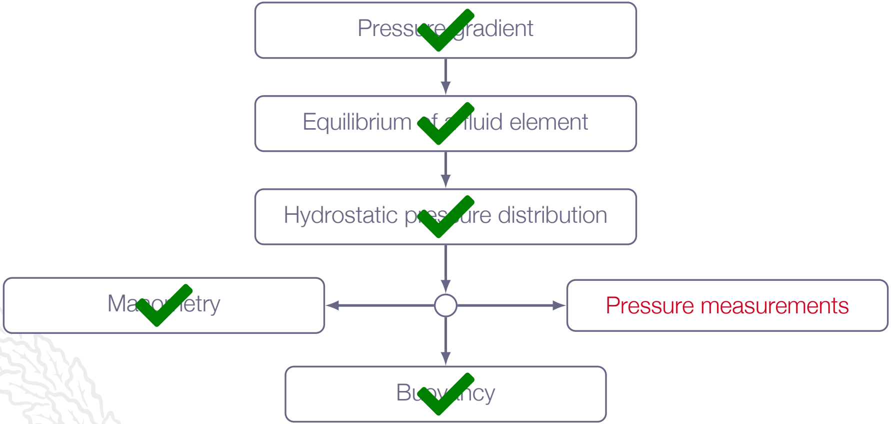
- Center of gravity G
- Center of buoyancy B
- Symmetry line
- Metacenter M



Note! the center of buoyancy (B) is, in this case, the centroid of the displaced volume of liquid

Buoyancy - Stability

Roadmap - Pressure Distribution in a Fluid

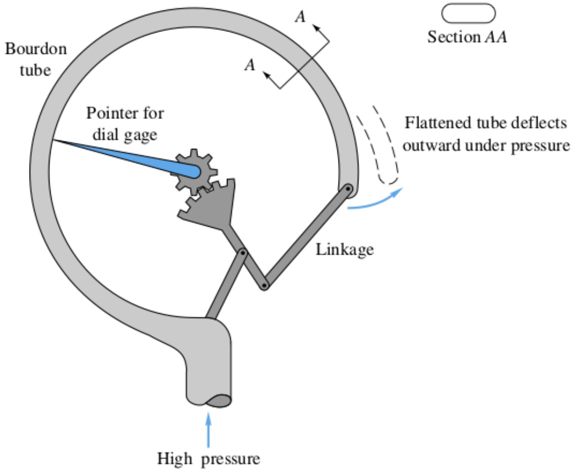


Pressure measurement

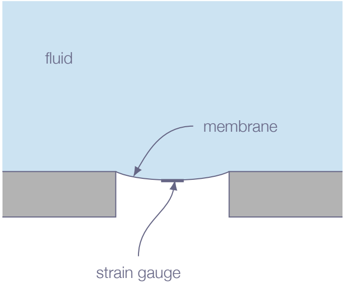
Pressure is a derived property

The force per unit area related to fluid molecular bombardment of a surface

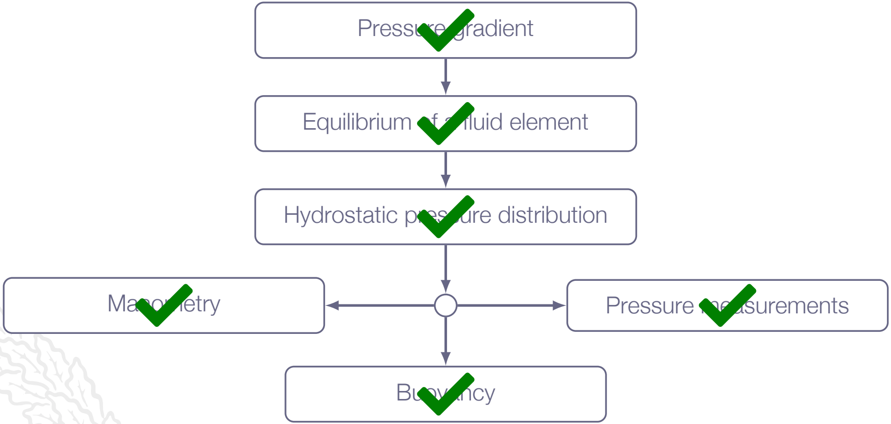
Pressure measurement



Pressure measurement



Roadmap - Pressure Distribution in a Fluid



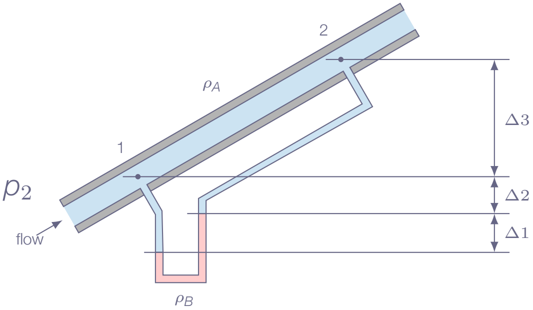
Manometer Example

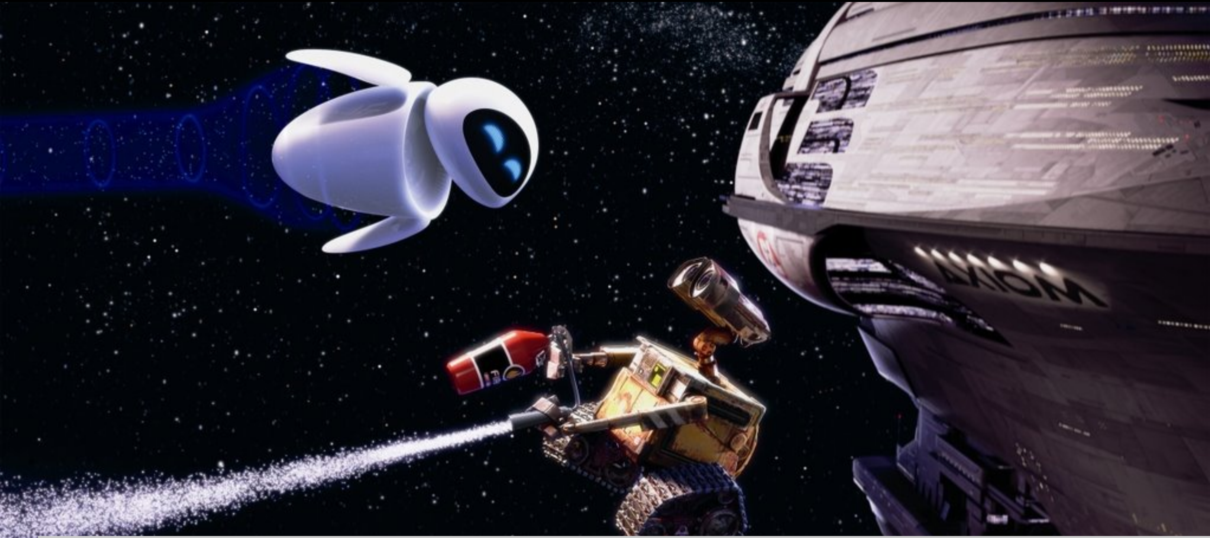
$$p_1 + \sum_{down} \rho_i g \Delta_i - \sum_{up} \rho_i g \Delta_i = p_2$$

$$p_1 + (\Delta_2 + \Delta_1)\rho_A g - \Delta_1 \rho_B g - (\Delta_2 + \Delta_3)\rho_A g = p_2$$

$$p_1 + (\Delta_2 + \Delta_1)\rho_A g - \Delta_1 \rho_B g - (\Delta_2 + z_2 - z_1)\rho_A g = p_2$$

$$\left(\frac{p_1}{\rho_A g} + z_1\right) - \left(\frac{p_2}{\rho_A g} + z_2\right) = \Delta_1\left(\frac{\rho_B}{\rho_A} - 1\right)$$





Chapter 3 - Integral Relations for a Control Volume

Overview

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Analysis
Differential
Relations
conser-
vation
relations
Flow
Relations
Bernoulli
Integral
Relations
conser-
vation
relations
Reynolds
transport
theorem
buoyancy
hydrostatic
forces
Pressure
velocity
field
Reynolds
number

Learning Outcomes

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

12 **Define** Reynolds transport theorem using the concepts control volume and system

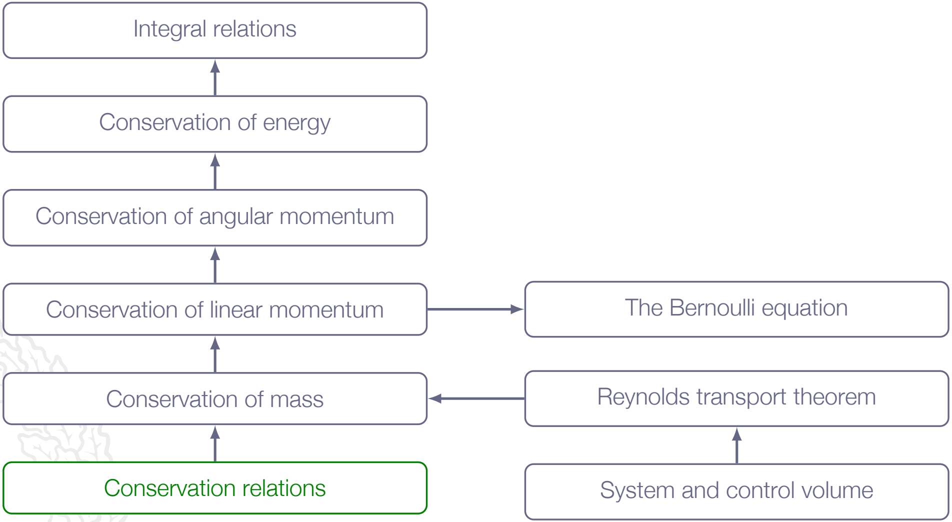
13 **Derive** the control volume formulation of the continuity, momentum, and energy equations using Reynolds transport theorem and solving problems using those relations

15 **Derive** and use the Bernoulli equation (using the relation includes having knowledge about its limitations)

we will derive methods suitable for estimation of forces and system analysis

fluid flow finally ...

Roadmap - Integral Relations

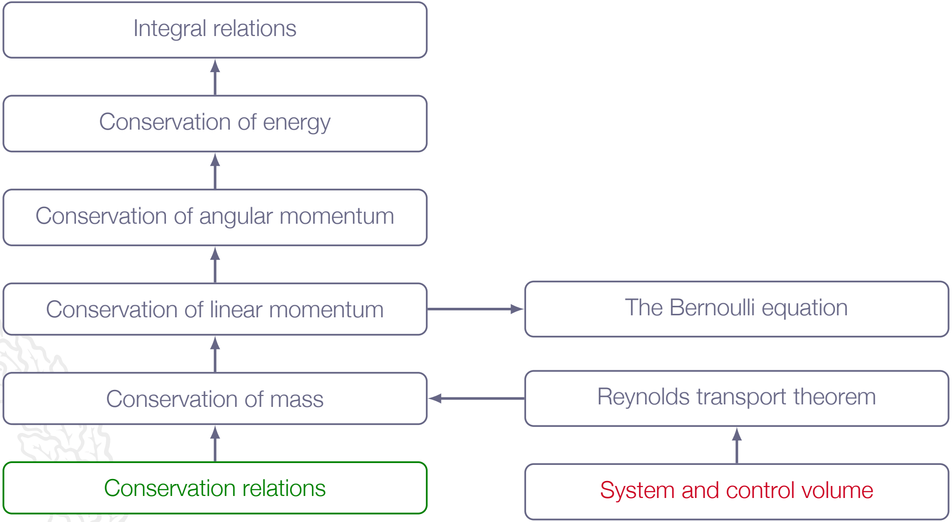


Motivation

Fluid motion analysis:

- differential approach (chapter 4):
  - describe the detailed flow pattern at every point in the flow
- control volume approach (chapter 3):
  - working with a finite region, balance in and out flow and determine gross flow effects (force, torque, energy exchange, ...)
  - gives useful engineering estimates

Roadmap - Integral Relations



System vs Control Volume

All laws of mechanics are written for a system:

- A system is an arbitrary quantity of mass of fixed identity m
- The system is separated from its surroundings by its boundaries
- Interaction between the system and its surroundings

System Mass

$$m_{syst} = const$$

$$\frac{dm}{dt} = 0$$

Obvious in solid mechanics but needs attention in fluid mechanics

Conservation Relations

1. Mass
2. Linear momentum
3. Angular momentum
4. Energy

Linear Momentum

If the surroundings exert a net force $\mathbf{F}$ on the system, the mass in the system will begin to accelerate

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{V}}{dt} = \frac{d}{dt}(m\mathbf{V})$$

Angular Momentum

If the surroundings exert a net moment $\mathbf{M}$ about the center of mass of the system, there will be a rotation effect

$$\mathbf{M} = \frac{d\mathbf{H}}{dt}$$

where $\mathbf{H} = \Sigma(\mathbf{r} \times \mathbf{V})\delta m$ is the angular momentum of the system about its center of mass

Energy

First law of thermodynamics

$$\delta Q - \delta W = dE$$

Second law of thermodynamics

$$dS \leq \frac{\delta Q}{T}$$

State Relations

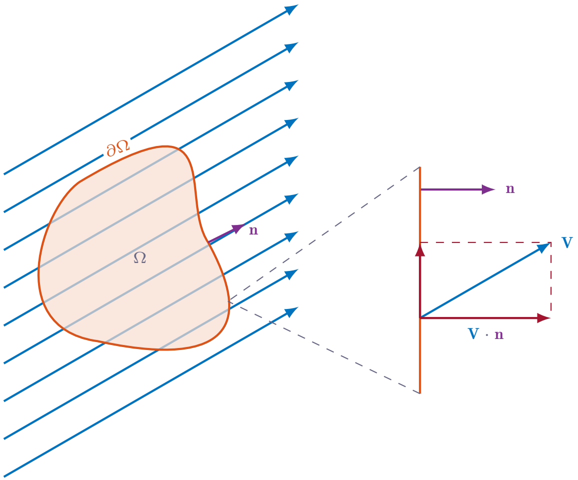
- The above-listed relations includes thermodynamic properties
- Needs to be supplemented by a **state relation**
- Remember: a thermodynamic property can be calculated from **any two other thermodynamics properties**

$$p = p(\rho, T),\; e = e(\rho, T)$$

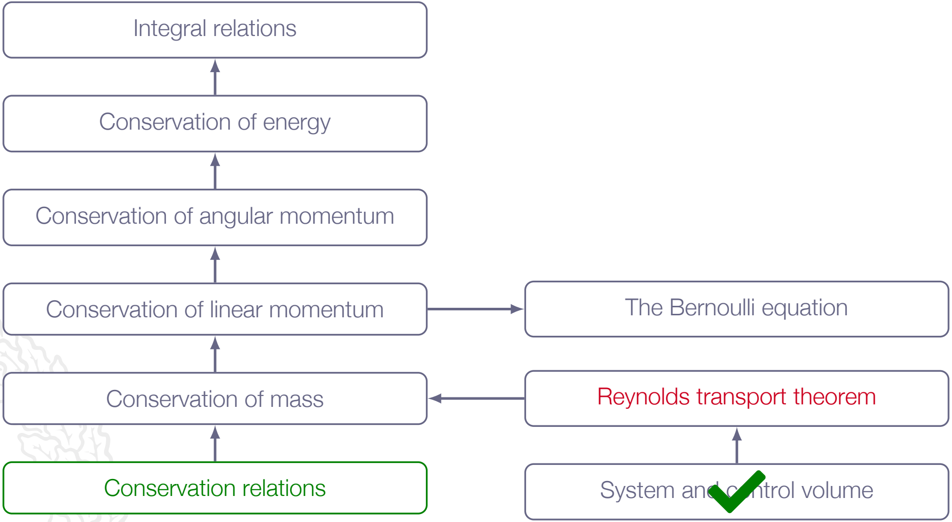
Volume and Mass Flow Rate

$$Q = \int_{\partial\Omega} (\mathbf{V} \cdot \mathbf{n}) dA$$

$$\dot{m} = \int_{\partial\Omega} \rho(\mathbf{V} \cdot \mathbf{n}) dA$$



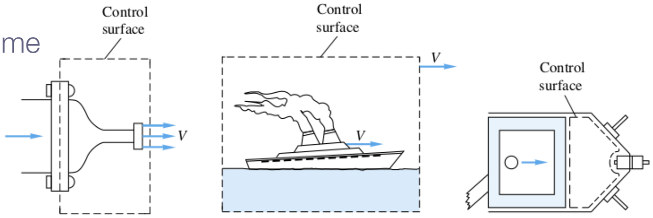
Roadmap - Integral Relations



Reynolds Transport Theorem

Converts mathematical relations for a specific system to relations for a specific region

- fixed control volume
- moving control volume
- deformable control volume



Reynolds Transport Theorem

Let B be any **extensive** property of the fluid (energy, momentum, enthalpy, ...)

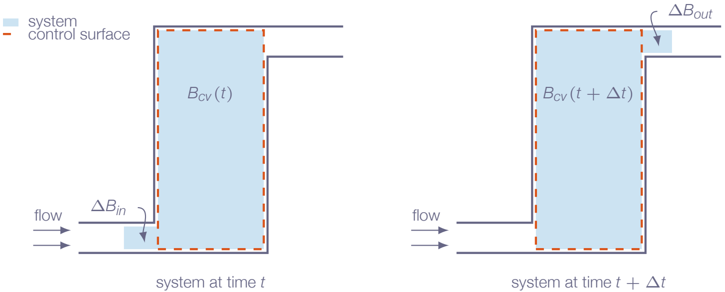
β is the corresponding **intensive** value (*the amount B per unit mass*)

The total amount of B in the control volume is

$$B_{CV} = \int_{CV} \beta dm = \int_{CV} \beta \rho d\mathcal{V}$$

where $\beta = \dfrac{dB}{dm}$

Reynolds Transport Theorem



$$B_{sys}(t) = B_{cv}(t) + \Delta B_{in}$$

$$B_{sys}(t+\Delta t) = B_{cv}(t+\Delta t) + \Delta B_{out}$$

Reynolds Transport Theorem

The rate of change of B for the system:

$$\frac{dB_{sys}}{dt} = \lim_{\Delta t \to 0} \left[\frac{B_{sys}(t + \Delta t) - B_{sys}(t)}{\Delta t} \right]$$

Apply relations from previous slide $\Rightarrow$

$$\frac{dB_{sys}}{dt} = \lim_{\Delta t \to 0} \left[\frac{B_{cv}(t + \Delta t) + \Delta B_{out} - B_{cv}(t) - \Delta B_{in}}{\Delta t} \right]$$

Reynolds Transport Theorem

Rewriting $\Rightarrow$

$$\frac{dB_{sys}}{dt} = \underbrace{\lim_{\Delta t \to 0} \left[\frac{B_{cv}(t+\Delta t) - B_{cv}(t)}{\Delta t} \right]}_{\frac{dB_{cv}}{dt}} + \underbrace{\underbrace{\lim_{\Delta t \to 0} \frac{\Delta B_{out}}{\Delta t}}_{\dot{B}_{out}} - \underbrace{\lim_{\Delta t \to 0} \frac{\Delta B_{in}}{\Delta t}}_{\dot{B}_{in}}}_{\dot{B}_{net}}$$

$$\frac{dB_{sys}}{dt} = \frac{dB_{cv}}{dt} + \dot{B}_{net}$$

Reynolds Transport Theorem

Rate of change of *B* within the control volume

$$\frac{d}{dt}\left(\int_{CV}\beta\rho d\mathcal{V}\right)$$

Net flux of *B* over the control volume surface

$$\int_{CS}\beta\rho(\mathbf{V}\cdot\mathbf{n})dA$$

Reynolds Transport Theorem

$$\underbrace{\frac{d}{dt}(B_{sys})}_{Lagrange} = \underbrace{\frac{d}{dt}\left(\int_{CV} \beta\rho d\mathcal{V}\right) + \int_{CS} \beta\rho(\mathbf{V}\cdot\mathbf{n})dA}_{Euler}$$

Reynolds Transport Theorem

For a fixed control volume (the volume does not change in time)

$$\frac{d}{dt}\left(\int_{CV}\beta\rho d\mathcal{V}\right)=\int_{CV}\frac{\partial}{\partial t}(\beta\rho)d\mathcal{V}$$

Reynolds Transport Theorem

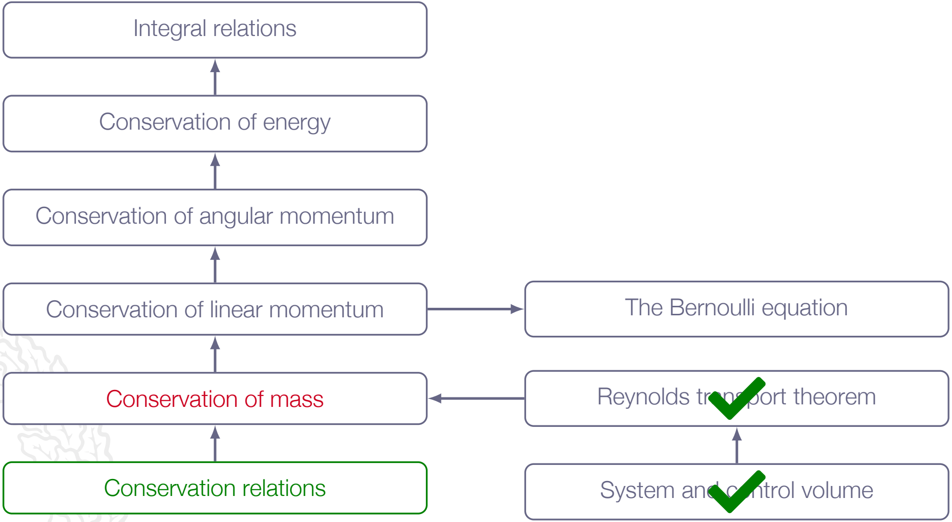
If the control volume moves with the constant velocity $\mathbf{V_s}$, the relative velocity of the fluid crossing the control volume surface $\mathbf{V_r}$ is

$$\mathbf{V_r} = \mathbf{V} - \mathbf{V_s}$$

and thus

$$\frac{d}{dt}(B_{sys}) = \frac{d}{dt}\left(\int_{CV} \beta\rho d\mathcal{V}\right) + \int_{CS} \beta\rho(\mathbf{V_r} \cdot \mathbf{n})dA$$

Roadmap - Integral Relations



Conservation of Mass

Reynolds transport theorem with $B = m$ and $\beta = dB/dm = dm/dm = 1$

$$\frac{d}{dt}(m_{sys}) = 0 = \frac{d}{dt}\left(\int_{CV} \rho d\mathcal{V}\right) + \int_{CS} \rho(\mathbf{V_r} \cdot \mathbf{n})dA$$

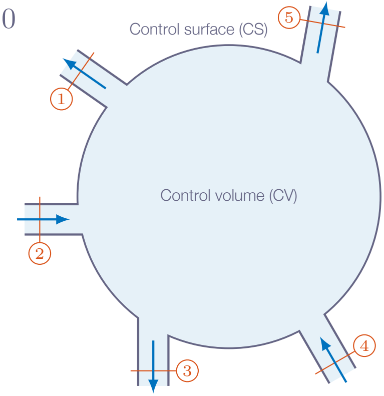
for a fixed control volume

$$\boxed{\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \int_{CS} \rho(\mathbf{V_r} \cdot \mathbf{n})dA = 0}$$

Conservation of Mass

for a control volume with a number of one-dimensional inlets and outlets

$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$



Conservation of Mass

Steady state $\Rightarrow \partial\rho/\partial t = 0$

$$\int_{CS} \rho(\mathbf{V_r} \cdot \mathbf{n})dA = 0$$

or

$$\sum_i (\rho_i A_i V_i)_{out} = \sum_i (\rho_i A_i V_i)_{in}$$

Conservation of Mass

Incompressible flow $\Rightarrow \partial\rho/\partial t = 0$

$$\int_{CS} (\mathbf{V_r} \cdot \mathbf{n}) dA = 0$$

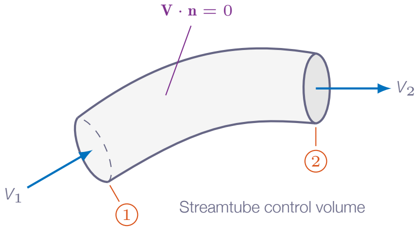
or

$$\sum_i (A_i V_i)_{out} = \sum_i (A_i V_i)_{in}$$

Conservation of Mass - Example 1

Steady flow through a streamtube

- steady state $\Rightarrow$ no changes in time
- streamtube $\Rightarrow$ only flow through the surfaces 1 and 2



$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2 = const$$

if the density is constant (incompressible flow)

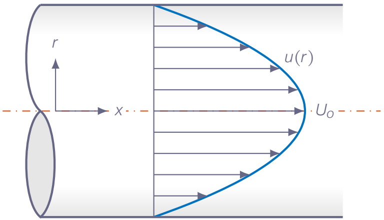
$$Q = A_1 V_1 = A_2 V_2 = const \Rightarrow V_2 = \frac{A_1}{A_2} V_1$$

Remember: a streamtube is constructed from a set of streamlines

Conservation of Mass - Example 2

Compute the average velocity for a steady **laminar** incompressible viscous flow through a circular tube with given axial velocity profile

$$u = U_o\left(1 - \left(\frac{r}{R}\right)^2\right)$$



Assumptions:

1. Laminar flow
2. Steady state ⇒ no changes in time
3. Incompressible ⇒ constant density

$$V_{av} = \frac{1}{A}\int u dA = \frac{1}{\pi R^2}\int_0^R U_o\left(1 - \left(\frac{r}{R}\right)^2\right)2\pi r dr = \frac{2U_o}{R^2}\int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right) r dr$$

Conservation of Mass - Example 2

$$V_{av} = \frac{2U_o}{R^2} \int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right) rdr = \frac{2U_o}{R^2} \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = \frac{U_o}{2}$$

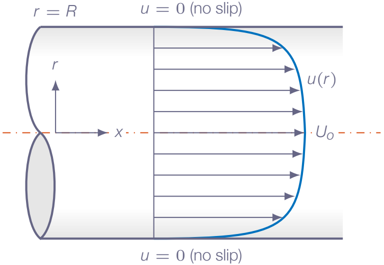
Thus, for **laminar** pipe flow

$$V_{av} = \frac{U_o}{2}$$

Conservation of Mass - Example 3

Compute the average velocity for a steady **turbulent** incompressible viscous flow through a circular tube with given axial velocity profile

$$u \approx U_o \left(1 - \frac{r}{R}\right)^m$$



Assumptions:

1. Turbulent flow: $1/5 \geq m \geq 1/9$
2. Steady state $\Rightarrow$ no changes in time
3. Incompressible $\Rightarrow$ constant density

$$V_{av} \approx \frac{1}{A}\int u dA = \frac{1}{\pi R^2}\int_0^R U_o \left(1 - \frac{r}{R}\right)^m 2\pi r dr = \frac{2U_o}{R^2}\int_0^R \left(1 - \frac{r}{R}\right)^m r dr$$

Conservation of Mass - Example 3

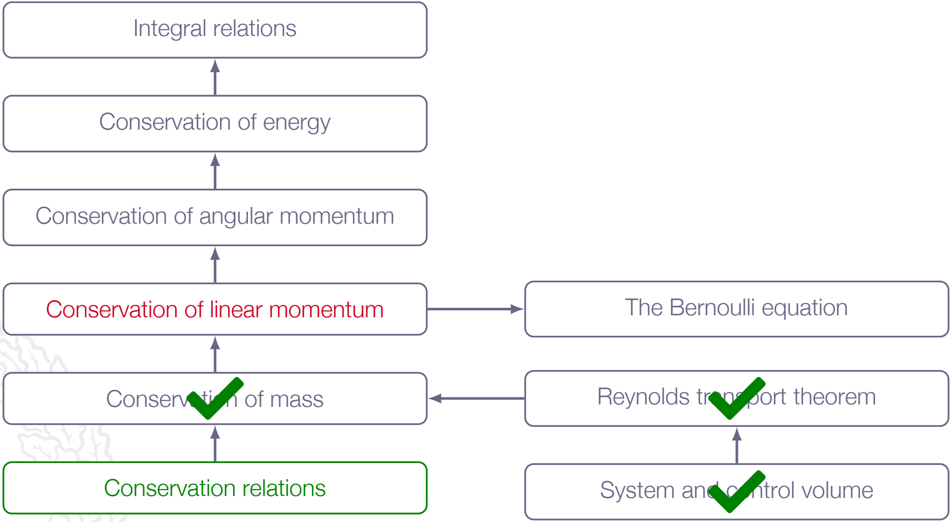
$$V_{av} \approx \frac{2U_o}{R^2}\int_0^R \left(1-\frac{r}{R}\right)^m rdr = \frac{2U_o}{R^2}\left[\frac{(r-R)\left(1-\frac{r}{R}\right)^m (mr+r+R)}{(m+1)(m+2)}\right]_0^R$$

Thus, for **turbulent** pipe flow

$$V_{av} \approx \frac{2U_o}{(m+1)(m+2)}$$

$m = 1/7 \Rightarrow V_{av} \approx 49U_o/60 \approx 0.82U_o$

Roadmap - Integral Relations



Conservation of Linear Momentum

Linear Momentum

Reynolds transport theorem with $B = m\mathbf{V}$ and $\beta = dB/dm = d(m\mathbf{V})/dm = \mathbf{V}$

$$\frac{d}{dt}(m\mathbf{V})_{sys} = \sum \mathbf{F} = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho d\mathcal{V}\right) + \int_{CS} \mathbf{V}\rho(\mathbf{V_r} \cdot \mathbf{n})dA$$

1. $\mathbf{V}$ is the velocity relative to an inertial (non-accelerating) coordinate system
2. $\sum \mathbf{F}$ is the vector sum of all forces on the system (surface forces and body forces)
3. the relation is a vector relation (three components)

Linear Momentum

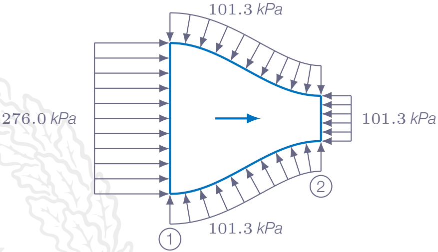
Forces:

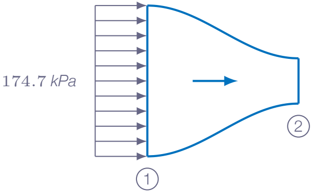
1. solid bodies that protrude through the control volume surface
2. forces due to pressure and viscous stresses of the surrounding fluid

Surface Pressure Force

$$\mathbf{F}_p = \int_{CS} p(-\mathbf{n})dA$$

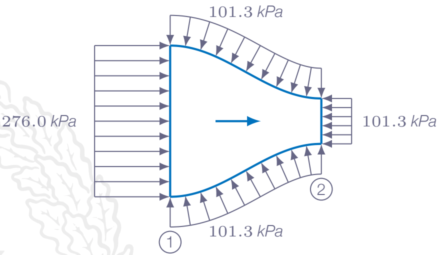
$$\mathbf{F}_p = \int_{CS} (p - p_{atm})(-\mathbf{n})dA = \int_{CS} p_{gage}(-\mathbf{n})dA$$

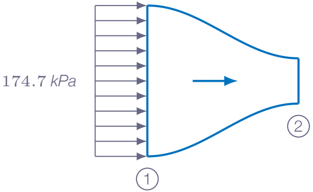




Surface Pressure Force

A free jet leaving a confined duct and exits into the ambient atmosphere will be at atmospheric pressure





Linear Momentum - Example

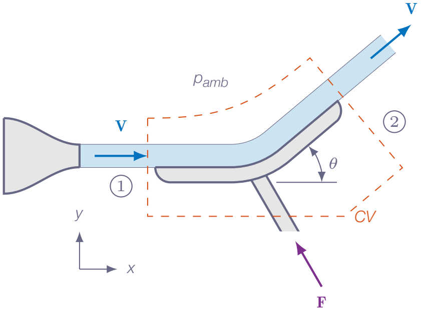
Steady-state flow: deflection av a water jet without changing its velocity magnitude

- steady-state
- water $\Rightarrow$ incompressible
- atmospheric pressure on all control volume surfaces
- neglect friction

$$\mathbf{F} = \dot{m}_2 \mathbf{V}_2 - \dot{m}_1 \mathbf{V}_1$$

$$|\mathbf{V}_1| = |\mathbf{V}_2| = V$$

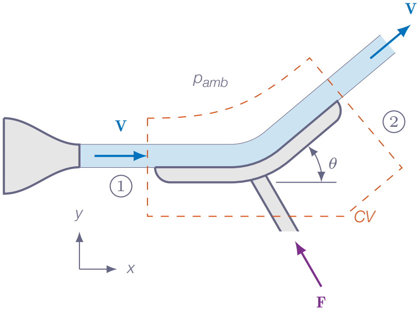
- mass conservation: $\dot{m}_1 = \dot{m}_2 = \dot{m} = \rho A V$



Linear Momentum - Example

$$F_x = \dot{m}V(\cos\theta - 1)$$
$$F_y = \dot{m}V\sin\theta$$

$$\mathbf{F} = \dot{m}V(\cos\theta - 1, \sin\theta, 0)$$



Momentum Flux Correction Factor

One-dimensional flow through inlets and outlets is of course not true in reality

Introducing the correction factor ζ

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \zeta V_{av}\dot{m}$$

where (for incompressible flow)

$$V_{av} = \frac{1}{A}\int u dA$$

Momentum Flux Correction Factor

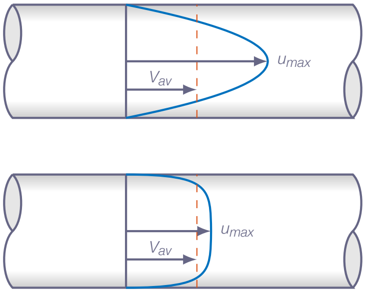
Laminar pipe flow:

Velocity profile:

$$u(r) = U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

From example 2 in the conservation-of-mass section we have that the average velocity for a laminar pipe flow is obtained as:

$$V_{av} = \frac{1}{2}U_{max}$$



Momentum Flux Correction Factor

$$u(r) = U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \int_0^R U_{max}^2\left(1 - \left(\frac{r}{R}\right)^2\right)\rho U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)2\pi r dr$$

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = 2\pi\rho U_{max}^2\int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right)^2 r dr$$

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{1}{3}\pi R^2\rho U_{max}^2$$

Momentum Flux Correction Factor

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{1}{3}\rho\pi R^2 U_{max}^2$$

the mass flow $\dot{m}$ can be obtained as:

$$\dot{m} = V_{av}\rho\pi R^2 = \frac{1}{2}U_{max}\rho\pi R^2$$

which gives

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{2U_{max}}{3}\dot{m}$$

from the definition of ζ

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \zeta V_{av}\dot{m} = \left\{V_{av} = \frac{1}{2}U_{max}\right\} = \frac{\zeta U_{max}}{2}\dot{m}$$

Momentum Flux Correction Factor

comparing the two expressions, we have that

$$\frac{2U_{max}}{3}\dot{m} = \frac{\zeta U_{max}}{2}\dot{m}$$

and thus $\zeta = 4/3$ for laminar incompressible flow

Momentum Flux Correction Factor

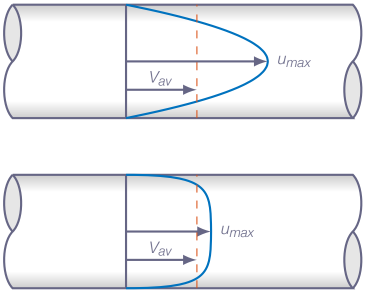
Turbulent pipe flow:

Velocity Profile:

$$u(r) \approx U_{max} \left(1 - \frac{r}{R}\right)^m, \; m \approx \frac{1}{7}$$

From example 3 in the conservation-of-mass section we have that the average velocity for a laminar pipe flow is obtained as:

$$V_{av} = \frac{2U_{max}}{(1+m)(2+m)}$$



Momentum Flux Correction Factor

$$\rho 2\pi U_{max}^2 \int_0^R \left(1-\frac{r}{R}\right)^{2m} rdr = \zeta\rho\pi R^2 V_{av}^2 = \zeta\frac{4\rho\pi R^2 U_{max}^2}{(1+m)^2(2+m)^2} \Rightarrow$$

$$2\left[\frac{(r-R)\left(1-\frac{r}{R}\right)^{2m}(2mr+r+R)}{2(1+2m)(1+m)}\right]_0^R = \zeta\frac{4R^2}{(1+m)^2(2+m)^2} \Rightarrow$$

$$\frac{R^2}{(1+2m)(1+m)} = \zeta\frac{4R^2}{(1+m)^3(2+m)^3} \Rightarrow \zeta = \frac{(1+m)^2(2+m)^2}{4(1+2m)(1+m)}$$

Momentum Flux Correction Factor

Laminar pipe flow:

$\zeta = 4/3$ should be used

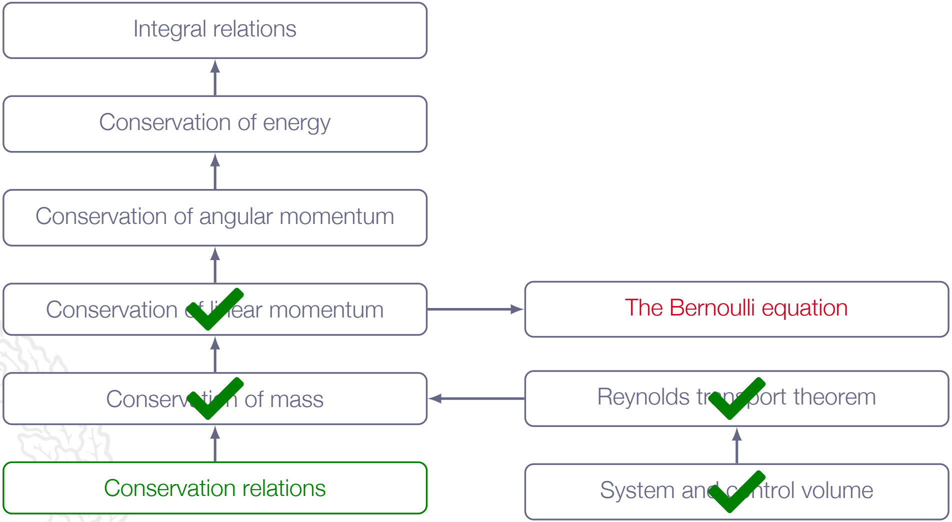
Turbulent pipe flow:

$$\zeta = \frac{(1+m)^2(2+m)^2}{4(1+2m)(1+m)}$$

m	1/5	1/6	1/7	1/8	1/9
ζ	1.037	1.027	1.020	1.016	1.013

($\zeta = 1.0$ is often a good approximation)

Roadmap - Integral Relations

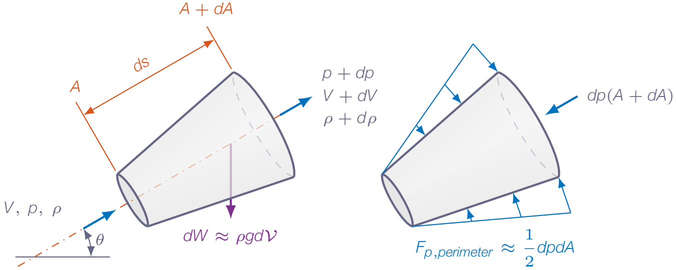


Daniel Bernoulli

The relation between pressure, velocity, and elevation in a frictionless flow

The Bernoulli Equation

Frictionless flow along a streamline (streamtube with infinitesimal cross section area)



conservation of mass:

$$\frac{d}{dt}\left(\int_{CV} \rho d\mathcal{V}\right) + \dot{m}_{out} - \dot{m}_{in} = 0 \approx \frac{\partial \rho}{\partial t} d\mathcal{V} + d\dot{m}$$

where $\dot{m} = \rho A V$ and $d\mathcal{V} \approx A ds$

$$d\dot{m} = d(\rho A V) = -\frac{\partial \rho}{\partial t} A ds$$

The Bernoulli Equation

linear momentum equation in the streamwise direction:

$$\sum dF_s = \frac{d}{dt}\left(\int_{CV} V\rho d\mathcal{V}\right) + (\dot{m}V)_{out} - (\dot{m}V)_{in} \approx \frac{\partial}{\partial t}(\rho V)Ads + d(\dot{m}V)$$

frictionless flow means: only pressure and gravity forces

$$dF_{s,p} \approx \frac{1}{2}dpdA - (A + dA)dp \approx -Adp$$

$$dF_{s,grav} = -dW\sin\theta = -(g\rho A)ds\sin\theta = -g\rho Adz$$

$$\sum dF_s = -g\rho Adz - Adp = \frac{\partial}{\partial t}(\rho V)Ads + d(\dot{m}V)$$

The Bernoulli Equation

$$-g\rho A dz - A dp = \frac{\partial \rho}{\partial t} V A ds + \frac{\partial V}{\partial t} \rho A ds + \dot{m} dV + V d\dot{m}$$

the continuity equation gives

$$V \left[\frac{\partial \rho}{\partial t} A ds + d\dot{m} \right] = 0$$

and thus

$$\frac{\partial V}{\partial t} \rho A ds + A dp + \dot{m} dV + g\rho A dz = 0$$

Now, divide by ρA

$$\frac{\partial V}{\partial t} ds + \frac{dp}{\rho} + V dV + g dz = 0$$

The Bernoulli Equation

Bernoulli's equation for unsteady frictionless flow along a streamline (the relation just derived) can be integrated between any two points along the streamline

$$\int_1^2 \frac{\partial V}{\partial t} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}\left(V_2^2 - V_1^2\right) + g\left(z_2 - z_1\right) = 0$$

The Bernoulli Equation

Steady ($\partial V/\partial t = 0$), incompressible (constant density) flow:

$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g z_2 = const$$

The Bernoulli Equation

Note! the following restrictive assumptions have been made in the derivation

1. steady flow

   many flows can be treated as steady at least when doing control volume type of analysis

2. incompressible flow

   low velocity gas flow without significant changes in pressure, liquid flow

3. frictionless flow

   friction is in general important

4. flow along a single streamline

   different streamlines in general have different constants, we shall see later that under specific circumstances all streamlines have the same constant

One should be aware of these restrictions when using the Bernoulli relation

Relation to the Energy Equation

$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g z_2 = const$$

- Derived from the momentum equation
- May be interpreted as a idealized energy equation (changes from 1 to 2)
  - reversible pressure work
  - kinetic energy change
  - potential energy change
  - no exchange due to viscous dissipation

Stagnation, Static, and Dynamic Pressures

In many flows, elevation changes are negligible

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2 = p_o$$

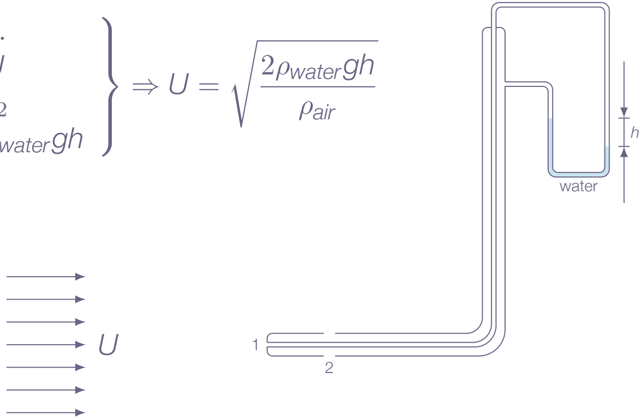
- Static pressure: p_1 and p_2
- Dynamic pressure: $\frac{1}{2}\rho V_1^2$ and $\frac{1}{2}\rho V_2^2$
- Stagnation (total) pressure: p_o

Pitot Static Tube

Pitot Static Tube

$$p_1 + \frac{1}{2}\rho_{air}U_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho_{air}U_2^2 + \rho g z_2$$

$$\left.\begin{aligned} U_1 &= 0. \\ U_2 &= U \\ z_1 &\approx z_2 \\ p_1 - p_2 &= \rho_{water} g h \end{aligned}\right\} \Rightarrow U = \sqrt{\frac{2\rho_{water} g h}{\rho_{air}}}$$



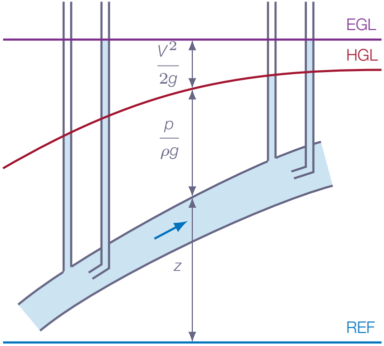
Hydraulic and Energy Grade Lines

EGL: $\dfrac{p}{\rho g} + \dfrac{V^2}{2g} + z$

constant if:

- no friction
- no heat transfer
- no work

HGL: $\dfrac{p}{\rho g} + z = \text{EGL} - \dfrac{V^2}{2g}$

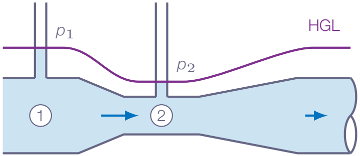


Venturi Tube

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2$$

$z_1 = z_2$ gives

$$V_2^2 - V_1^2 = \frac{2\Delta p}{\rho}$$



continuity:

$$A_1V_1 = A_2V_2 \Rightarrow V_1 = \frac{A_2}{A_1}V_2 = \frac{D_2^2}{D_1^2}V_2$$

inserted in the Bernoulli equation, this gives

$$V_2 = \left[\frac{2D_1^4\Delta p}{\rho(D_1^4 - D_2^4)}\right]^{1/2} \Rightarrow \dot{m} = \rho A_2 V_2 = \frac{\pi D_1^2 D_2^2}{4}\left[\frac{2\rho\Delta p}{D_1^4 - D_2^4}\right]^{1/2}$$

Tank Problem - Solution 1

conservation of mass:

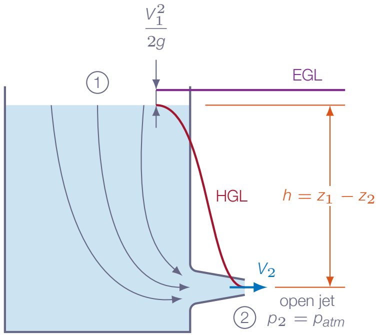
$$A_1V_1 = A_2V_2 \Rightarrow V_1 = \frac{A_2}{A_1}V_2$$

Bernoulli:

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2$$

$$p_1 = p_2 = p_{atm}$$

$$V_2^2 - V_1^2 = 2g(z_1 - z_2) = 2gh$$

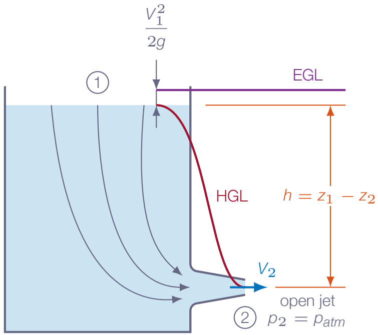


$$V_2 = \sqrt{\frac{2gh}{1 - \left(\frac{A_2}{A_1}\right)^2}}$$

$$A_2 \ll A_1 \Rightarrow V_2 \approx \sqrt{2gh}$$

Tank Problem - Solution 2

The outflow is very small in compared to the tank volume and thus the water surface hardly moves at all, *i.e.* $V_1 \approx 0$



Bernoulli:

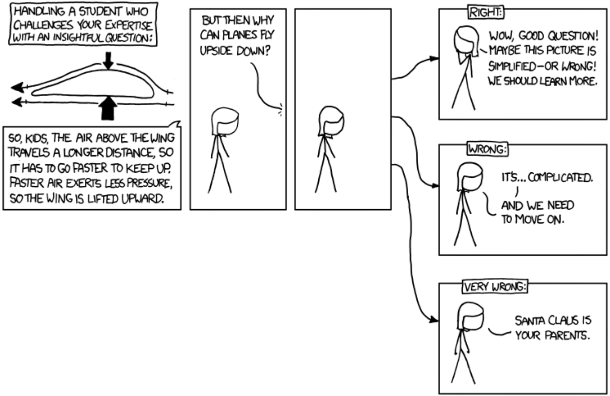
$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2$$

$$V_1 \approx 0, \; p_1 = p_2 = p_{atm}$$

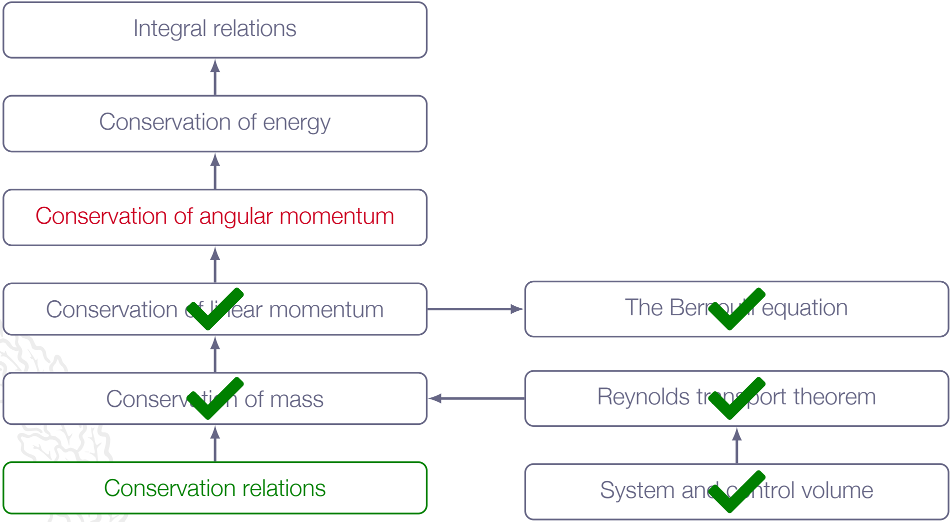
$$V_2^2 = 2g(z_1 - z_2) = 2gh$$

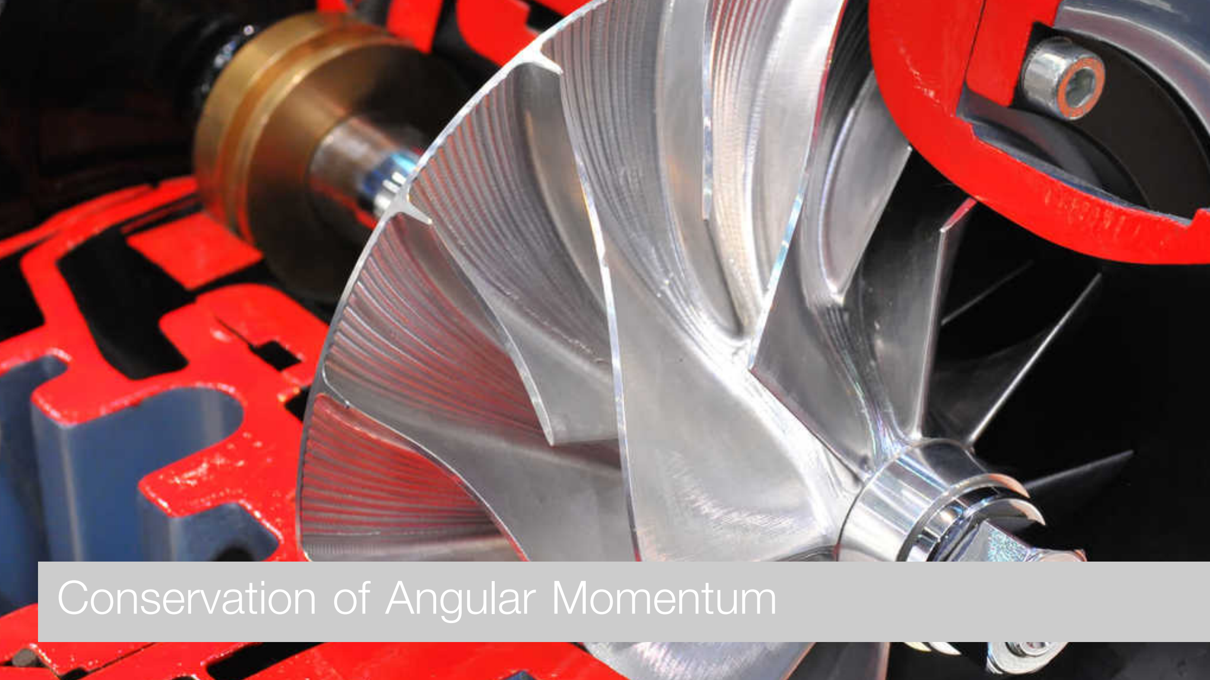
$$V_2 = \sqrt{2gh}$$

Airfoil


HANDLING A STUDENT WHO CHALLENGES YOUR EXPERTISE WITH AN INSIGHTFUL QUESTION:
SO, KIDS, THE AIR ABOVE THE WING TRAVELS A LONGER DISTANCE, SO IT HAS TO GO FASTER TO KEEP UP. FASTER AIR EXERTS LESS PRESSURE, SO THE WING IS LIFTED UPWARD.
BUT THEN WHY CAN PLANES FLY UPSIDE DOWN?
RIGHT:
WOW, GOOD QUESTION! MAYBE THIS PICTURE IS SIMPLIFIED—OR WRONG! WE SHOULD LEARN MORE.
WRONG:
IT'S... COMPLICATED.
AND WE NEED TO MOVE ON.
VERY WRONG:
SANTA CLAUS IS YOUR PARENTS.

Roadmap - Integral Relations





Conservation of Angular Momentum

Angular Momentum

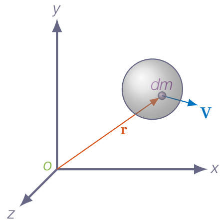
Angular momentum about a point o

$$\mathbf{H}_o = \int_{syst} (\mathbf{r} \times \mathbf{V}) dm = B$$

where $\mathbf{r}$ is the position vector from o to the element mass dm and $\mathbf{V}$ is the velocity of that element

The amount of angular momentum per unit mass

$$\beta = \frac{d\mathbf{H}_o}{dm} = \mathbf{r} \times \mathbf{V}$$



Angular Momentum

Reynold's transport theorem:

$$\left.\frac{d\mathbf{H}_o}{dt}\right|_{syst} = \frac{d}{dt}\left[\int_{CV}(\mathbf{r}\times\mathbf{V})\rho d\mathcal{V}\right] + \int_{CS}(\mathbf{r}\times\mathbf{V})\rho(\mathbf{V}_r\cdot\mathbf{n})dA$$

for inertial coordinate systems:

$$\frac{d\mathbf{H}_o}{dt} = \sum\mathbf{M}_o = \sum(\mathbf{r}\times\mathbf{F})_o$$

Angular Momentum

Non-deformable inertial control volume:

$$\sum \mathbf{M}_o = \frac{d}{dt}\left[\int_{CV}(\mathbf{r}\times\mathbf{V})\rho d\mathcal{V}\right] + \int_{CS}(\mathbf{r}\times\mathbf{V})\rho(\mathbf{V}\cdot\mathbf{n})dA$$

one-dimensional inlets and outlets

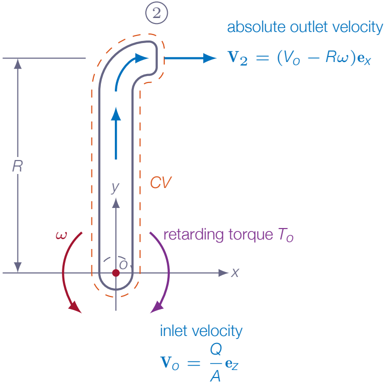
$$\int_{CS}(\mathbf{r}\times\mathbf{V})\rho(\mathbf{V}\cdot\mathbf{n})dA = \sum(\mathbf{r}\times\mathbf{V})_{out}\dot{m}_{out} - \sum(\mathbf{r}\times\mathbf{V})_{in}\dot{m}_{in}$$

Angular Momentum Example - Lawn Sprinkler

$$\sum \mathbf{M}_o = \frac{d}{dt}\left[\int_{CV} (\mathbf{r} \times \mathbf{V})\rho d\mathcal{V}\right] + \int_{CS} (\mathbf{r} \times \mathbf{V})\rho(\mathbf{V_r} \cdot \mathbf{n})dA$$

$$\begin{cases} \mathbf{V}_2 &= (V_o - R\omega, 0, 0) \\ \mathbf{V}_o &= (0, 0, V_o) \end{cases}$$

$$\begin{cases} \mathbf{r}_2 &= (0, R, 0) \\ \mathbf{r}_o &= (0, 0, 0) \end{cases}$$



Angular Momentum Example - Lawn Sprinkler

steady-state:

$$\frac{d}{dt}\left[\int_{CV}(\mathbf{r}\times\mathbf{V})\rho d\mathcal{V}\right]=0$$

Angular Momentum Example - Lawn Sprinkler

$$\sum \mathbf{M}_o = \int_{CS} (\mathbf{r} \times \mathbf{V}) \rho (\mathbf{V}_\mathbf{r} \cdot \mathbf{n}) dA$$

inlet:

$$(\mathbf{r}_o \times \mathbf{V}_o) = (0,0,0) \times (0,0,V_o) = (0,0,0)$$

$$(\mathbf{V}_{o,r} \cdot \mathbf{n}_o) = (0,0,V_o) \cdot (0,0,-1) = -V_o$$

$$(\mathbf{r}_o \times \mathbf{V}_o) \rho (\mathbf{V}_o \cdot \mathbf{n}_o) A_o = -(0,0,0) \rho V_o A_o = (0,0,0)$$

outlet:

$$(\mathbf{r}_2 \times \mathbf{V}_2) = (0,R,0) \times (V_o - R\omega, 0, 0) = (0,0,R^2\omega - RV_o)$$

$$(\mathbf{V}_{2,r} \cdot \mathbf{n}_2) = (V_o,0,0) \cdot (1,0,0) = V_o$$

$$(\mathbf{r}_2 \times \mathbf{V}_2) \rho (\mathbf{V}_2 \cdot \mathbf{n}_2) A_2 = (0,0,R^2\omega - RV_o) \rho V_o A_2 = \rho Q (0,0,R^2\omega - RV_o)$$

Angular Momentum Example - Lawn Sprinkler

$$\sum \mathbf{M}_o = (0, 0, -T_o) = \rho Q(0, 0, R^2\omega - RV_o)$$

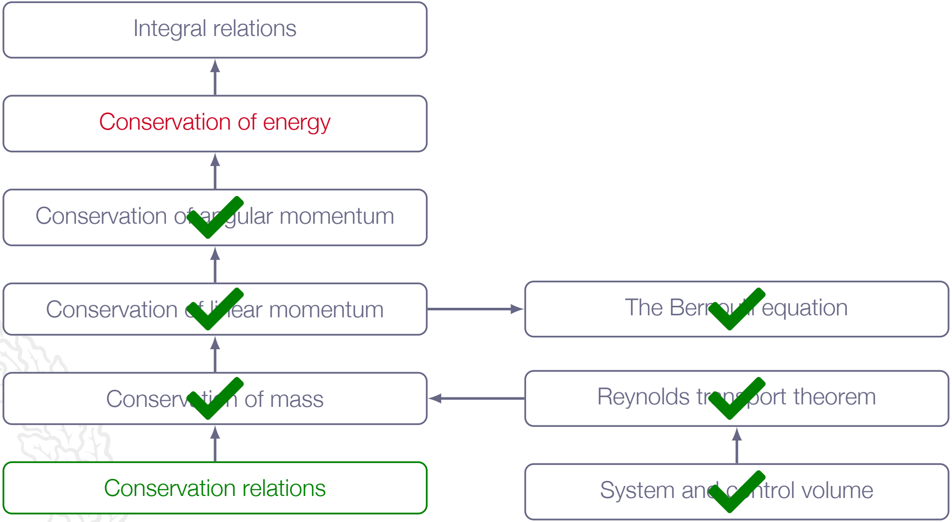
$$\omega = \frac{V_o}{R} - \frac{T_o}{\rho Q R^2}$$

Note:

with a negligible retarding torque, *i.e.* $T_o \approx 0.$, we get $\omega \approx \omega|_{T_o=0.} = \dfrac{V_o}{R}$ $[rad/s]$

the torque required to hold the sprinkler arm still is $T_o|_{\omega=0.} = \rho Q V_o R$ $[Nm]$

Roadmap - Integral Relations



The Energy Equation

Reynold's transport theorem applied the the **first law of thermodynamics** ($B = E$, $\beta = dE/dm = e$)

$$\frac{dQ_{sys}}{dt} - \frac{dW_{sys}}{dt} = \frac{dE_{sys}}{dt} = \frac{d}{dt}\left(\int_{CV} e\rho d\mathcal{V}\right) + \int_{CS} e\rho(\mathbf{V}\cdot\mathbf{n})dA$$

Recall:

positive Q_{sys}: heat added **to** the system

positive W_{sys}: work done **by** the system on its surroundings

The Energy Equation - Energy per Unit Mass

$$e = e_{internal} + e_{kinetic} + e_{potential} + e_{other}$$

e_{other} could be related to, for example, chemical reactions, nuclear reactions, or magnetic fields and will not be considered here

$$e = \hat{u} + \frac{1}{2}V^2 + gz$$

The Energy Equation - Work

The work term $\dot{W}$ can be divided into shaft work, pressure work, and work related to viscous forces

$$\dot{W} = \dot{W}_s + \dot{W}_p + \dot{W}_\nu$$

$$\dot{W}_p = \int_{CS} p(\mathbf{V} \cdot \mathbf{n})dA$$

$$\dot{W}_\nu = -\int_{CS} \tau \cdot \mathbf{V} dA$$

The Energy Equation - Pressure Work

$$\dot{W}_p = \int_{CS} p(\mathbf{V} \cdot \mathbf{n})dA$$

The rate of work done by **pressure forces** on the **control volume surfaces**

internal forces will always have an opposite force leading to cancelation

The Energy Equation - Viscous Work

$$\dot{W}_\nu = -\int_{CS} \tau \cdot \mathbf{V} dA$$

The rate of work related to **viscous stresses** on the **control volume surfaces**

important or not depending on flow situation

The Energy Equation - Control Volume Boundaries

Solid walls:

no-slip $\Rightarrow \dot{W}_\nu = 0$

Machine surfaces:

viscous work included implicitly in shaft work

Inlets/outlets:

flow aligned with surface normal (usually) and normal viscous stress components are in most cases very small

Streamlines:

viscous stresses may be significant *depending on streamline location*

The Energy Equation

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt}\left(\int_{CV} e\rho d\mathcal{V}\right) + \int_{CS} \rho e(\mathbf{V}\cdot\mathbf{n})dA + \int_{CS} p(\mathbf{V}\cdot\mathbf{n})dA$$

collecting surface integrals gives

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt}\left(\int_{CV} e\rho d\mathcal{V}\right) + \int_{CS} \left(e + \frac{p}{\rho}\right)\rho(\mathbf{V}\cdot\mathbf{n})dA$$

The Energy Equation

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt}\left(\int_{CV} e\rho d\mathcal{V}\right) + \int_{CS}\left(e + \frac{p}{\rho}\right)\rho(\mathbf{V}\cdot\mathbf{n})dA$$

or

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt}\left[\int_{CV}\left(\hat{u} + \frac{1}{2}V^2 + gz\right)\rho d\mathcal{V}\right] + \\ + \int_{CS}\left(\hat{h} + \frac{1}{2}V^2 + gz\right)\rho(\mathbf{V}\cdot\mathbf{n})dA$$

where $\hat{h}$ is the enthalpy defined as $\hat{h} = \hat{u} + p/\rho$

The Energy Equation

Steady state:

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \int_{CS} \left(\hat{h} + \frac{1}{2}V^2 + gz \right) \rho(\mathbf{V} \cdot \mathbf{n}) dA$$

Special case: one inlet and one outlet (both one-dimensional)

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = -\dot{m}_1 \left(\hat{h}_1 + \frac{1}{2}V_1^2 + gz_1 \right) + \dot{m}_2 \left(\hat{h}_2 + \frac{1}{2}V_2^2 + gz_2 \right)$$

continuity $\Rightarrow \dot{m}_1 = \dot{m}_2 = \dot{m}$, divide by $\dot{m}$ gives

$$\hat{h}_1 + \frac{1}{2}V_1^2 + gz_1 = \hat{h}_2 + \frac{1}{2}V_2^2 + gz_2 - q + w_s + w_\nu$$

The Energy Equation

$$\hat{h}_1 + \frac{1}{2}V_1^2 + gz_1 = \hat{h}_2 + \frac{1}{2}V_2^2 + gz_2 - q + w_s + w_\nu$$

all terms has the dimension $[m^2/s^2]$, divide by g $[m/s^2]$ to get dimension $[m]$

$$\frac{p_1}{\rho g} + \frac{\hat{u}_1}{g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{\hat{u}_2}{g} + \frac{V_2^2}{2g} + z_2 - \frac{q}{g} + h_s + h_\nu$$

$p/(\rho g)$: pressure head

$V^2/2g$: velocity head

The Energy Equation

- steady-state flow
- incompressible (low speed)
- pipe/duct that may or may not include turbines and pumps
- solid walls $\Rightarrow h_\nu = 0$

$$\underbrace{\left(\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1\right)}_{h_{o_1}}=\underbrace{\left(\frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2\right)}_{h_{o_2}}+\frac{\hat{u}_2-\hat{u}_1-q}{g}$$

where h_o is *available head* or *total head*

The Energy Equation

- friction head losses h_f (always positive)
- pump head input h_p
- turbine head extraction h_t

$$\left(\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1\right)_{in}=\left(\frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2\right)_{out}+h_f-h_p+h_t$$

Kinetic Energy Correction Factor

One-dimensional flow through inlets and outlets is of course not true in reality

Introducing the correction factor α

$$\int \frac{1}{2} V^2 \rho (\mathbf{V} \cdot \mathbf{n}) dA = \frac{\alpha {V_{av}}^2}{2} \dot{m}$$

where (for incompressible flow)

$$V_{av} = \frac{1}{A} \int u dA$$

Kinetic Energy Correction Factor

$$\left(\frac{p_1}{\rho g}+\frac{\alpha_1 V_1^2}{2g}+z_1\right)_{in}=\left(\frac{p_2}{\rho g}+\frac{\alpha_2 V_2^2}{2g}+z_2\right)_{out}+h_f-h_p+h_t$$

Kinetic Energy Correction Factor

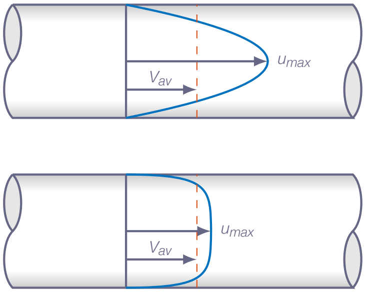
Laminar pipe flow:

Velocity profile:

$$u(r) = U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

From example 2 in the conservation-of-mass section we have that the average velocity for a laminar pipe flow is obtained as:

$$V_{av} = \frac{1}{2}U_{max}$$



Kinetic Energy Correction Factor

$$u(r) = U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

$$\int \frac{1}{2}V^2\rho(\mathbf{V}\cdot\mathbf{n})dA = \int_0^R \frac{1}{2}U_{max}^2\left(1 - \left(\frac{r}{R}\right)^2\right)^2 \rho U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right) 2\pi r dr$$

$$\int \frac{1}{2}V^2\rho(\mathbf{V}\cdot\mathbf{n})dA = \rho\pi U_{max}^3 \int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right)^3 r dr$$

$$\int \frac{1}{2}V^2\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{1}{8}\rho\pi R^2 U_{max}^3$$

Kinetic Energy Correction Factor

$$\int \frac{1}{2} V^2 \rho (\mathbf{V} \cdot \mathbf{n}) dA = \frac{1}{8} \rho \pi R^2 U_{max}^3$$

the mass flow $\dot{m}$ can be obtained as:

$$\dot{m} = V_{av} \rho \pi R^2 = \frac{1}{2} U_{max} \rho \pi R^2$$

which gives

$$\int \frac{1}{2} V^2 \rho (\mathbf{V} \cdot \mathbf{n}) dA = \frac{U_{max}^2}{4} \dot{m}$$

from the definition of α

$$\int \frac{1}{2} V^2 \rho (\mathbf{V} \cdot \mathbf{n}) dA = \frac{\alpha V_{av}^2}{2} \dot{m} = \left\{ V_{av} = \frac{1}{2} U_{max} \right\} = \frac{\alpha U_{max}^2}{8} \dot{m}$$

Kinetic Energy Correction Factor

comparing the two expressions, we have that

$$\frac{U_{max}^2}{4}\dot{m} = \frac{\alpha U_{max}^2}{8}\dot{m}$$

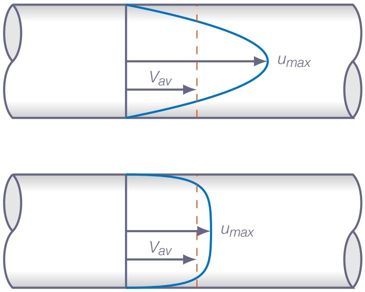
and thus $\alpha = 2$ for laminar incompressible flow

Kinetic Energy Correction Factor

Turbulent pipe flow:

Velocity Profile:

$$u(r) \approx U_{max}\left(1 - \frac{r}{R}\right)^m, \; m \approx \frac{1}{7}$$



From example 3 in the conservation-of-mass section we have that the average velocity for a laminar pipe flow is obtained as:

$$V_{av} = \frac{2U_{max}}{(1+m)(2+m)}$$

Kinetic Energy Correction Factor

$$\rho\pi U_{max}^3 \int_0^R \left(1-\frac{r}{R}\right)^{3m} rdr = \alpha\frac{1}{2}\rho\pi R^2 V_{av}^3 = \alpha\frac{4\rho\pi R^2 U_{max}^3}{(1+m)^3(2+m)^3} \Rightarrow$$

$$\left[\frac{(r-R)\left(1-\frac{r}{R}\right)^{3m}(3mr+r+R)}{(1+3m)(2+3m)}\right]_0^R = \alpha\frac{4R^2}{(1+m)^3(2+m)^3} \Rightarrow$$

$$\frac{R^2}{(1+3m)(2+3m)} = \alpha\frac{4R^2}{(1+m)^3(2+m)^3} \Rightarrow \alpha = \frac{(1+m)^3(2+m)^3}{4(1+3m)(2+3m)}$$

Kinetic Energy Correction Factor

Laminar pipe flow:

$\alpha = 2.0$ should be used

Turbulent pipe flow:

$$\alpha = \frac{(1+m)^3(2+m)^3}{4(1+3m)(2+3m)}$$

m	1/5	1/6	1/7	1/8	1/9
α	1.106	1.077	1.058	1.046	1.037

($\alpha = 1.0$ is often a good approximation)

The Energy Equation - Pump Example

Task:

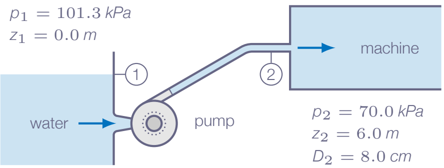
Calculate pump power if $\eta = 0.8$

Given:

1. Geometry and pressures from figure
2. The pump delivers water at a flow rate $Q = 0.04\ m^3/s$
3. Friction losses between 1 and 2 are given by $h_f = KV_2^2/(2g)$ where $K \approx 7.5$
4. $\alpha \approx 1.07$

Assumptions:

1. steady-state flow
2. negligible viscous work
3. large reservoir ($V_1 \approx 0$)



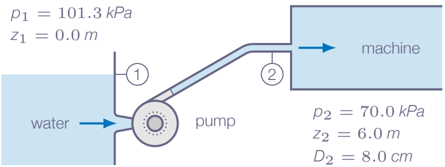
The Energy Equation - Pump Example

$$V_2 = \frac{Q}{A_2} = 7.96\ m/s$$

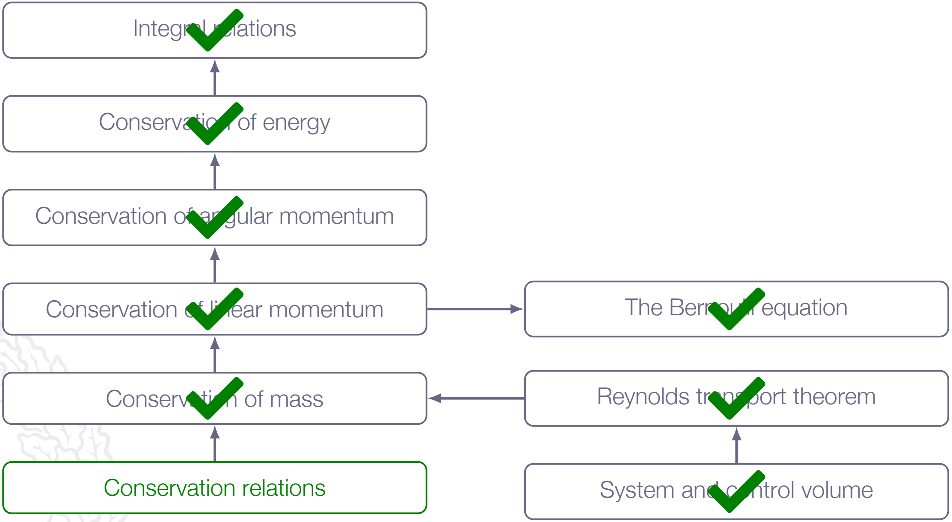
$$\left(\frac{p_1}{\rho g} + \frac{\alpha V_1^2}{2g} + z_1\right)_{in} = \left(\frac{p_2}{\rho g} + \frac{\alpha V_2^2}{2g} + z_2\right)_{out} + h_f - h_p + h_t$$

$$h_p = \frac{p_2 - p_1}{\rho g} + (z_2 - z_1) + (\alpha_2 + K)\frac{V_2^2}{2g} = 30.5\ m$$

$$P_{pump} = \frac{\rho g Q h_p}{\eta} = 14960\ W\ (or\ 20\ hp)$$



Roadmap - Integral Relations



Integral Relations - Considerations

1. control volume type: **non-deforming**?, **non-accelerating**?

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7. is there **heat transfer**, **shaft work** or **viscous work**

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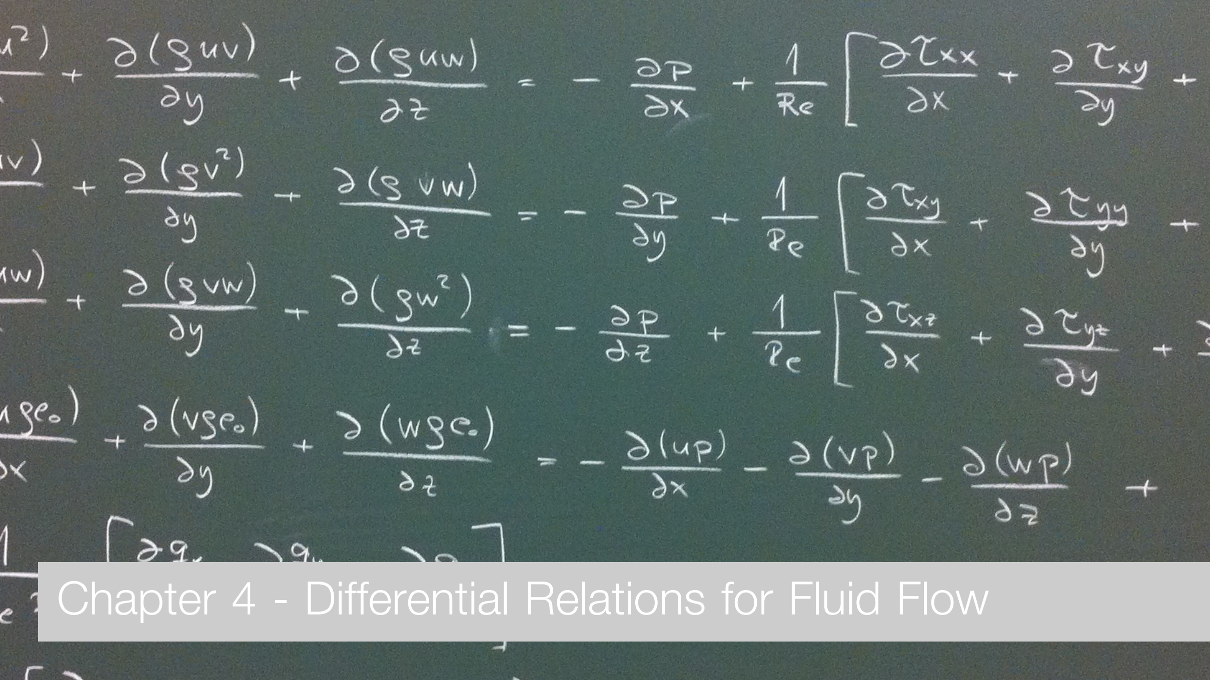
4. can the fluid be assumed to be **incompressible**?
5. if compressible, can the ideal gas law be used?
6. do we need to account for **body forces** (gravity etc)?
7. is there **heat transfer**, **shaft work** or **viscous work**
8. can inlets/outlets be assumed to be **one-dimensional**



Chapter 4 - Differential Relations for Fluid Flow

Overview

The Pi theorem
non-dimensional equations
rotation
stream function
modeling and similarity
Dimen-sional Analysis
conser-vation relations
Differential Relations
Bernoulli
Flow Relations
Integral Relations
conser-vation relations
hydrostatic forces
buoyancy

Learning Outcomes

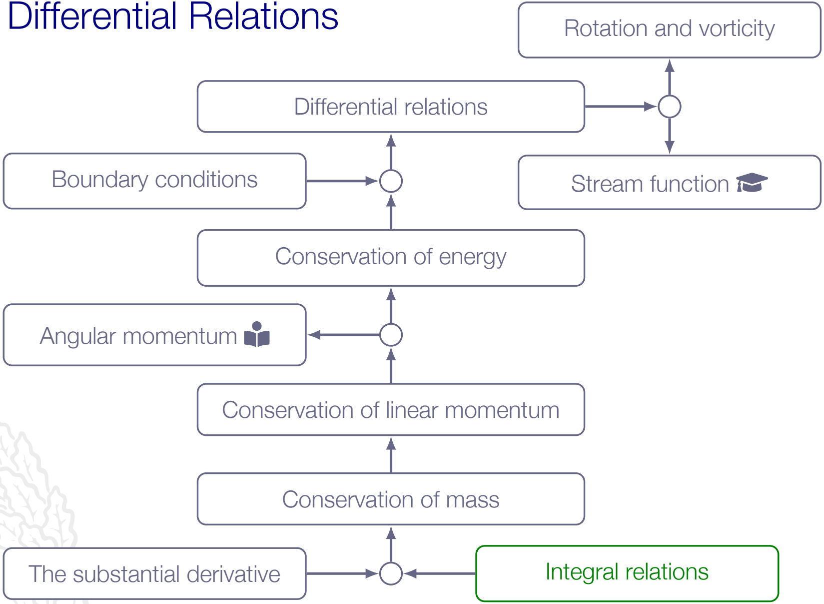
4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

14 **Derive** the continuity, momentum and energy equations on differential form

36 **Define** and explain vorticity

let's push the control volume approach to the limit ...

Roadmap - Differential Relations



Differential Relations

seeking the point-by-point details of a flow pattern by analyzing an infinitesimal region of the flow

Differential Relations

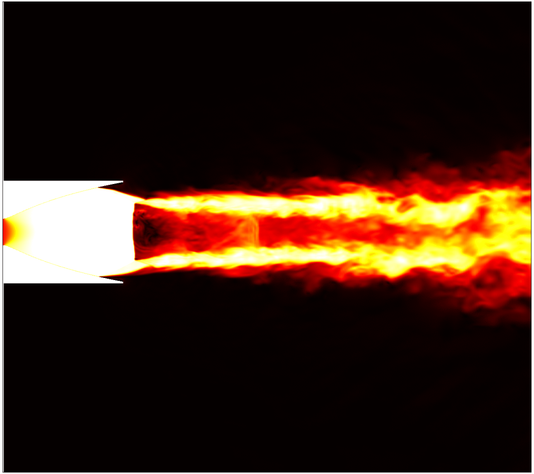
Apply the four basic conservation laws to an infinitesimal control volume

The differential relations are in general very difficult to solve

Analytical solutions exists for a few cases

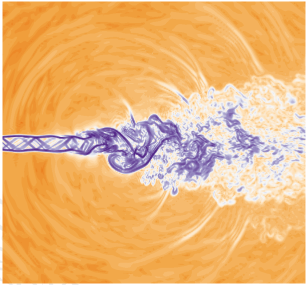
The differential relations form the basis for CFD software

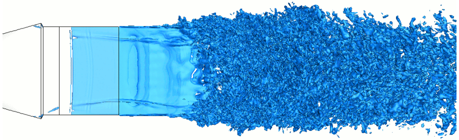
High-Speed Nozzle Flow



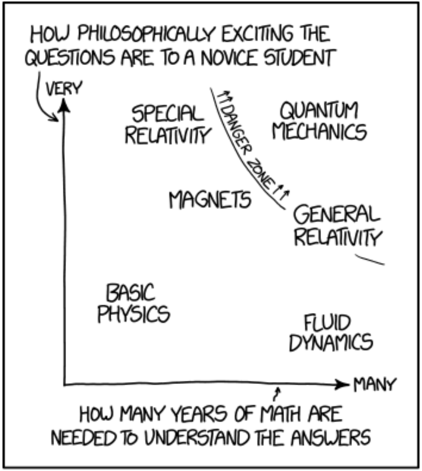
The Acoustic Signature of a Supersonic Jet

Screeching rectangular supersonic jet

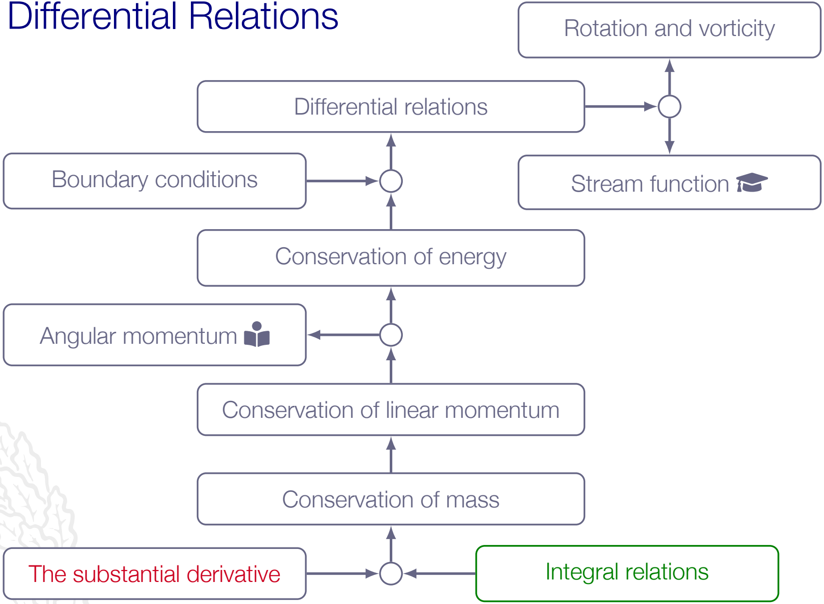




Lots of Equations Ahead



Roadmap - Differential Relations



Frame of Reference

Eulerian: observer fixed in space

Lagrangian: observer follows a fluid particle

recall the speedometer/traffic-camera analogy

Acceleration Field

In order to get to Newton's second law, we need the acceleration vector

Velocity field:

$$\mathbf{V}(\mathbf{r},t)=\mathbf{e}_x u(x,y,z,t)+\mathbf{e}_y v(x,y,z,t)+\mathbf{e}_z w(x,y,z,t)$$

Acceleration field:

$$\mathbf{a}=\frac{d\mathbf{V}}{dt}=\mathbf{e}_x\frac{du}{dt}+\mathbf{e}_y\frac{dv}{dt}+\mathbf{e}_z\frac{dw}{dt}$$

Acceleration Field

Each scalar component of the velocity vector (u, v, w) is a function of four variables (x, y, z, t) and thus

$$\frac{du(x,y,z,t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

By definition $dx/dt = u$, $dy/dt = v$, and $dz/dt = w$

$$\frac{du(x,y,z,t)}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\mathbf{V} \cdot \nabla)u$$

Acceleration Field

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\mathbf{V}\cdot\nabla)u$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{\partial v}{\partial t} + (\mathbf{V}\cdot\nabla)v$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} + (\mathbf{V}\cdot\nabla)w$$

$$\mathbf{a} = \underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{local} + \underbrace{\left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)}_{convective} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = \frac{D\mathbf{V}}{Dt}$$

Acceleration Field

$$\mathbf{a} = \underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{local} + \underbrace{\left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} \right)}_{convective} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = \frac{D\mathbf{V}}{Dt}$$

local acceleration: only in unsteady flows

convective acceleration: fluid particle that moves through regions of spatially varying velocity

Substantial derivative

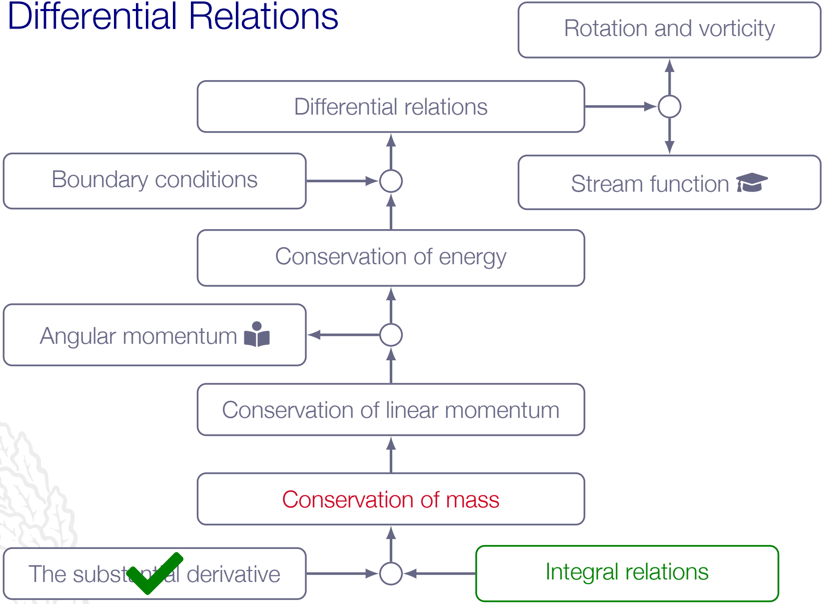
the sum of the **local** derivative and the **convective** derivative

follows a fluid particle but is expressed in an **Eulerian frame of reference**

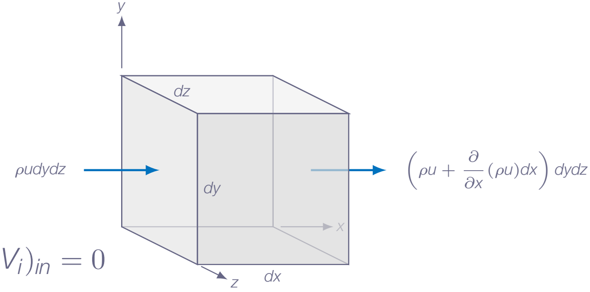
an **operator** that can be applied to any variable

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} = \frac{\partial p}{\partial t} + (\mathbf{V} \cdot \nabla)p$$

Roadmap - Differential Relations



Mass Conservation



$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$

$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} \approx \frac{\partial \rho}{\partial t} dxdydz$$

$$\frac{\partial \rho}{\partial t} dxdydz + \frac{\partial}{\partial x}(\rho u)dxdydz + \frac{\partial}{\partial y}(\rho v)dxdydz + \frac{\partial}{\partial z}(\rho w)dxdydz = 0$$

Mass Conservation

The result is the **continuity equation** - conservation of mass for an infinitesimal control volume

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

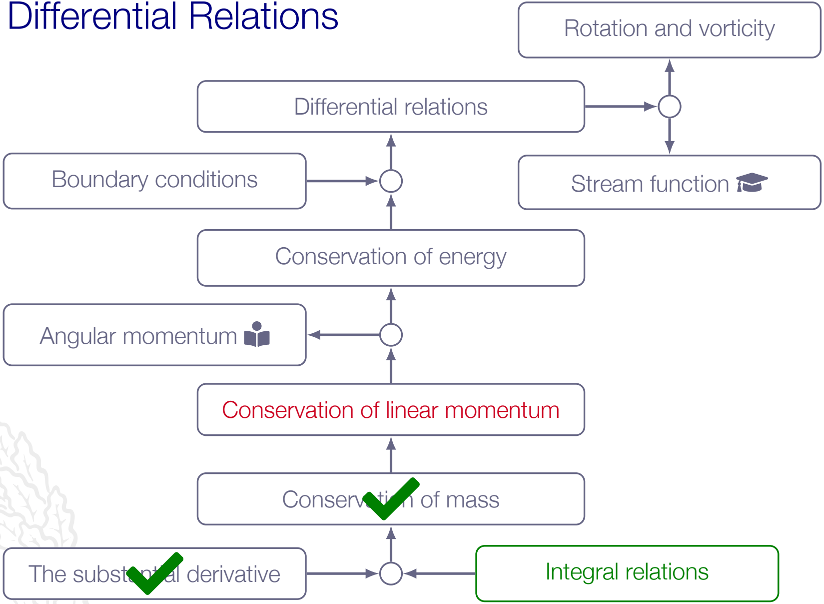
or in more compact form using vector notation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

Incompressible flow (constant density)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{V} = 0$$

Roadmap - Differential Relations

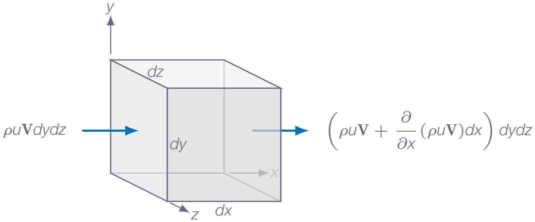


Linear Momentum

$$\sum \mathbf{F} = \int_{CV} \frac{\partial}{\partial t}(\mathbf{V}\rho)d\mathcal{V} + \sum(\dot{m}_i\mathbf{V}_i)_{out} - \sum(\dot{m}_i\mathbf{V}_i)_{in}$$

$$\frac{\partial}{\partial t}(\mathbf{V}\rho)d\mathcal{V} \approx \frac{\partial}{\partial t}(\mathbf{V}\rho)dxdydz$$

Linear Momentum



Face	Inlet momentum flux	Outlet momentum flux
x	$\rho u \mathbf{V} dydz$	$\left[\rho u \mathbf{V} + \frac{\partial}{\partial x}(\rho u \mathbf{V})dx\right] dydz$
y	$\rho v \mathbf{V} dxdz$	$\left[\rho v \mathbf{V} + \frac{\partial}{\partial y}(\rho v \mathbf{V})dy\right] dxdz$
z	$\rho w \mathbf{V} dxdy$	$\left[\rho w \mathbf{V} + \frac{\partial}{\partial z}(\rho w \mathbf{V})dz\right] dxdy$

Linear Momentum

$$\sum \mathbf{F} = \left[\frac{\partial}{\partial t}(\mathbf{V}\rho) + \frac{\partial}{\partial x}(\rho u\mathbf{V}) + \frac{\partial}{\partial y}(\rho v\mathbf{V}) + \frac{\partial}{\partial z}(\rho w\mathbf{V})\right] dxdydz$$

Linear Momentum

$$\underbrace{\frac{\partial}{\partial t}(\mathbf{V}\rho)}_{\mathbf{V}\frac{\partial\rho}{\partial t}+\rho\frac{\partial\mathbf{V}}{\partial t}} + \underbrace{\frac{\partial}{\partial x}(\rho u\mathbf{V})}_{\mathbf{V}\frac{\partial}{\partial x}(\rho u)+\rho u\frac{\partial\mathbf{V}}{\partial x}} + \underbrace{\frac{\partial}{\partial y}(\rho v\mathbf{V})}_{\mathbf{V}\frac{\partial}{\partial y}(\rho v)+\rho v\frac{\partial\mathbf{V}}{\partial y}} + \underbrace{\frac{\partial}{\partial z}(\rho w\mathbf{V})}_{\mathbf{V}\frac{\partial}{\partial z}(\rho w)+\rho w\frac{\partial\mathbf{V}}{\partial z}}$$

can be rewritten as

$$\underbrace{\mathbf{V}\left[\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{V})\right]}_{continuity\ equation} + \rho\underbrace{\left(\frac{\partial\mathbf{V}}{\partial t}+u\frac{\partial\mathbf{V}}{\partial x}+v\frac{\partial\mathbf{V}}{\partial y}+w\frac{\partial\mathbf{V}}{\partial z}\right)}_{\frac{\partial\mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}=\frac{D\mathbf{V}}{Dt}}$$

and thus

$$\sum\mathbf{F}=\rho\frac{D\mathbf{V}}{Dt}dxdydz$$

Linear Momentum - Forces

$$\sum \mathbf{F} = \rho \frac{D\mathbf{V}}{Dt} dxdydz$$

$\sum \mathbf{F}$:

body forces: gravity and other field forces
surface forces: pressure and viscous stresses

Linear Momentum - Gravity Force

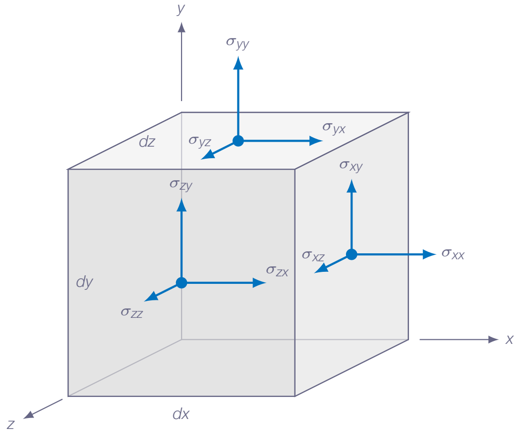
$$d\mathbf{F}_{gravity} = \rho\mathbf{g}dxdydz$$

if gravity is aligned with the negative z-direction

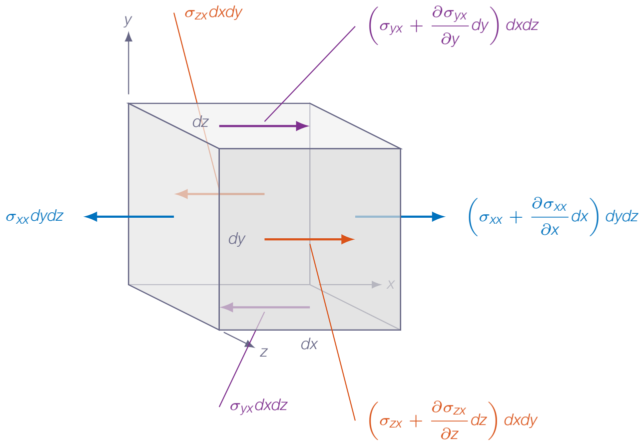
$$d\mathbf{F}_{gravity} = -\mathbf{e}_z\rho g dxdydz$$

Linear Momentum - Surface Forces

$$\sigma_{ij} = \begin{vmatrix} -p + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p + \tau_{zz} \end{vmatrix}$$



Linear Momentum - Surface Forces



$$dF_{x,surf} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right] dxdydz$$

Linear Momentum - Surface Forces

$$dF_{x,surf} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right] dxdydz$$

$$\sigma_{xx} = \tau_{xx} - p, \;\; \sigma_{yx} = \tau_{yx}, \;\; \sigma_{zx} = \tau_{zx}$$

$$dF_{x,surf} = \left[-\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx})\right] dxdydz$$

Linear Momentum - Surface Forces

$$dF_{x,surf} = \left[-\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx})\right] dxdydz$$

$$dF_{y,surf} = \left[-\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{zy})\right] dxdydz$$

$$dF_{z,surf} = \left[-\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\tau_{zz})\right] dxdydz$$

Linear Momentum - Surface Forces

$$d\mathbf{F} = \left[-\nabla p + \nabla \cdot \tau_{ij}\right] dxdydz$$

where

$$\tau_{ij} = \left|\begin{array}{ccc} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{array}\right|$$

is the viscous stress tensor

Linear Momentum - Forces

Now, inserting the forces into the momentum equation gives

$$\rho \mathbf{g} - \nabla p + \nabla \cdot \tau_{ij} = \rho \frac{D\mathbf{V}}{Dt}$$

Linear Momentum

vector notation is powerful, tensor notation is even better ...

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right)$$

Note! the convective term (RHS) is nonlinear

Linear Momentum

Recall:

"For a Newtonian fluid, the viscous stresses are proportional to the element strain and the viscosity"

For incompressible flow:

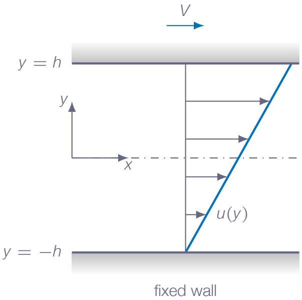
$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}, \; \tau_{yy} = 2\mu\frac{\partial v}{\partial y}, \; \tau_{zz} = 2\mu\frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \; \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Linear Momentum

Couette flow: $u = u(y)$



$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} = 0$$

$$\tau_{yy} = 2\mu \frac{\partial v}{\partial y} = 0$$

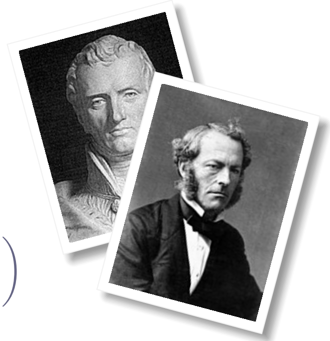
$$\tau_{zz} = 2\mu \frac{\partial w}{\partial z} = 0$$

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \mu \frac{\partial u}{\partial y}$$

$$\tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0$$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = 0$$

Linear Momentum

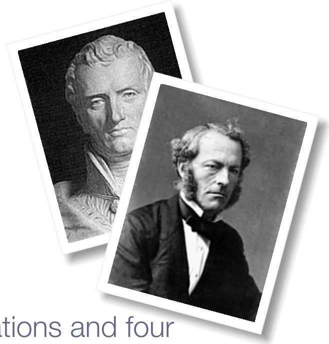


The incompressible Navier-Stokes equations

$$\rho\frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\frac{Dv}{Dt} = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\frac{Dw}{Dt} = \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$



The incompressible Navier-Stokes equations

- Non-linear equations
- Three equations and four unknowns (p, u, v, w)
- Combined with the continuity equations we have four equations and four unknowns

ABOUT PROGRAMS MILLENNIUM PROBLEMS PEOPLE PUBLICATIONS EVENTS EUCLID

Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

Navier–Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

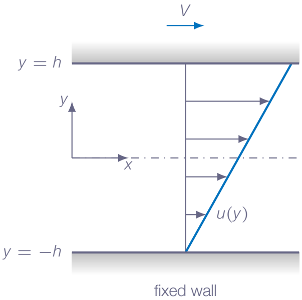
Poincaré Conjecture

In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold. This question, the Poincaré conjecture, was a special case of Thurston's geometrization conjecture. Perelman's proof tells us that every three manifold is built from a set of standard pieces, each with one of eight well-understood geometries.

Birch and Swinnerton-Dyer Conjecture

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of numbers into primes, and cryptography, to name three.

Example - Couette Flow



- incompressible ($\rho = const$)
- steady-state
- lower plate fixed, upper plate moving with the velocity V
- flow only in the x-direction $v = w = 0, \; u \neq 0$
- no pressure gradient

Example - Couette Flow

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \{v = w = 0\} \Rightarrow \frac{\partial u}{\partial x} = 0$$

momentum equation (x-direction):

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \Rightarrow \{v = w = 0,\ \frac{\partial p}{\partial x} = 0\} \Rightarrow \mu \frac{\partial^2 u}{\partial y^2} = 0$$

Example - Couette Flow

$$\frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u = ay + b$$

boundary conditions:

$$\left.\begin{array}{l} u(h) = V \\ u(-h) = 0 \end{array}\right\} \Rightarrow$$

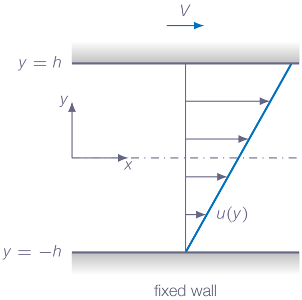
$$\begin{array}{rcrcr} & V & = & ah & + & b \\ + & 0 & = & -ah & + & b \\ \hline & V & = & & & 2b \end{array}$$

$$b = \frac{V}{2}$$

$$\begin{array}{rcrcr} & V & = & ah & + & b \\ - & 0 & = & -ah & + & b \\ \hline & V & = & 2ah & & \end{array}$$

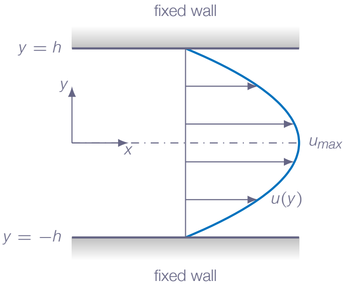
$$a = \frac{V}{2h}$$

Example - Couette Flow



$$u = \frac{V}{2}\left(\frac{y}{h} - 1\right)$$

Example - Poiseuille Flow



- incompressible ($\rho = const$)
- steady-state
- lower and upper plate fixed
- flow only in the x-direction $v = w = 0,\ u \neq 0$
- pressure gradient driven

Example - Poiseuille Flow

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \{v = w = 0\} \Rightarrow \frac{\partial u}{\partial x} = 0$$

momentum equation (x-direction):

$$\rho \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \Rightarrow \{v = w = 0\} \Rightarrow \mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}$$

Example - Poiseuille Flow

momentum equation (y-direction and z-direction):

$$
\left.\begin{aligned}
\frac{\partial p}{\partial y} &= 0 \\
\\
\frac{\partial p}{\partial z} &= 0
\end{aligned}\right\} \Rightarrow p = p(x)
$$

Example - Poiseuille Flow

$$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx} = const < 0$$

- Why constant?
  - RHS function of x only
  - LHS function of y only
  - RHS=LHS $\Rightarrow$ must be a constant
- Why < 0?
  - pressure must decrease in the flow direction

Example - Poiseuille Flow

$$\mu \frac{d^2 u}{dy^2} = \frac{dp}{dx} = const < 0$$

$$u = \frac{1}{2\mu} \frac{dp}{dx} y^2 + ay + b$$

Example - Poiseuille Flow

boundary conditions:

$$\left.\begin{aligned} u(h) &= 0 \\ u(-h) &= 0 \end{aligned}\right\} \Rightarrow$$

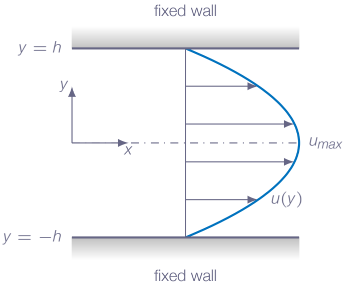
$$\begin{array}{rrcrrrrr} & 0 & = & \frac{1}{2\mu}\frac{dp}{dx}h^2 & + & ah & + & b \\ + & 0 & = & \frac{1}{2\mu}\frac{dp}{dx}h^2 & - & ah & + & b \\ \hline & 0 & = & \frac{1}{\mu}\frac{dp}{dx}h^2 & & & + & 2b \end{array}$$

$$b = -\frac{1}{2\mu}\frac{dp}{dx}h^2$$

$$\begin{array}{rrcrrrrr} & 0 & = & \frac{1}{2\mu}\frac{dp}{dx}h^2 & + & ah & + & b \\ - & 0 & = & \frac{1}{2\mu}\frac{dp}{dx}h^2 & - & ah & + & b \\ \hline & 0 & = & & & 2ah & & \end{array}$$

$$a = 0$$

Example - Poiseuille Flow

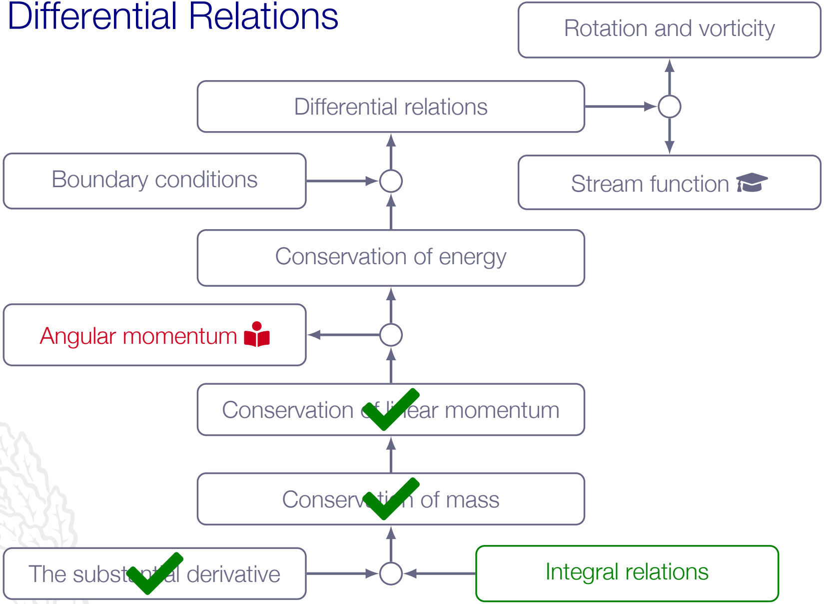


$$u = -\frac{h^2}{2\mu}\frac{dp}{dx}\left(1 - \left(\frac{y}{h}\right)^2\right)$$

$$\frac{du}{dy} = \frac{dp}{dx}\frac{y}{\mu} \Rightarrow \left.\frac{du}{dy}\right|_{y=0} = 0 \Rightarrow u_{max} = u(0) = -\frac{dp}{dx}\frac{h^2}{2\mu}$$

(remember: $dp/dx < 0$)

Roadmap - Differential Relations

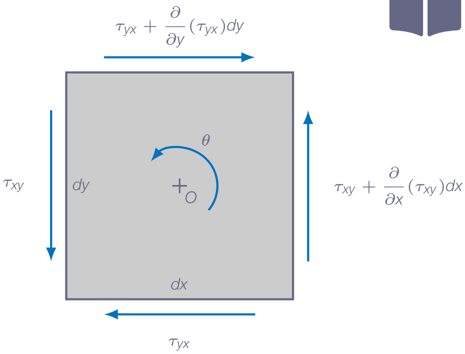


Angular Momentum

$$\sum \mathbf{M}_o = \frac{d}{dt}\left(\int_{CV} \rho(\mathbf{r} \times \mathbf{V}) d\mathcal{V}\right) + \int_{CS} (\mathbf{r} \times \mathbf{V}) \rho(\mathbf{V} \cdot \mathbf{n}) dA$$

Angular Momentum

- axis through o parallel to the z-axis
- axis through the centroid of the element
- θ angle of rotation about o



$$\left[\tau_{xy} - \tau_{yx} + \frac{1}{2}\frac{\partial}{\partial x}(\tau_{xy})dx - \frac{1}{2}\frac{\partial}{\partial y}(\tau_{yx})dy\right] dxdydz =$$

$$\frac{1}{12}\rho(dxdydz)(dx^2 + dy^2)\frac{d^2\theta}{dt^2}$$

Angular Momentum

$$\left[\tau_{xy} - \tau_{yx} + \frac{1}{2}\frac{\partial}{\partial x}(\tau_{xy})dx - \frac{1}{2}\frac{\partial}{\partial y}(\tau_{yx})dy\right] dxdydz =$$

$$\frac{1}{12}\rho(dxdydz)(dx^2 + dy^2)\frac{d^2\theta}{dt^2}$$

Neglect higher-order differential terms gives

$$\boxed{\tau_{xy} \approx \tau_{yx}}$$

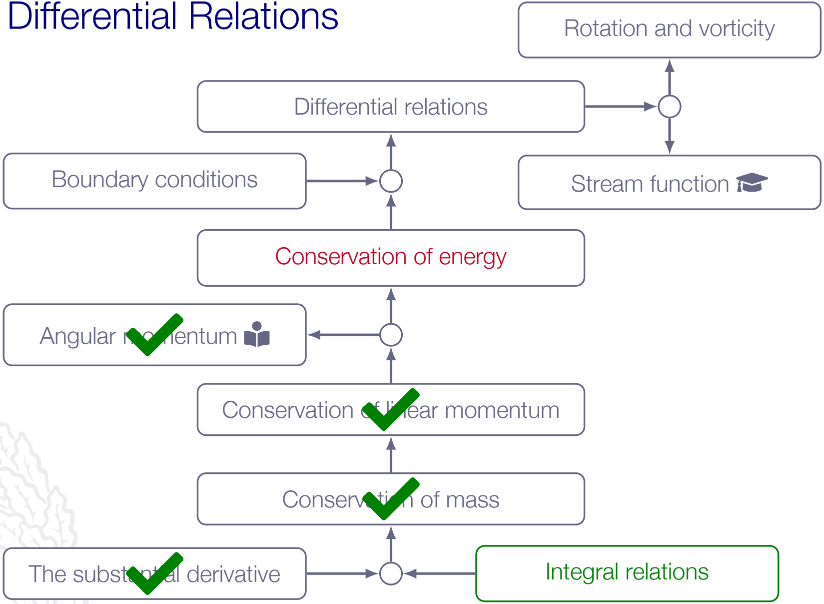
Analogously, we may obtain $\tau_{xz} \approx \tau_{zx}$ and $\tau_{zy} \approx \tau_{yz}$

Angular Momentum

Note! there is no differential angular momentum equation ...

the only result from this section is that shear stresses are symmetric: $\tau_{ij} = \tau_{ji}$

Roadmap - Differential Relations



The Energy Equation

Integral formulation:

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt}\left(\int_{CV} e\rho d\mathcal{V}\right) + \int_{CS}\left(e + \frac{p}{\rho}\right)\rho(\mathbf{V}\cdot\mathbf{n})dA$$

$$h = e + p/\rho$$

Differential form:

$$\dot{Q} - \dot{W}_\nu = \left[\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h) + \frac{\partial}{\partial z}(\rho w h)\right] dxdydz$$

$\dot{W}_s = 0$ *we can not have a infinitesimal shaft protruding the control volume*

The Energy Equation

$$\dot{Q} - \dot{W}_\nu = \left[\underbrace{\frac{\partial}{\partial t}(\rho e)}_{I} + \underbrace{\frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h) + \frac{\partial}{\partial z}(\rho w h)}_{II} \right] dxdydz$$

Part I.

$$\frac{\partial}{\partial t}(\rho e) = e\frac{\partial \rho}{\partial t} + \rho\frac{\partial e}{\partial t}$$

The Energy Equation

Part II.

$$\frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h) + \frac{\partial}{\partial z}(\rho w h) =$$

$$\underbrace{\frac{\partial}{\partial x}(\rho u e) + \frac{\partial}{\partial y}(\rho v e) + \frac{\partial}{\partial z}(\rho w e)}_{*} + \underbrace{\frac{\partial}{\partial x}(u p) + \frac{\partial}{\partial y}(v p) + \frac{\partial}{\partial z}(w p)}_{**}$$

Part II*

$$\frac{\partial}{\partial x}(\rho u e) + \frac{\partial}{\partial y}(\rho v e) + \frac{\partial}{\partial z}(\rho w e) =$$

$$e\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] + \rho\left[u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y} + w\frac{\partial e}{\partial z}\right]$$

The Energy Equation

Part II**

$$\frac{\partial}{\partial x}(up) + \frac{\partial}{\partial y}(vp) + \frac{\partial}{\partial z}(wp) =$$

$$p\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} =$$

$$p\nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p$$

The Energy Equation

reassemble and collect terms:

$$\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h) + \frac{\partial}{\partial z}(\rho w h) =$$

$$\underbrace{\rho \left[\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y} + w\frac{\partial e}{\partial z}\right]}_{\frac{\partial e}{\partial t}+(\mathbf{V}\cdot\nabla)e=\frac{De}{Dt}} +$$

$$\underbrace{e \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]}_{\text{continuity equation}} +$$

$$p\nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p$$

The Energy Equation

$$\dot{Q} - \dot{W}_\nu = \left[\rho \frac{De}{Dt} + p\nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p\right] dxdydz$$

The Energy Equation - Added Heat

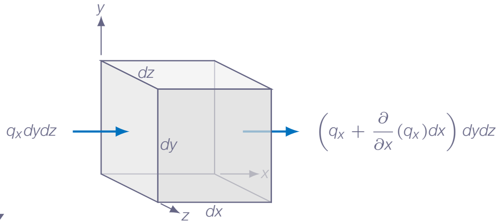
Now, let's have a look at the added heat term $\dot{Q}$

- Only **conduction** will be considered (no radiation)
- According the **Fourier's law** of conduction, the **heat flux** is proportional to the **temperature gradient**

$$\mathbf{q} = -k\nabla T$$

where k is the **thermal conductivity** and $\mathbf{q}$ is heat transfer per unit area

The Energy Equation - Added Heat



Face	Inlet heat flux	Outlet heat flux
x	$q_x dydz$	$\left[q_x + \frac{\partial q_x}{\partial x}dx\right] dydz \quad \text{where } q_x = -k\frac{\partial T}{\partial x}$
y	$q_y dxdz$	$\left[q_y + \frac{\partial q_y}{\partial y}dy\right] dxdz \quad \text{where } q_y = -k\frac{\partial T}{\partial y}$
z	$q_z dxdy$	$\left[q_z + \frac{\partial q_z}{\partial z}dz\right] dxdy \quad \text{where } q_z = -k\frac{\partial T}{\partial z}$

The Energy Equation - Added Heat

net added heat:

$$\dot{Q} = -\left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right] dxdydz = -\nabla \cdot \mathbf{q} dxdydz$$

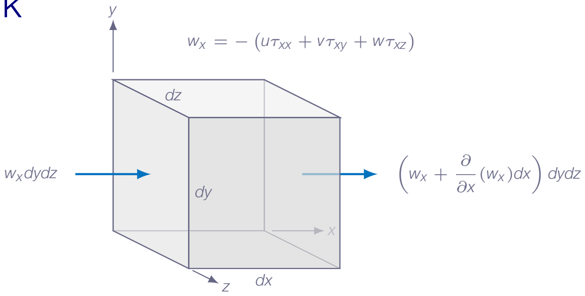
or

$$\dot{Q} = \nabla \cdot (k \nabla T) dxdydz$$

The Energy Equation - Viscous Work

The rate of work done by viscous stresses equals the product of the **stress component**, its corresponding **velocity component** and **surface area**

The Energy Equation - Viscous Work



$$\dot{W}_\nu = -\left[\frac{\partial}{\partial x}(u\tau_{xx} + v\tau_{xy} + w\tau_{xz}) + \frac{\partial}{\partial y}(u\tau_{yx} + v\tau_{yy} + w\tau_{yz}) + \frac{\partial}{\partial z}(u\tau_{zx} + v\tau_{zy} + w\tau_{zz})\right] dxdydz =$$

$$-\nabla \cdot (\mathbf{V} \cdot \tau_{ij})dxdydz$$

The Energy Equation

with the derived expressions for heat and viscous work we end up with

$$\nabla \cdot (k\nabla T) + \nabla \cdot (\mathbf{V} \cdot \tau_{ij}) = \rho\frac{De}{Dt} + p\nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p$$

The Energy Equation

Now, introducing the viscous-dissipation function ϕ for Newtonian fluids and incompressible flows

$$\nabla \cdot (\mathbf{V} \cdot \tau_{ij}) = \mathbf{V} \cdot (\nabla \cdot \tau_{ij}) + \phi$$

where

$$\phi = \mu \left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \right.$$

$$\left. \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 \right]$$

The Energy Equation

Note!

”All terms in the viscous-dissipation function are quadratic which means that in a viscous flow there will always be losses, which is in line with the second law of thermodynamics”

The Energy Equation

$$\nabla \cdot (k\nabla T) + \mathbf{V} \cdot (\nabla \cdot \tau_{ij}) + \phi = \rho \frac{De}{Dt} + p\nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p$$

Now, let's eliminate the term $\mathbf{V} \cdot (\nabla \cdot \tau_{ij})$ in the energy equation:

Momentum equation:

$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla p + \nabla \cdot \tau_{ij}$$

Multiply the momentum equation with the velocity vector (scalar product)

$$\mathbf{V} \cdot (\nabla \cdot \tau_{ij}) = \rho \mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \rho \mathbf{V} \cdot \mathbf{g} + \mathbf{V} \cdot \nabla p$$

The Energy Equation

Energy equation:

$$\rho\frac{De}{Dt} + \mathbf{V}\cdot\nabla p + p\nabla\cdot\mathbf{V} = \nabla\cdot(k\nabla T) + \mathbf{V}\cdot(\nabla\cdot\tau_{ij}) + \phi$$

eliminate $\mathbf{V}\cdot(\nabla\cdot\tau_{ij})$ using the result from previous slide

$$\rho\frac{De}{Dt} + \mathbf{V}\cdot\nabla p + p\nabla\cdot\mathbf{V} = \nabla\cdot(k\nabla T) + \rho\mathbf{V}\cdot\frac{D\mathbf{V}}{Dt} - \rho\mathbf{V}\cdot\mathbf{g} + \mathbf{V}\cdot\nabla p + \phi$$

Doesn't seem like a very wise move at this stage ...

The Energy Equation

As the next step, express energy per unit mass (e) as the sum of internal energy, kinetic energy, and potential energy (*as we did in Chapter 3*)

$$e = \hat{u} + \frac{1}{2}V^2 + gz$$

or in vector form:

$$e = \hat{u} + \frac{1}{2}\mathbf{V} \cdot \mathbf{V} - \mathbf{g}\mathbf{r}$$

where $\mathbf{g} = -(g_x \,, g_y \,, g_z)$ is the gravity vector and $\mathbf{r} = (x, \; y, \; z)$ is the location vector

The Energy Equation

$$e = \hat{u} + \frac{1}{2}\mathbf{V}\cdot\mathbf{V} - \mathbf{gr}$$

Now, apply the substantial derivative to e

$$\frac{De}{Dt} = \frac{D\hat{u}}{Dt} + \underbrace{\frac{1}{2}\frac{D}{Dt}(\mathbf{V}\cdot\mathbf{V})}_{=\mathbf{V}\cdot\frac{D\mathbf{V}}{Dt}*} - \underbrace{\frac{D}{Dt}(\mathbf{gr})}_{=\mathbf{V}\cdot\mathbf{g}*} = \frac{D\hat{u}}{Dt} + \mathbf{V}\cdot\frac{D\mathbf{V}}{Dt} - \mathbf{V}\cdot\mathbf{g}$$

* *details on the following slides*

The Energy Equation

$$\frac{1}{2}\frac{D}{Dt}(\mathbf{V}\cdot\mathbf{V}) = \frac{1}{2}\left[\underbrace{\frac{\partial}{\partial t}(\mathbf{V}\cdot\mathbf{V})}_{I} + \underbrace{(\mathbf{V}\cdot\nabla)(\mathbf{V}\cdot\mathbf{V})}_{II}\right]$$

I:

$$\frac{\partial}{\partial t}(\mathbf{V}\cdot\mathbf{V}) = \frac{\partial}{\partial t}(u^2+v^2+w^2) = \frac{\partial u^2}{\partial t} + \frac{\partial v^2}{\partial t} + \frac{\partial w^2}{\partial t} =$$

$$2u\frac{\partial u}{\partial t} + 2v\frac{\partial v}{\partial t} + 2w\frac{\partial w}{\partial t} = 2\mathbf{V}\cdot\frac{\partial \mathbf{V}}{\partial t}$$

The Energy Equation

$$\frac{1}{2}\frac{D}{Dt}(\mathbf{V}\cdot\mathbf{V}) = \frac{1}{2}\left[2\mathbf{V}\cdot\frac{\partial\mathbf{V}}{\partial t} + \underbrace{(\mathbf{V}\cdot\nabla)(\mathbf{V}\cdot\mathbf{V})}_{II}\right]$$

II:

$$(\mathbf{V}\cdot\nabla)(\mathbf{V}\cdot\mathbf{V}) = (\mathbf{V}\cdot\nabla)(u^2+v^2+w^2) =$$

$$= u\frac{\partial u^2}{\partial x} + u\frac{\partial v^2}{\partial x} + u\frac{\partial w^2}{\partial x} + v\frac{\partial u^2}{\partial x} + v\frac{\partial v^2}{\partial x} + v\frac{\partial w^2}{\partial x} + w\frac{\partial u^2}{\partial x} + w\frac{\partial v^2}{\partial x} + w\frac{\partial w^2}{\partial x} =$$

$$= 2\left[u^2\frac{\partial u}{\partial x} + uv\frac{\partial v}{\partial x} + uw\frac{\partial w}{\partial x} + uv\frac{\partial u}{\partial x} + v^2\frac{\partial v}{\partial x} + vw\frac{\partial w}{\partial x} + uw\frac{\partial u}{\partial x} + vw\frac{\partial v}{\partial x} + w^2\frac{\partial w}{\partial x}\right] =$$

$$= 2\mathbf{V}\cdot(\mathbf{V}\cdot\nabla)\mathbf{V}$$

The Energy Equation

$$\frac{1}{2}\frac{D}{Dt}(\mathbf{V}\cdot\mathbf{V}) = \frac{1}{2}\left[2\mathbf{V}\cdot\frac{\partial\mathbf{V}}{\partial t} + 2\mathbf{V}\cdot(\mathbf{V}\cdot\nabla)\mathbf{V}\right] = \mathbf{V}\cdot\frac{D\mathbf{V}}{Dt}$$

The Energy Equation

$$\frac{D}{Dt}(\mathbf{gr}) = \frac{\partial}{\partial t}(\mathbf{gr}) + (\mathbf{V}\cdot\nabla)(\mathbf{gr}) = \underbrace{\frac{\partial}{\partial t}(g_x x,\ g_y y,\ g_z z)}_{=(0,0,0)} + (\mathbf{V}\cdot\nabla)(g_x x,\ g_y y,\ g_z z) =$$

$$\mathbf{V}\cdot\underbrace{\left(x\frac{\partial g_x}{\partial x}+g_x\frac{\partial x}{\partial x},\ y\frac{\partial g_y}{\partial y}+g_y\frac{\partial y}{\partial y},\ z\frac{\partial g_z}{\partial z}+g_z\frac{\partial z}{\partial z}\right)}_{\frac{\partial g_x}{\partial x}=\frac{\partial g_y}{\partial y}=\frac{\partial g_z}{\partial z}=0 \text{ and } \frac{\partial x}{\partial x}=\frac{\partial y}{\partial y}=\frac{\partial z}{\partial z}=1} = \mathbf{V}\cdot(g_x,\ g_y,\ g_z) = \mathbf{V}\cdot\mathbf{g}$$

The Energy Equation

Now, insert

$$\frac{De}{Dt} = \frac{D\hat{u}}{Dt} + \mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \mathbf{V} \cdot \mathbf{g}$$

in the energy equation

$$\rho \frac{D\hat{u}}{Dt} + \rho\mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \rho\mathbf{V} \cdot \mathbf{g} + \mathbf{V} \cdot \nabla p + p\nabla \cdot \mathbf{V} =$$

$$\nabla \cdot (k\nabla T) + \rho\mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \rho\mathbf{V} \cdot \mathbf{g} + \mathbf{V} \cdot \nabla p + \phi$$

The highlighted terms cancel each other

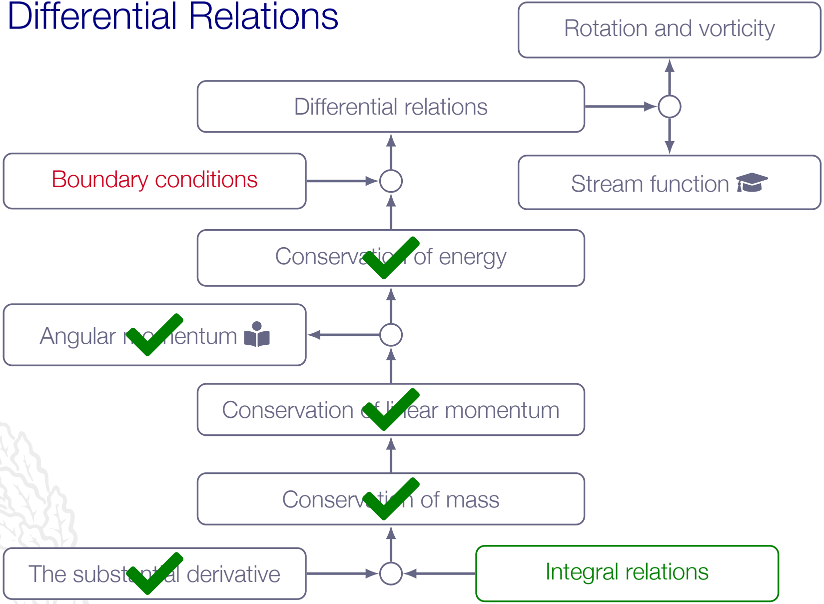
Ok, this was why momentum equation was used here ...

The Energy Equation

$$\rho \frac{D\hat{u}}{Dt} + p\nabla \cdot \mathbf{V} = \nabla \cdot (k\nabla T) + \phi$$

Local and convective changes of internal energy are balanced by pressure work, heat addition and viscous dissipation – viscous dissipation will always increase the internal energy of the fluid

Roadmap - Differential Relations



Flow Equations on Differential Form

Continuity: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

Momentum: $$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla p + \nabla \cdot \tau_{ij}$$

Energy: $$\rho \frac{D\hat{u}}{Dt} + p\nabla \cdot \mathbf{V} = \nabla \cdot (k\nabla T) + \phi$$

five equations and seven unknowns $(\rho, u, v, w, p, \hat{u}, T) \Rightarrow$ two additional relations needed:

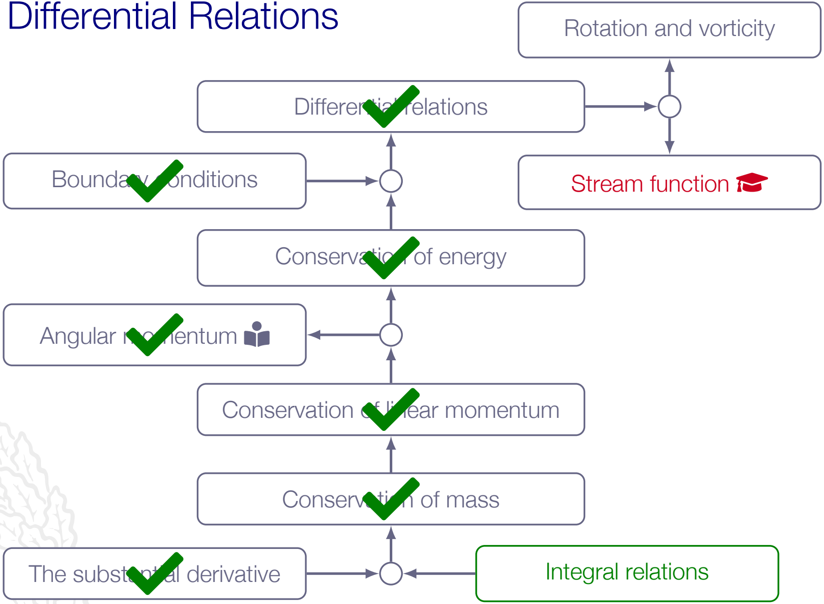
$$\rho = \rho(p, T),\ \hat{u} = \hat{u}(p, T)$$

Flow Equations on Differential Form

Boundary conditions:

- solid wall: no slip, no temperature jump
- inlet, outlet
- liquid-gas interface

Roadmap - Differential Relations



The Stream Function (*for the interested*)

fulfill the continuity equation and solve the momentum equation directly for the single variable ψ

The Stream Function (*for the interested*)

incompressible, two-dimensional flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

define $\psi(x, y)$ such that

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0$$

and thus

$$u = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial x}$$

or

$$\mathbf{V} = \left[\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right]$$

The Stream Function (*for the interested*)

The rotation of the flow field is calculated using the curl operator

$$\mathit{curl}(\mathbf{V}) = \nabla \times \mathbf{V} = -\nabla^2 \psi \mathbf{e}_z$$

The Stream Function (*for the interested*)

Now, apply the curl operator to the momentum equation

$$\nabla \times \frac{D\mathbf{V}}{Dt} = \underbrace{\nabla \times \mathbf{g}}_{=0} - \frac{1}{\rho}\underbrace{\nabla \times \nabla p}_{=0} + \nu\nabla \times \nabla^2\mathbf{V} = \nu\nabla \times \nabla^2\mathbf{V}$$

$$\nabla \times \frac{\partial\mathbf{V}}{\partial t} + \nabla \times (\mathbf{V}\cdot\nabla)\mathbf{V} = \nu\nabla \times \nabla^2\mathbf{V}$$

$$\left.\begin{array}{l} \dfrac{\partial\mathbf{V}}{\partial t} = 0 \; (\textit{steady}) \\[2ex] \nu\nabla \times \nabla^2\mathbf{V} = \nu\nabla^2(\nabla \times \mathbf{V}) \end{array}\right\} \Rightarrow \nabla \times (\mathbf{V}\cdot\nabla)\mathbf{V} = \nu\nabla^2(\nabla \times \mathbf{V})$$

The Stream Function (*for the interested*)

$$(\mathbf{V}\cdot\nabla)\mathbf{V}=\frac{1}{2}\nabla(\mathbf{V}\cdot\mathbf{V})-\mathbf{V}\times(\nabla\times\mathbf{V})=\nabla\left(\frac{V^2}{2}\right)-\mathbf{V}\times(\nabla\times\mathbf{V})$$

and thus

$$\nabla\times(\mathbf{V}\cdot\nabla)\mathbf{V}=\underbrace{\nabla\times\nabla\left(\frac{V^2}{2}\right)}_{=0}-\nabla\times\mathbf{V}\times(\nabla\times\mathbf{V})=\nabla\times(\nabla\times\mathbf{V})\times\mathbf{V}$$

$$\nabla\times(\nabla\times\mathbf{V})\times\mathbf{V}=$$

$$(\mathbf{V}\cdot\nabla)(\nabla\times\mathbf{V})-((\nabla\times\mathbf{V})\cdot\nabla)\mathbf{V}+\underbrace{(\nabla\times\mathbf{V})(\nabla\cdot\mathbf{V})}_{=0\ (\textit{incompressible})}+\mathbf{V}\underbrace{(\nabla\cdot(\nabla\times\mathbf{V}))}_{=0}=$$

$$(\mathbf{V}\cdot\nabla)(\nabla\times\mathbf{V})-((\nabla\times\mathbf{V})\cdot\nabla)\mathbf{V}$$

The Stream Function (*for the interested*)

$$(\mathbf{V}\cdot\nabla)(\nabla\times\mathbf{V}) - ((\nabla\times\mathbf{V})\cdot\nabla)\mathbf{V} = \nu\nabla^2(\nabla\times\mathbf{V})$$

insert the stream function

$$(\mathbf{V}\cdot\nabla)(\nabla\times\mathbf{V}) = (\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0)\cdot(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(0, 0, -\nabla^2\psi)$$

$$((\nabla\times\mathbf{V})\cdot\nabla)\mathbf{V} = (0, 0, -\nabla^2\psi)\cdot(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0) = 0$$

$$\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi) - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi) = \nu\nabla^2(\nabla^2\psi)$$

Stream Function (*for the interested*)

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi) = \nu\nabla^2(\nabla^2\psi)$$

+ one equation for ψ that fulfills both the momentum and continuity equations
+ scalar equation
- contains fourth-order derivatives

Stream Function (*for the interested*)

Definition of a streamline in two dimensions:

$$\frac{dx}{u} = \frac{dy}{v}$$

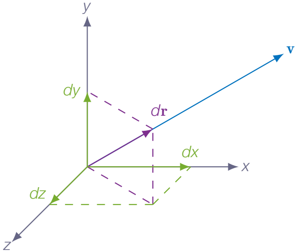
or

$$udy - vdx = 0$$

and thus

$$\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = d\psi = 0$$

or ψ is constant along a streamline ...



$d\mathbf{r}$ is aligned with $\mathbf{v}$

$dx \propto u$

$dy \propto v$

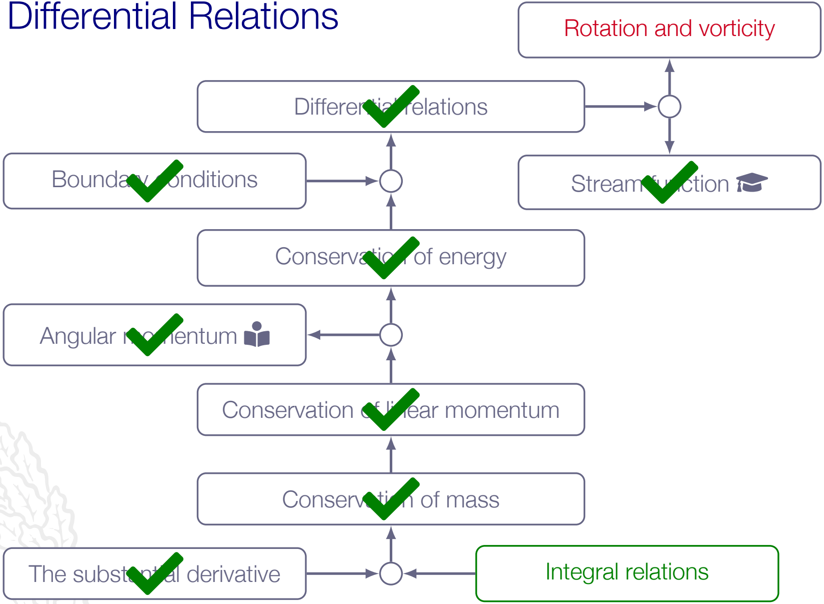
$dz \propto w$

Stream Function (*for the interested*)

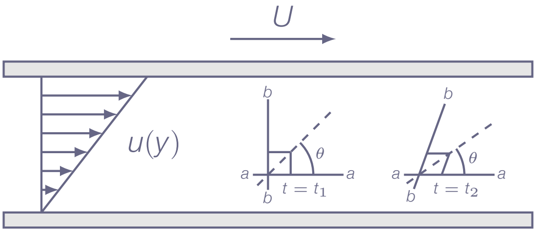
Implication:

- Lines of constant ψ are streamlines of the flow
- If we know $\psi(x, y)$, lines of constant ψ will be streamlines of the flow

Roadmap - Differential Relations

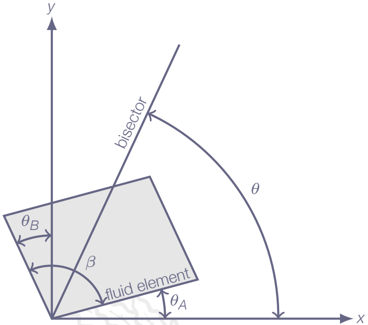


Flow Rotation



- Is the Couette flow irrotational?
- Note the change of the fluid element bisector angle θ

Flow Rotation



$$\beta = \frac{\pi}{2} + \theta_B - \theta_A$$

$$\theta = \frac{\beta}{2} + \theta_A = \frac{\pi}{4} + \frac{1}{2}(\theta_A + \theta_B)$$

the angular velocity of the bisector:

$$\dot{\theta} = \frac{1}{2}\left(\dot{\theta}_A + \dot{\theta}_B\right)$$

Flow Rotation

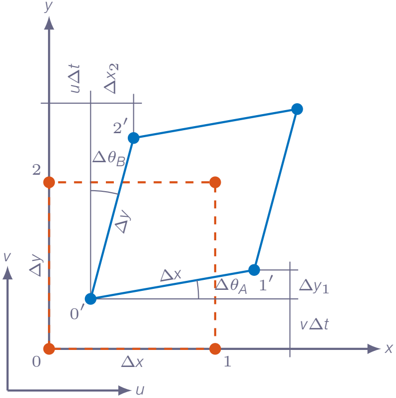
$$\sin(\Delta\theta_A) = \frac{\Delta y_1}{\Delta x} \approx \frac{\left(v + \frac{\partial v}{\partial x}\Delta x\right)\Delta t - v\Delta t}{\Delta x} = \frac{\partial v}{\partial x}\Delta t$$

$\sin(\Delta\theta_A) \approx \Delta\theta_A$ for small angles

$$\Rightarrow \underbrace{\frac{\Delta\theta_A}{\Delta t}}_{=\dot{\theta}_A} \approx \frac{\partial v}{\partial x}$$

in the same way $\dot{\theta}_B \approx -\dfrac{\partial u}{\partial y}$

the angular velocity of the bisector: $\dot{\theta} = \dfrac{1}{2}\left(\dot{\theta}_A + \dot{\theta}_B\right) = \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right)$



Flow Rotation

From previous slide we get the angular velocity about the z axis

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Using the same reasoning, we can get the angular velocities about the x and y axes

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

Flow Rotation

$$\omega = \frac{1}{2} \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \frac{1}{2} curl(\mathbf{V})$$

The flow **vorticity** ζ is defined as:

$$\zeta = 2\omega = curl(\mathbf{V})$$

Flows with **zero vorticity** are called **irrotational**

Frictionless Irrotational Flow

If the flow is both **frictionless** and **irrotational**:

1. the momentum equation reduces to Euler's equation

$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla p$$

2. the acceleration term can be simplified

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}$$

where we can use the vector identity

$$(\mathbf{V} \cdot \nabla)\mathbf{V} = \nabla(\frac{1}{2}V^2) + \zeta \times \mathbf{V}$$

Doesn't seem like a simplification but let's try ...

Frictionless Flow

1. combine Euler's equation with the modified acceleration term
2. divide by ρ
3. dot product between the entire equation and an arbitrary displacement vector $d\mathbf{r}$

$$\left[\frac{\partial \mathbf{V}}{\partial t} + \nabla(\frac{1}{2}V^2) + \zeta \times \mathbf{V} + \frac{1}{\rho}\nabla p - \mathbf{g}\right] \cdot d\mathbf{r} = 0$$

Frictionless Flow

Now we want to get rid of the term $(\zeta \times \mathbf{V}) \cdot d\mathbf{r}$

1. $\mathbf{V} = 0$; no flow - not interesting
2. $\zeta = 0$; irrotational flow
3. $d\mathbf{r}$ perpendicular to $(\zeta \times \mathbf{V})$; strange
4. $d\mathbf{r}$ parallel to $\mathbf{V}$; integrate along a streamline

Frictionless Flow

Fourth alternative: integrate along a streamline:

$$\left[\frac{\partial \mathbf{V}}{\partial t} + \nabla(\frac{1}{2}V^2) + \frac{1}{\rho}\nabla p - \mathbf{g}\right] \cdot d\mathbf{r} = 0$$

performing the scalar products gives

$$-\mathbf{g} \cdot d\mathbf{r} = \{\mathbf{g} = -g\mathbf{e}_z\} = gdz$$

$$\nabla p \cdot d\mathbf{r} = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = dp$$

$$\nabla(\frac{1}{2}V^2) \cdot d\mathbf{r} = \frac{1}{2}\left(\frac{\partial V^2}{\partial x}dx + \frac{\partial V^2}{\partial y}dy + \frac{\partial V^2}{\partial z}dz\right) = \frac{1}{2}d\left(V^2\right)$$

Frictionless Flow

$$\frac{\partial \mathbf{V}}{\partial t} \cdot d\mathbf{r} + \frac{1}{2} d\left(V^2\right) + \frac{dp}{\rho} + g dz = 0$$

Frictionless Flow

Integrate between any two points along the streamline

$$\int_1^2 \frac{\partial V}{\partial t} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0$$

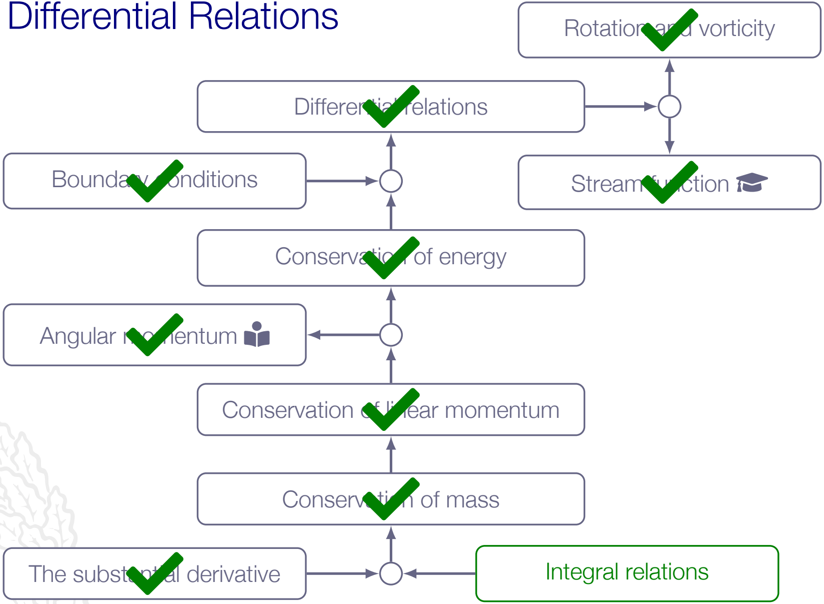
The Bernoulli equation for frictionless unsteady flow

Steady incompressible flow gives

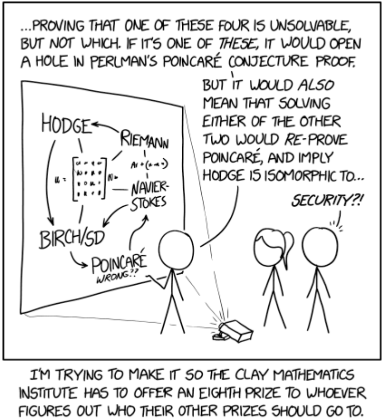
$$\frac{p}{\rho} + \frac{1}{2}V^2 + gz = constant$$

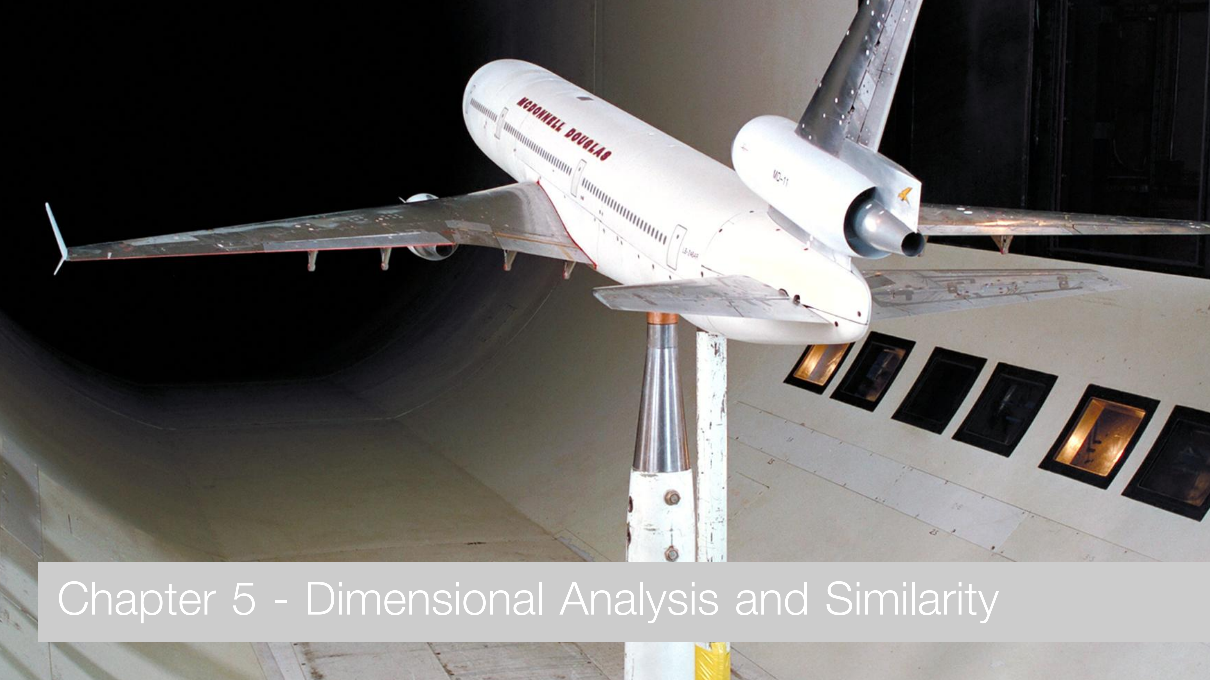
Note! for irrotational flow this last results holds in the entire flow field with the same constant

Roadmap - Differential Relations



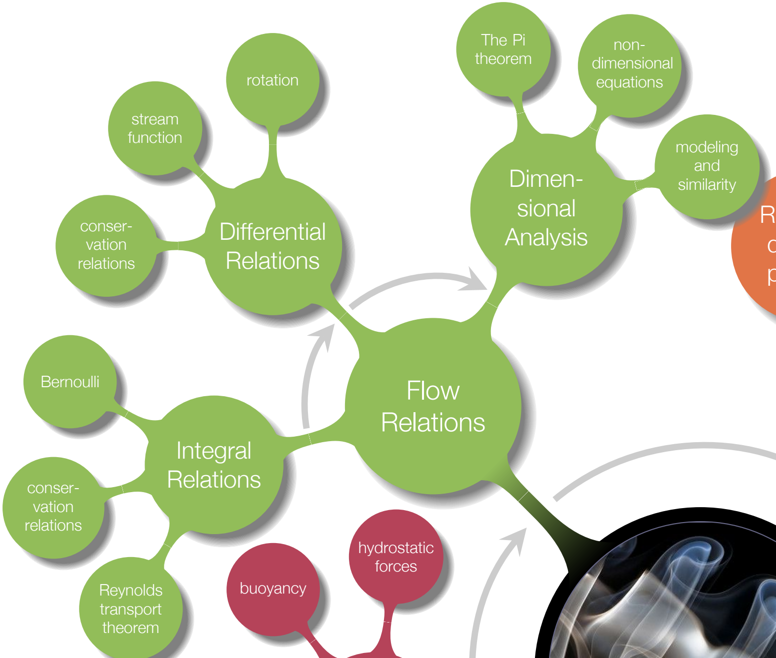
Millennium Problems





Chapter 5 - Dimensional Analysis and Similarity

Overview



Flow Relations
Differential Relations
rotation
stream function
conser-vation relations
Dimen-sional Analysis
The Pi theorem
non-dimensional equations
modeling and similarity
Integral Relations
Bernoulli
conser-vation relations
Reynolds transport theorem
hydrostatic forces
buoyancy

Learning Outcomes

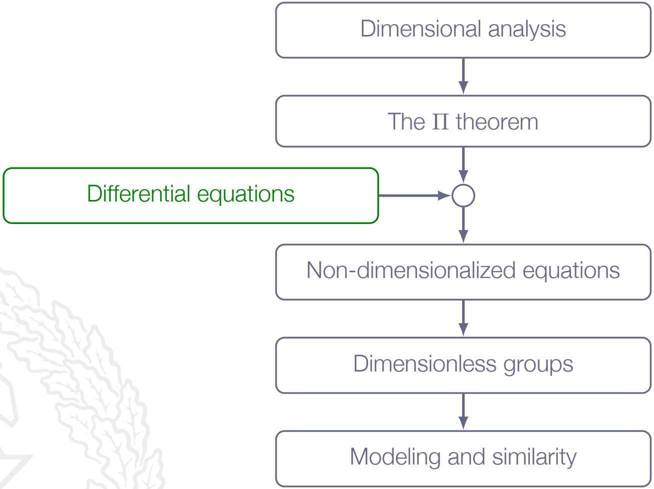
3 **Define** the Reynolds number

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

17 **Explain** about how to use non-dimensional numbers and the Π theorem

we will learn about how to plan experiments and compare experimental data using dimensionless numbers

Roadmap - Dimensional Analysis and Similarity



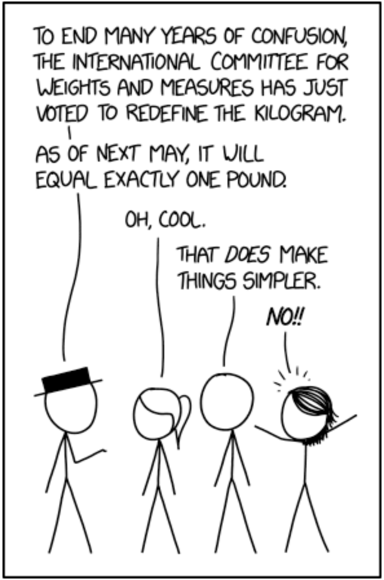
Motivation

"Most practical fluid flow problems are too complex, both geometrically and physically, to be solved analytically. They must be tested by experiments or approximated by CFD"

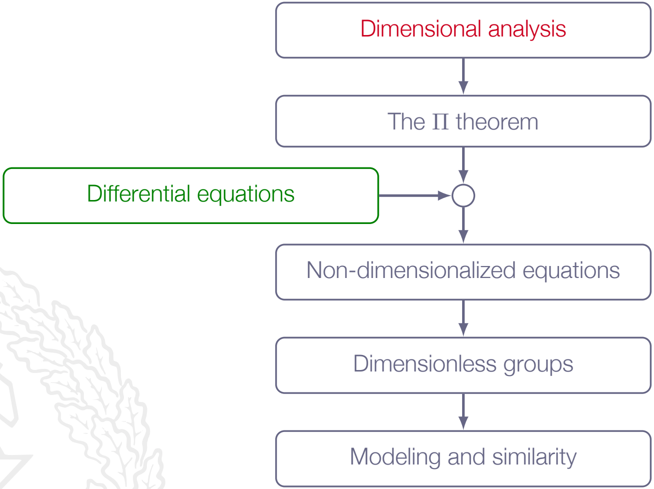
Dimensional analysis:

- Large data sets may be represented by a **few curves** or even a single curve
- A systematic tool for **data reduction**
- Experimental/simulation data are **more general** in **dimensionless** form

Dimensions


TO END MANY YEARS OF CONFUSION, THE INTERNATIONAL COMMITTEE FOR WEIGHTS AND MEASURES HAS JUST VOTED TO REDEFINE THE KILOGRAM.
AS OF NEXT MAY, IT WILL EQUAL EXACTLY ONE POUND.
OH, COOL.
THAT *DOES* MAKE THINGS SIMPLER.
NO!!

Roadmap - Dimensional Analysis and Similarity



Dimensional Analysis - What is it?

Dimensional analysis is a tool for systematic

1. **planning** of experiments
   similarity between model and prototype
2. **presentation** of experimental data
   insight into physical relationships
3. **interpretation** of measurements
   identify important and unimportant parameters

Dimensional Analysis - What is it?

General description:

"If a phenomenon depends on n dimensional variables, dimensional analysis will reduce the problem to only k dimensionless variables, where the reduction $n-k$ depends on the problem complexity"

"Generally, $n-k$ equals the number of primary dimensions"

Dimensional Analysis - Example Problem

Problem definition:

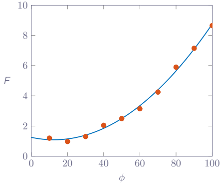
Suppose that we know that the force F on a particular body shape in a fluid flow depends on

1. The length of the body L
2. The flow freestream velocity V
3. The fluid density ρ
4. The fluid viscosity μ

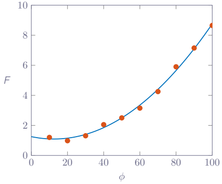
$$\Rightarrow \quad F = f(L, V, \rho, \mu)$$

Dimensional Analysis - Example Problem

Let's say that we need ten data points to define a curve



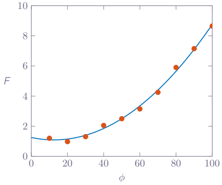
Dimensional Analysis - Example Problem



Let’s say that we need ten data points to define a curve

We need to test 10 lengths and for each of those, 10 velocities, ...

Dimensional Analysis - Example Problem

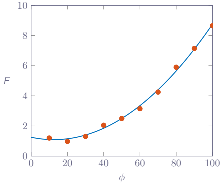


Let’s say that we need ten data points to define a curve

We need to test 10 lengths and for each of those, 10 velocities, ...

For our example problem we need to do **10000 experiments!!**

Dimensional Analysis - Example Problem



Let's say that we need ten data points to define a curve

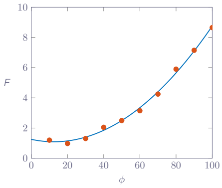
We need to test 10 lengths and for each of those, 10 velocities, ...

For our example problem we need to do **10000 experiments!!**

With dimensional analysis, the problem can be reduced as follows

$$\frac{F}{\rho V^2 L^2} = g\left(\frac{\rho V L}{\mu}\right) \text{ or } C_F = g(Re) \textit{ where g is an unknown function}$$

Dimensional Analysis - Example Problem



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With dimensional analysis, the problem can be reduced as follows

$$\frac{F}{\rho V^2 L^2} = g\left(\frac{\rho V L}{\mu}\right) \text{ or } C_F = g(Re) \textit{ where g is an unknown function}$$

The number of experiments needed have been **reduced by a factor of 1000!!**

Similarity - Model and Prototype

Let's go back to the example problem from before

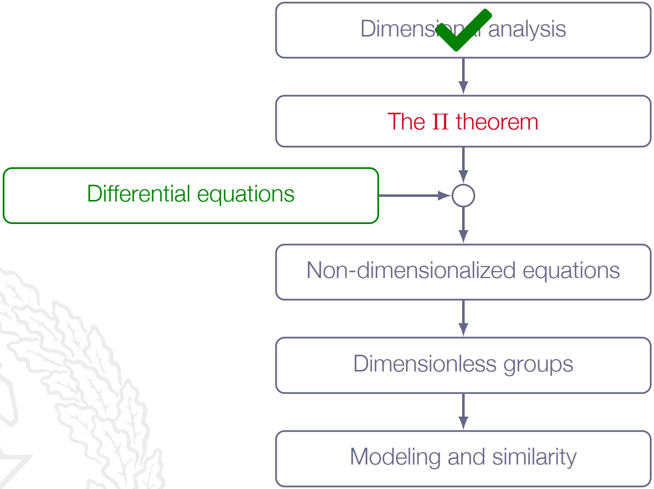
$$C_F = g(Re)$$

so if $Re_m = Re_p$ that means that $C_{F,m} = C_{F,p}$ (where m is model and p prototype)

$$C_{F,m} = \frac{F_m}{\rho_m V_m^2 L_m^2} \text{ and } C_{F,p} = \frac{F_p}{\rho_p V_p^2 L_p^2}$$

$$\frac{F_m}{\rho_m V_m^2 L_m^2} = \frac{F_p}{\rho_p V_p^2 L_p^2} \Rightarrow \frac{F_p}{F_m} = \frac{\rho_p}{\rho_m}\left(\frac{V_p}{V_m}\right)^2\left(\frac{L_p}{L_m}\right)^2$$

Roadmap - Dimensional Analysis and Similarity



The Buckingham Π-theorem

Systematic identification of non-dimensional numbers (Π-groups):

"If there is a physically meaningful equation involving a certain number n of physical variables, then the original equation can be rewritten in terms of a set of k dimensionless parameters Π_1, Π_2, ..., Π_k. The reduction, $j = n - k$, equals the number of variables that do not form a Π among themselves and is always less than or equal to the number of physical dimensions involved"

The Buckingham Π-theorem

Systematic identification of non-dimensional numbers (Π-groups):

1. List and count the **number of variables** in the problem n
2. List the **dimensions** for each of the n variables
3. Count **number of dimensions** m
4. Find the **reduction** j
   4.1 initial guess: j equals the **number of dimensions** m
   4.2 look for j variables that do not form a Π
   4.3 if not possible reduce j by one and go back to 4.2
5. Select j **scaling parameters**
6. Add one of the other variables to your j **repeating variables** and form a power product
7. Algebraically, find exponents that make the product dimensionless

The Buckingham Π-theorem - Example

$$F = f(L, U, \rho, \mu)$$

number of variables: $n = 5$

F	L	U	ρ	μ
$\{MLT^{-2}\}$	$\{L\}$	$\{LT^{-1}\}$	$\{ML^{-3}\}$	$\{ML^{-1}T^{-1}\}$

number of dimensions: m=3

reduction: $j \leq 3$

number of dimensionless groups: $k = n - j \geq 2$

The Buckingham Π-theorem - Example

1. Inspecting the variables, we see that L, U, and ρ cannot form a **Π-group**

   only ρ contains M (mass)

   only U contains T (time)

2. L, U, and ρ are selected as the j **repeating variables**

3. The **reduction** will be $j = 3$ and thus $k = n - j = 2$

4. One of the **Π-groups** will contain F and the other will contain μ

The Buckingham Π-theorem - Example

$$\Pi_1 = L^a U^b \rho^c F \Rightarrow (L)^a (LT^{-1})^b (ML^{-3})^c (MLT^{-2}) = M^0 L^0 T^0$$

$$\begin{array}{rccccccccc} L: & a & + & b & - & 3c & + & 1 & = & 0 \\ M: & & & & & c & + & 1 & = & 0 \\ T: & & - & b & & & - & 2 & = & 0 \end{array}$$

which gives

$$a = -2,\ b = -2,\ c = -1$$

and thus

$$\Pi_1 = \frac{F}{\rho U^2 L^2} = C_F$$

The Buckingham Π-theorem - Example

$$\Pi_2 = L^a U^b \rho^c \mu^{-1} \Rightarrow (L)^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^{-1} = M^0 L^0 T^0$$

$$\begin{array}{rcccccccccc} L: & a & + & b & - & 3c & + & 1 & = & 0 \\ M: & & & & & c & - & 1 & = & 0 \\ T: & & - & b & & & + & 1 & = & 0 \end{array}$$

which gives

$$a = b = c = 1$$

and thus

$$\Pi_2 = \frac{\rho U L}{\mu} = Re$$

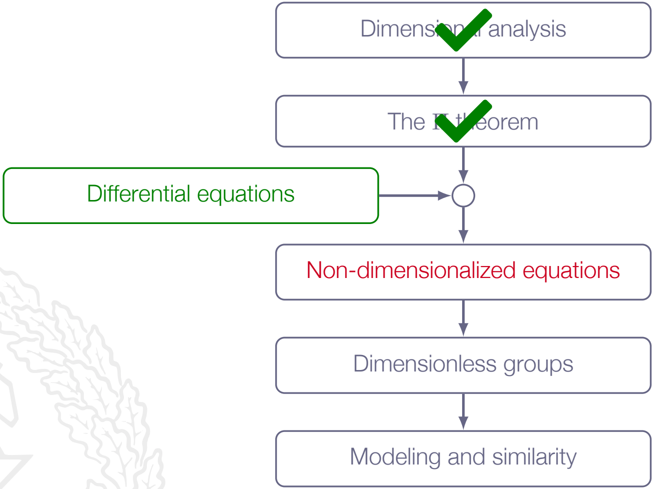
The Buckingham Π-theorem - Example

If $F = f(L, V, \rho, \mu)$, the theorem guaranties that, in this case, $\Pi_1 = g(\Pi_2)$

$$\frac{F}{\rho U^2 L^2} = g\left(\frac{\rho U L}{\mu}\right) \text{ or } C_F = g(Re)$$

where g is an unknown function

Roadmap - Dimensional Analysis and Similarity



Non-dimensionalized Equations

Why would one want to make the governing equations non-dimensional?

- Understand flow physics
- Gives information about under what conditions terms are negligible
- A way to find important non-dimensional groups for a specific flow

Non-dimensionalized Equations

The incompressible flow continuity and momentum equations and corresponding boundary conditions:

Continuity: $\nabla \cdot \mathbf{V} = 0$

Navier-Stokes: $\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{V}$

Solid surface: no-slip ($\mathbf{V} = 0$ if fixed surface)

Inlet/outlet: known velocity and pressure

Non-dimensionalized Equations

The variables in the continuity and momentum equations contain **three primary dimensions**; M, L, and T

All variables included $(\rho, \mathbf{V}, p, x, y, z)$ can be made non-dimensional using three constants:

1. density: ρ
2. reference velocity: U
3. reference length: L

reference properties are constants characteristic for a specific flow

Non-dimensionalized Equations

non-dimensional variables are denoted by an asterisk:

$$\mathbf{V}^* = \frac{\mathbf{V}}{U}$$

$$t^* = \frac{tU}{L}$$

$$\nabla^* = L\nabla$$

$$p^* = \frac{p - \rho\mathbf{gr}}{\rho U^2}$$

$$(x^*,y^*,z^*) = \frac{1}{L}(x,\ y,\ z)$$

Non-dimensionalized Equations

$$\rho\frac{D\mathbf{V}}{Dt}=\rho\mathbf{g}-\nabla p+\mu\nabla^2\mathbf{V}$$

$$\frac{D\mathbf{V}^*}{Dt^*}=\frac{\partial\mathbf{V}^*}{\partial t^*}+u^*\frac{\partial\mathbf{V}^*}{\partial x^*}+v^*\frac{\partial\mathbf{V}^*}{\partial y^*}+w^*\frac{\partial\mathbf{V}^*}{\partial z^*}=\frac{L}{U^2}\frac{D\mathbf{V}}{Dt}$$

Non-dimensionalized Equations

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{V}$$

$$\nabla^* p^* = \frac{L}{\rho U^2}\nabla\left(p - \rho\mathbf{g}\mathbf{r}\right) = \frac{L}{\rho U^2}\left(\nabla p - \rho\nabla\mathbf{g}\mathbf{r}\right)$$

$$\nabla\mathbf{g}\mathbf{r} = \nabla\left(g_x x, g_y y, g_z z\right) = \left(g_x\frac{\partial x}{\partial x} + x\frac{\partial g_x}{\partial x},\ g_y\frac{\partial y}{\partial y} + y\frac{\partial g_y}{\partial y},\ g_z\frac{\partial z}{\partial z} + z\frac{\partial g_z}{\partial z}\right) =$$

$$= \left(g_x,\ g_y,\ g_z\right) = \mathbf{g} \Rightarrow -\nabla^* p^* = \frac{L}{\rho U^2}\left(\rho\mathbf{g} - \nabla p\right)$$

Non-dimensionalized Equations

$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{V}$$

$$\nabla^{*^2} \mathbf{V} = \frac{L^2}{U} \nabla^2 \mathbf{V}$$

Non-dimensionalized Equations

Continuity: $\nabla^* \cdot \mathbf{V}^* = 0$

Navier-Stokes: $$\frac{D\mathbf{V}^*}{Dt^*} = -\nabla^* p^* + \frac{\mu}{\rho U L} \nabla^{*2} \mathbf{V}^*$$

Solid surface: no-slip ($\mathbf{V}^* = 0$ if fixed surface)

Inlet/outlet: known velocity and pressure ($\mathbf{V}^*$, p^*)

Non-dimensionalized Equations

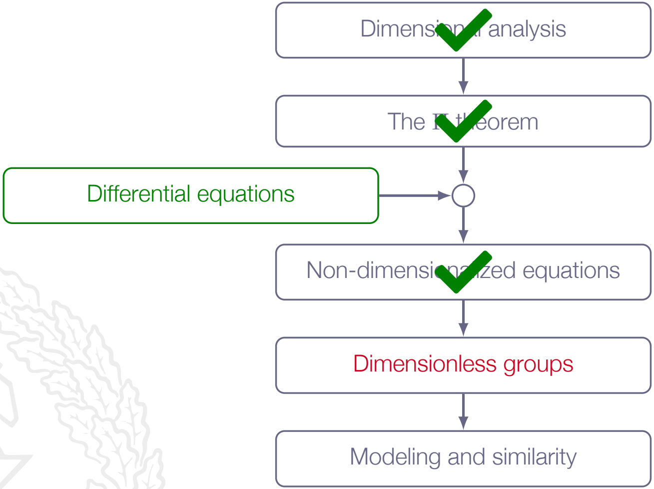
The **Reynolds number** appears in the non-dimensional Navier-Stokes equations

$$\frac{D\mathbf{V}^*}{Dt^*} = -\nabla^* p^* + \frac{\mu}{\rho U L} \nabla^{*2} \mathbf{V}^*$$

$$Re = \frac{\rho U L}{\mu}$$

Reynolds number - ratio of inertia and viscosity

Roadmap - Dimensional Analysis and Similarity



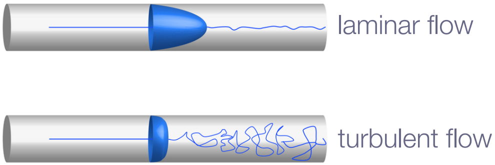
Dimensionless Groups

Definitions and interpretations of non-dimensional groups frequently used in fluid mechanics

parameter	definition	interpretation	importance
Reynolds number	$Re = \frac{\rho U L}{\mu}$	$\frac{\text{inertia}}{\text{viscosity}}$	almost always
Mach number	$M = \frac{U}{a}$	$\frac{\text{flow speed}}{\text{speed of sound}}$	compressible flow
Froude number	$Fr = \frac{U^2}{gL}$	$\frac{\text{inertia}}{\text{gravity}}$	free-surface flow
Weber number	$We = \frac{\rho U^2 L}{\Upsilon}$	$\frac{\text{inertia}}{\text{surface tension}}$	free-surface flow
Prandtl number	$Pr = \frac{\mu C_p}{k}$	$\frac{\text{dissipation}}{\text{conduction}}$	heat convection
specific heat ratio	$\gamma = \frac{C_p}{C_v}$	$\frac{\text{enthalpy}}{\text{internal energy}}$	compressible flow
Strouhal number	$St = \frac{\omega L}{U}$	$\frac{\text{oscillation}}{\text{mean flow speed}}$	oscillating flow
roughness ratio	$\frac{\varepsilon}{L}$	$\frac{\text{wall roughness}}{\text{body length}}$	turbulent flow
pressure coefficient	$C_p = \frac{p - p_\infty}{0.5\rho U^2}$	$\frac{\text{static pressure}}{\text{dynamic pressure}}$	aerodynamics
lift coefficient	$C_L = \frac{F_L}{0.5\rho U^2 A}$	$\frac{\text{lift force}}{\text{dynamic force}}$	aerodynamics
drag coefficient	$C_D = \frac{F_D}{0.5\rho U^2 A}$	$\frac{\text{drag force}}{\text{dynamic force}}$	aerodynamics
skin friction coefficient	$C_f = \frac{\tau_{wall}}{0.5\rho U^2}$	$\frac{\text{wall-shear stress}}{\text{dynamic pressure}}$	boundary layers

The Reynolds Number

$$Re = \frac{\rho U L}{\mu} = \frac{U L}{\nu}$$



Compressible Flow

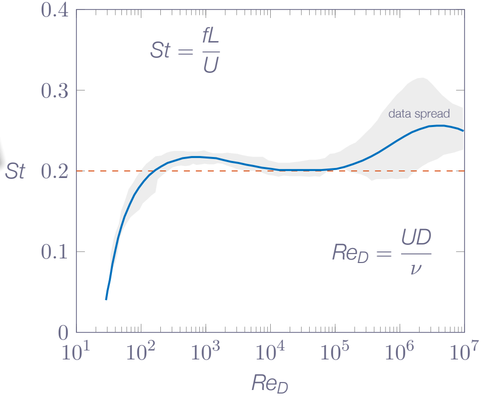
https://www.youtube.com/watch?v=wRaDPnpnx04

$$Ma = \frac{U}{a} = \frac{U}{\sqrt{\gamma RT}}$$

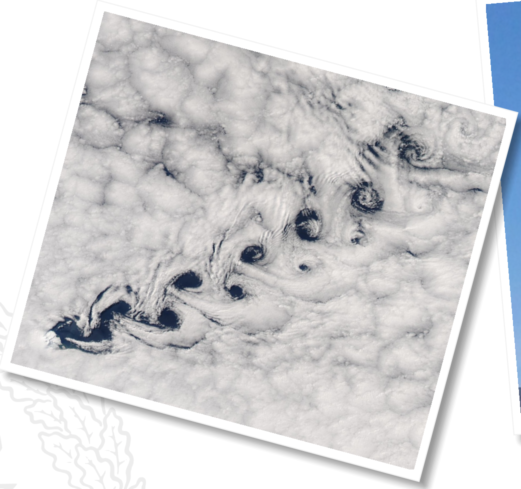
$$\gamma = \frac{C_p}{C_v}$$

Oscillating Flows

Von Kármán vortex street



Oscillating Flows



Oscillating Flows

https://www.youtube.com/watch?v=XggxeuFDaDU

Tacoma bridge collapse 1940

oscillating frequency close to the natural vibration frequency of the bridge structure

Oscillating Flows

https://www.youtube.com/watch?v=ptYrbQGk6DQ

Example of Successful Dimensional Analysis

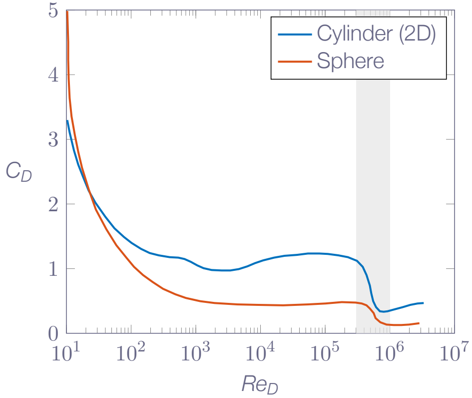
collection of data from a large number of experiments

cylinder: $C_D = \frac{F_D}{\frac{1}{2}\rho U^2 L d}$

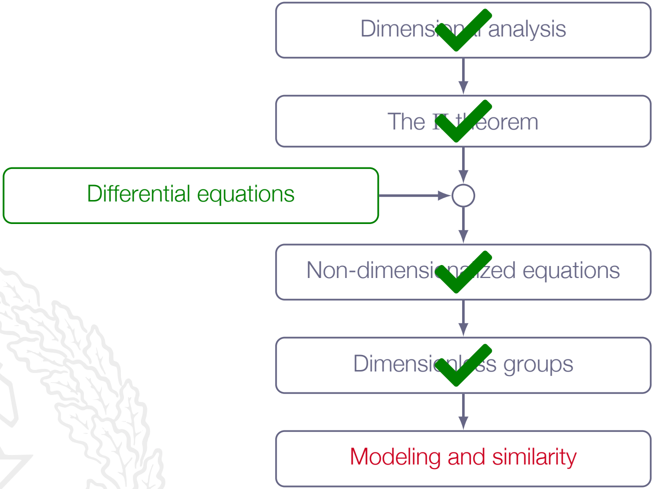
sphere: $C_D = \frac{F_D}{\frac{1}{2}\rho U^2 \frac{1}{4}\pi d^2}$

general: $C_D = \frac{F_D}{\frac{1}{2}\rho U^2 A_p}$

A_p is the projected area



Roadmap - Dimensional Analysis and Similarity



Modeling and Similarity

Scaling of experimental results from **model** scale to **prototype** scale:

> *"Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for the model and the prototype"*

Geometric Similarity

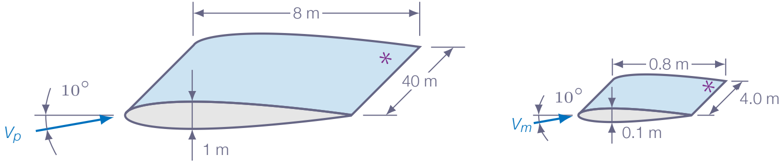
”A model and prototype are geometrically similar if and only if all body dimensions in all three coordinates have the same linear-scale ratio”

”All angles are preserved in geometric similarity. All flow directions are preserved. The orientations of model and prototype with respect to the surroundings must be identical”

Geometric Similarity

Geometric Similarity

Homologous points - points that with the same relative location



1. all dimensions should be scaled with the same linear scaling ratio
2. angle of attach should be the same
3. scaled nose radius
4. scaled surface roughness

Kinematic Similarity

"The motions of two systems are kinematically similar if homologous particles lie at homologous points at homologous times"

Geometric similarity is probably not sufficient to establish time-scale equivalence

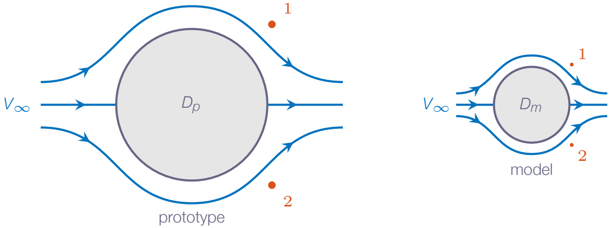
Dynamic considerations:

1. Reynolds number equivalence
2. Mach number equivalence

Kinematic Similarity

"Incompressible frictionless low-speed flows without free surfaces are kinematically similar with independent length and time scales"

$$
\begin{aligned}
D_m &= \alpha D_p \\
V_{\infty_m} &= \beta V_{\infty_p} \\
V_{1_m} &= \beta V_{1_p} \\
V_{2_m} &= \beta V_{2_p}
\end{aligned}
$$



Dynamic Similarity

"Dynamic similarity is achieved when the model and prototype have the same length scale ratio, time scale ratio, and force scale ratio"

Compressible flow:

1. Reynolds number equivalence
2. Mach number equivalence
3. specific-heat ratio equivalence

Incompressible flow without free surfaces:

1. Reynolds number equivalence

Incompressible flow with free surfaces:

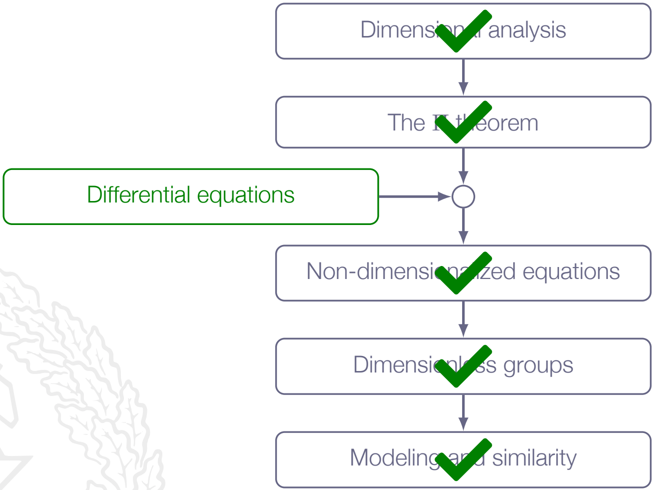
1. Reynolds number equivalence
2. Froude number equivalence (*and if necessary Weber number and/or cavitation number*)

Dynamic Similarity

$$\mathbf{F}_{inertia} = \mathbf{F}_{pressure} + \mathbf{F}_{gravity} + \mathbf{F}_{friction}$$

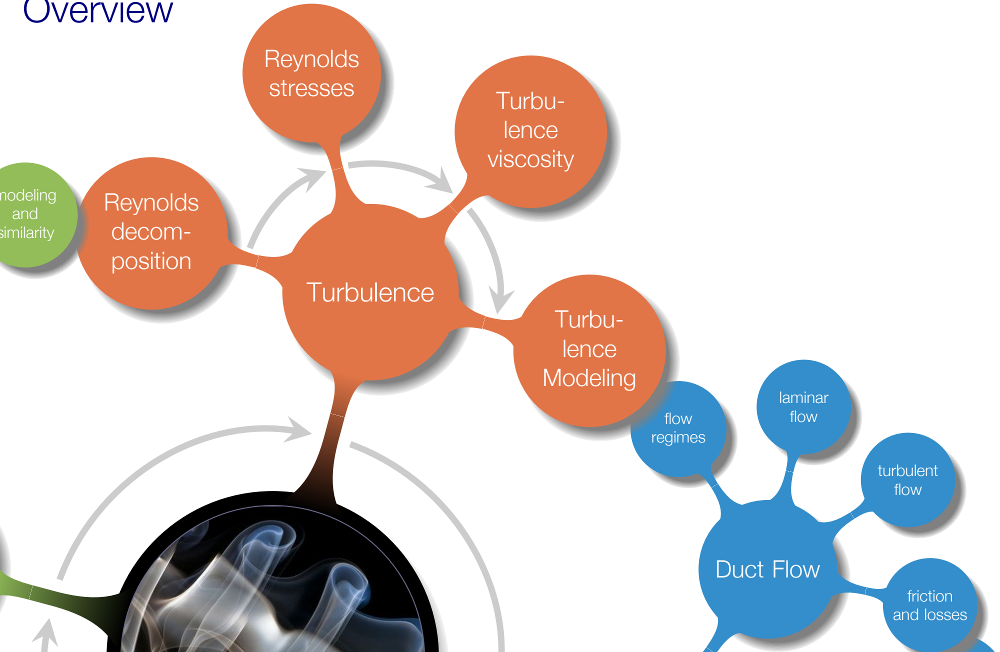
"Dynamic similarity ensures that each of the force components will be in the same ratio and have the same directions for model and prototype"

Roadmap - Dimensional Analysis and Similarity



Chapter 6 - Viscous Flow in Ducts

Overview



Reynolds stresses
Turbu- lence viscosity
modeling and similarity
Reynolds decom- position
Turbulence
Turbu- lence Modeling
flow regimes
laminar flow
turbulent flow
Duct Flow
friction and losses

Learning Outcomes

3 **Define** the Reynolds number

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

6 **Explain** what a boundary layer is and when/where/why it appears

8 **Understand** and be able to **explain** the concept shear stress

18 **Explain** losses appearing in pipe flows

19 **Explain** the difference between laminar and turbulent pipe flow

20 **Solve** pipe flow problems using Moody charts

24 **Explain** what is characteristic for a turbulent flow

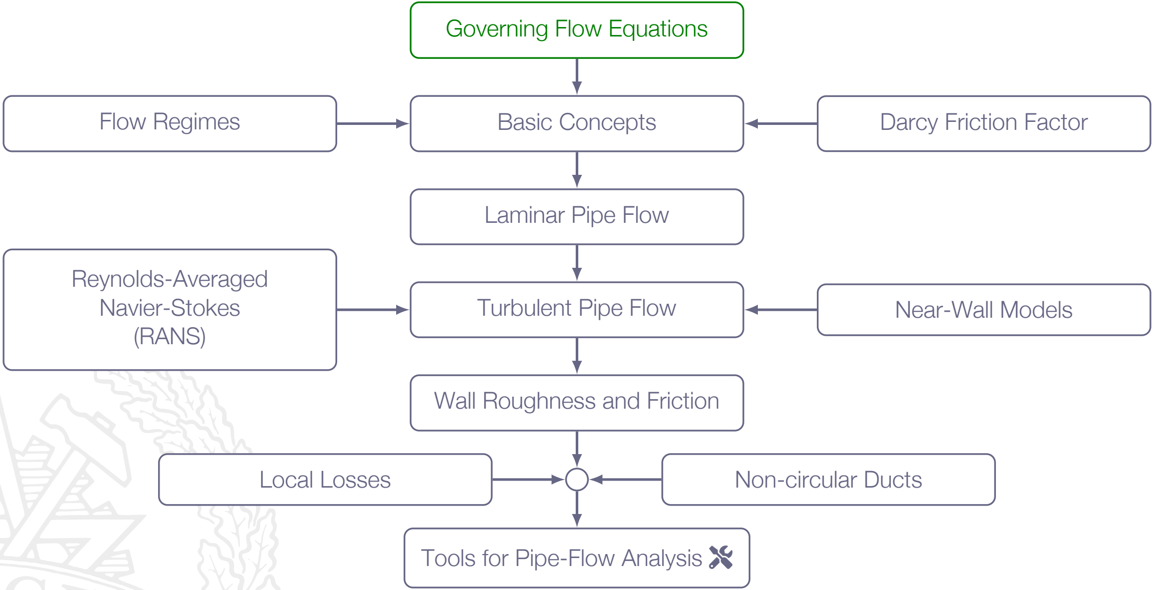
25 **Explain** Reynolds decomposition and derive the RANS equations

26 **Understand** and **explain** the Boussinesq assumption and turbulent viscosity

27 **Explain** the difference between the regions in a boundary layer and what is characteristic for each of the regions (viscous sub layer, buffer region, log region)

if you think about it, pipe flows are everywhere (a pipe flow is not a flow of pipes)

Roadmap - Viscous Flow in Ducts

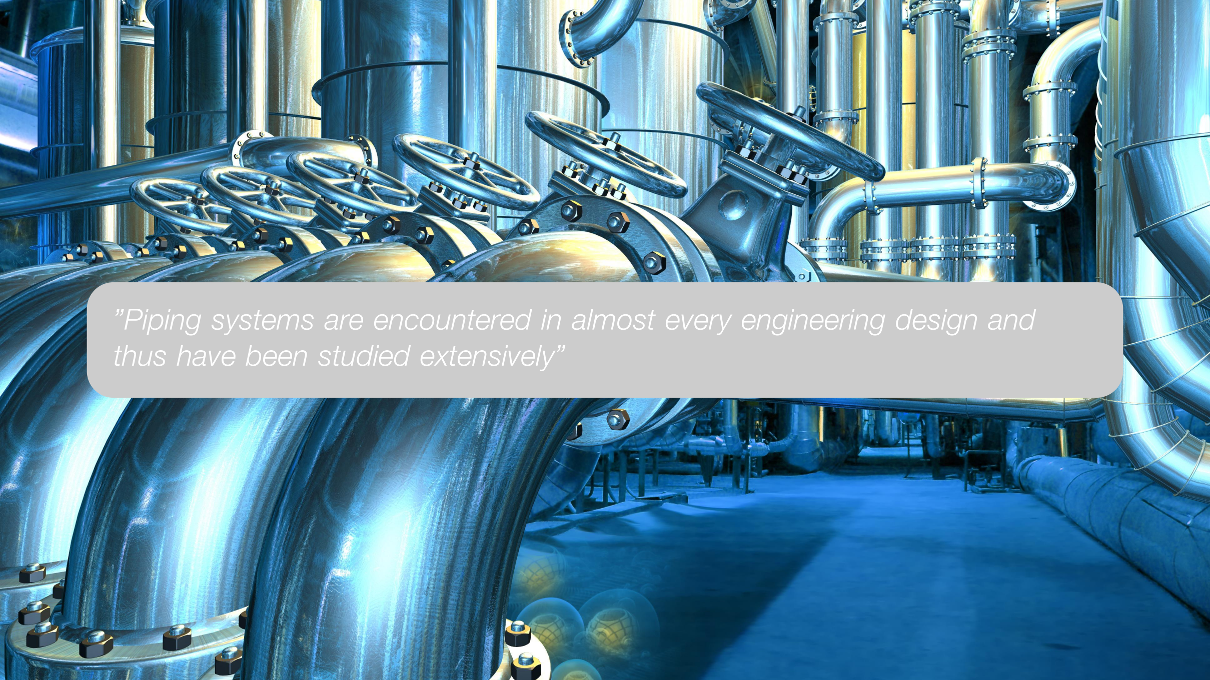


Complementary Course Material

These lecture notes covers chapter 6 in the course book and additional course material that you can find in the following documents

`MTF053_Equation-for-Boundary-Layer-Flows.pdf`

`MTF053_Turbulence.pdf`


”Piping systems are encountered in almost every engineering design and thus have been studied extensively”

Typical Pipe-Flow Problems

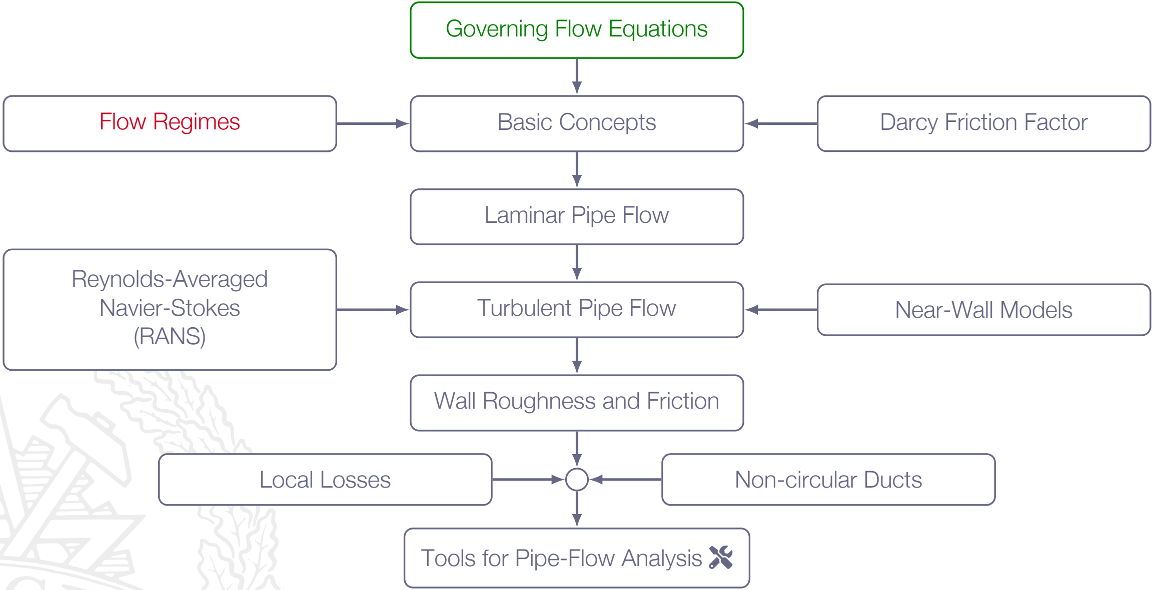
Example I:

Given pipe geometry, fluid properties, flow rate, and locations of valves, bends, diffusers etc - estimate the pressure drop needed to drive the flow

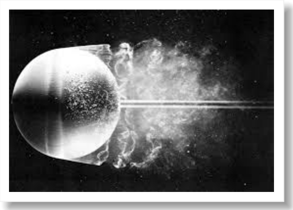
Example II:

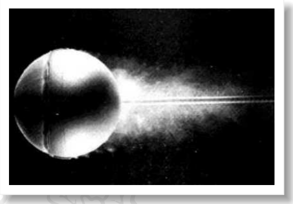
Given the pressure drop available from a pump - what flow rate can be expected

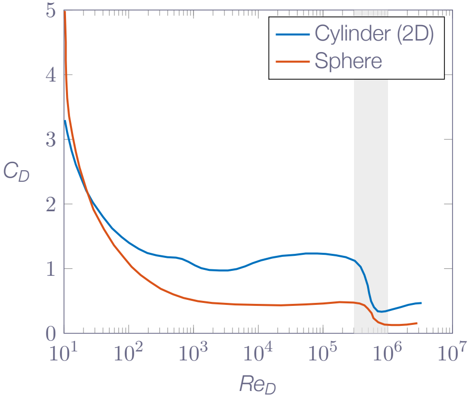
Roadmap - Viscous Flow in Ducts



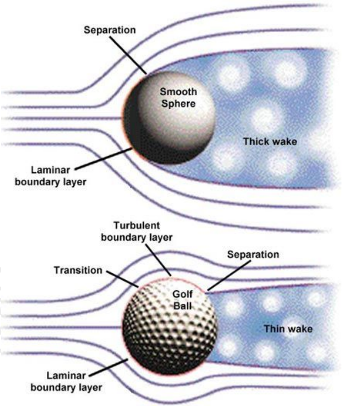
Transition to Turbulence

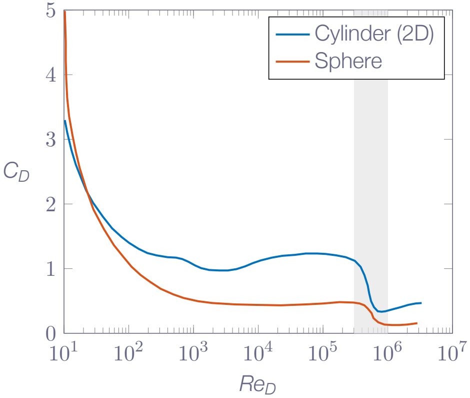






Transition to Turbulence



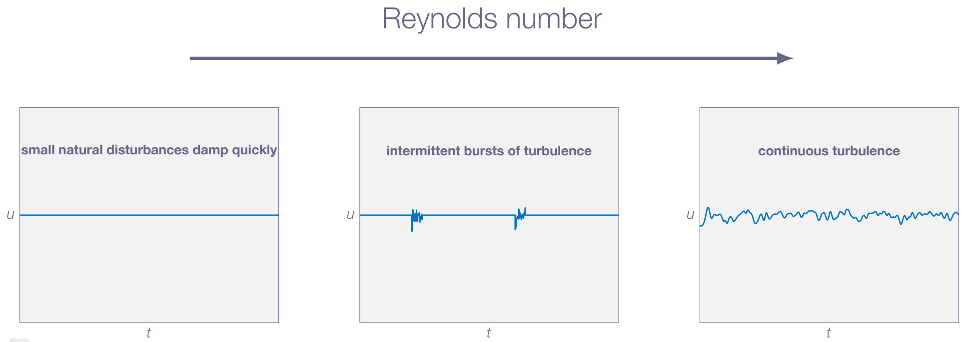


Transition to Turbulence

Factors that affects the transition to turbulent flow:

- Wall roughness
- Fluctuations in incoming flow
- Reynolds number

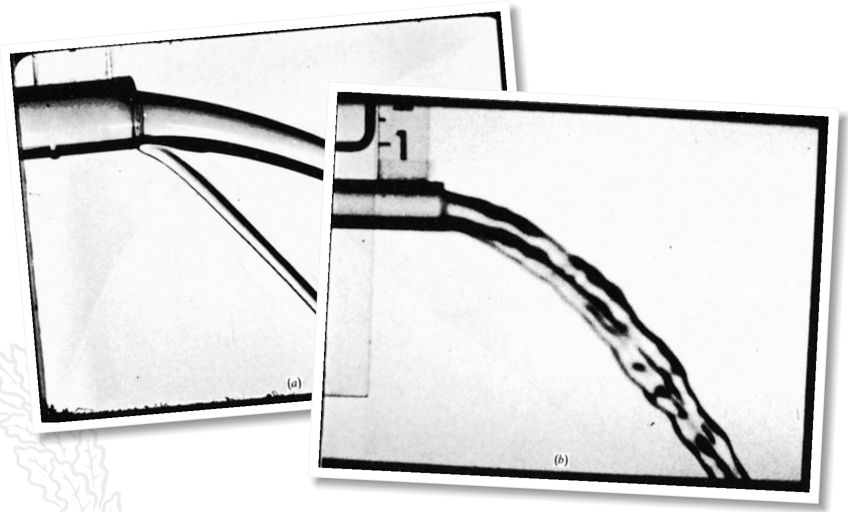
Transition to Turbulence



Fluctuations in the fully turbulent flow velocity signal:

- typically 1% to 20% of the average velocity
- not periodic
- random
- continuous range (spectrum) of frequencies

Transition to Turbulence



Transition to Turbulence - Viscous Flow in Ducts

0	$< Re <$	1	highly viscous laminar ”creeping” motion
1	$< Re <$	100	laminar, strong Reynolds number dependence
100	$< Re <$	10^3	laminar, boundary layer theory useful
10^3	$< Re <$	10^4	transition to turbulence
10^4	$< Re <$	10^6	turbulent, moderate Reynolds number dependence
10^6	$< Re <$	∞	turbulent, slight Reynolds number dependence

Note! The ranges will vary somewhat with geometry and surface roughness

Transition to Turbulence - Viscous Flow in Ducts

An accepted design value for **pipe flow transition** is

$$Re_{d,crit} \approx 2300$$

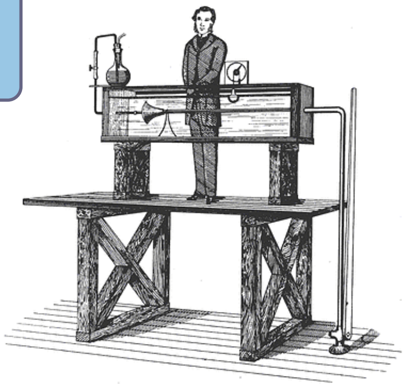
Note!

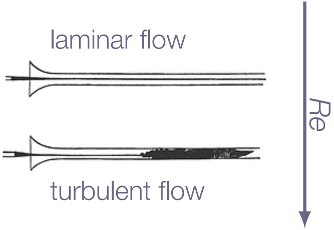
1. this value is **for pipe flows**, other applications have different transition Reynolds numbers
2. by careful design the Reynolds number can be pushed to higher values

The great majority of our analyses are concerned with laminar flow or with turbulent flow, and one should not normally design a flow operation in the transition region.

Transition to Turbulence - Osborne Reynolds (1842-1912)

$$Re = \frac{\rho U D}{\mu}$$

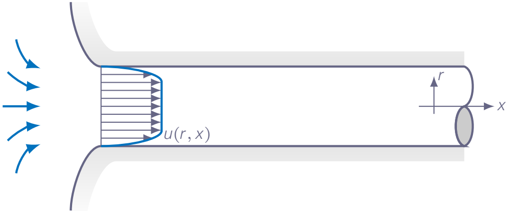




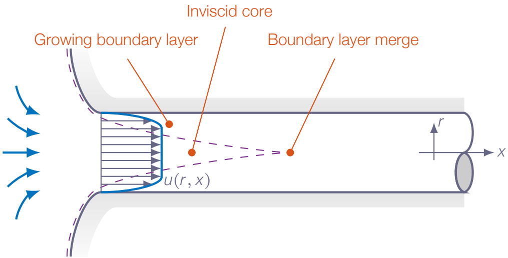
Internal Flows

- **Wall-bounded flows** - constrained by bounding walls
- Boundary layers grows and meet at the center

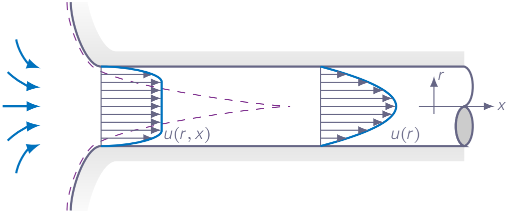
Velocity Profile Development



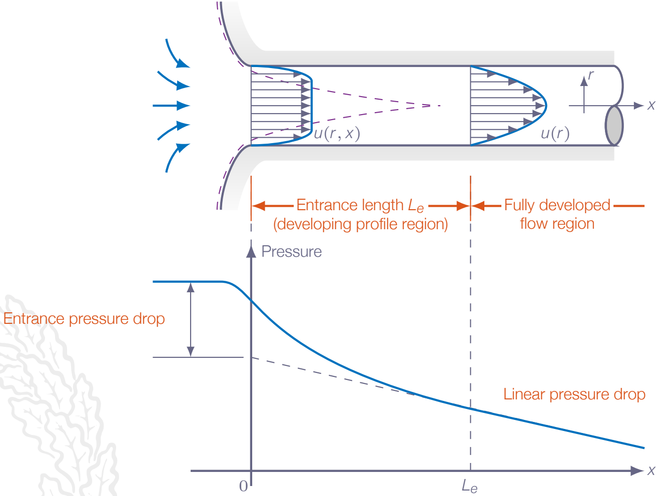
Velocity Profile Development



Velocity Profile Development



Velocity Profile Development



Velocity Profile Development

$$L_e = f(d, V, \rho, \mu),\ V = \frac{Q}{A},\ Q = \int u dA = const$$

Dimensional analysis gives:

$$\frac{L_e}{d} = g\left(\frac{\rho V d}{\mu}\right) = g(Re_d)$$

Velocity Profile Development

Laminar flow:

$$\frac{L_e}{d} \approx 0.06 Re_d$$

The maximum laminar entrance length, at $Re_d = Re_{d,crit} = 2300$, is $L_e = 138d$, which is the longest development length possible

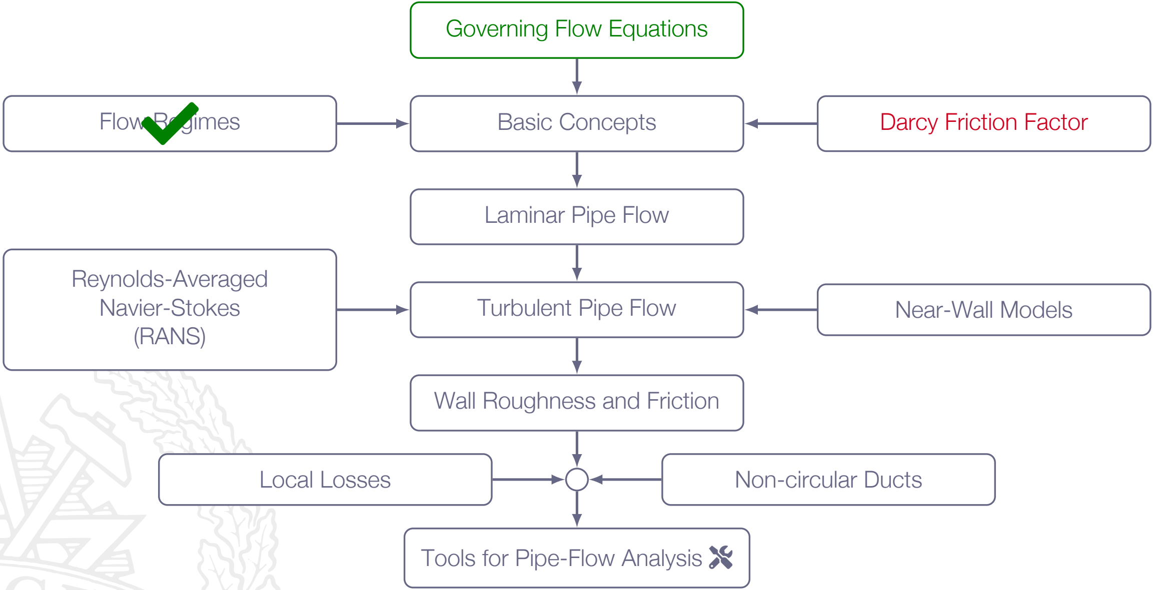
Velocity Profile Development

Turbulent flow ($Re_d \leq 10^7$):

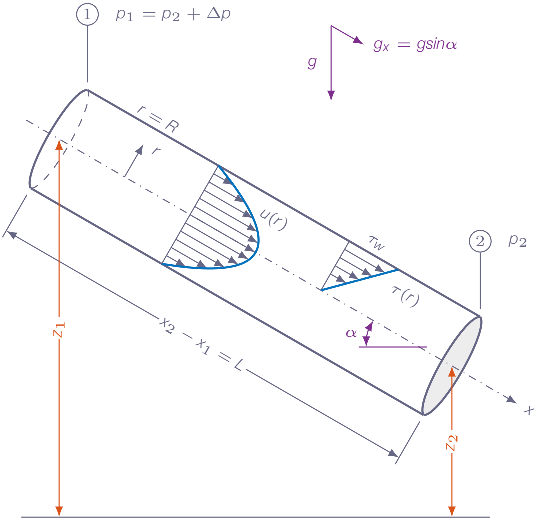
$$\frac{L_e}{d} \approx 1.6 Re_d^{1/4}$$

Re_d	4.0×10^3	1.0×10^4	1.0×10^5	1.0×10^6	1.0×10^7
L_e/d	13	16	28	51	90

Roadmap - Viscous Flow in Ducts



Head Loss



1
$p_1 = p_2 + \Delta p$
$g_x = g sin\alpha$
g
$r = R$
r
$u(r)$
τ_w
2
p_2
$\tau(r)$
α
z_1
$x_2 - x_1 = L$
z_2
x

Head Loss

$$Q_1 = Q_2 = Q, \; V_1 = V_2 = V$$

Steady-flow energy equation:

$$\left(\frac{p}{\rho g} + \alpha\frac{V^2}{2g} + z\right)_1 = \left(\frac{p}{\rho g} + \alpha\frac{V^2}{2g} + z\right)_2 + h_f$$

- No pumps or turbines between 1 and 2
- Fully developed flow ($\alpha_1 = \alpha_2$)

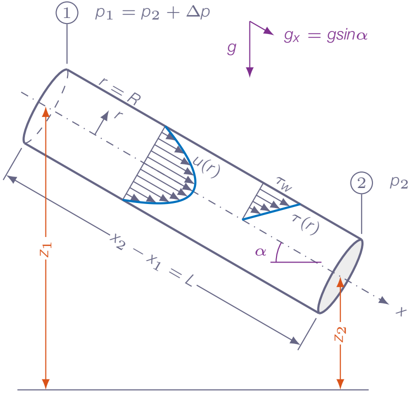
$$h_f = (z_1 - z_2) + \left(\frac{p_1 - p_2}{\rho g}\right) = \Delta z + \frac{\Delta p}{\rho g}$$

Head Loss

Apply the momentum equation along the pipe:

$$\sum F_x = \Delta p(\pi R^2) + \rho g(\pi R^2)L\sin\alpha - \tau_w(2\pi R)L$$

$$\sum F_x = \dot{m}(V_2 - V_1) = 0$$



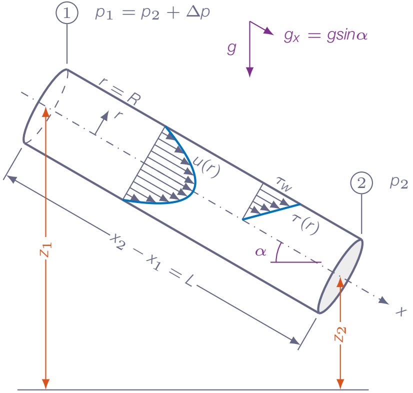
Head Loss

$$\Delta p(\pi R^2) + \rho g(\pi R^2)L\sin\alpha = \tau_w(2\pi R)L$$

$$\frac{\Delta p}{\rho g} + L\sin\alpha = \frac{2\tau_w}{\rho g}\frac{L}{R}$$

$$\frac{\Delta p}{\rho g} + \Delta z = \frac{4\tau_w}{\rho g}\frac{L}{d}$$

$$h_f = \frac{4\tau_w}{\rho g}\frac{L}{d}$$



Friction Factor

Henry Darcy 1803-1858

$$h_f = f_D \frac{L}{d} \frac{V^2}{2g}$$

where

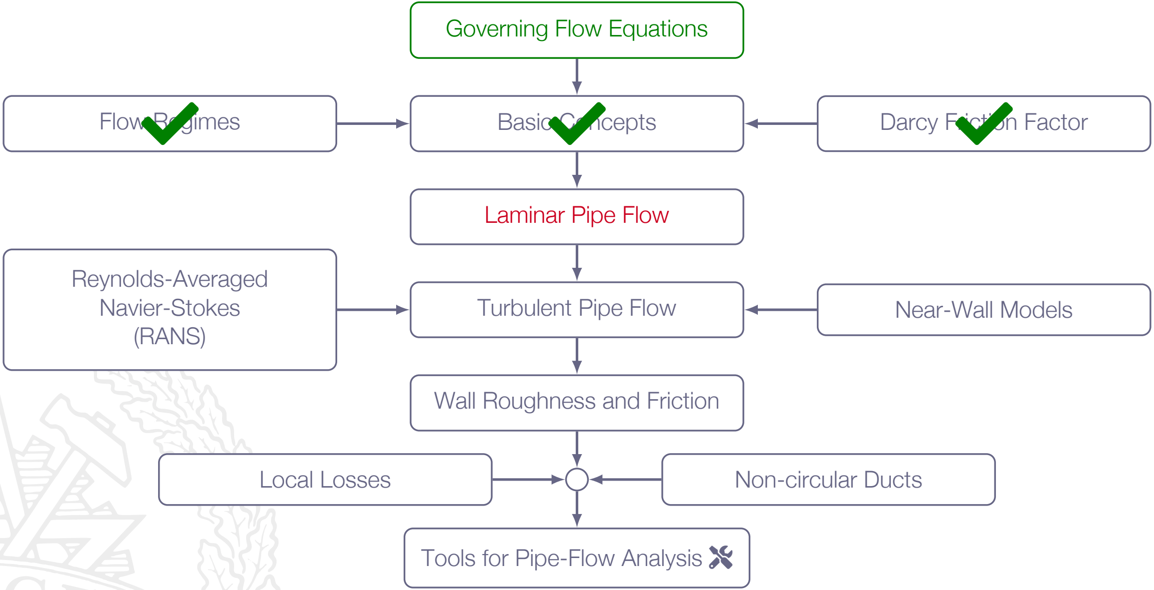
$$f_D = f(Re_d, \varepsilon/d, duct\ shape)$$

is the **Darcy friction factor**

$$\frac{4\tau_w}{\rho g} \frac{L}{d} = f_D \frac{L}{d} \frac{V^2}{2g} \Rightarrow f_D = \frac{8\tau_w}{\rho V^2}$$

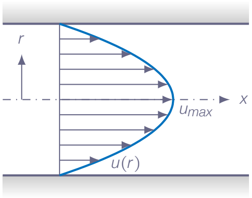
Note! for non-circular pipes, τ_w is an average value around the duct perimeter

Roadmap - Viscous Flow in Ducts



Fully-Developed Laminar Pipe Flow

- Fully developed
- circular pipe with the diameter D and radius R
- Pressure driven (Poiseuille flow)



$$u(r) = u_{max}\left(1 - \left(\frac{r}{R}\right)^2\right) \Rightarrow \frac{du}{dr} = -2u_{max}\frac{r}{R^2} = \left\{V = \frac{u_{max}}{2}\right\} = -4V\frac{r}{R^2}$$

$$\tau_w = \mu\left|\frac{du}{dr}\right|_{r=R} = \frac{4\mu V}{R} = \frac{8\mu V}{D}$$

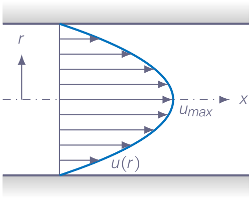
Fully-Developed Laminar Pipe Flow

For laminar flow:

$$f_D = \frac{8\tau_w}{\rho V^2} = \left\{\tau_w = \frac{8\mu V}{D}\right\} = \frac{64\mu}{\rho V D} = \frac{64}{Re_D}$$

Note! in laminar flow, the friction factor is inversely proportional to the Reynolds number

Fully-Developed Laminar Pipe Flow



- Fully developed
- circular pipe with the diameter D and radius R
- Pressure driven (Poiseuille flow)

$$u(r) = u_{max}\left(1 - \left(\frac{r}{R}\right)^2\right) \text{ where } u_{max} = -\frac{dp}{dx}\frac{R^2}{4\mu}$$

Fully-Developed Laminar Pipe Flow

$$-\frac{dp}{dx} = \left(\frac{\Delta p + \rho g \Delta z}{L}\right) \Rightarrow u_{max} = \left(\frac{\Delta p + \rho g \Delta z}{L}\right) \frac{R^2}{4\mu}$$

$$V = \frac{u_{max}}{2} = \left(\frac{\Delta p + \rho g \Delta z}{L}\right) \frac{R^2}{8\mu}$$

$$Q = \int u dA = VA = V\frac{\pi D^2}{4} = \left(\frac{\Delta p + \rho g \Delta z}{L}\right) \frac{\pi D^4}{128\mu}$$

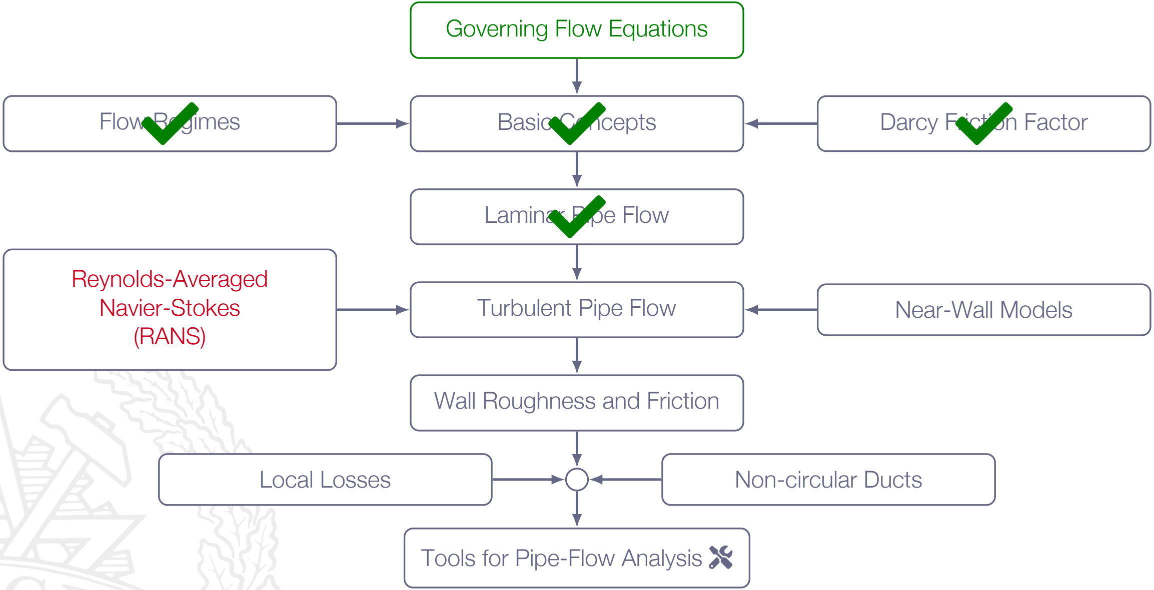
Fully-Developed Laminar Pipe Flow

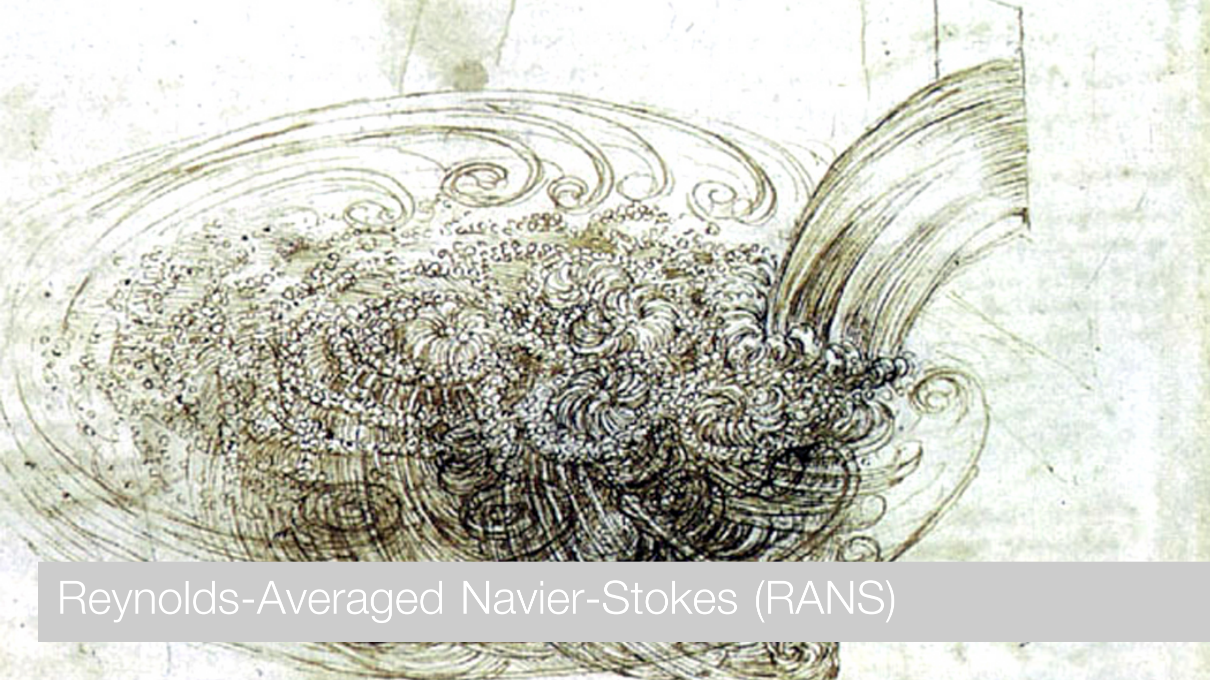
We can now calculate the head loss according to

$$h_f = f_D \frac{L}{D} \frac{V^2}{2g} \text{ where } f_D = \frac{8\tau_w}{\rho V^2}$$

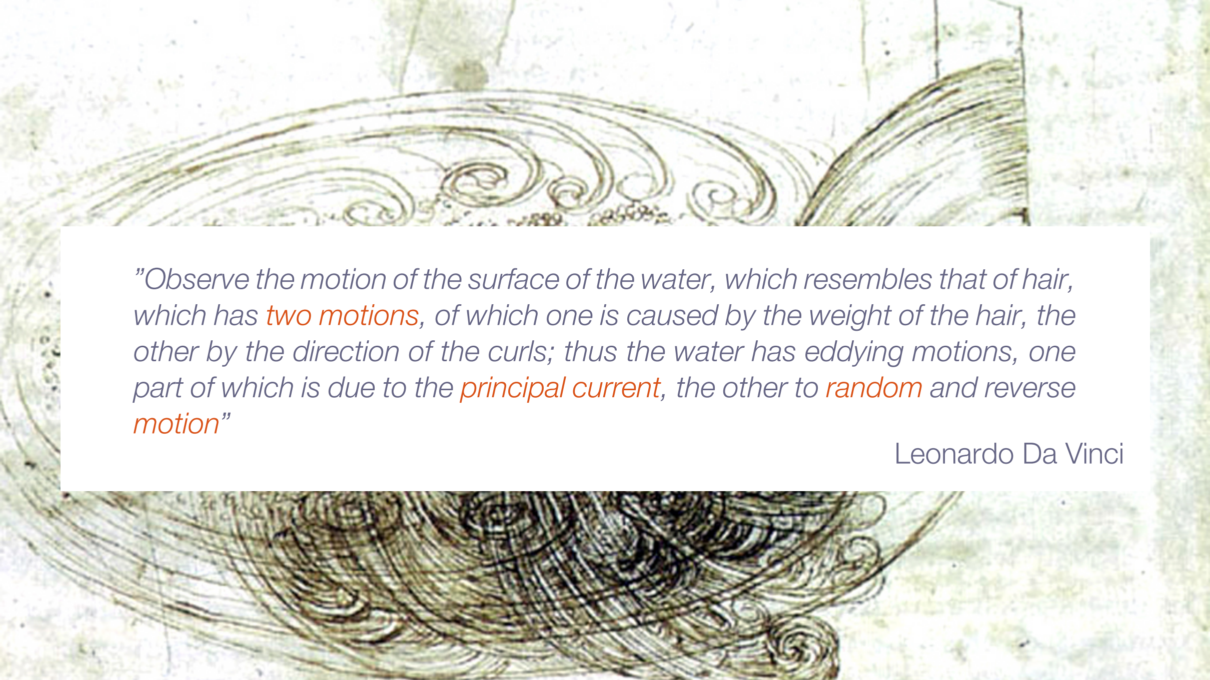
$$h_f = \frac{4\tau_w L}{\rho g D} = \left\{ \tau_w = \frac{8\mu V}{D} \right\} = \frac{16\mu V L}{\rho g D R} = \frac{32\mu V L}{\rho g D^2} = \left\{ V = \frac{4Q}{\pi D^2} \right\} = \frac{128\mu Q L}{\pi \rho g D^4}$$

Roadmap - Viscous Flow in Ducts





Reynolds-Averaged Navier-Stokes (RANS)



"Observe the motion of the surface of the water, which resembles that of hair, which has two motions, of which one is caused by the weight of the hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the principal current, the other to random and reverse motion"

Leonardo Da Vinci

Governing Equations

Assumptions:

1. constant density and viscosity
2. no thermal interaction

Flow equations:

continuity: $\nabla \cdot \mathbf{V} = 0$

momentum: $\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V}$

Governing Equations

The differential energy equation is not included here but let's have a look at it anyway

$$\rho \frac{D\hat{u}}{Dt} + p\nabla \cdot \mathbf{V} = \nabla \cdot (k\nabla T) + \phi$$

Pressure work:

pressure drives the flow through the duct

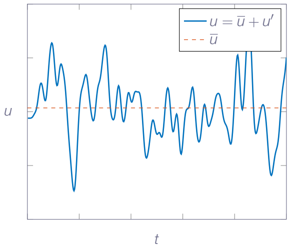
Viscous work:

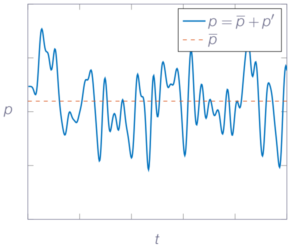
no-slip condition $\Rightarrow$ zero velocity at the walls $\Rightarrow$ no work done by wall shear stress

So, where does the energy go?

pressure work is balanced by **viscous dissipation** in the interior of the flow

Reynolds' Decomposition

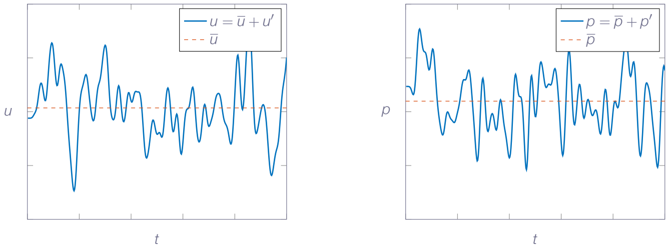




Not possible to solve analytically

Often, the time-averaged quantities are what we are looking for

Reynolds' Decomposition



$$\overline{u} = \frac{1}{T}\int_0^T u dt$$

$$u' = u - \overline{u}$$

$$\overline{u'} = \frac{1}{T}\int_0^T (u - \overline{u})dt = \overline{u} - \overline{u} = 0$$

Reynolds' Decomposition

The mean square of the fluctuations are, however, not zero

$$\overline{u'^2} = \frac{1}{T}\int_0^T u'^2 dt \neq 0$$

measure of turbulence intensity

Mean of fluctuation products are generally not zero ($\overline{u'v'}$, $\overline{u'p'}$)

Reynolds' Decomposition

Reynolds' idea was to split all properties into mean and fluctuating parts:

$$u = \overline{u} + u',\ v = \overline{v} + v',\ w = \overline{w} + w',\ p = \overline{p} + p'$$

1. insert into the governing equations
2. time average the equations

Reynolds' Decomposition

Continuity:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Momentum (x-component):

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

Reynolds' Decomposition

Continuity:

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

time averaging the equation gives

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0$$

and as a consequence

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

Reynolds' Decomposition

Momentum (x-component):

$$\rho\left(\frac{\partial \overline{u}}{\partial t}+\frac{\partial u'}{\partial t}\right)+$$
$$\rho\left(\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{u}\frac{\partial u'}{\partial x}+u'\frac{\partial \overline{u}}{\partial x}+u'\frac{\partial u'}{\partial x}\right)+$$
$$\rho\left(\overline{v}\frac{\partial \overline{u}}{\partial y}+\overline{v}\frac{\partial u'}{\partial y}+v'\frac{\partial \overline{u}}{\partial y}+v'\frac{\partial u'}{\partial y}\right)+$$
$$\rho\left(\overline{w}\frac{\partial \overline{u}}{\partial z}+\overline{w}\frac{\partial u'}{\partial z}+w'\frac{\partial \overline{u}}{\partial z}+w'\frac{\partial u'}{\partial z}\right)=$$
$$-\frac{\partial \overline{p}}{\partial x}-\frac{\partial p'}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 \overline{u}}{\partial x^2}+\frac{\partial^2 \overline{u}}{\partial y^2}+\frac{\partial^2 \overline{u}}{\partial z^2}+\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 u'}{\partial y^2}+\frac{\partial^2 u'}{\partial z^2}\right)$$

Reynolds' Decomposition

Momentum (x-component):

time averaging the equation gives:

$$\rho\left(\frac{\partial\overline{u}}{\partial t}+\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}+\overline{w}\frac{\partial\overline{u}}{\partial z}+\overline{u'\frac{\partial u'}{\partial x}}+\overline{v'\frac{\partial u'}{\partial y}}+\overline{w'\frac{\partial u'}{\partial z}}\right)=$$
$$-\frac{\partial\overline{p}}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}+\frac{\partial^2\overline{u}}{\partial z^2}\right)$$

The highlighted terms can be rewritten as:

$$\overline{u'\frac{\partial u'}{\partial x}}+\overline{v'\frac{\partial u'}{\partial y}}+\overline{w'\frac{\partial u'}{\partial z}}=\frac{\partial\overline{u'u'}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}-\overline{u'\underbrace{\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)}_{=0}}$$

Reynolds' Decomposition

the continuity equation reduces to

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0$$

the axial component of the momentum equation:

$$\rho \frac{D\overline{u}}{Dt} = -\frac{\partial \overline{p}}{\partial x} + \rho g_x + \frac{\partial}{\partial x}\left(\mu \frac{\partial \overline{u}}{\partial x} - \rho \overline{u'^2}\right) +$$

$$\frac{\partial}{\partial y}\left(\mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{u'v'}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial \overline{u}}{\partial z} - \rho \overline{u'w'}\right)$$

Reynolds' Decomposition

By applying Reynolds' decomposition to our governing equations, we have introduced a number of new unknowns

The number of equations is the same as before, which means problems

Our new problem has a name

The closure problem

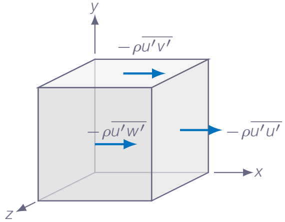
Reynolds' Decomposition

The three correlation terms $-\rho\overline{u'^2}$, $-\rho\overline{u'v'}$, and $-\rho\overline{u'w'}$ are called Reynolds stresses or turbulent stresses

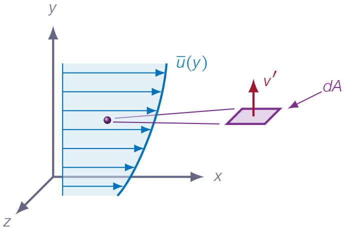
In duct and boundary layer flow, the stress $-\rho\overline{u'v'}$, associated with the direction normal to the wall, is dominant

$$\rho\frac{D\bar{u}}{Dt} \approx -\frac{\partial\bar{p}}{\partial x} + \rho g_x + \frac{\partial\tau}{\partial y}$$

$$\tau = \mu\frac{\partial\bar{u}}{\partial y} - \rho\overline{u'v'} = \tau_{lam} + \tau_{turb}$$



Reynolds' Decomposition



mass flow through surface element: $\dot{m}_y = \rho v' dA$

momentum balance in x-direction: $F_x = \dot{m}_y u = \rho v' \left(\overline{u} + u'\right) dA$

$$\tau_{dA} = -\frac{\overline{F}_x}{dA} = -\overline{\rho v' \left(\overline{u} + u'\right)} = -\rho\overline{v'\overline{u}} - \rho\overline{u'v'} = \left\{\overline{v'\overline{u}} = \overline{v'}\overline{u} = 0\right\} = -\rho\overline{u'v'}$$

$\Rightarrow -\rho\overline{u'v'}$ can be interpreted as a shear stress

Reynolds' Decomposition

Introducing **turbulent viscosity** μ_t defined such that

$$-\rho\overline{u'v'} = \mu_t \frac{\partial \overline{u}}{\partial y}$$

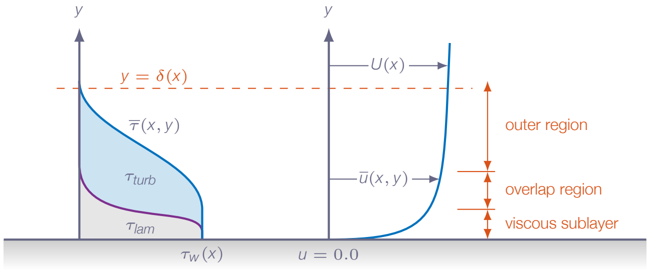
Boussinesq's assumption

With the turbulent viscosity, the total shear stress τ becomes:

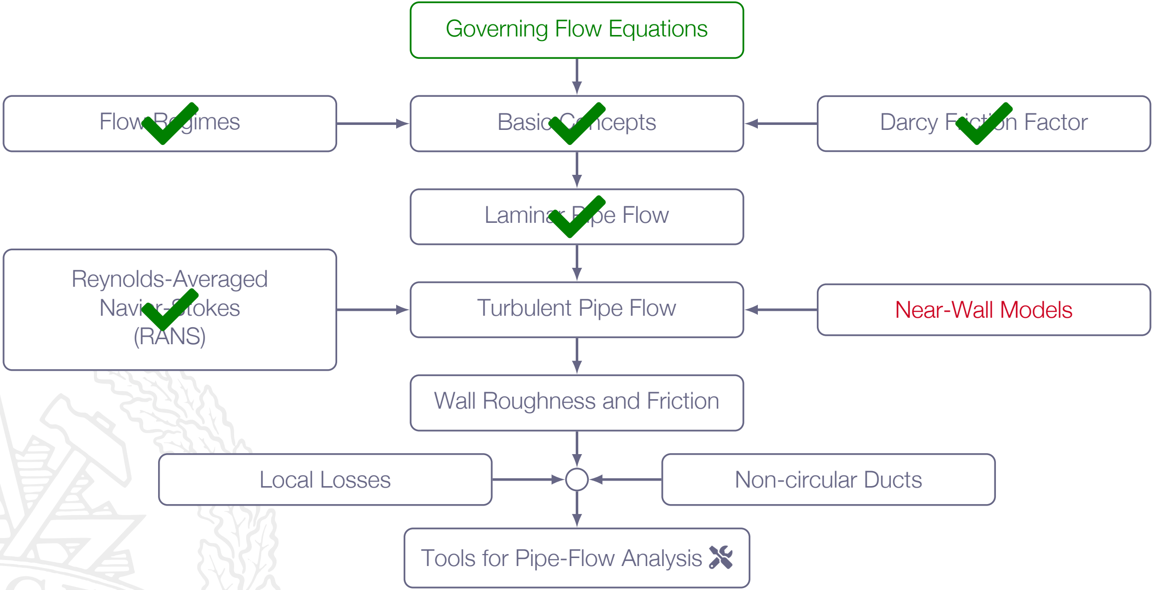
$$\tau = \mu \frac{\partial \overline{u}}{\partial y} - \rho\overline{u'v'} = (\mu + \mu_t)\frac{\partial \overline{u}}{\partial y}$$

Reynolds' Decomposition

- laminar shear (τ_{lam}) dominates in the near-wall region
- turbulent shear (τ_{turb}) dominates in the outer region
- both are important in the overlap layer



Roadmap - Viscous Flow in Ducts



Turbulent Pipe Flow - Boundary-Layer Equations

Momentum equation (x-component)

$$\rho \frac{D\bar{u}}{Dt} \approx -\frac{\partial \bar{p}}{\partial x} + \rho g_x + \frac{\partial \tau}{\partial y}$$

where

$$\tau = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'} = (\mu + \mu_t)\frac{\partial \bar{u}}{\partial y}$$

For boundary-layer flows

$$\rho \left(\bar{u}\frac{\partial \bar{u}}{dx} + \bar{v}\frac{\partial \bar{u}}{dy} \right) = -\frac{d\bar{p}}{dx} + \rho g_x + (\mu + \mu_t)\frac{\partial \bar{u}}{\partial y}$$

(will be discussed in more detail in later lectures)

Turbulent Pipe Flow - Boundary-Layer Equations

$$\rho\left(\overline{u}\frac{\partial\overline{u}}{dx}+\overline{v}\frac{\partial\overline{u}}{dy}\right)=-\frac{d\overline{p}}{dx}+\rho g_x+\frac{\partial\tau}{\partial y}$$

$$y\rightarrow 0\Rightarrow\begin{cases}\overline{u}\rightarrow 0\\ \overline{v}\rightarrow 0\end{cases}\Rightarrow$$

$$\frac{\partial\tau}{\partial y}=\frac{d\overline{p}}{dx}-\rho g_x$$

Turbulent Pipe Flow - Boundary-Layer Equations

$$\frac{\partial \tau}{\partial y} = \frac{d\overline{p}}{dx} - \rho g_x$$

$$\tau(y) = \left(\frac{d\overline{p}}{dx} - \rho g_x\right) y + C$$

$$\tau(0) = C = \tau_w \Rightarrow \tau(y) = \left(\frac{d\overline{p}}{dx} - \rho g_x\right) y + \tau_w$$

Note! with a negative pressure gradient, the shear stress will reduce with increasing distance from the wall

Turbulent Pipe Flow - Boundary-Layer Equations

$$\tau(y) = \left(\frac{d\overline{p}}{dx} - \rho g_x\right) y + \tau_w$$

At the wall, the shear stress is equal to the wall-shear stress

$$y \to 0 \Rightarrow \tau(y) \to \tau_w$$

In fact, assuming that the **shear stress** (τ) is **constant** and equal to the wall-shear stress (τ_w) is a valid assumption in the **near-wall region** (some distance from the wall but still close) as long as the pressure gradient is moderate.

Outside of the near-wall region, inertial effects has to be accounted for, i.e., $D\overline{u}/Dt$ will not be zero and thus the shear stress will not be equal to the wall-shear stress.

Turbulent Boundary Layers

A turbulent boundary layer may be divided into different regions where the physical processes leading to shear stress are clearly distinguishable

The viscous sublayer
: the shear stress is dominated by molecular viscosity (μ)

The buffer region
: molecular viscosity (μ) and turbulent viscosity (μ_t) are equally important

The log layer
: the shear stress is dominated by turbulent viscosity (μ_t)

The outer region
: inertial effects must be accounted for

In the following we will discuss two turbulent boundary layer regions in detail:

- **The viscous sublayer** - the region closest to the wall
- **The log region** - outside of the viscous sublayer but still in the near-wall region

Viscous Sublayer

At the wall

$$\tau = \tau_w = \mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{u'v'}$$

$$y \rightarrow 0 \Rightarrow \begin{cases} u' \rightarrow 0 \\ v' \rightarrow 0 \end{cases} \Rightarrow$$

$$\tau = \mu \frac{\partial \overline{u}}{\partial y}$$

Viscous Sublayer

$$\tau = \mu \frac{\partial \overline{u}}{\partial y} \Rightarrow \overline{u}(y) = \frac{\tau_w}{\mu} y + C$$

$$\overline{u}(0) = 0 \Rightarrow C = 0 \Rightarrow$$

$$\overline{u}(y) = \frac{\tau_w}{\mu} y$$

Note! in the viscous sublayer, the average velocity increase linearly with the wall distance

Viscous Sublayer

Introducing friction velocity defined as

$$u^* = \sqrt{\frac{\tau_w}{\rho}}$$

and thus

$$\bar{u}(y) = \frac{\tau_w}{\mu} y = \frac{\rho {u^*}^2 y}{\mu} = \frac{{u^*}^2 y}{\nu}$$

which can be rewritten as:

$$\underbrace{\frac{\bar{u}}{u^*}}_{u^+} = \underbrace{\frac{u^* y}{\nu}}_{y^+} \text{ valid for } y^+ \leq 5 - 10$$

The Log Region

Now, let's move a bit further out from the wall

1. $\tau = const = \tau_w$ still (*we have not moved that far out from the wall*)
2. outside of the viscous sublayer $\mu_t \gg \mu$ and thus

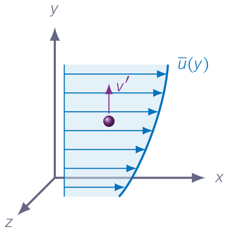
$$\tau = \tau_w = \mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{u'v'} \approx -\rho \overline{u'v'} = \mu_t \frac{\partial \overline{u}}{\partial y}$$

We need an estimate of μ_t to be able to solve this ...

The Log Region

Let’s first examine the relation between u' and v' (the velocity fluctuations in the x and y directions)

The illustration below shows a fluid particle in a boundary-layer flow

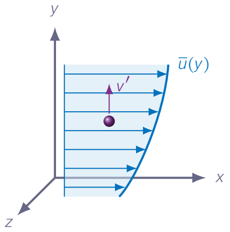


The Log Region

A positive v' fluctuation will lead to a vertical transport of the fluid particle

The fluid particle will end up in a position in the flow where the axial velocity is higher than where it came from, thus leading to a negative fluctuation in the axial velocity at that position $(u' < 0)$

In the same way, a negative v' fluctuation will lead to $u' > 0$

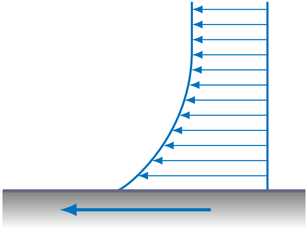


The product $u'v'$ will **always** be negative if $\partial\overline{u}/\partial y$ is positive in the wall-normal direction

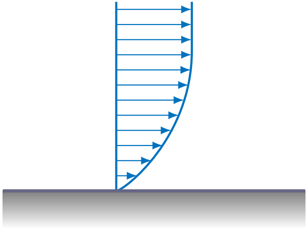
Thus $\tau_{turb} = -\rho\overline{u'v'} = \mu_t \dfrac{\partial\overline{u}}{\partial y}$ is positive

The Log Region

What about other type of boundary layers such as for example the flow over a moving surface



moving wall

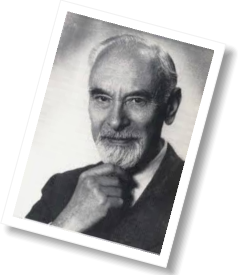


frame of reference of the wall

The Log Region

Prandtl's mixing length concept

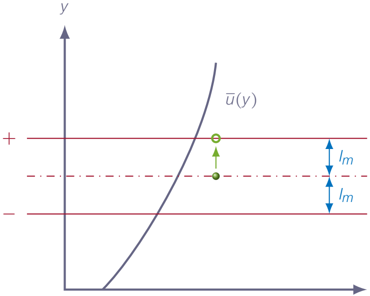
"the average distance that a small mass of fluid will travel before it exchanges its momentum with another mass of fluid"



Ludwig Prandtl 1875-1953

$$\bar{u}(y + l_m) = \bar{u}(y) + l_m \frac{\partial \bar{u}}{\partial y}$$

$$\bar{u}(y - l_m) = \bar{u}(y) - l_m \frac{\partial \bar{u}}{\partial y}$$



Prandtl assumed $u' \approx l_m \frac{\partial \bar{u}}{\partial y}$

He further assumed v' to be of the same size as u'

The Log Region

Prandtl's mixing length concept

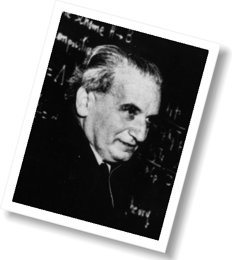
$$\tau_t = -\rho\overline{u'v'} \approx \rho l_m^2 \left(\frac{\partial \overline{u}}{\partial y}\right)^2$$

$$-\rho\overline{u'v'} \approx \mu_t \frac{\partial \overline{u}}{\partial y} \Rightarrow \mu_t \approx \rho l_m^2 \left|\frac{\partial \overline{u}}{\partial y}\right|$$

$$\nu_t = \frac{\mu_t}{\rho} \approx l_m^2 \left|\frac{\partial \overline{u}}{\partial y}\right|$$

The Log Region

Prandtl's mixing length concept



Theodore von Kármán 1881-1963

So, how do we estimate the mixing length l_m

$$l_m(y) = a_o + a_1 y + a_2 y^2 + \dots$$

1. $y \rightarrow 0 \Rightarrow l_m \rightarrow 0 \Rightarrow a_o = 0$
2. small values of y (*we are still very close to the wall*) $\Rightarrow l_m = a_1 y$

$$l_m = \kappa y$$

where κ is Kármán's constant $\kappa \approx 0.41$

The Log Region

$$\mu_t \approx \rho l_m^2 \left|\frac{\partial \overline{u}}{\partial y}\right| = \rho\kappa^2 y^2 \left|\frac{\partial \overline{u}}{\partial y}\right|$$

$$\tau_w = \mu_t \frac{\partial \overline{u}}{\partial y} = \rho\kappa^2 y^2 \left(\frac{\partial \overline{u}}{\partial y}\right)^2 = \rho u^{*2}$$

$$\kappa^2 y^2 \left(\frac{\partial \overline{u}}{\partial y}\right)^2 = u^{*2} \Rightarrow$$

$$\frac{\partial \overline{u}}{\partial y} = \frac{u^*}{\kappa y}$$

The Log Region

$$\frac{\partial \overline{u}}{\partial y} = \frac{u^*}{\kappa y} \Rightarrow$$

$$\overline{u}(y) = \frac{u^*}{\kappa} \ln(y) + C$$

or in non-dimensional form

$$\underbrace{\frac{\overline{u}(y)}{u^*}}_{u^+} = \frac{1}{\kappa} \ln \underbrace{\left(\frac{y u^*}{\nu}\right)}_{y^+} + \underbrace{\frac{C}{u^*} - \ln\left(\frac{u^*}{\nu}\right)}_{B}$$

The Log Region

$$u^+ = \frac{1}{\kappa} \ln\left(y^+\right) + B$$

valid for $30 \lesssim y^+ \lesssim 1000$

From experiments we have:

$\kappa \approx 0.41$ and $4.9 < B < 5.5$

flow over a flat plate (external flow): $B \approx 4.9$
duct flow (internal flow): $B \approx 5.3$
White: $B \approx 5.0$

Outer region

In the outer region it has been found that

$$\frac{U - \overline{u}}{u^*} = f\left(\frac{y}{\delta}\right)$$

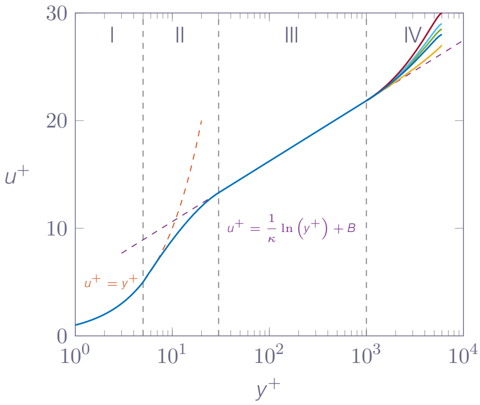
where δ is the thickness of the outer layer and U the velocity at the edge of the outer layer

Regions in a Turbulent Boundary Layer

between the viscous sublayer and the log region, none of the models works

in the outer region, inertial forces needs to be included

$$\rho\left(\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}\right)\neq 0$$



I: viscous sublayer
II: buffer layer
III: log-law region
IV: outer layer

Example - Pipe Flow Boundary Layer

Air at 20° flows through a 14-cm-diameter pipe. The flow is fully developed and the centerline velocity is 5.0 m/s

From the provided data, estimate the friction velocity (u^*) and the wall-shear stress (τ_w)

Air @ 20° $\Rightarrow \rho = 1.2\ kg/m^3,\ \mu = 1.8 \times 10^{-5}\ kg/(ms)$

$D = 0.14\ m$

$U_{max} = 5.0\ m/s$

Example - Pipe Flow Boundary Layer

Assume turbulent flow:

$$V_{av} = \frac{2U_{max}}{(1+m)(2+m)}$$

$m = 1/7$ gives $V_{av} = 4.08\ m/s$

$$Re_D = \frac{\rho V_{av} D}{\mu} \approx 38000 \gg Re_{D_{critical}} = 2300$$

The flow is turbulent

Example - Pipe Flow Boundary Layer

Assume that the log-law is valid all the way to the center of the pipe

$$u^+ = \frac{1}{\kappa}\ln(y^+) + B \Leftrightarrow 0 = \frac{1}{\kappa}\ln(y^+) + B - u^+$$

or (at the center of the pipe where $y = R$ and $u = U_{max}$)

$$0 = \frac{1}{\kappa}\ln\left(\frac{Ru^*}{\nu}\right) + B - \frac{U_{max}}{u^*}$$

where $\kappa = 0.41$ and $B = 5.0$

$$u^* = \sqrt{\frac{\tau_w}{\rho}}$$

Example - Pipe Flow Boundary Layer

Find estimates of u^* and τ_w using a Newton-Raphson solver

Using the definitions of y^+, u^+, and u^*, we can get a function $f(\tau_w)$

$$f(\tau_w) = \frac{1}{\kappa}\ln\left(\frac{R\sqrt{\tau_w}}{\sqrt{\rho}\nu}\right) + B - \frac{U_{max}\sqrt{\rho}}{\sqrt{\tau_w}}$$

The derivative of $f(\tau_w)$ is obtained as (*details on next slide*)

$$f'(\tau_w) = \frac{(1/\kappa)\sqrt{\tau_w} + U_{max}\sqrt{\rho}}{2\tau_w^{3/2}} = \frac{(1/\kappa) + u^+}{2\tau_w}$$

Example - Pipe Flow Boundary Layer

$$f(\tau_w) = \frac{1}{\kappa}\ln\left(\frac{R\sqrt{\tau_w}}{\sqrt{\rho}\nu}\right) + B - \frac{U_{max}\sqrt{\rho}}{\sqrt{\tau_w}}$$

$$f'(\tau_w) = \frac{\partial}{\partial\tau_w}\left(\frac{1}{\kappa}\ln\left(\frac{R\sqrt{\tau_w}}{\sqrt{\rho}\nu}\right)\right) - \frac{\partial}{\partial\tau_w}\left(\frac{U_{max}\sqrt{\rho}}{\sqrt{\tau_w}}\right) =$$

$$= \frac{\partial}{\partial\tau_w}\left(\frac{1}{\kappa}\left[\ln\left(\frac{R}{\sqrt{\rho}\nu}\right) + \ln\left(\sqrt{\tau_w}\right)\right]\right) - \left(-\frac{1}{2}\right)\frac{U_{max}\sqrt{\rho}}{\tau_w^{3/2}} =$$

$$= \frac{\partial}{\partial\tau_w}\left(\frac{1}{\kappa}\left[\ln\left(\frac{R}{\sqrt{\rho}\nu}\right) + \frac{1}{2}\ln\left(\tau_w\right)\right]\right) + \frac{U_{max}\sqrt{\rho}}{2\tau_w^{3/2}} =$$

$$= \left(\frac{1}{\kappa}\right)\frac{1}{2\tau_w} + \frac{U_{max}\sqrt{\rho}}{2\tau_w^{3/2}} = \frac{(1/\kappa)\sqrt{\tau_w} + U_{max}\sqrt{\rho}}{2\tau_w^{3/2}} = \frac{(1/\kappa) + u^+}{2\tau_w}$$

Example - Pipe Flow Boundary Layer

With the functions $f(\tau_w)$ and $f'(\tau_w)$ defined, we can set up an iterative Newton-Raphson solver to find τ_w using

$$\tau_{w_{n+1}} = \tau_{w_n} - \frac{f(\tau_{w_n})}{f'(\tau_{w_n})}$$

where $n+1$ and n are iteration numbers. Iterate until converged with the following convergence criterium:

$$\left|\frac{f(\tau_{w_n})}{f'(\tau_{w_n})}\right| \leq \tau_w \times 10^{-4}$$

Example - Pipe Flow Boundary Layer

```python
import numpy as np

def calc_yplus_uplus(rho,mu,tau_w,y,U):
    nu=mu/rho
    ustar=np.sqrt(tau_w/rho)
    yplus=y*ustar/nu
    uplus=U/ustar
    return yplus,uplus,ustar

mu    =  1.8e-5 # fluid viscosity (dynamic viscosity)
rho   =  1.2    # fluid density
u_max =  5.0    # centerline velocity
R     =  0.07   # pipe radius
kappa =  0.41   # von Kármán constant
B     =  5.0    # integration constant in the log-law
```

Example - Pipe Flow Boundary Layer

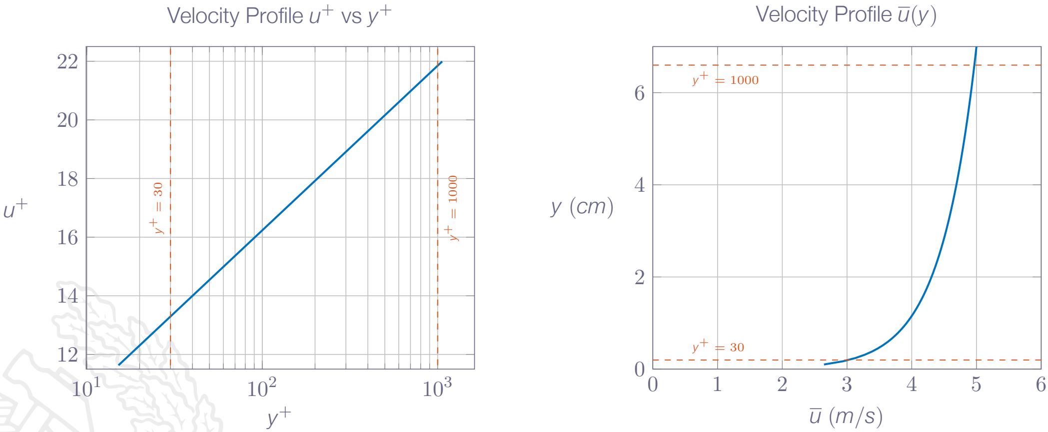
```
tau_w = mu*u_max/R # initial guess

yplus,uplus,ustar=calc_yplus_uplus(rho,mu,tau_w,R,u_max)

dtau_w = 10.*tau_w

while( abs(dtau_w) > 0.0001*tau_w ):
  f      = (1./kappa)*np.log(yplus)-uplus+B
  df     = 0.5*((1./kappa)+uplus)/tau_w
  dtau_w = -f/df
  tau_w  = tau_w+dtau_w
  yplus,uplus,ustar=calc_yplus_uplus(rho,mu,tau_w,R,u_max)
```

Example - Pipe Flow Boundary Layer

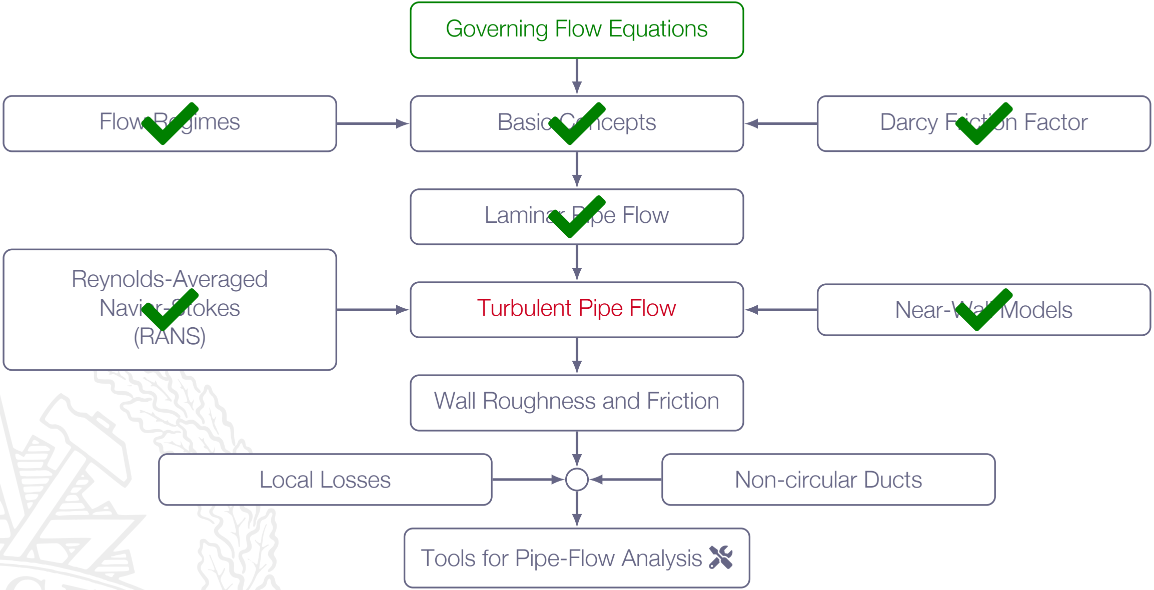
variable	dimension	value
y^+		1061
u^*	m/s	0.227
τ_w	N/m^2	0.062

Note! $y^+ = 1061$ is actually outside the range of y^+ values for which the log-law is valid - but it is very close to the limit...

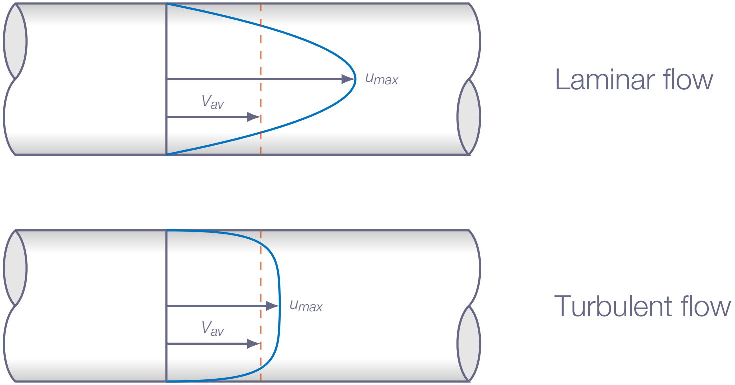
Example - Pipe Flow Boundary Layer



Roadmap - Viscous Flow in Ducts



Turbulent Pipe Flow



Turbulent Pipe Flow

As we did for laminar pipe flow, we will now obtain the friction factor for turbulent pipe flow

$$\tau_w = f_D \frac{\rho V^2}{8} = \rho u^{*2} \Rightarrow f_D = 8 \left(\frac{V}{u^*} \right)^{-2}$$

So, what we need now is an estimate of the average flow velocity in the pipe (V) ...

There are different ways to do this and here is one example:

1. Assume that we can use the log-law all the way across the pipe
2. Integrate to get the average velocity
3. Insert the calculated average velocity into the relation above

Turbulent Pipe Flow

$$f_D = 8\left(\frac{V}{u^*}\right)^{-2}$$

$$\left.\begin{aligned}\frac{\overline{u}(r)}{u^*} &\approx \frac{1}{\kappa}\ln\frac{(R-r)u^*}{\nu}+B \\ V = \frac{Q}{A} &= \frac{1}{\pi R^2}\int_0^R \overline{u}(r)2\pi r dr\end{aligned}\right\} \Rightarrow \frac{V}{u^*} \approx \frac{1}{\pi R^2}\int_0^R\left[\frac{1}{\kappa}\ln\frac{(R-r)u^*}{\nu}+B\right] r dr$$

with $\kappa = 0.41$ and $B = 5.0$ we get

$$\frac{V}{u^*} \approx 2.44\ln\frac{Ru^*}{\nu}+1.34$$

details on next slide

Turbulent Pipe Flow

$$\frac{V}{u^*} = \frac{2}{R^2}\int_0^R \left[\frac{r}{\kappa}\ln\left(\frac{(R-r)u^*}{\nu}\right) + Br\right]dr = \frac{2}{\kappa R^2}\int_0^R \left[\ln(R-r) + \ln\left(\frac{u^*}{\nu}\right) + B\kappa\right] rdr =$$

$$= \frac{1}{\kappa}\left(\ln\left(\frac{u^*}{\nu}\right) + B\kappa\right) + \frac{2}{\kappa R^2}\int_0^R r\ln(R-r)dr =$$

$$= \frac{1}{\kappa}\ln\left(\frac{u^*}{\nu}\right) + B + \frac{2}{\kappa R^2}\left[\frac{1}{4}\left(-2(R^2-r^2)\ln(R-r) - r(2R+r)\right)\right]_0^R =$$

$$= \frac{1}{\kappa}\ln\left(\frac{Ru^*}{\nu}\right) + B - \frac{3}{2\kappa} = \{\kappa = 0.41,\ B = 5.0\} = 2.44\ln\frac{Ru^*}{\nu} + 1.34$$

Turbulent Pipe Flow

$$\frac{V}{u^*} \approx 2.44 \ln \frac{Ru^*}{\nu} + 1.34$$

The argument of the logarithm can be rewritten as

$$\frac{Ru^*}{\nu} = \frac{VD}{2\nu}\frac{u^*}{V} = \left\{ Re_D = \frac{VD}{\nu},\ f_D = 8\left(\frac{u^*}{V}\right)^2 \right\} = \frac{1}{2} Re_D \left(\frac{f_D}{8}\right)^{1/2}$$

and thus:

$$\boxed{\frac{1}{\sqrt{f_D}} \approx 2.0 \log_{10}(Re_D \sqrt{f_D}) - 0.8}$$

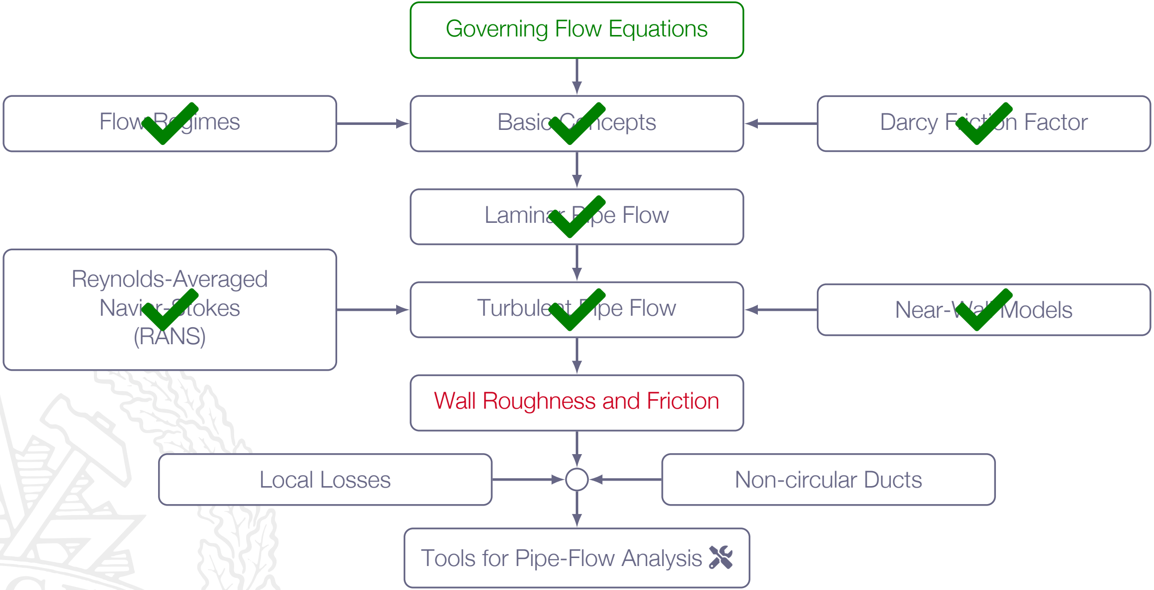
Turbulent Pipe Flow

Alternative 2:

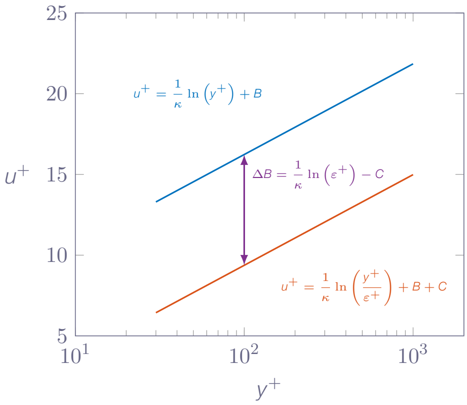
If we assume that $\frac{\bar{u}}{u^*} = 8.3 \left(\frac{u^* y}{\nu}\right)^{1/7}$ applies all over the cross section we get

$$f_D = \frac{0.3164}{Re_D^{1/4}}$$

Roadmap - Viscous Flow in Ducts



Wall Roughness



Effects of surface roughness on friction:

- Negligible for laminar pipe flow
- Significant for turbulent flow
  - breaks up the viscous sublayer
  - modifies the log law (changes the value of the integration constant B)

$$\Delta B \propto (1/\kappa)\ln \epsilon^+ \quad \text{where } \epsilon^+ = \frac{\epsilon u^*}{\nu}$$

Wall Roughness

$$\frac{\epsilon u^*}{\nu} < 5$$

hydraulically smooth

no effects of roughness

$$5 \leq \frac{\epsilon u^*}{\nu} \leq 70$$

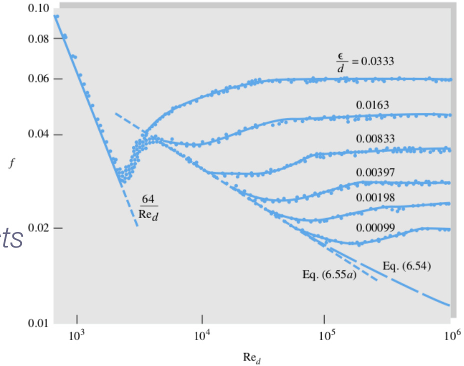
transitional

moderate Reynolds number effects

$$\frac{\epsilon u^*}{\nu} > 70$$

fully rough

sublayer totally broken up
independent of Reynolds number



Wall Roughness

$$\frac{\epsilon u^*}{\nu} < 5$$

hydraulically smooth

no effects of roughness

$$5 \leq \frac{\epsilon u^*}{\nu} \leq 70$$

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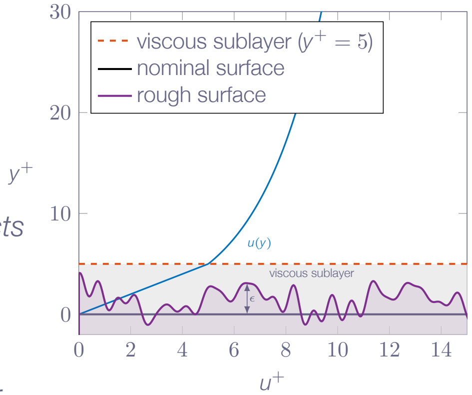
moderate Reynolds number effects

$$\frac{\epsilon u^*}{\nu} > 70$$

fully rough

sublayer totally broken up
independent of Reynolds number



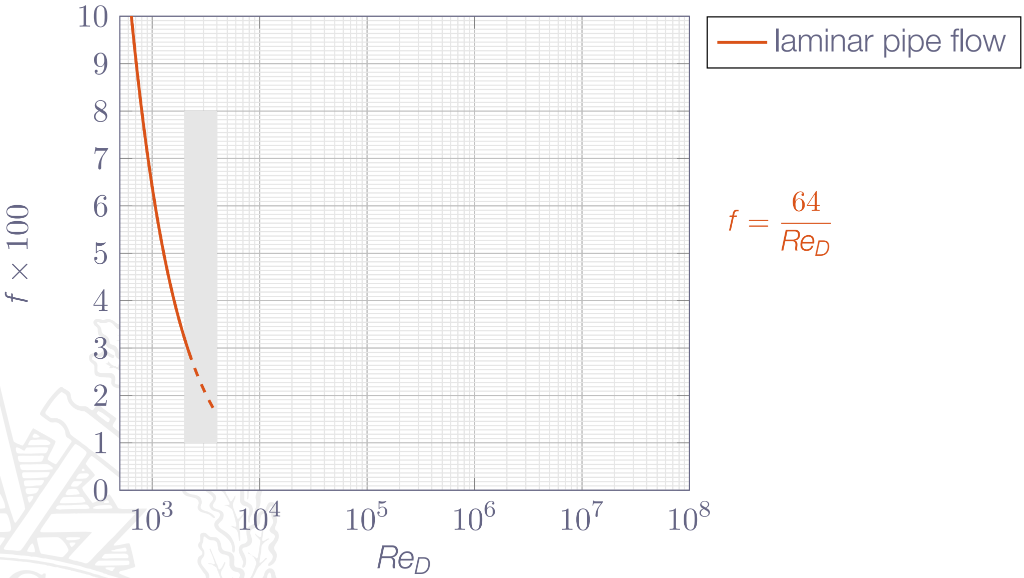
Wall Roughness

Ra	Roughness Average	arithmetic average of the absolute values of the profile heights
Rq	RMS Roughness	root mean square average of the profile heights
Rp	Maximum Profile Peak Height	distance between the highest point of the profile and the mean line
Rpm	Average Maximum Profile Peak Height	average of the successive values of **Rp**
Rv	Maximum Profile Valley Depth	distance between the deepest valley of the profile and the mean line
Rt	Maximum Height of the Profile	vertical distance between the highest and lowest points of the profile
Rz	Average Maximum Height of the Profile	average of the successive values of **Rt**

Wall Roughness

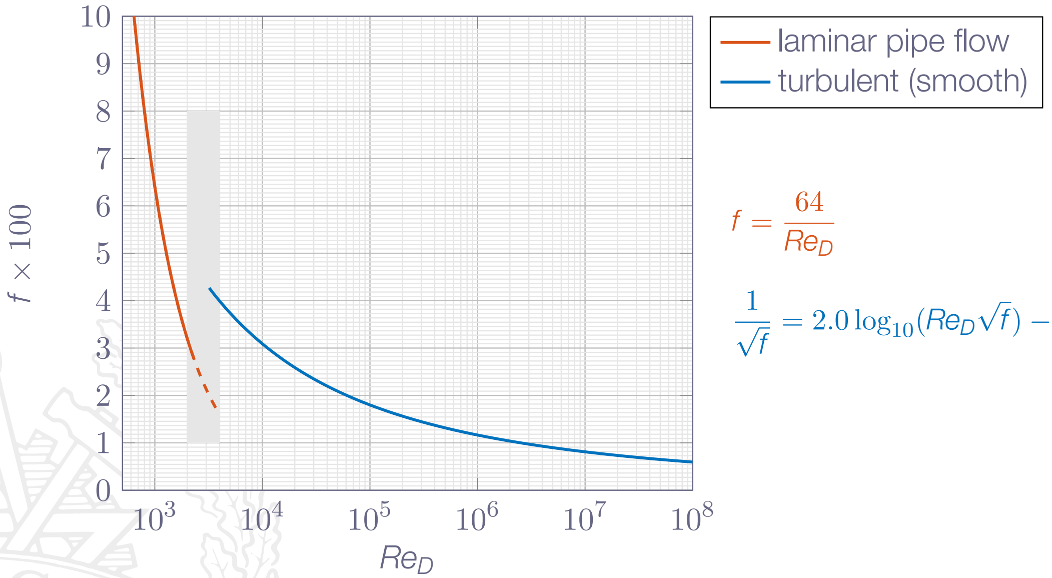
$$\frac{1}{\sqrt{f_D}} = -2.0 \log_{10} \left(\frac{\epsilon/D}{3.7} + \frac{2.5}{Re_D \sqrt{f_D}} \right)$$

The Moody Chart



$$f = \frac{64}{Re_D}$$

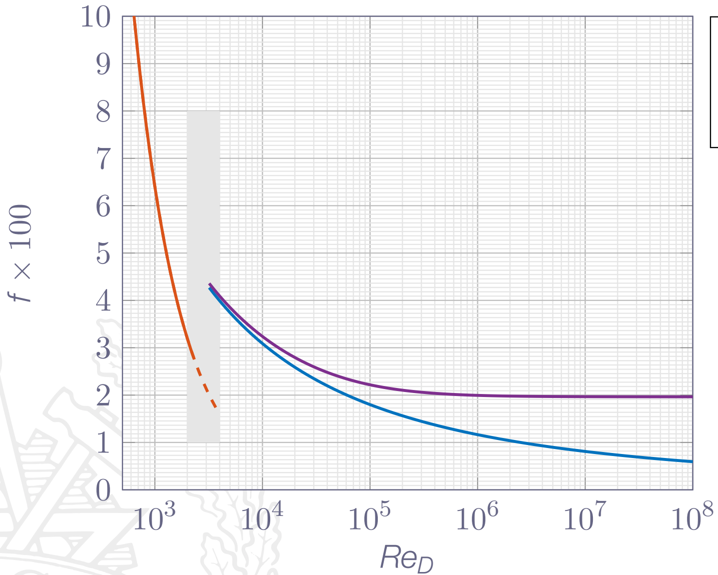
The Moody Chart



$$f = \frac{64}{Re_D}$$

$$\frac{1}{\sqrt{f}} = 2.0 \log_{10}(Re_D \sqrt{f}) - 0.8$$

The Moody Chart

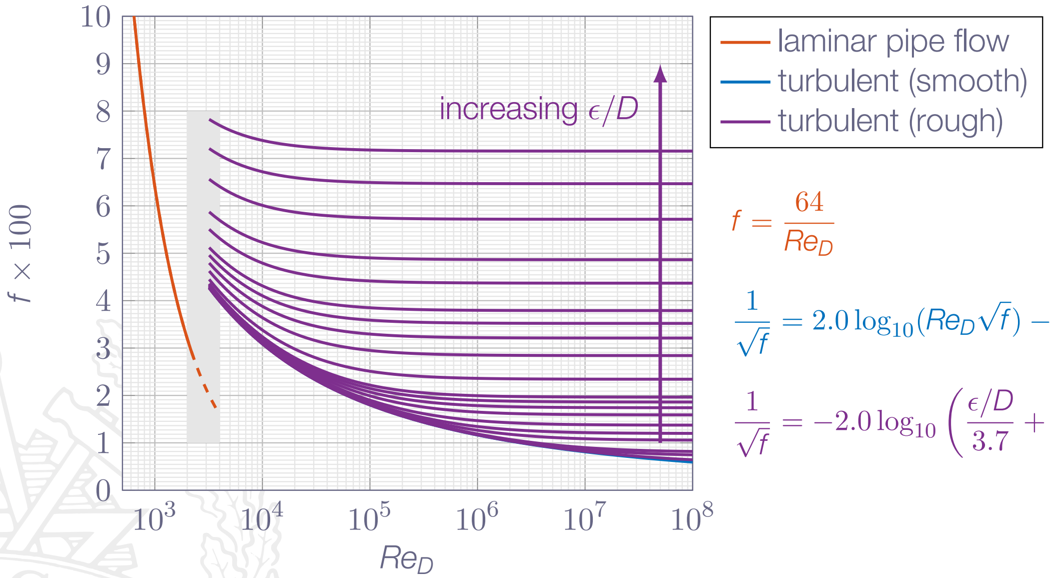


$$f = \frac{64}{Re_D}$$

$$\frac{1}{\sqrt{f}} = 2.0 \log_{10}(Re_D\sqrt{f}) - 0.8$$

$$\frac{1}{\sqrt{f}} = -2.0 \log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.5}{Re_D\sqrt{f}}\right)$$

The Moody Chart



$$f = \frac{64}{Re_D}$$

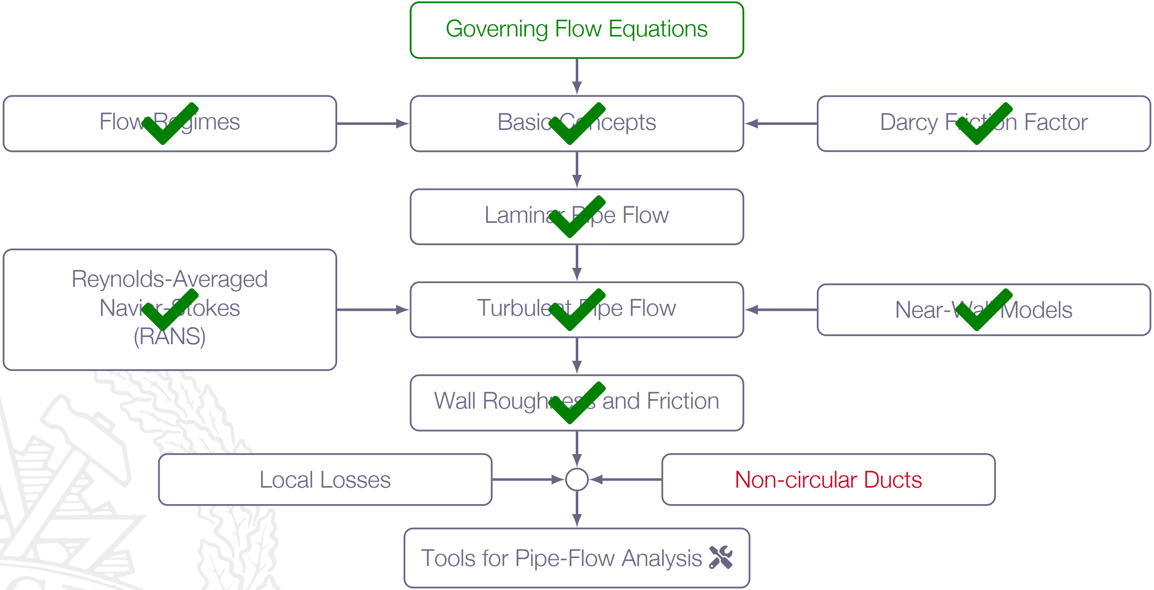
$$\frac{1}{\sqrt{f}} = 2.0 \log_{10}(Re_D\sqrt{f}) - 0.8$$

$$\frac{1}{\sqrt{f}} = -2.0 \log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.5}{Re_D\sqrt{f}}\right)$$

Wall Roughness

Material	Condition	ϵ $[mm]$	Uncertainty [%]
Steel	Sheet metal (new)	0.05	$\pm$ 60
	Stainless (new)	0.002	$\pm$ 50
	Commercial (new)	0.046	$\pm$ 30
	Riveted	3.0	$\pm$ 70
	Rusted	2.0	$\pm$ 50
Iron	Cast (new)	0.26	$\pm$ 50
	Wrought (new)	0.046	$\pm$ 20
	Galvanized (new)	0.15	$\pm$ 40
	Asphalted cast	0.12	$\pm$ 50
Brass	Drawn (new)	0.002	$\pm$ 50
Plastic	Drawn tubing	0.0015	$\pm$ 60
Glass	-	smooth	
Concrete	Smoothed	0.04	$\pm$ 60
	Rough	2.0	$\pm$ 50
Rubber	Smoothed	0.01	$\pm$ 60
Wood	Stave	0.5	$\pm$ 40

Roadmap - Viscous Flow in Ducts



Non-circular Ducts

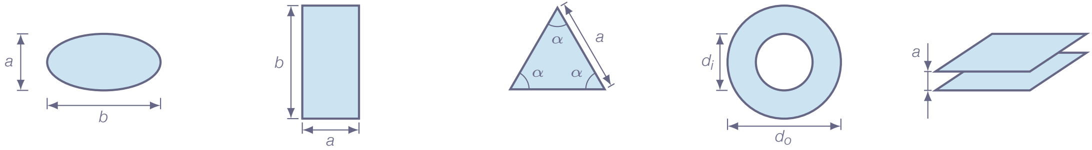
Use the same formulas of the Moody chart but replace the pipe diameter D with the hydraulic diameter D_h

$$D_h = \frac{4A}{\mathcal{P}}$$

where A is the cross section area and $\mathcal{P}$ is the wetter perimeter

$$\Delta p_f = f_D \frac{L}{D_h} \frac{\rho V^2}{2}, \; Re_{D_h} = \frac{VD_h}{\nu}, \; \frac{\epsilon}{D_h}$$

Non-circular Ducts



a/b	D_h	C
0.7	$1.17a$	65.0
0.5	$1.30a$	68.0
0.3	$1.44a$	73.0
0.2	$1.50a$	78.0
0.1	$1.55a$	79.0

b/a	D_h	C
1.0	$1.00a$	57.0
1.25	$1.11a$	57.6
2.0	$1.33a$	62.0
3.0	$1.50a$	69.0
4.0	$1.60a$	73.0
5.0	$1.67a$	78.0
8.0	$1.78a$	83.0
10.0	$1.82a$	85.0

D_h	C
$0.58a$	53.0

d_i/d_o	C
$\frac{d_i}{d_o} = 0.10$	89.2
$\frac{d_i}{d_o} = 0.25$	94.0
$0.5 < \frac{d_i}{d_o} < 1.0$	96.0

$$D_h = d_o - d_i$$

D_h	C
$2.0a$	96.0

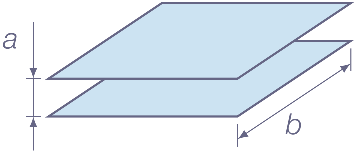
Non-circular Ducts

Laminar flow:

$$f_D = \frac{C}{Re_{D_h}}$$

(for circular pipes: $C = 64$ and $D_h = D$)

Non-circular Ducts

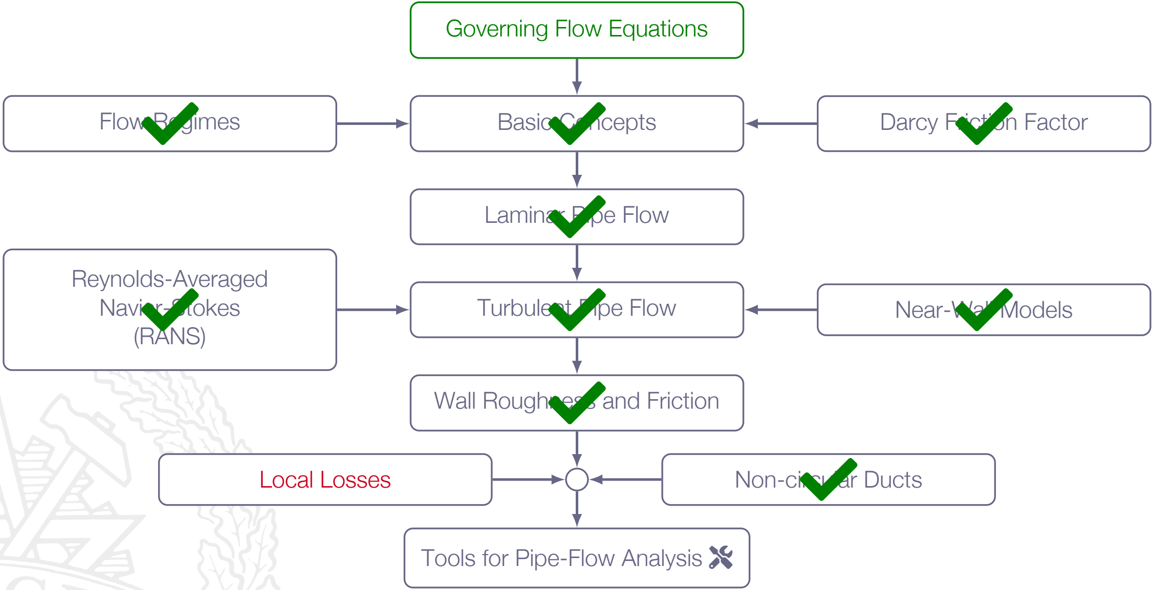


Flow between parallel plates

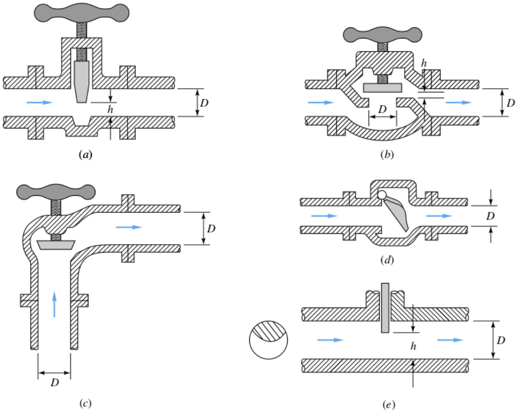
- vertical distance between plates: a
- plate width: b

$$D_h = \frac{4A}{\mathcal{P}} = \left.\frac{4ab}{2a+2b}\right|_{b\to\infty} = \frac{4ab}{2b} = 2a$$

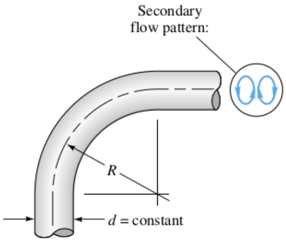
Roadmap - Viscous Flow in Ducts



Local Losses



Local Losses



Swirl generated by:

- Inlets or outlets
- Sudden area changes
- Bends
- Valves
- Gradual expansions or contractions

$$\Delta p_f = K \frac{\rho V^2}{2}$$

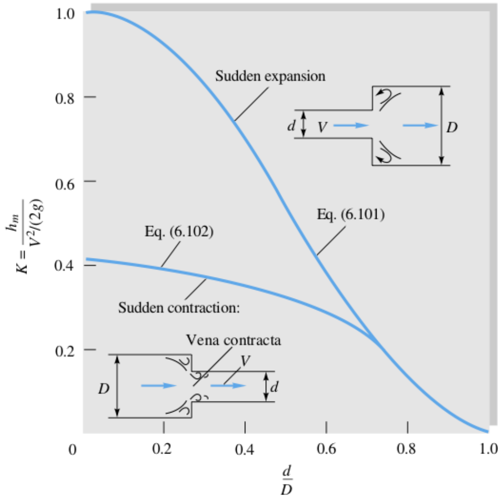
$$\Delta p_{f_{tot}} = \sum_i f_{D_i} \frac{L_i}{D_i} \frac{\rho V_i^2}{2} + \sum_j K_j \frac{\rho V_j^2}{2}$$

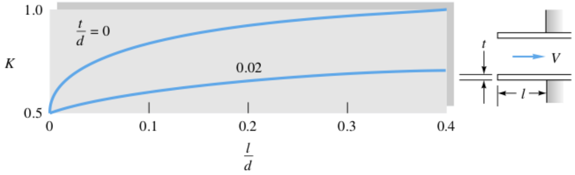
Local Losses

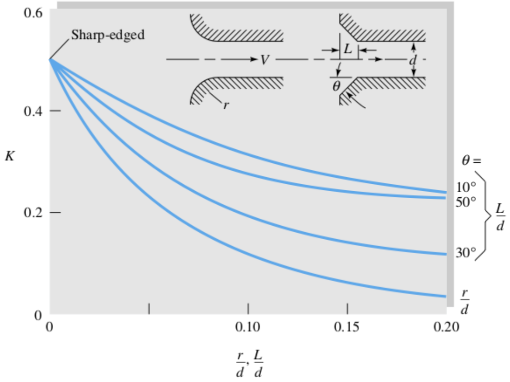
Generated swirl will be damped out by inner friction

Kinetic energy is converted to internal energy, which results in a pressure loss

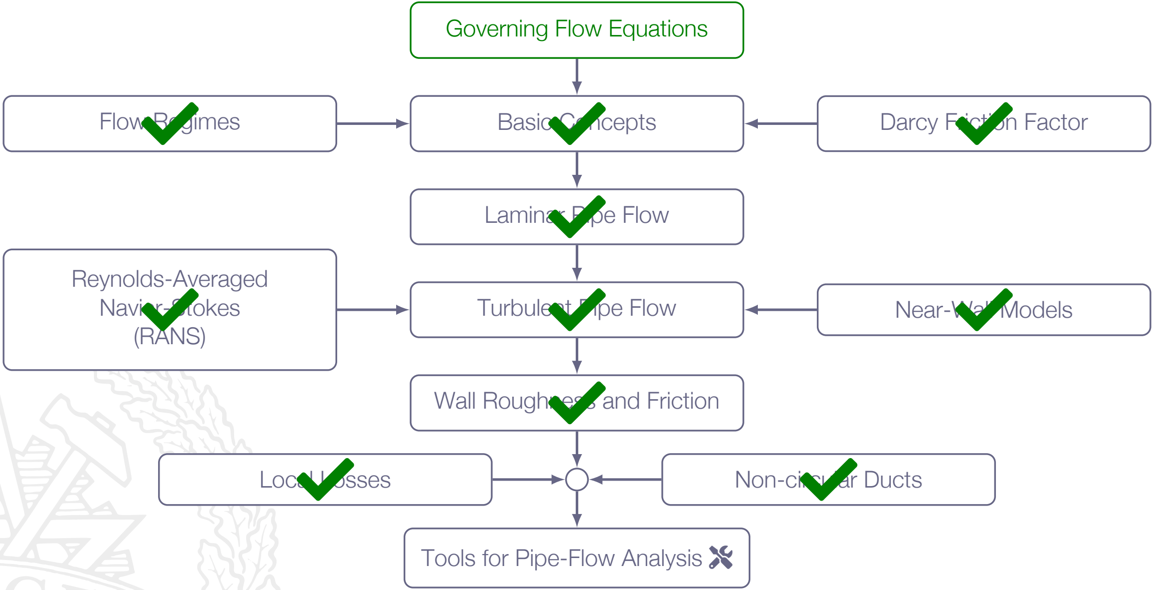
Local Losses







Roadmap - Viscous Flow in Ducts



Pipe Flow Example 1: Find Flow Rate (Rough Pipe)

Given data:
Oil with the density $\rho = 950.0\ kg/m^3$ and viscosity $\nu = 2.0 \times 10^{-5}\ m^2/s$ flows through a $L = 100\ m$ long pipe with the diameter $D = 0.3\ m$. The roughness ratio is $\varepsilon/D = 2.0 \times 10^{-4}$ and the head loss is $h_f = 8.0\ m$.

Assumptions:
steady-state, fully developed, turbulent, incompressible pipe flow

Task:
Find the average flow velocity (V) and the flow rate (Q)

Pipe Flow Example 1: Find Flow Rate (Rough Pipe)

We are given a measure of the head loss (h_f) for the pipe

The definition of the **Darcy friction factor** gives a relation between head loss (h_f) and the average velocity (V)

$$h_f = f \frac{V^2}{2g} \frac{L}{D}$$

To be able to calculate the average velocity (V), we need the friction factor (f)

Pipe Flow Example 1: Find Flow Rate (Rough Pipe)

The flow in the pipe is assumed to be turbulent and fully developed

For turbulent flows in rough pipes, **Colebrook's formula** gives a relation between friction factor (*f*) and average flow velocity (*V*)

$$\frac{1}{\sqrt{f}} = -2.0 \log \left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{Re_D \sqrt{f}} \right)$$

Use an iterative approach to find the friction factor (*f*) using Colebrook's relation and

$$Re_D = \frac{VD}{\nu}, \text{ where } V = \sqrt{\frac{2h_f gD}{L}}$$

Pipe Flow Example 1: Find Flow Rate (Rough Pipe)

```
import numpy as np

def GetVelocity(hf,f,D,L):
  return np.sqrt((2.*9.81*hf*D)/(f*L))

def GetReynoldsNumber(D,V,nu):
  return D*V/nu

def Colebrook(f,D,nu,eps,V):
  # Colebrook friction factor
  return -2.0*np.log10(((eps/D)/3.7)+(2.51/(GetReynoldsNumber(D,V,nu)*np.
    sqrt(f))))-1./np.sqrt(f)

def GetFlowRate(V,D):
  return (V*np.pi*D**2)/4.
```

Pipe Flow Example 1: Find Flow Rate (Rough Pipe)

```
nu    = 2.0e-5    # fluid viscosity [m^2/s]
D     = 3.0e-1    # pipe diameter [m]
L     = 1.0e2     # pipe length [m]
hf    = 8.0       # head loss [m]
eps   = 2.0e-4*D  # surface roughness [m]
f     = 1.5e-2    # friction factor (inital guess)

# Newton-Raphson solver
f_old = 1.0e3
df    = 1.0e-6
while np.abs(f-f_old)>1.0e-6*f:
  f_old = f
  V   = GetVelocity(hf,f,D,L)
  ff  = Colebrook(f,D,nu,eps,V)
  dff = (Colebrook(f+df,D,nu,eps,V)-Colebrook(f-df,D,nu,eps,V))/(2.*df)
  f   = f_old-(ff/dff)
```

Pipe Flow Example 1: Find Flow Rate (Rough Pipe)

Result:

Average flow velocity	V	4.84	m/s
Flow rate	Q	0.342	m^3/s
Reynolds number	Re_D	72585	
Friction factor	f	0.0201	

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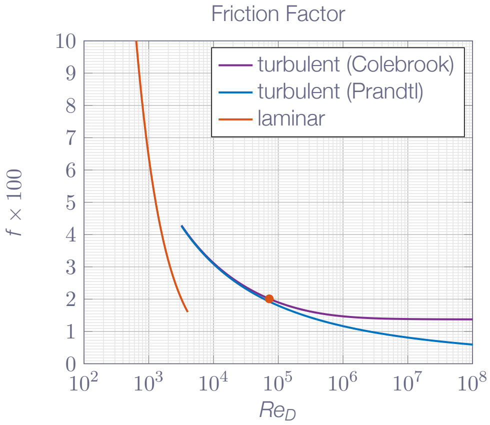


Pipe Flow Example 2: Find Pipe Diameter (Rough Pipe)

Given data:
Oil with the density $\rho = 950.0\ kg/m^3$ and viscosity $\nu = 2.0 \times 10^{-5}\ m^2/s$ flows through a $L = 100\ m$ long pipe at a flow rate of $Q = 0.342\ m^3/s$. The surface roughness is $\varepsilon = 0.06\ mm$ and the head loss is $h_f = 8.0\ m$.

Assumptions:
steady-state, fully developed, turbulent, incompressible pipe flow

Task:
Find the pipe diameter (D)

Pipe Flow Example 2: Find Pipe Diameter (Rough Pipe)

We are given a measure of the head loss (h_f) for the pipe

The definition of the **Darcy friction factor** gives a relation between head loss (h_f) and the pipe diameter (D)

$$h_f = f\frac{V^2}{2g}\frac{L}{D} = \left\{Q = V\frac{\pi D^2}{4}\right\} = f\frac{8Q^2L}{\pi^2 g D^5}$$

To be able to calculate the pipe diameter (D), we need the friction factor (f)

Pipe Flow Example 2: Find Pipe Diameter (Rough Pipe)

The flow in the pipe is assumed to be turbulent and fully developed

For turbulent flows in rough pipes, **Colebrook's formula** gives a relation between friction factor (*f*) and pipe diameter (*D*)

$$\frac{1}{\sqrt{f}} = -2.0 \log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{Re_D\sqrt{f}}\right)$$

Use an iterative approach to find the friction factor (*f*) using Colebrook's relation and

$$Re_D = \frac{VD}{\nu}, \text{ where } V = \frac{4Q}{\pi D^2}$$

Pipe Flow Example 2: Find Pipe Diameter (Rough Pipe)

```python
import numpy as np

def GetDiameter(hf,f,L,Q):
  return ((8.*f*Q**2*L)/(9.81*np.pi**2*hf))**(1./5.)

def GetReynoldsNumber(D,V,nu):
  return D*V/nu

def Colebrook(f,D,nu,eps,V):
  # Colebrook friction factor
  return -2.0*np.log10(((eps/D)/3.7)+(2.51/(GetReynoldsNumber(D,V,nu)*np.
    sqrt(f))))-1./np.sqrt(f)

def GetVelocity(Q,D):
  return 4.*Q/(np.pi*D**2)
```

Pipe Flow Example 2: Find Pipe Diameter (Rough Pipe)

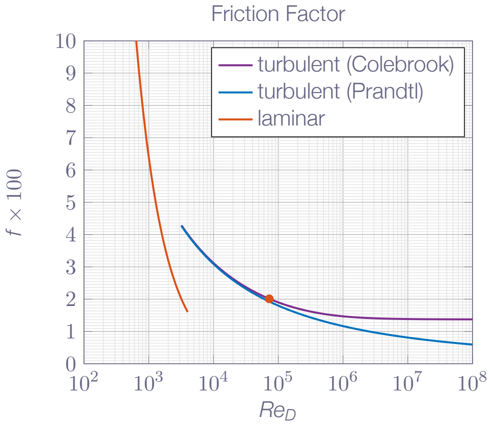
```
nu    = 2.0e-5    # fluid viscosity [m^2/s]
L     = 1.0e2     # pipe length [m]
hf    = 8.0       # head loss [m]
eps   = 6.0e-5    # surface roughness [m]
Q     = 3.42e-1   # flow rate [m^3/s]
f     = 1.5e-2    # friction factor (inital guess)

# Newton-Raphson solver
f_old = 1.0e3
df    = 1.0e-6
while np.abs(f-f_old)>1.0e-6*f:
  f_old = f
  D   = GetDiameter(hf,f,L,Q)
  V   = GetVelocity(Q,D)
  ff  = Colebrook(f,D,nu,eps,V)
  dff = (Colebrook(f+df,D,nu,eps,V)-Colebrook(f-df,D,nu,eps,V))/(2.*df)
  f   = f_old-(ff/dff)
```

Pipe Flow Example 2: Find Pipe Diameter (Rough Pipe)

Result:

Pipe diameter	D	0.299	m
Average flow velocity	V	4.84	m/s
Reynolds number	Re_D	72579	
Friction factor	f	0.0201	



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Pipe Flow Example 3: Find Pipe Diameter (Smooth Pipe)

Given data:
A smooth plastic pipe is to be designed to carry $Q = 0.25 \ m^3/s$ of water at $20°C$ through a $L = 300 \ m$ horizontal pipe with the exit at atmospheric pressure. The pressure drop is approximated to be $\Delta p = 1.7 \ MPa$.

Water @ $20°C$: $\rho = 998 \ kg/m^3$ and $\mu = 0.001 \ kg/(ms)$ ($\nu = 1.002 \times 10^{-6} \ m^2/s$)

Assumptions:
steady-state, fully developed, turbulent, incompressible pipe flow

Task:
Find a suitable pipe diameter (D)

Pipe Flow Example 3: Find Pipe Diameter (Smooth Pipe)

The energy equation on integral form gives us a relation between the pressure drop Δp and the pipe head loss h_f

$$\left(\frac{p}{\rho g}+\frac{\alpha V^2}{2g}+z\right)_1=\left(\frac{p}{\rho g}+\frac{\alpha V^2}{2g}+z\right)_2+h_t-h_p+h_f$$

1. Steady-state, incompressible flow ($Q_1=Q_2=Q$) in a constant-diameter pipe ($D_1=D_2=D$) $\Rightarrow V_1=V_2=V$
2. Fully-developed turbulent pipe flow with constant average velocity $\Rightarrow$ $\alpha_1=\alpha_2=\alpha$
3. No information about elevation change is given so we will assume that $z_1=z_2=z$
4. There are no turbines or pumps in the pipe $\Rightarrow h_t=h_p=0$.

$$\frac{p_1-p_2}{\rho g}=\frac{\Delta p}{\rho g}=h_f$$

Pipe Flow Example 3: Find Pipe Diameter (Smooth Pipe)

Again, we will use the definition of the **Darcy friction factor** (f) to get a relation between the losses and the pipe diameter

$$h_f = f\frac{V^2}{2g}\frac{L}{D} \Rightarrow \left\{h_f = \frac{\Delta p}{\rho g}, Q = V\frac{\pi D^2}{4}\right\} \Rightarrow f = \frac{\pi^2 \Delta p}{8Q^2 L \rho} D^5$$

To be able to calculate the pipe diameter (D), we need the friction factor (f)

Pipe Flow Example 3: Find Pipe Diameter (Smooth Pipe)

The flow in the pipe is assumed to be turbulent and fully developed

For turbulent flows in smooth pipes, **Prandtl's formula** gives a relation between friction factor (*f*) and pipe diameter (*D*)

$$\frac{1}{\sqrt{f}} = 2.0\log\left(Re_D\sqrt{f}\right) - 0.8$$

Use an iterative approach to find the friction factor (*f*) using Prandtl's relation and

$$Re_D = \frac{VD}{\nu}, \text{ where } V = \frac{4Q}{\pi D^2}$$

Pipe Flow Example 3: Find Pipe Diameter (Smooth Pipe)

```python
import numpy as np

def GetDiameter(Dp,rho,f,L,Q):
  return ((8.*f*Q**2*L*rho)/(np.pi**2*Dp))**(1./5.)

def GetReynoldsNumber(D,V,nu):
  return D*V/nu

def Prandtl(f,D,nu,V):
  # Prandtl friction factor
  return 2.0*np.log10(GetReynoldsNumber(D,V,nu)*np.sqrt(f))-0.8-(1./np.sqrt(f));

def GetVelocity(Q,D):
  return 4.*Q/(np.pi*D**2)
```

Pipe Flow Example 3: Find Pipe Diameter (Smooth Pipe)

```
rho   = 998.0     # fluid density [kg/m^3]
mu    = 1.0e-3    # fluid viscosity [kg/ms]
nu    = mu/rho    # fluid viscosity [m^2/s]
L     = 3.0e2     # pipe length [m]
Dp    = 1.7e6     # pressure drop [Pa]
Q     = 2.5e-1    # flow rate [m^3/s]
f     = 1.5e-2    # friction factor (inital guess)

# Newton-Raphson solver
f_old = 1.0e3
df    = 1.0e-6
while np.abs(f-f_old)>1.0e-6*f:
  f_old = f
  D   = GetDiameter(Dp,rho,f,L,Q)
  V   = GetVelocity(Q,D)
  ff  = Prandtl(f,D,nu,V)
  dff = (Prandtl(f+df,D,nu,V)-Prandtl(f-df,D,nu,V))/(2.*df)
  f   = f_old-(ff/dff)
```

Pipe Flow Example 3: Find Pipe Diameter (Smooth Pipe)

Result:

Pipe diameter	D	0.156	m
Average flow velocity	V	13.1	m/s
Reynolds number	Re_D	2036821	
Friction factor	f	0.01034	

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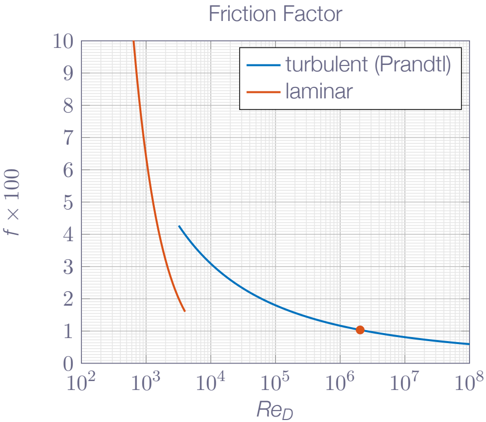
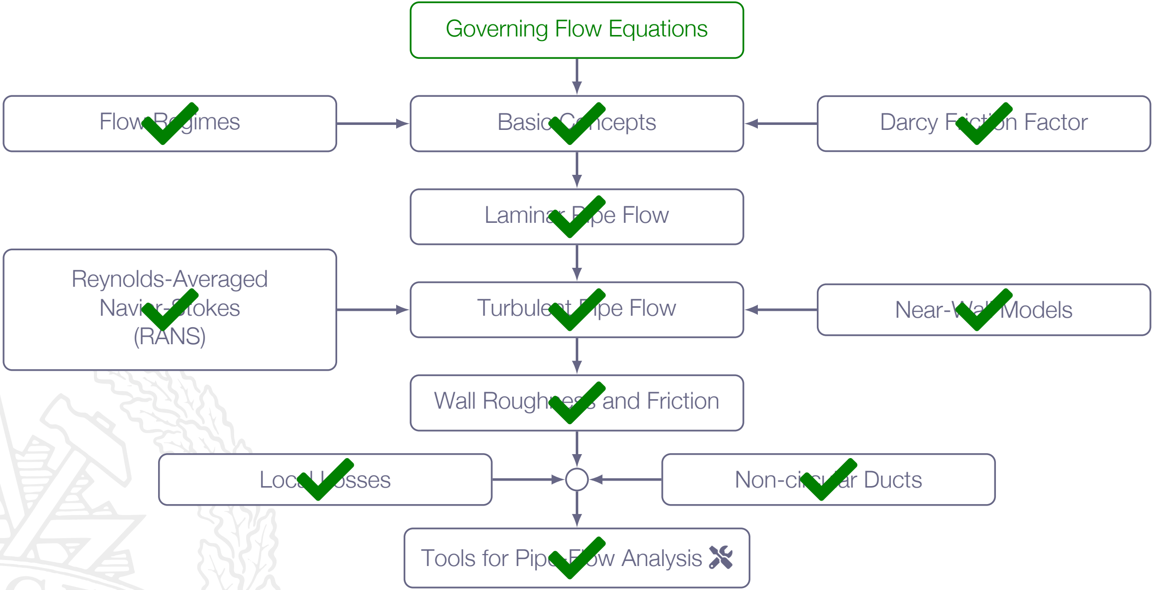
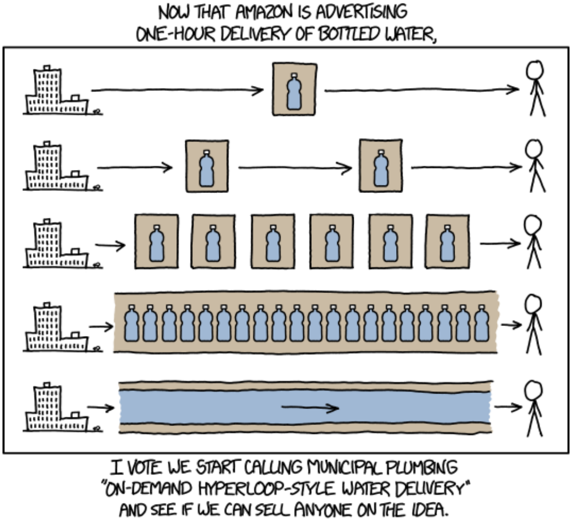


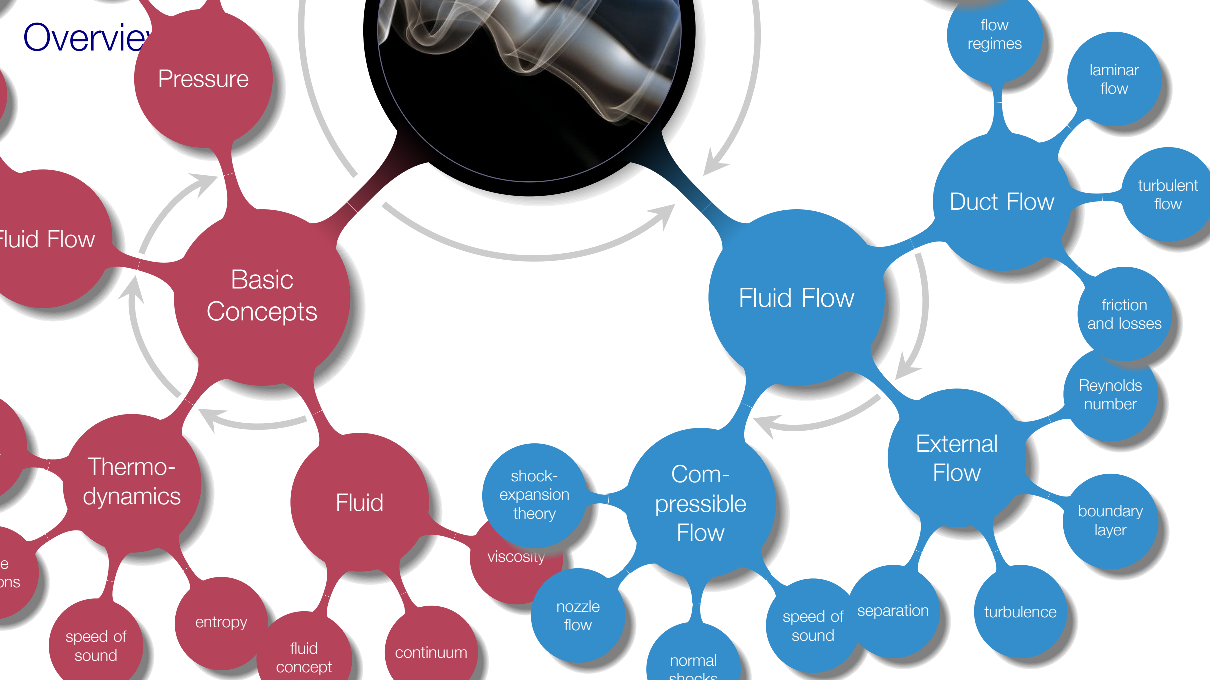
Roadmap - Viscous Flow in Ducts



On-Demand Hyperloop-Style Water Delivery



Chapter 7 - Flow Past Immersed Bodies


Pressure
Basic Concepts
Thermo-dynamics
Fluid
speed of sound
entropy
fluid concept
continuum
viscosity
Fluid Flow
Duct Flow
flow regimes
laminar flow
turbulent flow
friction and losses
Reynolds number
External Flow
boundary layer
turbulence
separation
Com-pressible Flow
shock-expansion theory
nozzle flow
normal
speed of sound

Learning Outcomes

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

6 **Explain** what a boundary layer is and when/where/why it appears

21 **Explain** how the flat plate boundary layer is developed (transition from laminar to turbulent flow)

22 **Explain** and use the Blasius equation

23 **Define** the Reynolds number for a flat plate boundary layer

24 **Explain** what is characteristic for a turbulent flow

29 **Explain** flow separation (separated cylinder flow)

30 **Explain** how to delay or avoid separation

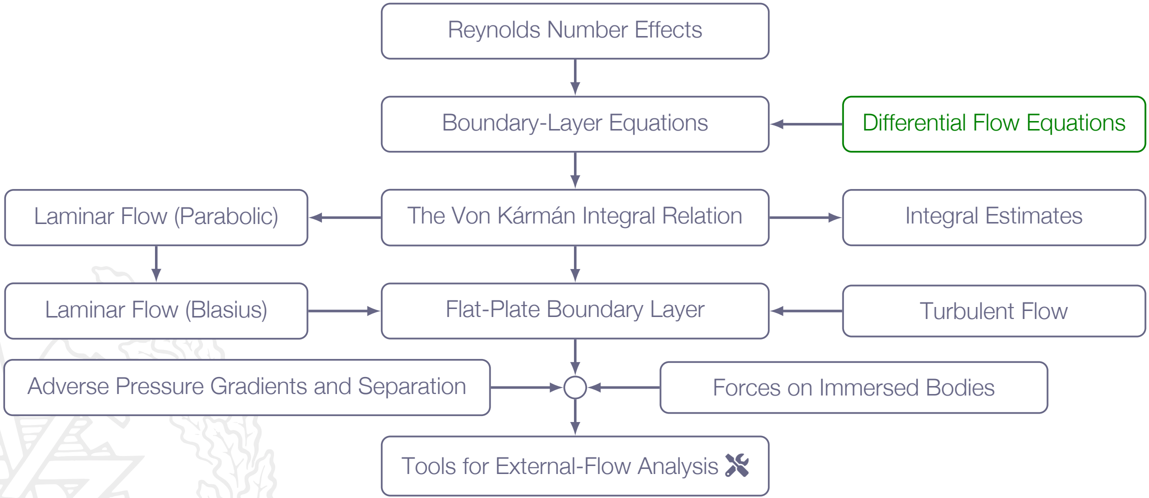
31 **Derive** the boundary layer formulation of the Navier-Stokes equations

32 **Understand** and explain displacement thickness and momentum thickness

33 **Understand**, **explain** and **use** the concepts drag, friction drag, pressure drag, and lift

Let's take a deep dive into boundary-layer theory

Roadmap - Flow Past Immersed Bodies



Complementary Course Material

These lecture notes covers chapter 7 in the course book and additional course material that you can find in the following documents

`MTF053_Equation-for-Boundary-Layer-Flows.pdf`

`MTF053_Turbulence.pdf`

”Understanding the mechanisms behind flow-related forces is a key factor to success in many engineering applications”
S MB 7177

External Flow

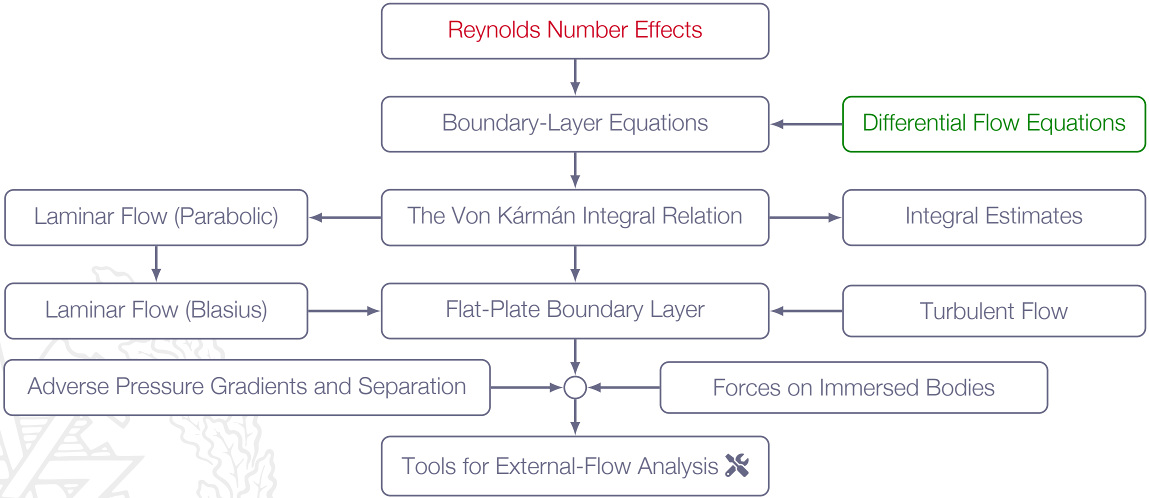
Significant viscous effects near the surface of an **immersed body**

Nearly inviscid far from the body

Unconfined - boundary layers are free to grow

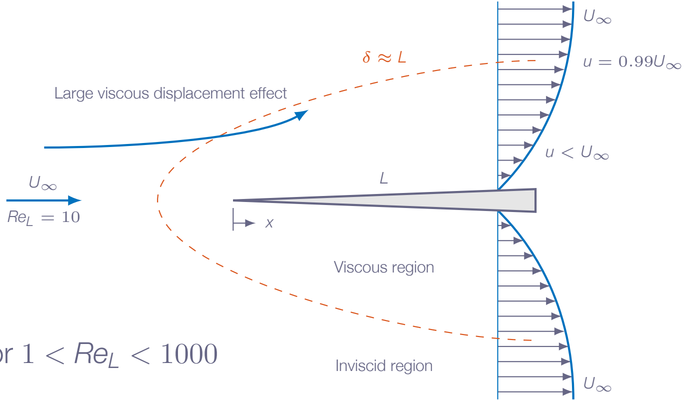
Most often CFD or experiments are needed to analyze an external flow unless the geometry is very simple

Roadmap - Flow Past Immersed Bodies

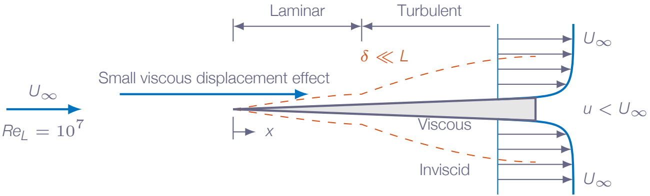


Reynolds Number Effects

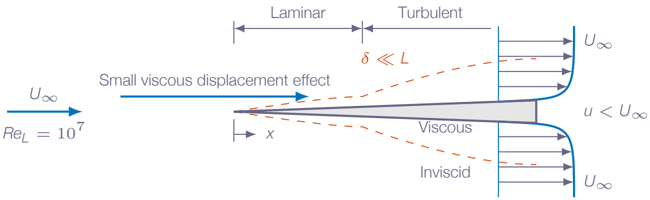
$$Re_L = \frac{U_\infty L}{\nu}$$



Note: no simple theory exists for $1 < Re_L < 1000$



Reynolds Number Effects



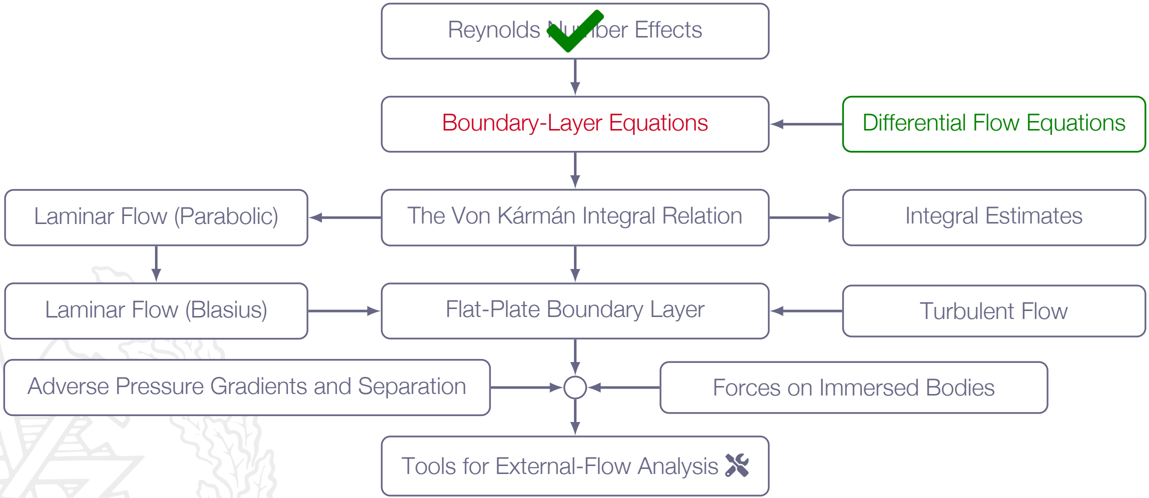
$$\frac{\delta}{x} \approx \begin{cases} \dfrac{5.0}{{Re_x}^{1/2}} & \text{laminar} \quad 10^3 < Re_x < 10^6 \\ \dfrac{0.16}{{Re_x}^{1/7}} & \text{turbulent} \quad 10^6 < Re_x \end{cases}$$

$$Re_L = \frac{U_\infty L}{\nu}$$

$$Re_x = \frac{U_\infty x}{\nu}$$

Note! Re_L and the **local Reynolds number** Re_x are not the same

Roadmap - Flow Past Immersed Bodies



Boundary Layer Equations

We will derive a set of equations suitable for **boundary-layer flow analysis**

Starting point: the **non-dimensional equations** derived in Chapter 5

We will assume two-dimensional, incompressible, steady-state flow

We will do an **order-of-magnitude** comparison of all the terms in the governing equations on non-dimensional form and identify terms that can be neglected in a **thin-boundary-layer** flow

Boundary Layer Equations

continuity:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

x-momentum:

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re_L}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

y-momentum:

$$u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re_L}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right)$$

$$\begin{aligned} x^* &= \frac{x}{L} \\ y^* &= \frac{y}{L} \\ u^* &= \frac{u}{U_\infty} \\ v^* &= \frac{v}{U_\infty} \\ p^* &= \frac{p}{\rho U_\infty^2} \\ Re_L &= \frac{U_\infty L}{\nu} \end{aligned}$$

Boundary Layer Equations

To be able to find the relative sizes of different terms in the equations, we will first have a look at the flow parameters and operators

$$u^* = u/U_\infty \sim 1$$

$$x^* = x/L \sim 1$$

$$y^* = y/L \sim \delta^*$$

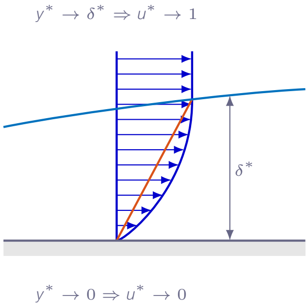
δ *denotes boundary layer thickness and* $\delta^* = \delta/L$

Note! here, u^* is **not** the friction velocity and δ^* is **not** the displacement thickness

Boundary Layer Equations

What about derivatives?

$$\frac{\partial u^*}{\partial y^*} \sim \frac{1-0}{\delta^*} = \frac{1}{\delta^*}$$



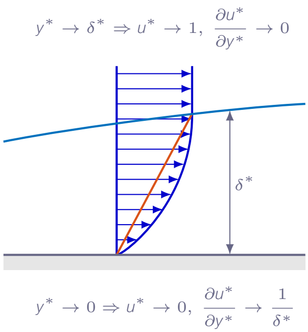
Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

What about derivatives?

$$\frac{\partial u^*}{\partial y^*} \sim \frac{1-0}{\delta^*} = \frac{1}{\delta^*}$$

$$\frac{\partial^2 u^*}{\partial y^{*2}} = \frac{\partial}{\partial y^*}\frac{\partial u^*}{\partial y^*} \sim \frac{|0-1/\delta^*|}{\delta^*} = \frac{1}{\delta^{*2}}$$



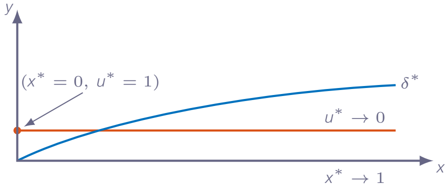
Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

$$x^* \to 0 \Rightarrow u^* \to 0$$

$$x^* \to 1 \Rightarrow u^* \to 1$$

$$\frac{\partial u^*}{\partial x^*} \sim \frac{|0-1|}{1-0} = 1$$

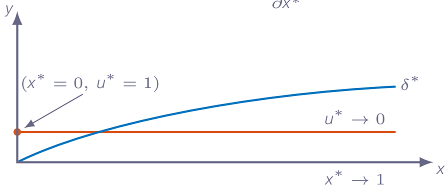


Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

$$x^* \to 0 \Rightarrow u^* \to 0, \ \frac{\partial u^*}{\partial x^*} \to 0$$

$$x^* \to 1 \Rightarrow u^* \to 1, \ \frac{\partial u^*}{\partial x^*} \to 1$$



$$\frac{\partial u^*}{\partial x^*} \sim \frac{|0-1|}{1-0} = 1$$

$$\frac{\partial^2 u^*}{\partial x^{*2}} = \frac{\partial}{\partial x^*}\frac{\partial u^*}{\partial x^*} \sim \frac{1-0}{1-0} = 1$$

Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

continuity:

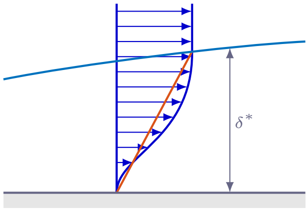
$$\underbrace{\frac{\partial u^*}{\partial x^*}}_{\sim\frac{1}{1}} + \underbrace{\frac{\partial v^*}{\partial y^*}}_{\sim\frac{?}{\delta^*}} = 0 \Rightarrow v^* \sim \delta^*$$

$\frac{\partial v^*}{\partial y^*}$ must be of the same order of magnitude as $\frac{\partial u^*}{\partial x^*}$ in order to fulfill the continuity equation

Boundary Layer Equations

$$\frac{\partial v^*}{\partial y^*} \sim \frac{\delta^*}{\delta^*} = 1$$

$y^* \to \delta^* \Rightarrow v^* \to \delta^*$



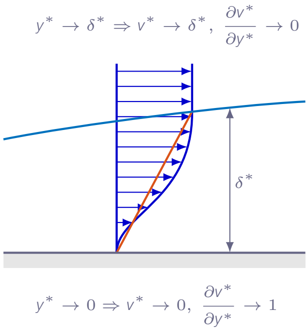
$y^* \to 0 \Rightarrow v^* \to 0$

Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

$$\frac{\partial v^*}{\partial y^*} \sim \frac{\delta^*}{\delta^*} = 1$$

$$\frac{\partial^2 v^*}{\partial {y^*}^2} = \frac{\partial}{\partial y^*}\frac{\partial v^*}{\partial y^*} \sim \frac{|0-1|}{\delta^*} = \frac{1}{\delta^*}$$



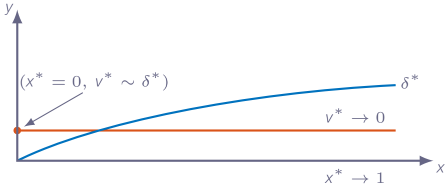
Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

$$x^* \to 0 \Rightarrow v^* \to \delta^*$$

$$x^* \to 1 \Rightarrow v^* \to 0$$

$$\frac{\partial v^*}{\partial x^*} \sim \frac{|0-\delta^*|}{1-0} = \delta^*$$

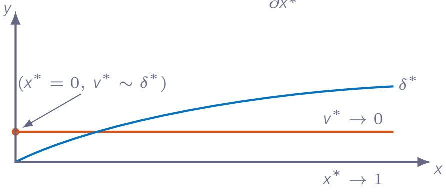


Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

$$x^* \to 0 \Rightarrow v^* \to \delta^*, \ \frac{\partial v^*}{\partial x^*} \to 0$$

$$x^* \to 1 \Rightarrow v^* \to 0, \ \frac{\partial v^*}{\partial x^*} \to \delta^*$$



$$\frac{\partial v^*}{\partial x^*} \sim \frac{|0 - \delta^*|}{1 - 0} = \delta^*$$

$$\frac{\partial^2 v^*}{\partial x^{*2}} = \frac{\partial}{\partial x^*}\frac{\partial v^*}{\partial x^*} \sim \frac{\delta^* - 0}{1 - 0} = \delta^*$$

Note! The sign of terms is not important here, we are only interested in the **order of magnitude**

Boundary Layer Equations

x-momentum:

$$\underbrace{u^* \frac{\partial u^*}{\partial x^*}}_{\sim 1 \times 1 = 1} + \underbrace{v^* \frac{\partial u^*}{\partial y^*}}_{\sim \delta^* \frac{1}{\delta^*} = 1} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re_L} \left(\underbrace{\frac{\partial^2 u^*}{\partial x^{*2}}}_{\sim 1} + \underbrace{\frac{\partial^2 u^*}{\partial y^{*2}}}_{\sim \frac{1}{\delta^{*2}}} \right)$$

the boundary layer is assumed to be very thin $\Rightarrow \delta^* \ll 1$ and thus

$$\frac{\partial^2 u^*}{\partial x^{*2}} \ll \frac{\partial^2 u^*}{\partial y^{*2}}$$

assuming the inertial forces to be of the same size as the friction forces in the boundary layer we get: $1/Re_L \sim \delta^{*2}$

Boundary Layer Equations

y-momentum:

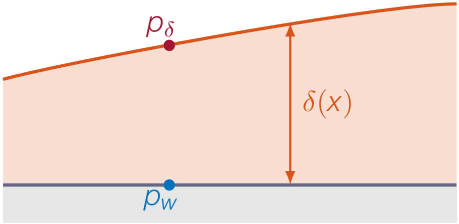
$$\underbrace{u^*\frac{\partial v^*}{\partial x^*}}_{\sim 1\times\delta^*=\delta^*} + \underbrace{v^*\frac{\partial v^*}{\partial y^*}}_{\sim\delta^*\frac{\delta^*}{\delta^*}=\delta^*} = -\frac{\partial p^*}{\partial y^*} + \underbrace{\frac{1}{Re_L}}_{\sim{\delta^*}^2}\left(\underbrace{\frac{\partial^2 v^*}{\partial {x^*}^2}}_{\sim\delta^*} + \underbrace{\frac{\partial^2 v^*}{\partial {y^*}^2}}_{\sim\frac{1}{\delta^*}}\right)$$

examining the equation we see that all terms are at most of size $\delta^* \Rightarrow \dfrac{\partial p^*}{\partial y^*} \sim \delta^*$

δ^* is small $\Rightarrow$ p is independent of y

Boundary Layer Equations

The pressure can be assumed to be constant in the vertical direction through the boundary layer and thus $p = p(x)$



$$|p_\delta^* - p_w^*| \approx \frac{\partial p^*}{\partial y^*}\delta^* \sim {\delta^*}^2$$

Boundary Layer Equations

With the knowledge gained, we now move back to the dimensional equations

laminar

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

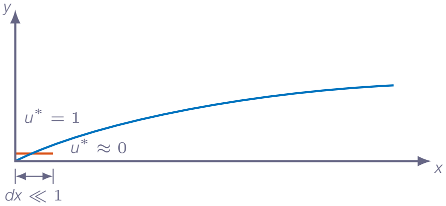
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

turbulent

$$\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}=0$$

$$\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{v}\frac{\partial \overline{u}}{\partial y}=-\frac{1}{\rho}\frac{d\overline{p}}{dx}+\nu\frac{\partial^2 \overline{u}}{\partial y^2}-\frac{\partial}{\partial y}\overline{u'v'}$$

Boundary Layer Equations



Limitations

1. The boundary layer equations **do not apply close to the start of the boundary layer** where $\dfrac{\partial u^*}{\partial x^*} \gg 1$
2. The equations are derived assuming a **thin boundary layer**

Boundary Layer Equations

The pressure derivative can be replaced with a velocity derivative

Outside of the boundary layer the flow is inviscid $\Rightarrow$ we can use the Bernoulli equation

$$p + \frac{1}{2}\rho U_\infty^2 = const \Rightarrow \frac{dp}{dx} + \rho U_\infty \frac{dU_\infty}{dx} = 0 \Rightarrow -\frac{1}{\rho}\frac{dp}{dx} = U_\infty \frac{dU}{dx}$$

Boundary Layer Equations

laminar boundary layer

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

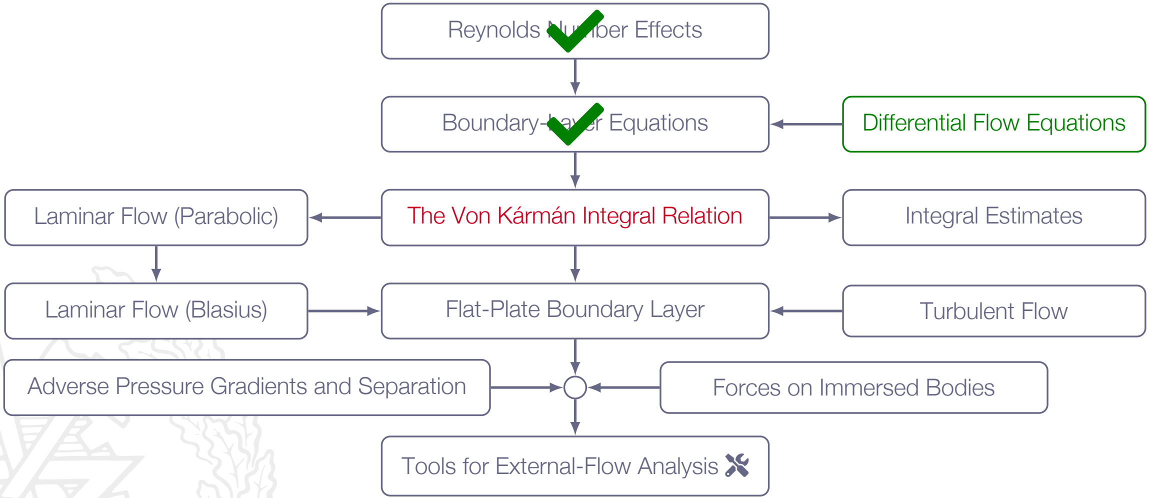
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + \nu \frac{\partial^2 u}{\partial y^2}$$

Two equations and two unknowns $\Rightarrow$ possible to solve ☺

Boundary Layer Equations

Note! the boundary layer equations can be used for curved surfaces if the boundary layer thickness δ is small compared to the curvature radius r

Roadmap - Flow Past Immersed Bodies



The Von Kármán Integral Relation

- Approximate solutions for $\delta(x)$ and $\tau_w(x)$
- Control volume approach applied to a boundary layer
- Assuming steady-state incompressible flow

$$\sum \mathbf{F} = \int_{cs} \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA$$

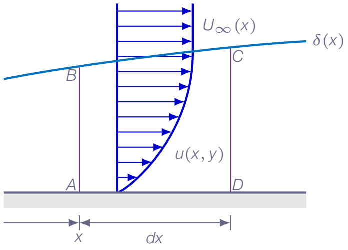
The Von Kármán Integral Relation

Massflow

$$\dot{m}_{AB} = \rho \int_0^{\delta} u dy$$

$$\dot{m}_{CD} = \rho \int_0^{\delta} u dy + \frac{d}{dx}\left[\rho \int_0^{\delta} u dy\right] dx$$

$$\dot{m}_{BC} = \rho \frac{d}{dx}\left[\int_0^{\delta} u dy\right] dx$$



The Von Kármán Integral Relation

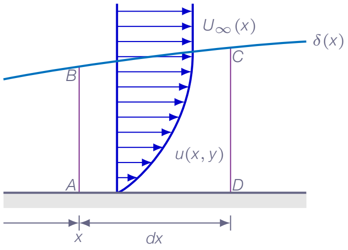
Momentum

$$\dot{I}_{AB} = \rho \int_0^{\delta} u^2 dy$$

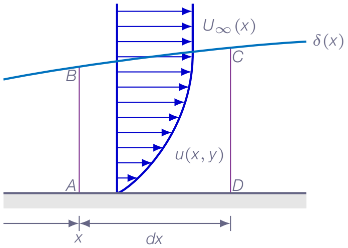
$$\dot{I}_{CD} = \rho \int_0^{\delta} u^2 dy + \frac{d}{dx}\left[\rho \int_0^{\delta} u^2 dy\right] dx$$

$$\dot{I}_{BC} = U\dot{m}_{BC} = \rho U_{\infty} \frac{d}{dx}\left[\int_0^{\delta} u dy\right] dx$$

$$\dot{I}_{CD} - \dot{I}_{AB} - \dot{I}_{BC} = \rho \frac{d}{dx}\left[\int_0^{\delta} u^2 dy\right] dx - \rho U_{\infty} \frac{d}{dx}\left[\int_0^{\delta} u dy\right] dx$$



The Von Kármán Integral Relation



Pressure forces in the x-direction

AB: $p\delta$

CD: $-\left(p + \frac{dp}{dx}dx\right)\left(\delta + \frac{d\delta}{dx}dx\right)$

BC: $\approx \left(p + \frac{1}{2}\frac{dp}{dx}dx\right)\frac{d\delta}{dx}dx$

Shear forces in the x-direction

AD: $-\tau_w dx$

The Von Kármán Integral Relation

Forces

$$dF_x =$$

$$-\tau_w dx + p\delta - \left[p\delta + p\frac{d\delta}{dx}dx + \delta\frac{dp}{dx}dx + \frac{dp}{dx}\frac{d\delta}{dx}dxdx\right] + p\frac{d\delta}{dx}dx + \frac{1}{2}\frac{dp}{dx}\frac{d\delta}{dx}dxdx$$

products of infinitesimal quantities can be regarded to be zero and thus

$$dF_x = -\tau_w dx - \delta\frac{dp}{dx}dx$$

The Von Kármán Integral Relation

Momentum equation

Now we have all components of the momentum equation defined

$$\rho \frac{d}{dx}\left[\int_0^\delta u^2 dy\right] - \rho U_\infty \frac{d}{dx}\left[\int_0^\delta u dy\right] = -\tau_w - \delta \frac{dp}{dx}$$

The momentum equation for boundary layers or **Von Kármán's integral relation**

Note! the relation is valid for laminar and turbulent flows (for turbulent flows use time-averaged quantities)

The Von Kármán Integral Relation

Outside of the boundary layer the flow is inviscid $\Rightarrow$ we can use Bernoulli

$$p + \frac{1}{2}\rho U_\infty^2 = const \Rightarrow \frac{dp}{dx} + \rho U_\infty \frac{dU_\infty}{dx} = 0 \Rightarrow -\frac{1}{\rho}\frac{dp}{dx} = U_\infty \frac{dU_\infty}{dx}$$

The Von Kármán Integral Relation

$$\frac{1}{\rho}\frac{dp}{dx} = -U_\infty \frac{dU_\infty}{dx} \Rightarrow \frac{\tau_w}{\rho} - \delta U_\infty \frac{dU_\infty}{dx} = U_\infty \frac{d}{dx}\left[\int_0^\delta u dy\right] - \frac{d}{dx}\left[\int_0^\delta u^2 dy\right]$$

$$\delta U_\infty \frac{dU_\infty}{dx} = U_\infty \frac{dU_\infty}{dx}\int_0^\delta dy$$

$$U_\infty \frac{d}{dx}\left[\int_0^\delta u dy\right] = \frac{d}{dx}\left[U_\infty \int_0^\delta u dy\right] - \frac{dU_\infty}{dx}\int_0^\delta u dy$$

$$\frac{\tau_w}{\rho} - U_\infty \frac{dU_\infty}{dx}\int_0^\delta dy = \frac{d}{dx}\left[U_\infty \int_0^\delta u dy\right] - \frac{dU_\infty}{dx}\int_0^\delta u dy - \frac{d}{dx}\left[\int_0^\delta u^2 dy\right]$$

The Von Kármán Integral Relation

$$\frac{\tau_w}{\rho} - U_\infty \frac{dU_\infty}{dx} \int_0^\delta dy = \frac{d}{dx}\left[U_\infty \int_0^\delta u dy\right] - \frac{dU_\infty}{dx} \int_0^\delta u dy - \frac{d}{dx}\left[\int_0^\delta u^2 dy\right]$$

$$U_\infty \frac{dU_\infty}{dx} \int_0^\delta dy - \frac{dU_\infty}{dx} \int_0^\delta u dy = \frac{dU_\infty}{dx} \int_0^\delta (U_\infty - u) dy$$

$$\frac{d}{dx}\left[U_\infty \int_0^\delta u dy\right] - \frac{d}{dx}\left[\int_0^\delta u^2 dy\right] = \frac{d}{dx} \int_0^\delta u(U_\infty - u) dy$$

The Von Kármán Integral Relation

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^{\delta} u(U_\infty - u)dy + \frac{dU_\infty}{dx}\int_0^{\delta}(U_\infty - u)dy$$

The Von Kármán Integral Relation

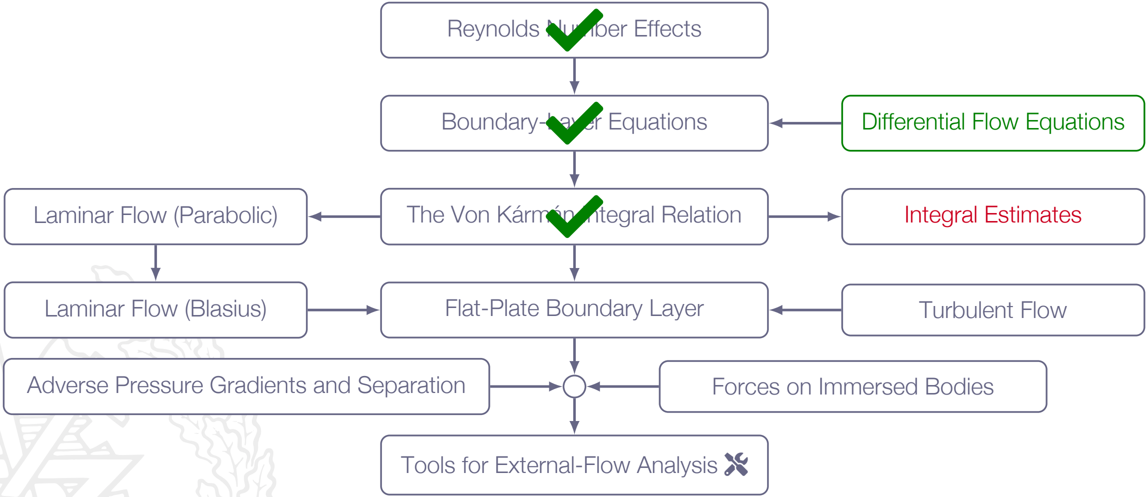
$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^{\delta} u(U_\infty - u)dy + \frac{dU_\infty}{dx}\int_0^{\delta}(U_\infty - u)dy$$

Constant freestream velocity gives

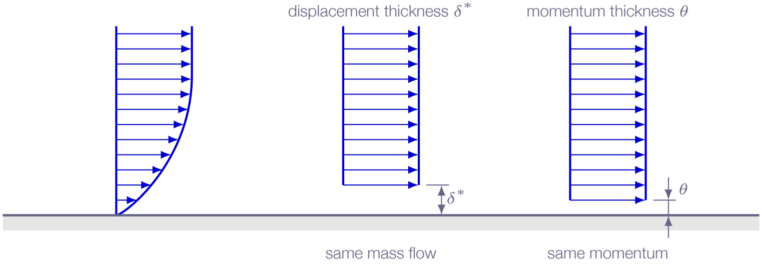
$$\frac{dU_\infty}{dx} = 0 \Rightarrow \frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^{\delta} u(U_\infty - u)dy$$

Ok, but what does this mean??

Roadmap - Flow Past Immersed Bodies

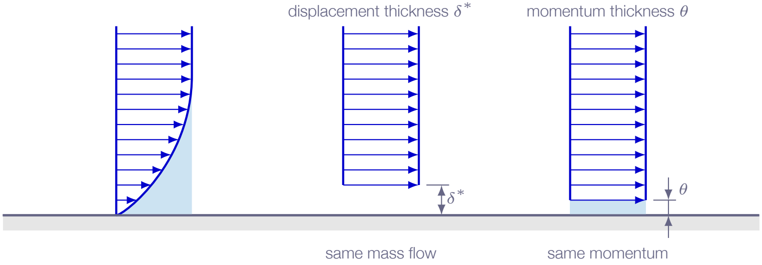


Momentum Integral Estimates



"The presence of a boundary layer will result in a small but finite displacement of the flow streamlines"

Momentum Thickness



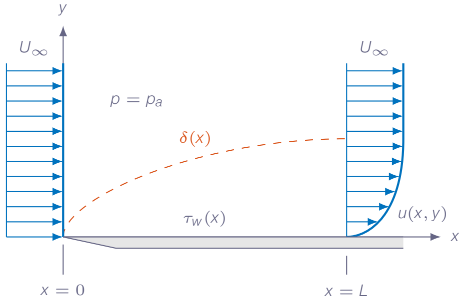
$$\int_0^{\delta} \rho u(U_\infty - u) b dy = \rho U_\infty^2 b\theta \Rightarrow \theta = \int_0^{\delta} \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

Note! *b is the width of the flat plate*

Momentum Thickness

The drag D for a plate of width b

$$D(x) = b \int_0^x \tau_w(x)dx \Rightarrow \frac{dD}{dx} = b\tau_w$$



from before we have

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^\delta u(U_\infty - u)dy = \frac{d}{dx}U_\infty^2 \underbrace{\int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy}_{\theta} = U_\infty^2 \frac{d\theta}{dx}$$

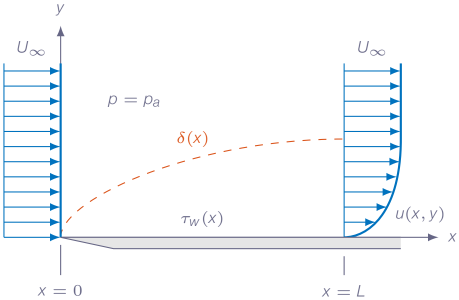
and thus

$$\frac{dD}{dx} = b\rho U_\infty^2 \frac{d\theta}{dx} \Rightarrow D(x) = \rho b U_\infty^2 \theta$$

Momentum Thickness

$$D(x) = \rho b U_\infty^2 \theta, \; \tau_w = \rho U_\infty^2 \frac{d\theta}{dx}$$

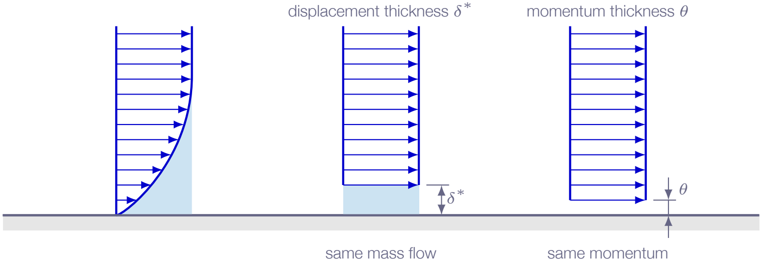
$$\theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$



Note!

1. the momentum thickness θ is a measure of the total drag
2. can be used both for laminar and turbulent flows
3. no assumption about velocity profile shape made

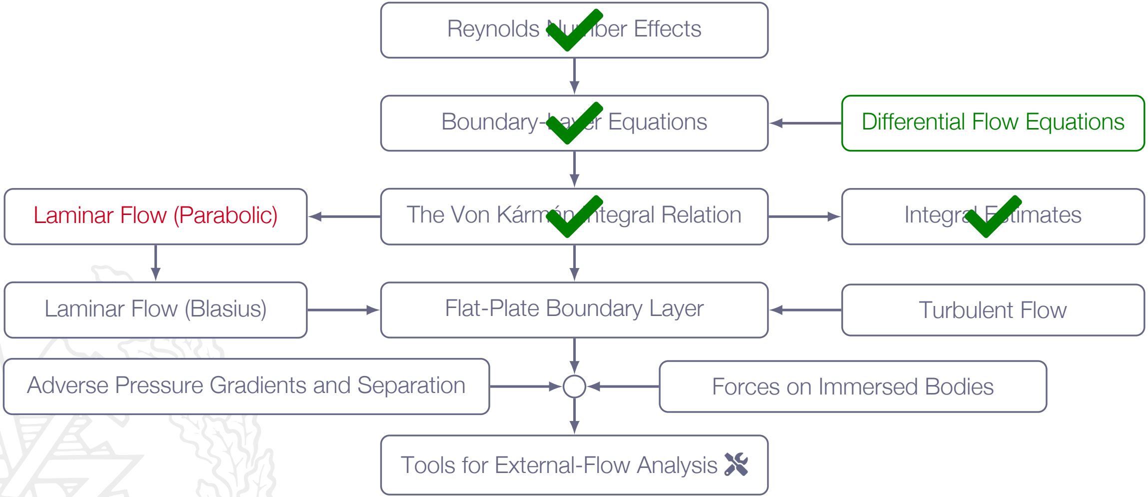
Displacement Thickness



$$\int_0^{\delta} \rho(U_\infty - u)b\,dy = \rho U_\infty b\delta^* \Rightarrow \delta^* = \int_0^{\delta} \left(1 - \frac{u}{U_\infty}\right) dy$$

δ^* is an estimate of the displacement in the wall-normal direction of streamlines in the outer part of the boundary layer due to the deficit of massflow caused by the no-slip condition at the wall - a measure of the boundary-layer thickness

Roadmap - Flow Past Immersed Bodies



Laminar Boundary Layer

The Von Kármán integral relation gives us the wall shear stress (τ_w) as a function of the velocity profile ($u(y)$) and the boundary-layer thickness (δ)

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^{\delta} u(U_\infty - u)dy$$

So now we need a velocity profile $u = u(y)$ to continue ...

Laminar Boundary Layer

Assumptions:

1. Boundary layer over a flat plate
2. Constant freestream velocity $U_\infty = const \Rightarrow \dfrac{dU_\infty}{dx} = 0$
3. Laminar flow
4. Parabolic velocity profile

Laminar Boundary Layer - Parabolic Velocity Profile

$$u(y) = A + By + Cy^2$$

The constants A, B, and C are defined using boundary conditions

1. no slip:

   $$u(0) = 0 \Rightarrow A = 0$$

2. constant velocity at $y = \delta$:

   $$\left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0 \Rightarrow B + 2C\delta = 0 \Rightarrow B = -2\delta C$$

3. freestream velocity:

   $$u(\delta) = U_\infty \Rightarrow B\delta + C\delta^2 = U_\infty \Rightarrow \{B = -2\delta C\} \Rightarrow -C\delta^2 = U_\infty \Rightarrow C = -\frac{U_\infty}{\delta^2}$$

Laminar Boundary Layer - Parabolic Velocity Profile

$$u(y) = A + By + Cy^2$$

$$A = 0, \quad B = \frac{2U_\infty}{\delta}, \quad C = -\frac{U_\infty}{\delta^2}$$

$$u(y) = U_\infty \left(\frac{2}{\delta} y - \frac{1}{\delta^2} y^2 \right)$$

Laminar Boundary Layer - Parabolic Velocity Profile

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^{\delta} u(U_\infty - u)dy$$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu\frac{2U_\infty}{\delta}$$

$$\int_0^{\delta} u(U_\infty - u)dy = \int_0^{\delta} U_\infty^2\left(\frac{2}{\delta}y - \frac{1}{\delta^2}y^2\right) - U_\infty^2\left(\frac{4}{\delta^2}y^2 - \frac{4}{\delta^3}y^3 + \frac{1}{\delta^4}y^4\right)dy = \frac{2}{15}U_\infty^2\delta$$

$$\frac{\mu}{\rho}\frac{2U_\infty}{\delta} = \frac{d}{dx}\left(\frac{2}{15}U_\infty^2\delta\right) \Rightarrow \frac{\nu}{\delta} = \frac{U_\infty}{15}\frac{d\delta}{dx} \Rightarrow \delta d\delta = 15\frac{\nu}{U_\infty}dx$$

Laminar Boundary Layer - Parabolic Velocity Profile

$$\delta d\delta = 15\frac{\nu}{U_\infty}dx$$

$$\frac{\delta^2}{2} = 15\frac{\nu}{U_\infty}x + C = \{x = 0 \Rightarrow \delta = 0 \Rightarrow C = 0\} = 15\frac{\nu}{U_\infty}x$$

$$\delta = \sqrt{\frac{30\nu x}{U_\infty}} \Rightarrow \frac{\delta}{x} = \sqrt{\frac{30\nu}{U_\infty x}} \approx \frac{5.5}{\sqrt{Re_x}}$$

Laminar Boundary Layer - Parabolic Velocity Profile

$$\tau_w = \mu \frac{2U_\infty}{\delta} = \frac{2\mu U_\infty}{\sqrt{\dfrac{30\nu x}{U_\infty}}} = \frac{2}{\sqrt{30}} \frac{\rho U_\infty^2}{\sqrt{\dfrac{U_\infty x}{\nu}}} \approx \frac{0.365}{\sqrt{Re_x}} \rho U_\infty^2$$

Introducing the **skin friction coefficient** c_f

$$c_f = \frac{2\tau_w}{\rho U_\infty^2} \approx \frac{0.73}{\sqrt{Re_x}}$$

Laminar Boundary Layer - Parabolic Velocity Profile

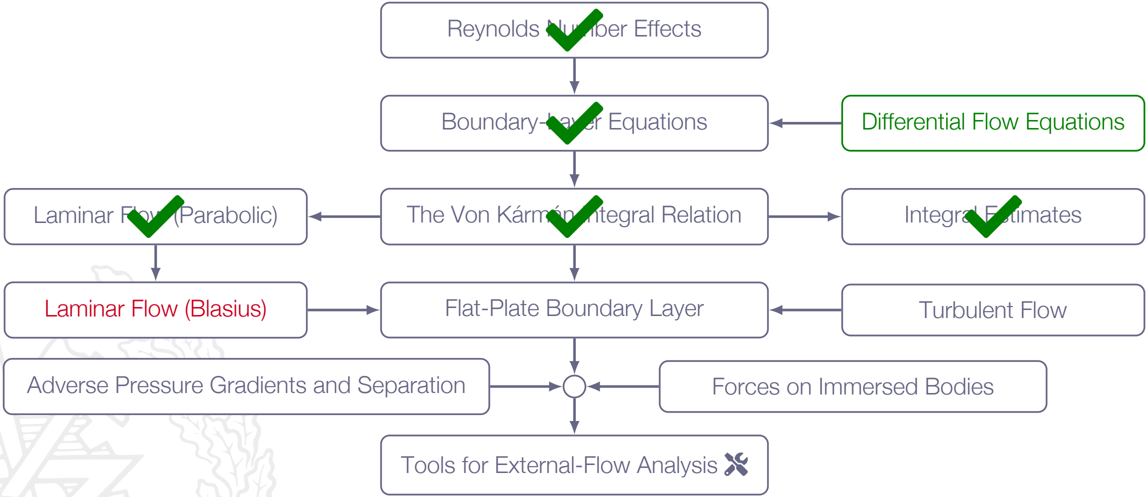
Note! *more accurate solutions for laminar flat plate boundary layers exists:*

$$c_f \approx \frac{0.664}{\sqrt{Re_x}}, \ \frac{\delta}{x} \approx \frac{5.0}{\sqrt{Re_x}}$$

Ok, so where did we go wrong?

For external (unconfined) boundary layers, the velocity profile is not parabolic – but quite close to parabolic ...

Roadmap - Flow Past Immersed Bodies



The Blasius Velocity Profile

For laminar flow, the boundary layer equations can be solved for u and v

Blasius presented a solution 1908 where he had used a coordinate transformation and showed that $\frac{u}{U_\infty}$ is a function of a single dimensionless variable $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$

The coordinate transformation corresponds to a scaling of the y coordinate with the boundary layer thickness δ

$$\frac{\delta}{x} \propto \frac{1}{\sqrt{Re_x}} \Rightarrow \frac{y}{\delta} \propto \frac{y}{x/\sqrt{Re_x}} = \frac{y}{x}\sqrt{\frac{U_\infty x}{\nu}} = y\sqrt{\frac{U_\infty}{\nu x}} = \eta$$

The Blasius Velocity Profile

1. Rewrite the boundary layer equations using the stream function (Chapter 4)
2. Rewrite the equation again $\Psi = f(\eta)\sqrt{\nu U_\infty x}$ where η is the scaled wall-normal coordinate and $f(\eta)$ is a non-dimensional stream function
3. Lots of math ....

The Navier-Stokes equations are reduced to an ordinary differential equation (ODE)

$$f''' + \frac{1}{2}ff'' = 0$$

with the boundary conditions

$$\begin{cases} f(0) = f'(0) = 0 \\ f'_{\eta\to\infty} \to 1.0 \end{cases}$$

The Blasius Velocity Profile

$$\frac{u}{U_\infty} = f'(\eta)$$

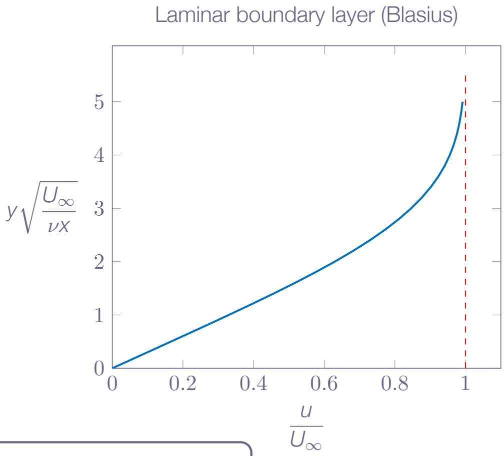
Note! $u/U_\infty \to 1$ as $y \to \infty$ and therefore δ is usually defined as the distance from the wall where $u/U_\infty = 0.99$

The Blasius Velocity Profile

$$\frac{u}{U_\infty} = f'(\eta)$$

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}}$$

$$\delta_{99\%}\sqrt{\frac{U_\infty}{\nu x}} \approx 5.0$$



$$\delta_{99\%}\sqrt{\frac{U_\infty}{\nu x}} \approx 5.0 \quad \text{or} \quad \frac{\delta}{x} \approx \frac{5.0}{\sqrt{Re_x}}$$

The Blasius Velocity Profile

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu U_\infty \left[\frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)\frac{d\eta}{dy}\right]_{\eta=0}$$

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \Rightarrow \frac{d\eta}{dy} = \sqrt{\frac{U_\infty}{\nu x}} \Rightarrow \tau_w = \mu U_\infty \sqrt{\frac{U_\infty}{\nu x}} \frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)_{\eta=0} = \frac{\rho U_\infty^2}{\sqrt{Re_x}} \frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)_{\eta=0}$$

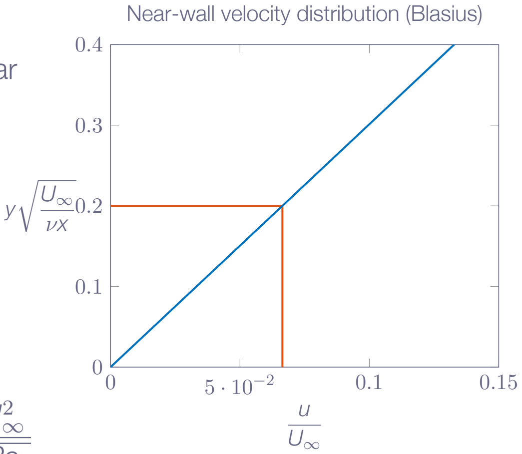
The Blasius Velocity Profile

close the the wall the velocity profile is linear

$$\eta = 0.2 \Rightarrow \frac{u}{U_\infty} \approx 0.0664$$

$$\frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)_{\eta=0} \approx \frac{0.0664}{0.2} = 0.332$$

$$\tau_w = \frac{\rho U_\infty^2}{\sqrt{Re_x}} \frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)_{\eta=0} \approx 0.332 \frac{\rho U_\infty^2}{\sqrt{Re_x}}$$



Laminar Boundary Layer - Blasius

$$\tau_w(x) \approx \frac{0.332\rho^{1/2}\mu^{1/2}U_\infty^{3/2}}{x^{1/2}}$$

Note! the wall shear stress drops off with increasing distance due to the boundary layer growth

Recall *for pipe flow, the wall shear stress is independent of x – pipe flow is confined and the boundary layer height is restricted*

Laminar Boundary Layer - Blasius

wall shear stress:

$$\tau_w(x) \approx \frac{0.332\rho^{1/2}\mu^{1/2}U_\infty^{3/2}}{x^{1/2}}$$

drag force:

$$D(x) = b\int_0^x \tau_w(x)dx \approx 0.664b\rho^{1/2}\mu^{1/2}U_\infty^{3/2}x^{1/2}$$

drag coefficient:

$$C_D = \frac{2D(L)}{\rho U_\infty^2 bL} \approx \frac{1.328}{\sqrt{Re_L}}$$

Laminar Boundary Layer - Blasius

From before we have $D(x) = \rho b \int_0^{\delta(x)} u(U_\infty - u)dy$

$$D(x) = \rho b U_\infty^2 \underbrace{\int_0^{\delta(x)} \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy}_{\theta(x)} = \rho b U_\infty^2 \theta(x)$$

$$b\int_0^x \tau_w(x)dx = \rho b U_\infty^2 \theta(x) \approx 0.664 b \rho^{1/2}\mu^{1/2}U_\infty^{3/2}x^{1/2} \Rightarrow \theta(x) \approx \frac{0.664\mu^{1/2}x^{1/2}}{\rho^{1/2}U_\infty^{1/2}}$$

$$\Rightarrow \theta(x) \approx \frac{0.664\mu^{1/2}x}{\rho^{1/2}U_\infty^{1/2}x^{1/2}} \text{ and thus } \frac{\theta(x)}{x} \approx \frac{0.664}{\sqrt{Re_x}}$$

Laminar Boundary Layer - Blasius

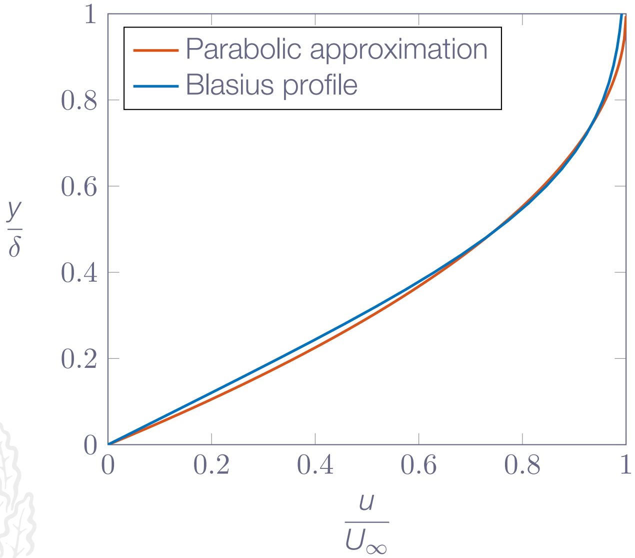
Displacement thickness:

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) dy$$

$$\frac{\delta^*}{x} \approx \frac{1.721}{Re_x^{1/2}}$$

Note! *since δ^* is much smaller than x for large values of Re_x, the velocity component in the wall-normal direction will be much smaller than the velocity parallel to the plate*

Laminar Boundary Layer



Laminar Boundary Layer

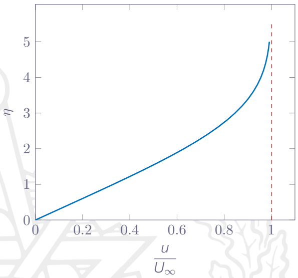
description	variable	laminar flow (Blasius)	turbulent flow (Prandtl)
boundary layer thickness	$\frac{\delta}{x}$	$\frac{5.0}{\sqrt{Re_x}}$	
displacement thickness	$\frac{\delta^*}{x}$	$\frac{1.721}{\sqrt{Re_x}}$	
momentum thickness	$\frac{\theta}{x}$	$\frac{0.664}{\sqrt{Re_x}}$	
shape factor	$H = \frac{\delta^*}{\theta}$	2.59	
wall shear stress	τ_w	$0.332\frac{\rho U_\infty^2}{\sqrt{Re_x}}$	
local skin friction coefficient	$c_f = \frac{2\tau_w}{\rho U_\infty^2}$	$\frac{0.664}{\sqrt{Re_x}}$	
drag coefficient	C_D	$\frac{1.328}{\sqrt{Re_L}}$	

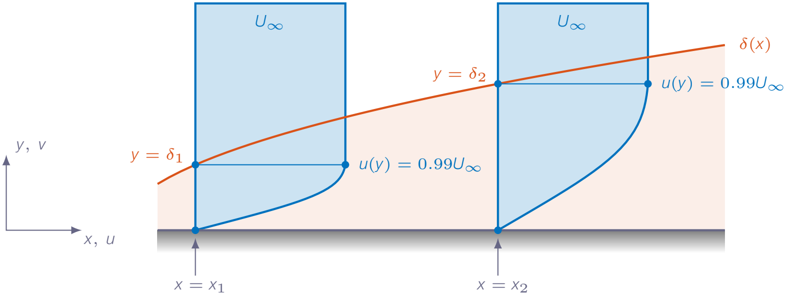
The Blasius Velocity Profile - Self Similarity

From before:

$$\eta(x,y) = y\sqrt{\frac{U_\infty}{\nu x}}$$

$$\frac{u}{U_\infty} = 0.99 \Rightarrow \eta \approx 5.0$$

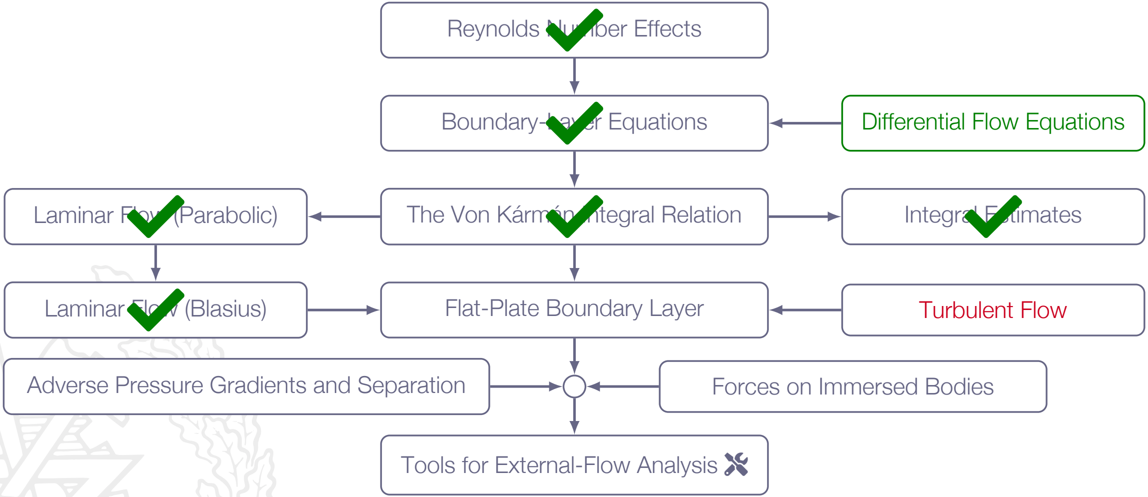




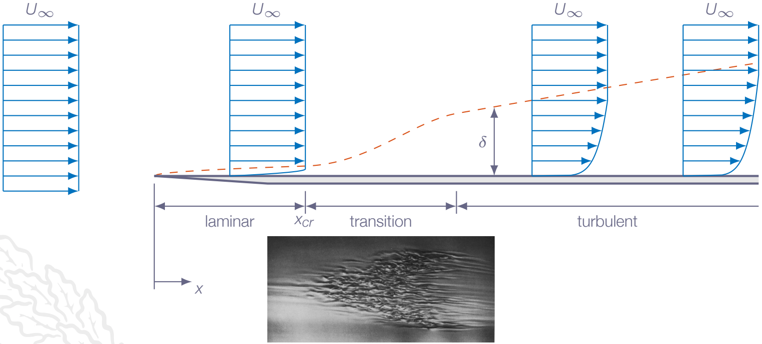
$$\eta(x_1,\delta_1) = \eta(x_2,\delta_2) \Rightarrow \delta_1\sqrt{\frac{U_\infty}{\nu x_1}} = \delta_2\sqrt{\frac{U_\infty}{\nu x_2}}$$

$$x_1 < x_2 \Rightarrow \sqrt{\frac{U_\infty}{\nu x_1}} > \sqrt{\frac{U_\infty}{\nu x_2}} \Rightarrow \delta_1 < \delta_2$$

Roadmap - Flow Past Immersed Bodies



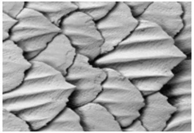
Boundary Layer Transition



- - - $u = 0.99U_\infty$

$$Re_x = \frac{U_\infty x}{\nu}$$

$$Re_{x_{cr}} = \frac{U_\infty x_{cr}}{\nu} \approx 5.0 \times 10^5$$



Boundary Layer Transition

- For low Re_x, disturbances in the flow are damped out by viscous forces
- For somewhat higher Reynolds numbers, friction forces are less important and the flow becomes unstable
- The transition region is short - can be treated as a point (the transition point)

Boundary Layer Transition

The onset of transition from laminar to turbulent is affected by a number of factors such as:

- Turbulence in the freestream
- Surface roughness
- Pressure gradient

With a smooth surface, no turbulence in the freestream, and zero pressure gradient, the onset of transition can be pushed up to $Re_x \approx 3.0 \times 10^6$

As a rule of thumb, we can assume $Re_{x_{cr}} \approx 5.0 \times 10^5$

Boundary Layer Transition

Freestream turbulence:

- frestream turbulence reduces the critical Reynolds number
- with high turbulence intensity in the freestream, the transition can start already at $Re_x \approx 3.0 \times 10^5$ or lower

Boundary Layer Transition

Surface roughness:

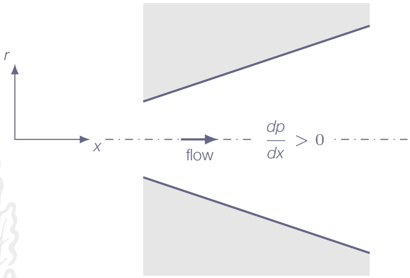
- surface roughness does not affect transition significantly if $Re_\epsilon = \dfrac{U_\infty \epsilon}{\nu} < 680$
- if $Re_\epsilon > 680$, the extent of the laminar region can be shortened significantly ($Re_x \approx 3.0 \times 10^5$)

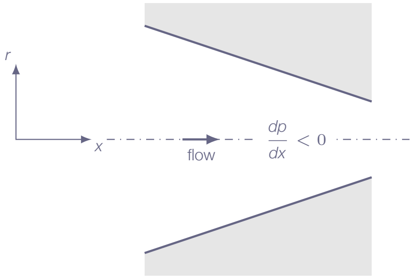
Note! rule of thumb

Boundary Layer Transition

Negative pressure gradient:

- decreasing pressure in the flow direction has a stabilizing effect on the flow and can delay transition from laminar to turbulent flow





Boundary Layer Transition

Forced transition:

- a trip wire or added surface roughness can make the transition to turbulence really fast
- the critical Reynolds number is not meaningful if the boundary layer is forced to transition

Flat Plate - Turbulent Boundary Layer

A turbulent boundary layer grows faster than a laminar boundary layer

- the velocity fluctuations (u', v', w') leads to increased exchange of momentum
- increased shear stress compared to the laminar case where we only have forces related to molecular viscosity
- larger portion of the fluid will be decelerated close to the wall

Flat Plate - Turbulent Boundary Layer

The Von Kármán integral relation and the integral estimates are valid for both laminar and turbulent boundary layers

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^{\delta} u(U_\infty - u)dy$$

$$\theta = \int_0^{\delta} \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) d$$

$$\delta^* = \int_0^{\delta} \left(1 - \frac{u}{U_\infty}\right) dy$$

We need a velocity profile $u(y)$ for turbulent boundary layers to be able to calculate τ_w, θ, and δ^*

- Approach 1: the log law
- Approach 2: Prandtl's power law approximation

Flat Plate - Turbulent Boundary Layer

Approach 1: the log law

$$\frac{u}{u^*} \approx \frac{1}{\kappa} \ln\left(\frac{yu^*}{\nu}\right) + B \quad \text{where } \kappa = 0.41 \text{ and } B = 5.0$$

u^* is the **friction velocity** defined as $u^* = \sqrt{\frac{\tau_w}{\rho}}$

at the edge of the boundary layer $u = U_\infty$ and $y = \delta$ and thus

$$\frac{U_\infty}{u^*} \approx \frac{1}{\kappa} \ln\left(\frac{\delta u^*}{\nu}\right) + B$$

Flat Plate - Turbulent Boundary Layer

Approach 1: the log law

The **skin friction coefficient** c_f is defined as $c_f = \dfrac{2\tau_w}{\rho U_\infty^2} \Rightarrow \tau_w = c_f \dfrac{1}{2}\rho U_\infty^2$

the **friction velocity** can be expressed as $u^* = \sqrt{\dfrac{\tau_w}{\rho}} = U_\infty \sqrt{\dfrac{c_f}{2}}$

insert in the **log-law** and we get

$$\sqrt{\frac{2}{c_f}} \approx \frac{1}{\kappa} \ln \left(Re_\delta \sqrt{\frac{c_f}{2}} \right) + B$$

rather difficult to work with ...

Flat Plate - Turbulent Boundary Layer

Approach 2: Prandtl's power law approximation

Prandtl suggested the following relations:

$$c_f \approx 0.02 Re_\delta^{-1/6}$$
$$\frac{u}{U_\infty} \approx \left(\frac{y}{\delta}\right)^{1/7}$$

from before we have the following relation: $\tau_w = \rho U_\infty^2 \frac{d\theta}{dx} \Rightarrow c_f = 2\frac{d\theta}{dx}$

calculate the **momentum thickness** $\theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy = \frac{7}{72}\delta$

Flat Plate - Turbulent Boundary Layer

Approach 2: Prandtl's power law approximation

Now, combining the two **skin friction coefficient** relations we see that

$$0.02Re_\delta^{-1/6} = 2\frac{d}{dx}\left(\frac{7}{72}\delta\right)$$

and thus $Re_\delta^{-1/6} \approx 9.72\frac{d\delta}{dx} = 9.72\frac{d(Re_\delta)}{d(Re_x)}$

integration gives $Re_\delta \approx 0.16Re_x^{6/7}$ or $\frac{\delta}{x} \approx \frac{0.16}{Re_x^{1/7}}$

Note! the turbulent boundary layer grows significantly faster than the laminar
$\delta_{turb} \propto x^{6/7}$ vs $\delta_{lam} \propto x^{1/2}$

Flat Plate - Turbulent Boundary Layer

Approach 2: Prandtl's power law approximation

$$c_f \approx \frac{0.027}{Re_x^{1/7}}$$

$$\tau_{w_{turb}} \approx \frac{0.0135\mu^{1/7}\rho^{6/7}U_\infty^{13/7}}{x^{1/7}}$$

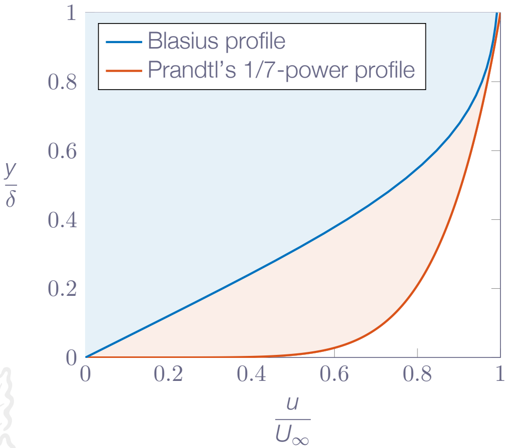
Note! friction drops slowly with x, increases nearly as ρ and U_∞^2, and is rather insensitive to viscosity

Flat Plate - Turbulent Boundary Layer

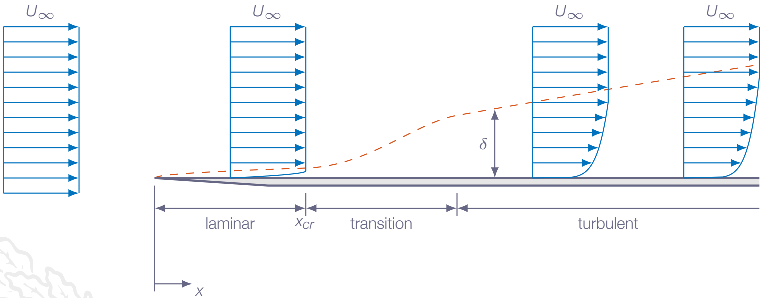
description	variable	laminar flow (Blasius)	turbulent flow (Prandtl)
boundary layer thickness	$\frac{\delta}{x}$	$\frac{5.0}{\sqrt{Re_x}}$	$\frac{0.16}{Re_x^{1/7}}$
displacement thickness	$\frac{\delta^*}{x}$	$\frac{1.721}{\sqrt{Re_x}}$	$\frac{0.02}{Re_x^{1/7}}$
momentum thickness	$\frac{\theta}{x}$	$\frac{0.664}{\sqrt{Re_x}}$	$\frac{0.016}{Re_x^{1/7}}$
shape factor	$H = \frac{\delta^*}{\theta}$	2.59	1.29
wall shear stress	τ_w	$0.332\frac{\rho U_\infty^2}{\sqrt{Re_x}}$	$0.0135\frac{\rho U_\infty^2}{Re_x^{1/7}}$
local skin friction coefficient	$c_f = \frac{2\tau_w}{\rho U_\infty^2}$	$\frac{0.664}{\sqrt{Re_x}}$	$\frac{0.027}{Re_x^{1/7}}$
drag coefficient	C_D	$\frac{1.328}{\sqrt{Re_L}}$	$\frac{0.031}{Re_L^{1/7}}$

Flat Plate - Turbulent Boundary Layer

The velocity profile in a turbulent boundary layer is quite far from the Blasius profile used for laminar boundary layers



Flat Plate Boundary Layer



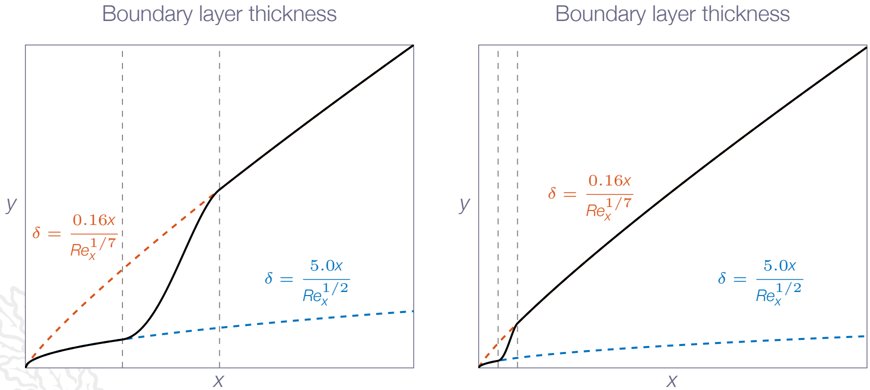
$u = 0.99U_\infty$

$$Re_x = \frac{U_\infty x}{\nu}$$

$$Re_{x_{cr}} = \frac{U_\infty x_{cr}}{\nu} \approx 5.0 \times 10^5$$

$$D = b\frac{1}{2}\rho U^2 \left[\int_{>0}^{x_{cr}} \frac{0.664}{Re_x^{1/2}}dx + \int_{x_{cr}}^{L} \frac{0.027}{Re_x^{1/7}}dx\right]$$

Flat Plate Boundary Layer



For a long boundary layer the length of the laminar region becomes relatively short in comparison with the length of the turbulent region

Wall Roughness

laminar:

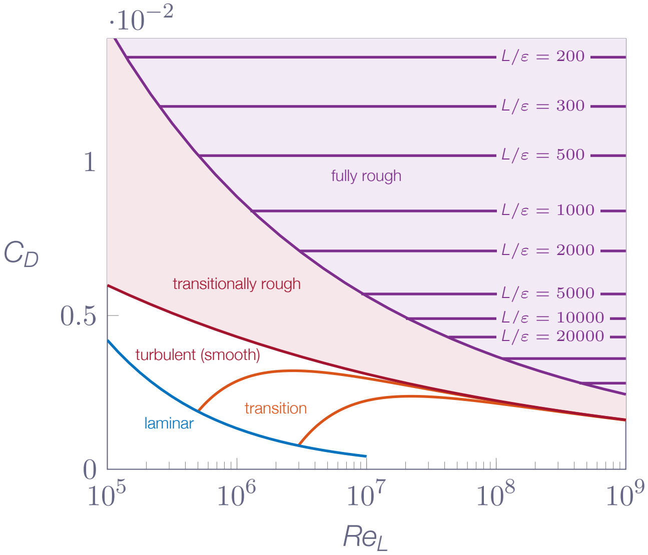
$$C_D = \frac{1.328}{Re_L^{1/2}}$$

turbulent (smooth):

$$C_D = \frac{0.031}{Re_L^{1/7}}$$

turbulent (fully rough):

$$C_D = (1.89 + 1.62 \log(L/\varepsilon))^{-2.5}$$



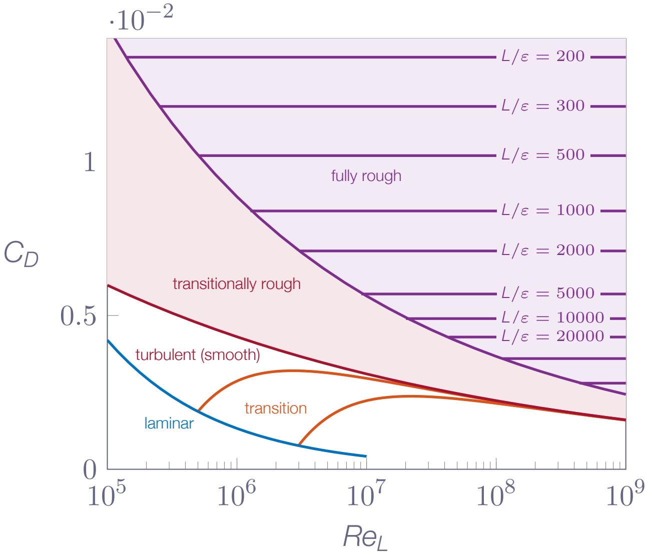
Wall Roughness

transition ($Re_{trans} = 5.0 \times 10^5$):

$$C_D = \frac{0.031}{Re_L^{1/7}} - \frac{1440}{Re_L}$$

transition ($Re_{trans} = 3.0 \times 10^6$):

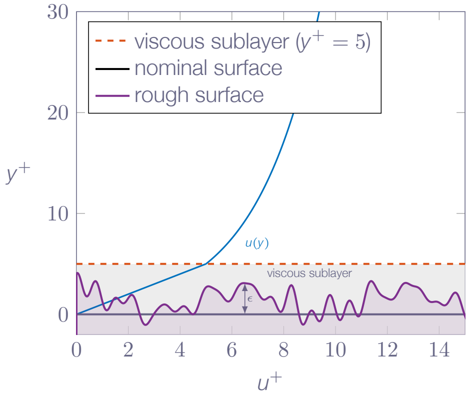
$$C_D = \frac{0.031}{Re_L^{1/7}} - \frac{8700}{Re_L}$$



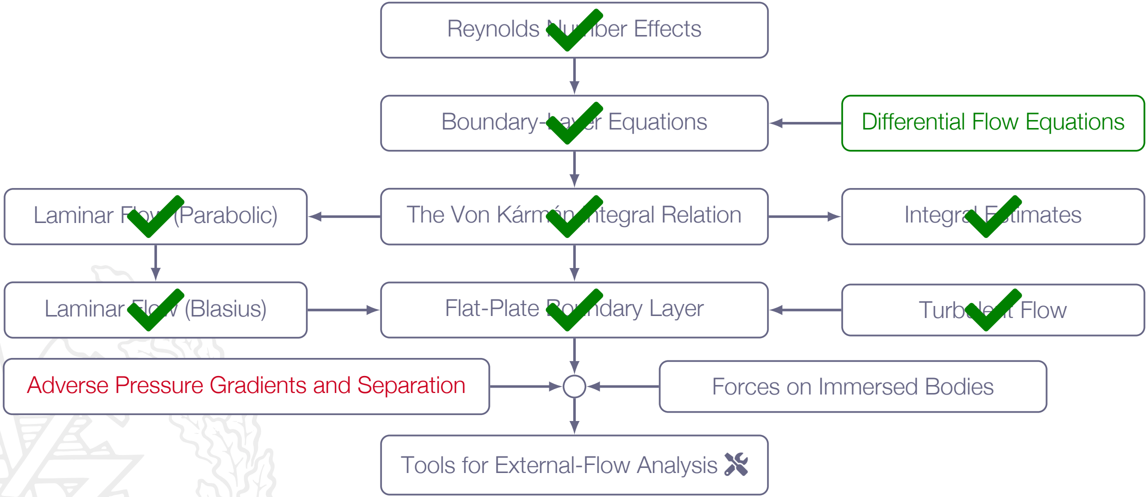
Wall Roughness

Recall: smooth surface:

Surface roughness (ϵ) within the viscous sublayer



Roadmap - Flow Past Immersed Bodies



Pressure Gradient

Adverse pressure gradient

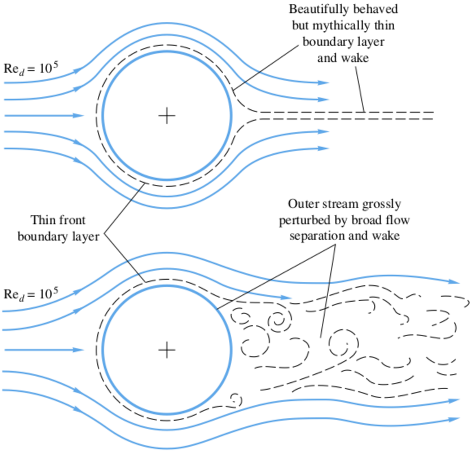
- pressure increases in the flow direction
- may lead to separation

Favorable pressure gradient

- pressure decreases in the flow direction
- the flow will not separate

Separation mechanism

- loss of momentum near the wall
- adverse pressure gradient
- decelerated fluid will force flow to separate from the body



Pressure Gradient

Boundary layer formulation of the momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

with $u = v = 0$ close at the wall, we get

$$\left.\frac{\partial \tau}{\partial y}\right|_{wall} = \mu\left.\frac{\partial^2 u}{\partial y^2}\right|_{wall} \Rightarrow \left.\frac{\partial^2 u}{\partial y^2}\right|_{wall} = \frac{1}{\mu}\frac{dp}{dx}$$

Note! applies both for laminar and turbulent flow

Pressure Gradient

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{wall} = \frac{1}{\mu}\frac{dp}{dx}$$

Adverse pressure gradient ($\frac{dp}{dx} > 0$):

$\frac{\partial^2 u}{\partial y^2} > 0$ at the wall

$\frac{\partial^2 u}{\partial y^2} < 0$ at the outer layer $y = \delta$

thus $\frac{\partial^2 u}{\partial y^2} = 0$ somewhere in the boundary layer

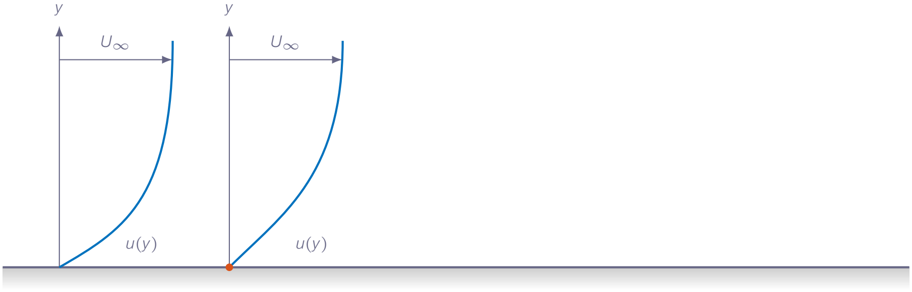
Pressure Gradient

Favorable gradient
$(dp/dx < 0)$

Point of inflection:
inside wall

No separation

Pressure Gradient



Favorable gradient
$(dp/dx < 0)$

Point of inflection:
inside wall

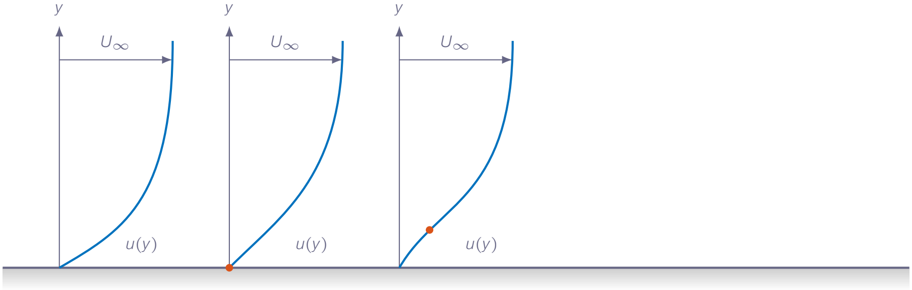
No separation

Zero gradient
$(dp/dx = 0)$

Point of inflection:
at the wall

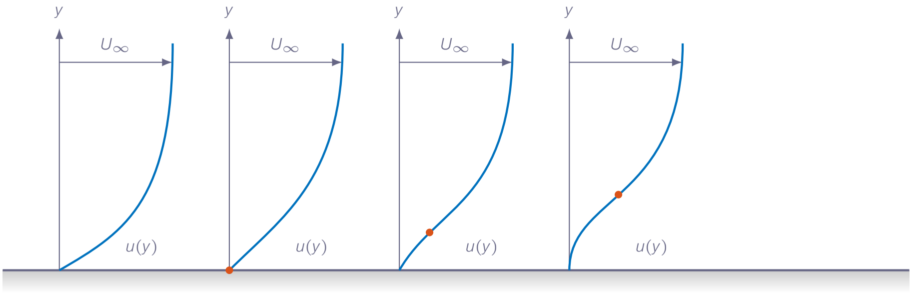
No separation

Pressure Gradient



Favorable gradient ($dp/dx < 0$)	Zero gradient ($dp/dx = 0$)	Weak adverse gradient ($dp/dx > 0$)
Point of inflection: inside wall	Point of inflection: at the wall	Point of inflection: in the flow
No separation	No separation	No separation

Pressure Gradient



Favorable gradient $(dp/dx < 0)$

Point of inflection: inside wall

No separation

Zero gradient $(dp/dx = 0)$

Point of inflection: at the wall

No separation

Weak adverse gradient $(dp/dx > 0)$

Point of inflection: in the flow

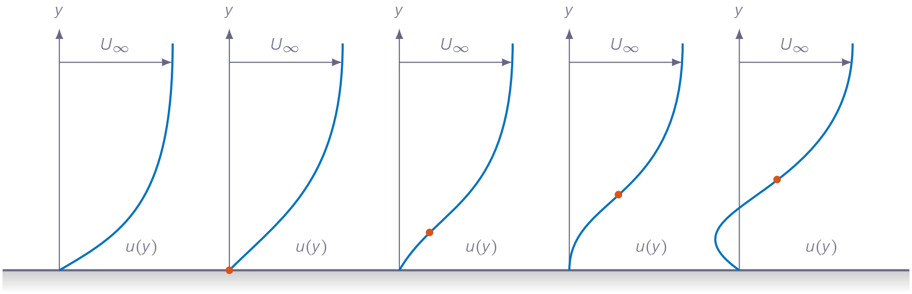
No separation

Critical adverse gradient $(dp/dx > 0)$

Point of inflection: in the flow

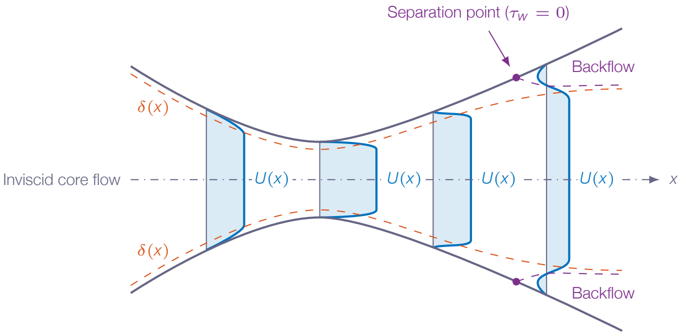
Separation
zero slope at wall
$\tau_w = 0$

Pressure Gradient



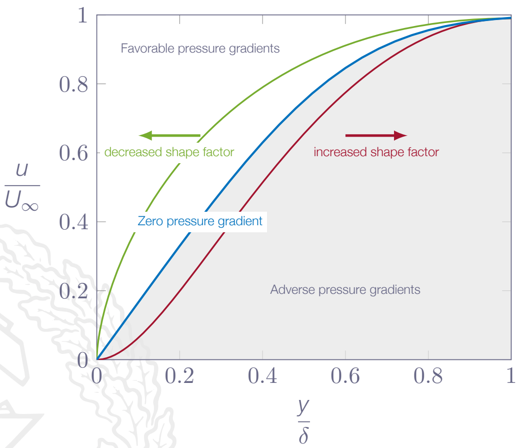
Favorable gradient ($dp/dx < 0$)	Zero gradient ($dp/dx = 0$)	Weak adverse gradient ($dp/dx > 0$)	Critical adverse gradient ($dp/dx > 0$)	Excessive adverse gradient ($dp/dx > 0$)
Point of inflection: inside wall	Point of inflection: at the wall	Point of inflection: in the flow	Point of inflection: in the flow	Point of inflection: in the flow
No separation	No separation	No separation	Separation zero slope at wall $\tau_w = 0$	Separated flow backflow at wall

Pressure Gradient



Nozzle	**Throat**	**Diffuser**
decreasing area	minimum area	increasing area
favorable pressure gradient	zero pressure gradient	adverse pressure gradient
$dp/dx < 0$	$dp/dx = 0$	$dp/dx > 0$
$dU/dx > 0$	$dU/dx = 0$	$dU/dx < 0$

Shape Factor



Shape factor: $H = \frac{\delta^*}{\theta}$

Laminar flow:

No pressure gradient: $H \approx 2.6$

Separation: $H \approx 3.5$

Turbulent flow:

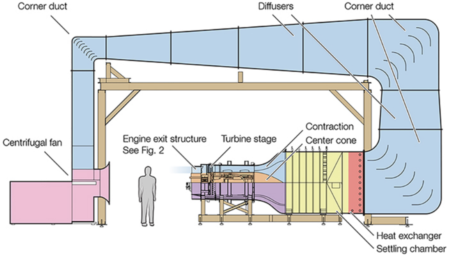
No pressure gradient: $H \approx 1.3$

Separation: $H \approx 2.4$

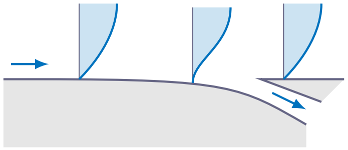
Avoid or Delay Separation

Decrease magnitude of adverse pressure gradient

- Guide vanes
- Streamlining



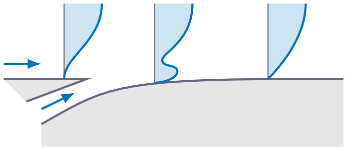
Avoid or Delay Separation



Remove decelerated fluid

Boundary layer suction

Avoid or Delay Separation



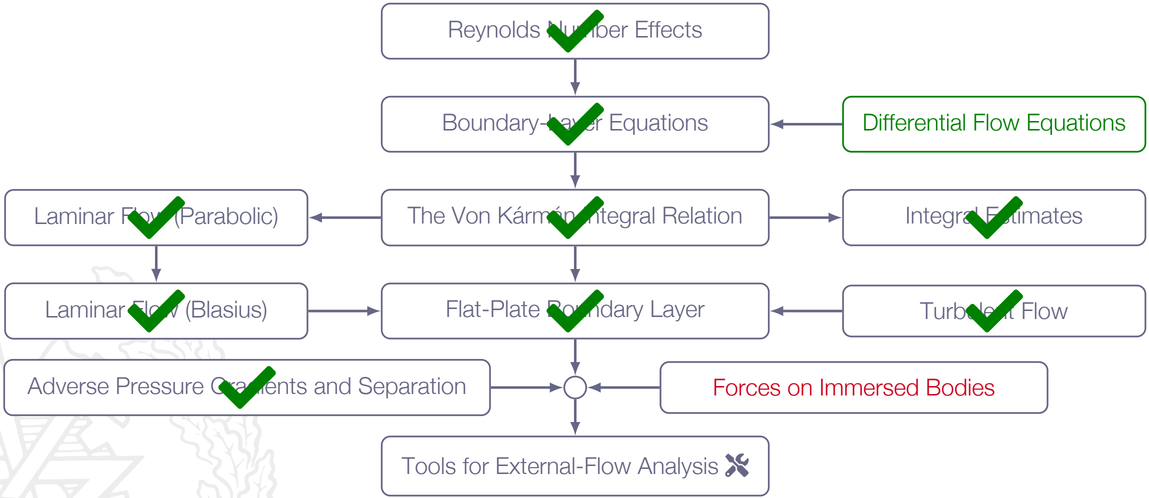
Increase near-wall momentum

Forced **transition to turbulence**

- surface roughness
- surface irregularities (dimples on the surface of a golf ball)
- trip wires

Negative consequence: comes with **increased friction**

Roadmap - Flow Past Immersed Bodies



Drag of Immersed Bodies

$$C_D = \frac{drag}{\frac{1}{2}\rho U_\infty^2 A} = f\left(\frac{U_\infty L}{\nu}\right)$$

Characteristic area A:

1. Frontal area

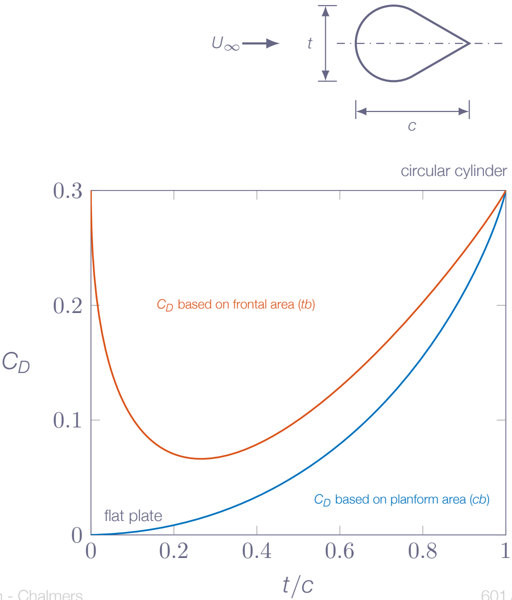
   blunt objects: *cylinders, cars*

2. Planform area

   wide flat bodies: *wings, hydrofoils*

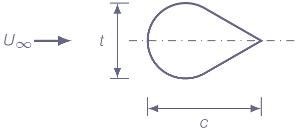
3. Wetted area

   ships

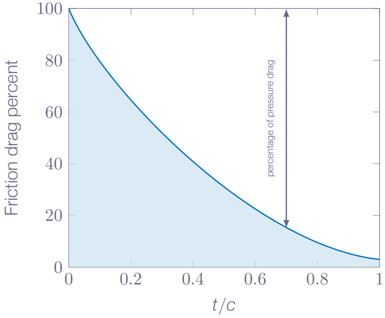


Drag of Immersed Bodies

$$C_D = C_{D_{pressure}} + C_{D_{friction}}$$



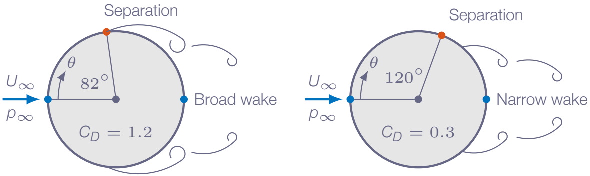
- Pressure drag
  - difference between the high front stagnation pressure and the low wake pressure on the backside of the body
  - often larger than the friction drag



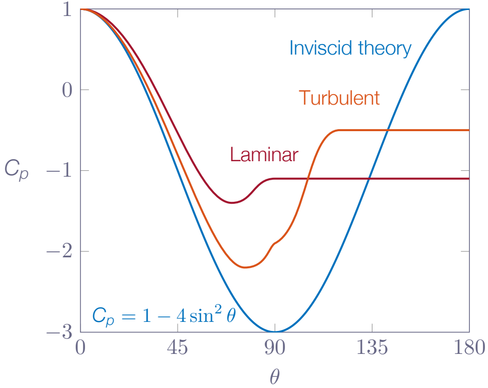
- The relative importance of friction and pressure drag depends on
  - shape
  - surface roughness

Note! for a cylinder, friction drag can be as low as a few percent of the total drag

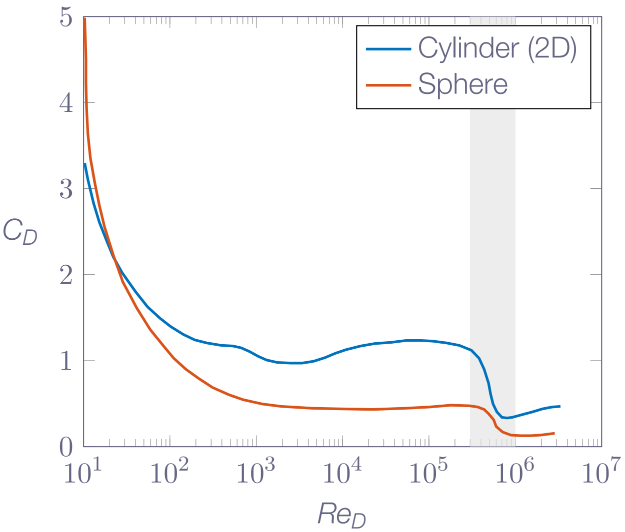
Cylinder Surface Pressure



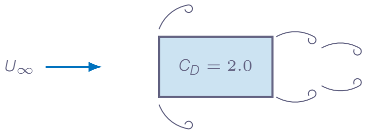
$$C_p = \frac{p - p_\infty}{\rho U_\infty^2/2}$$



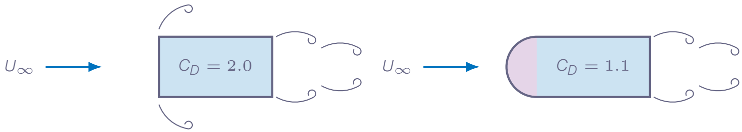
Cylinder Drag



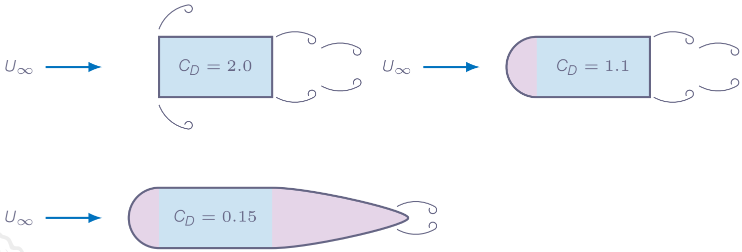
Streamlining



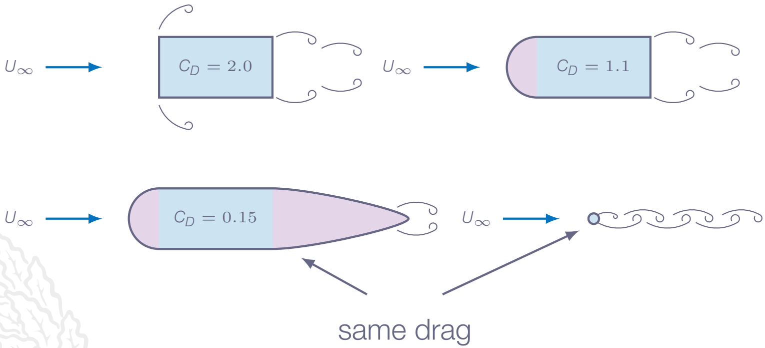
Streamlining



Streamlining



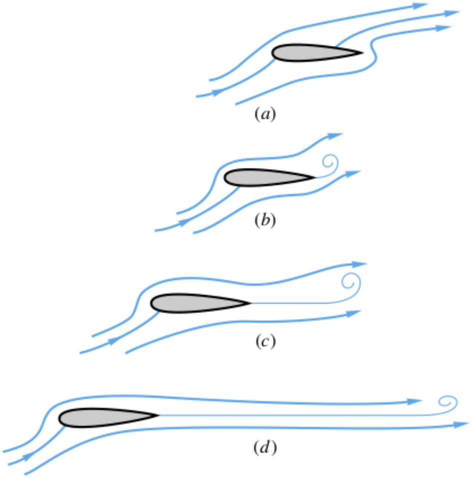
Streamlining



Drag Prediction

- No reliable theory for drag prediction (with the exception of flat plates)
- The separation point can be predicted with some accuracy but not the wake flow
- CFD or experiments needed

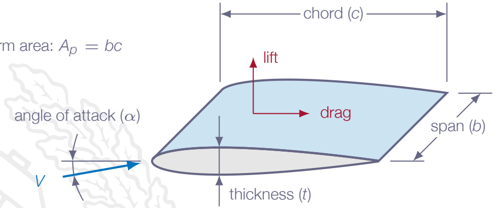
Wing Lift and Drag

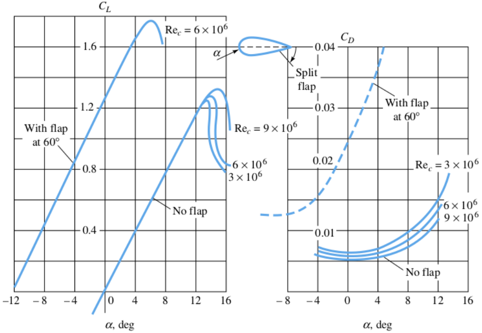


Wing Lift and Drag

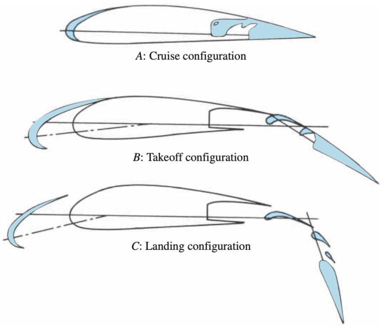
$$C_D = \frac{F_D}{\frac{1}{2}\rho U_\infty^2 A_p} \qquad C_L = \frac{F_L}{\frac{1}{2}\rho U_\infty^2 A_p}$$

planform area: $A_p = bc$

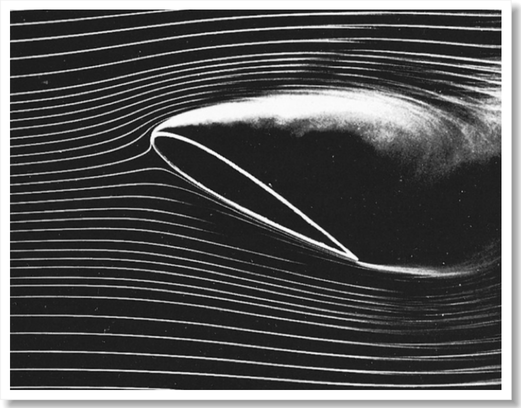




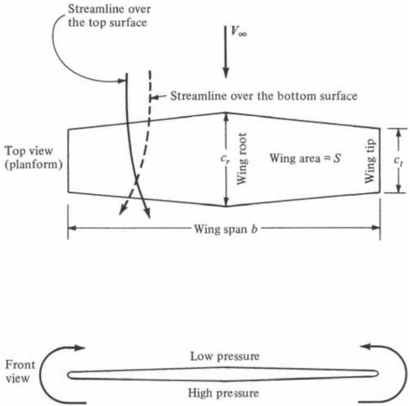
Wing Lift and Drag - High-Lift Devices

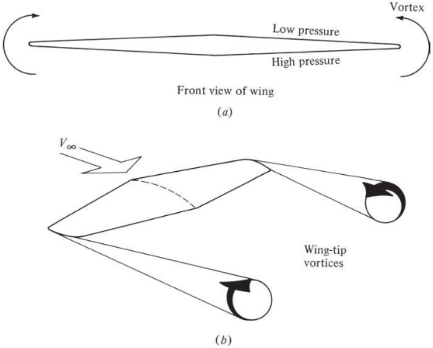


Wing Lift and Drag - Wing Stall

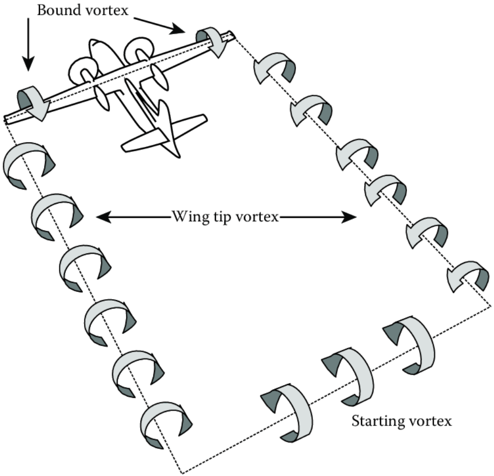


Wing Lift and Drag - Induced Drag



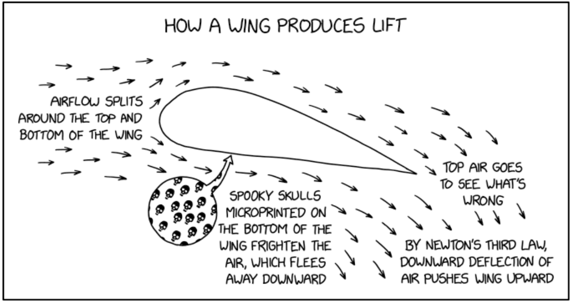


Wing Lift and Drag - Induced Drag

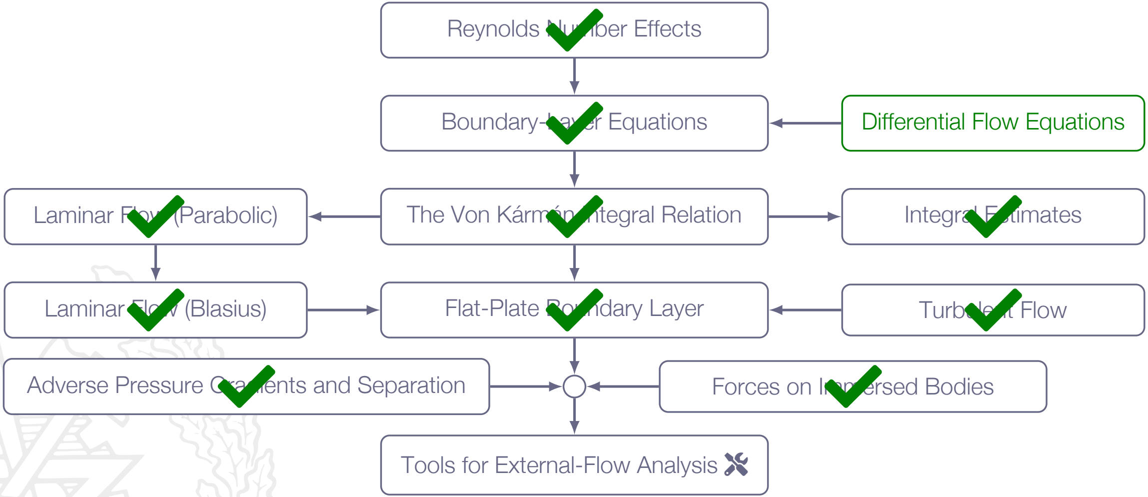


Wing Lift and Drag - Induced Drag

Wing Lift and Drag

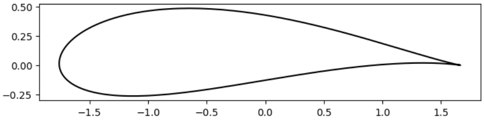


Roadmap - Flow Past Immersed Bodies



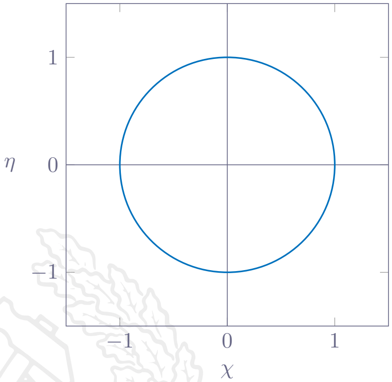
Joukowsky Transform

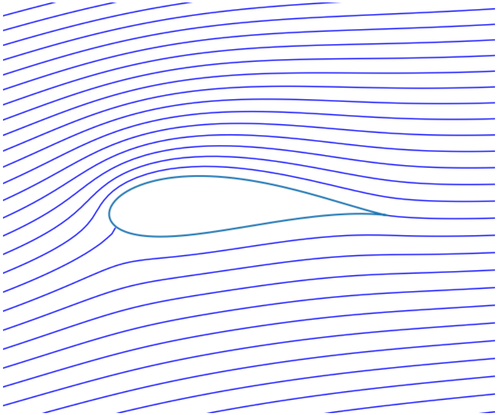
A Joukowsky wing is generated in the complex plane by applying the Joukowsky transform to a cylinder



Since the potential flow around a cylinder is well known it is by using so-called conformal mapping possible to get the flow around the wing profile from the cylinder solution

Joukowsky Transform

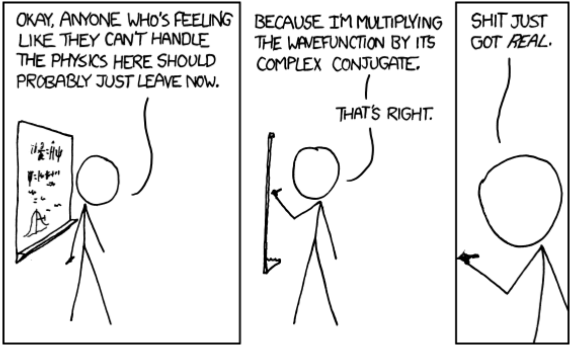




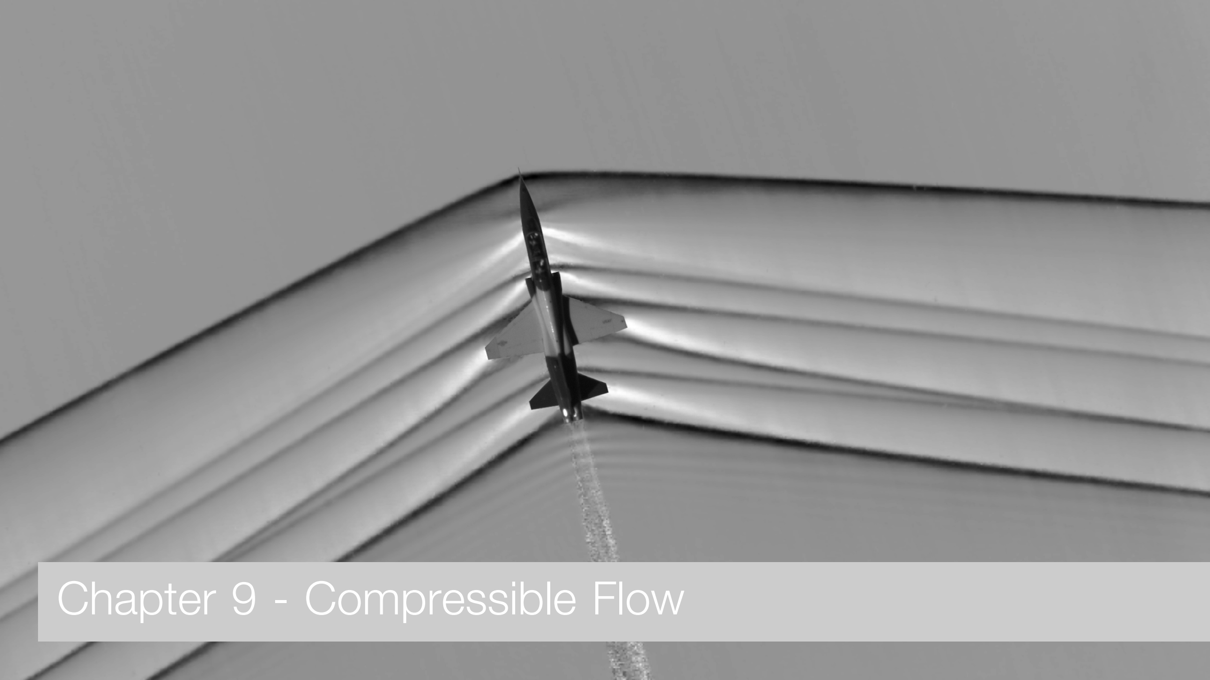
$$\zeta = \chi + i\eta$$

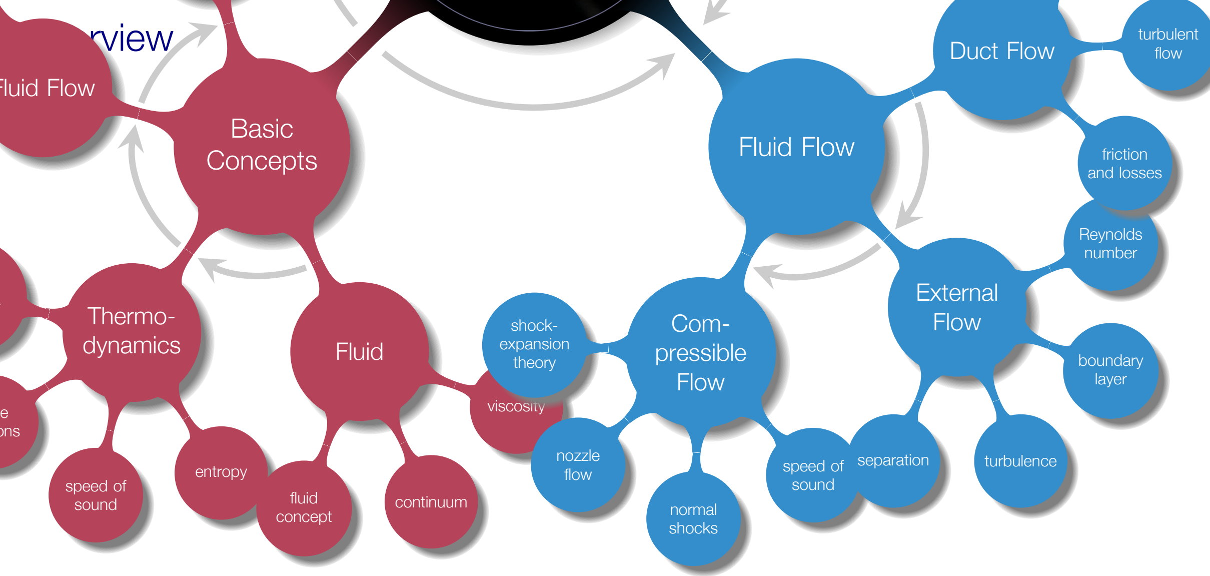
$$z = \zeta + \frac{1}{\zeta} = x + iy$$

Complex Conjugate



Chapter 9 - Compressible Flow





rview
luid Flow
Basic Concepts
Thermo-dynamics
Fluid
speed of sound
entropy
fluid concept
continuum
viscosity
Fluid Flow
Duct Flow
turbulent flow
friction and losses
Reynolds number
External Flow
boundary layer
separation
turbulence
Com-pressible Flow
shock-expansion theory
nozzle flow
normal shocks
speed of sound

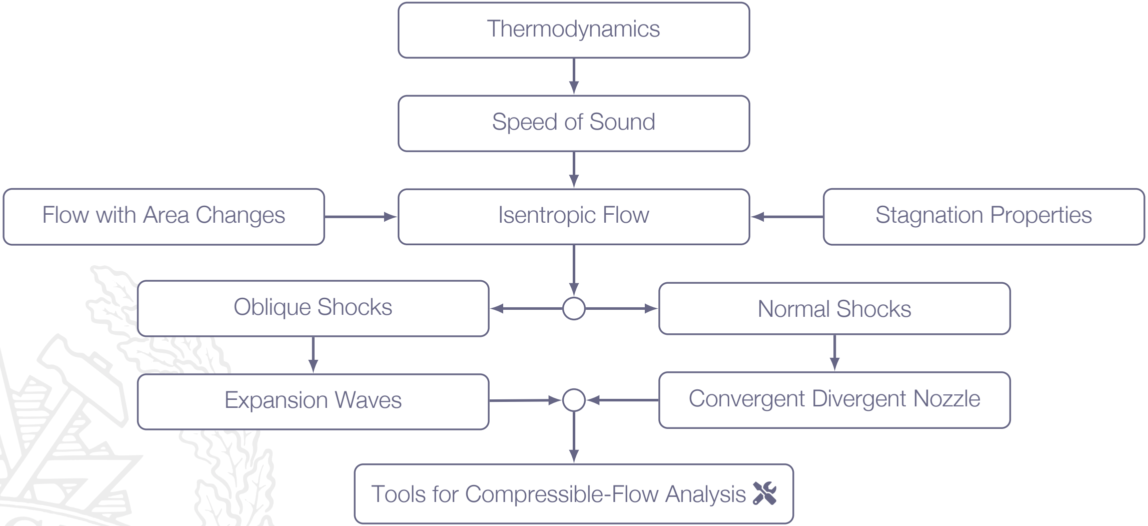
Learning Outcomes

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

37 **Understand and explain** basic concepts of compressible flows (the gas law, speed of sound, Mach number, isentropic flow with changing area, normal shocks, oblique shocks, Prandtl-Meyer expansion)

Let's go supersonic ...

Roadmap - Compressible Flow



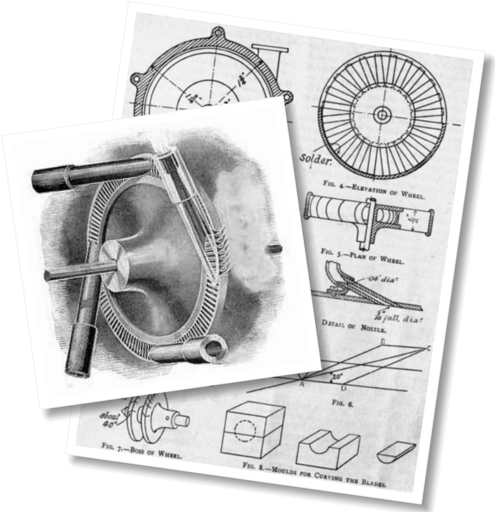
Motivation

Compressible flow:

- flows where variations in density are significant
- most often high-speed gas flows (gas dynamics)
- fluids moving at speeds comparable to the speed of sound
- not common in liquids (would require very high pressures)

Historical Milestones

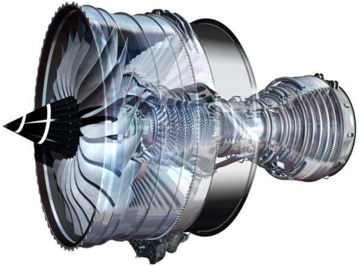
First supersonic flight - Charles Yeager 1947



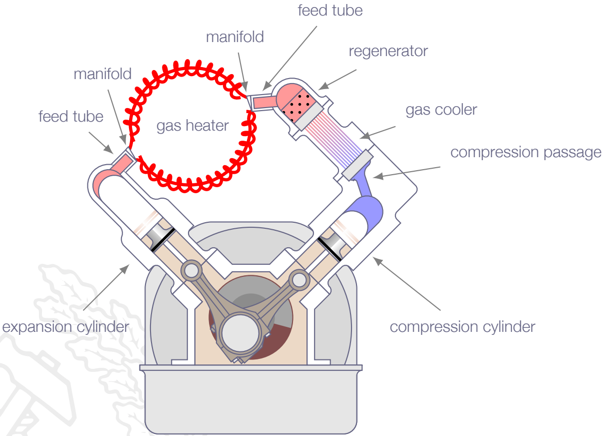
Steam turbine with convergent-divergent nozzles - Carl Gustav de Laval 1893

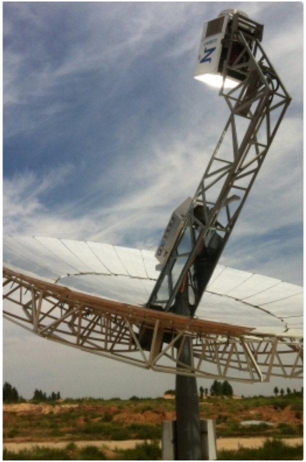
Compressible Flow Applications

Compressible Flow Applications



Compressible Flow Applications





Governing Equations

- With significant density changes follows substantial changes in pressure and temperature
- The energy equation must be included
- Four equations:
  1. Continuity
  2. Momentum
  3. Energy
  4. Equation of state
- Unknowns: ρ, p, T, and $\mathbf{V}$
- The four equations must be solved simultaneously

Mach Number Regimes

Incompressible flow

- insignificant density changes

Subsonic flow

- local and global Mach number less than unity

Transonic flow

- subsonic flow with regions of supersonic flow (local Mach number can be higher than one)
- supersonic flow with regions of subsonic flow (local Mach number can be less than one)

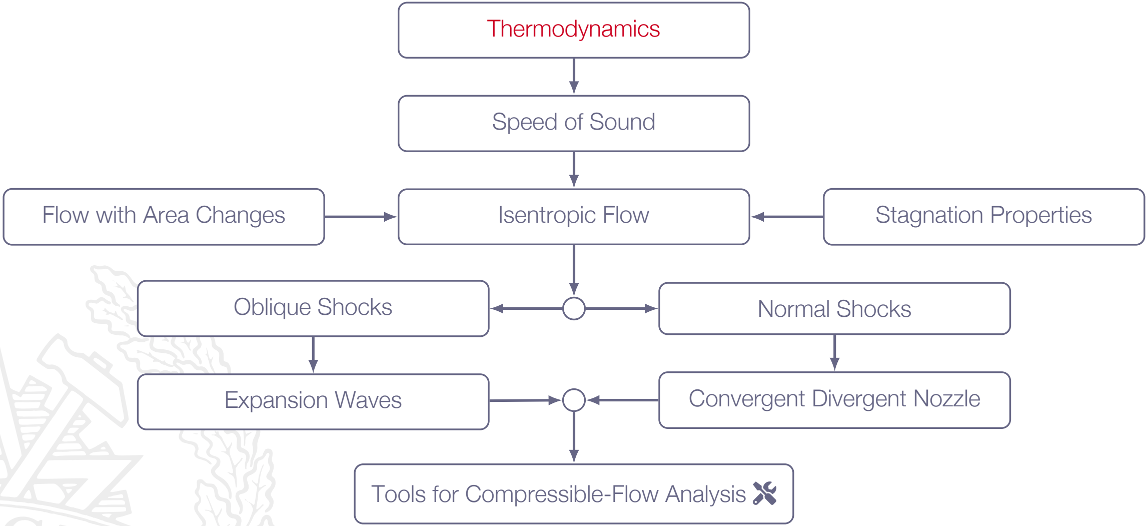
Supersonic flow

- local and global Mach number higher than one

Hypersonic flow

- Mach number higher than 5.0

Roadmap - Compressible Flow



Ratio of Specific Heats

- The ratio of specific heats is important in compressible flow

$$\gamma = \frac{C_p}{C_v}$$

- γ is a fluid property
- For moderate temperatures γ is a constant
- For higher temperatures γ varies with temperature
- For air, $\gamma = 1.4$

Equation of State

In the following, we will assume that the ideal gas law is applicable and that the specific heats are constants:

$$p = \rho RT$$

$$R = C_p - C_v = const$$

$$\gamma = \frac{C_p}{C_v} = const$$

Auxiliary relations:

$$C_v = \frac{R}{\gamma - 1}, C_p = \frac{\gamma R}{\gamma - 1}$$

Internal Energy and Enthalpy

Constant specific heats:

$$d\hat{u} = C_v dT$$

$$dh = C_p dT$$

Variable specific heats:

$$\hat{u} = \int C_v dT$$

$$h = \int C_p dT$$

Isentropic Relations

First law of thermodynamics

$$\delta q + \delta w = de$$

For reversible processes: $\delta w = -pd\upsilon$ (where $\upsilon = p/\rho$)

$$h = e + \frac{p}{\rho} = e + p\upsilon \Rightarrow dh = de + pd\upsilon + \upsilon dp$$

$$\delta q = dh - \upsilon dp$$

Second law of thermodynamics

$$ds = \frac{\delta q_{rev}}{T} = \frac{\delta q}{T} + ds_{irev} \Rightarrow ds \geq \frac{\delta q}{T}$$

Isentropic Relations

compute entropy change from the first and second law of thermodynamics (assuming reversible heat addition)

$$Tds = dh - \frac{dp}{\rho}$$

for perfect gases, $dh = C_p dT$

$$\int_1^2 ds = \int_1^2 C_p \frac{dT}{T} - R \int_1^2 \frac{dp}{p}$$

for constant specific heats (calorically perfect)

$$s_2 - s_1 = C_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} = C_v \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1}$$

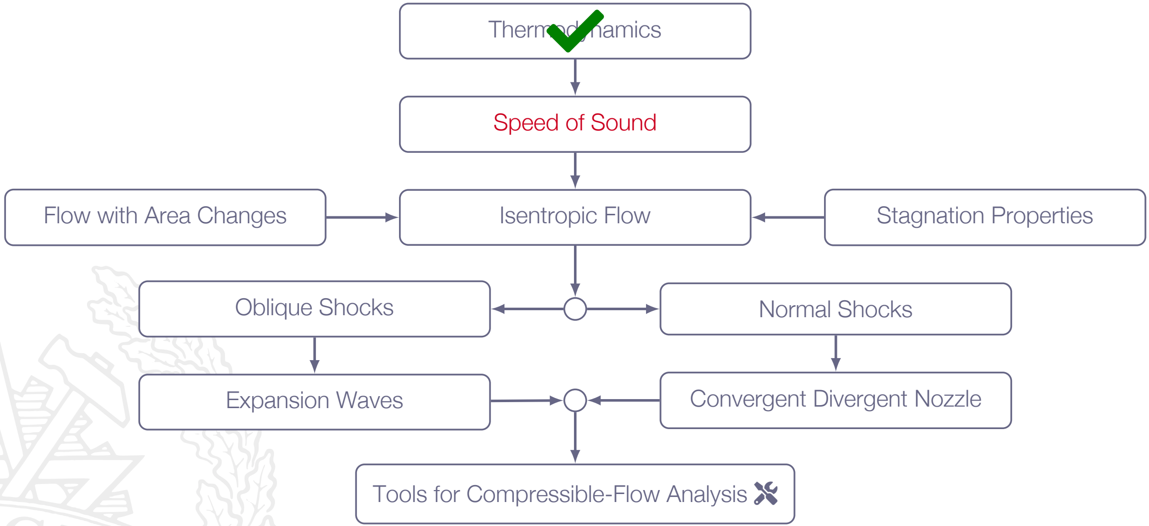
Isentropic Relations

$$s_2 - s_1 = C_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} = C_v \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1}$$

for isentropic flow ($s_2 = s_1$) we get

$$\boxed{\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}}$$

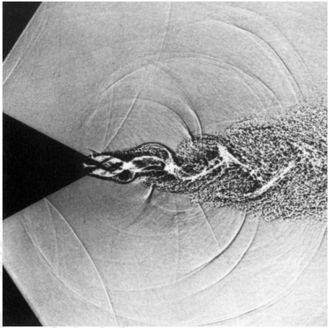
Roadmap - Compressible Flow

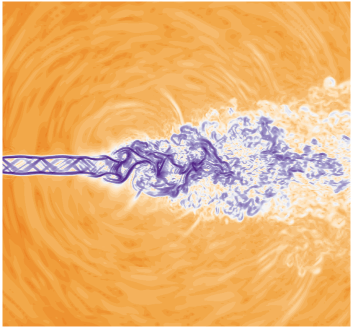


Speed of Sound

- The rate of propagation of a pressure pulse of infinitesimal strength through a fluid at rest
- Related to the molecular activity of the fluid
- A thermodynamic property

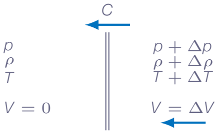
Speed of Sound



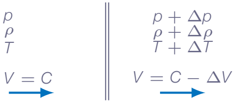


Speed of Sound

frame of reference fixed to fluid



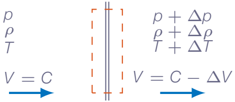
frame of reference following the wave



Speed of Sound

frame of reference following the wave

continuity:



$$\rho A C = (\rho + \Delta\rho) A (C - \Delta V)$$

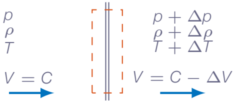
$$\Delta V = C \frac{\Delta\rho}{\rho + \Delta\rho}$$

Note! there are no gradients in the flow so viscous effects are confined to the interior of the wave

Speed of Sound

frame of reference following the wave

momentum:



$$pA - (p + \Delta p)A = (\rho AC)(C - \Delta V - C) \Rightarrow \Delta p = \rho C \Delta V$$

with ΔV from the continuity equation we get

$$C^2 = \frac{\Delta p}{\Delta \rho}\left(1 + \frac{\Delta \rho}{\rho}\right)$$

Note! the larger $\Delta\rho/\rho$, the higher the propagation velocity

Speed of Sound

In the limit of infinitesimal strength $\Delta\rho \to 0$ and thus

$$C^2 = a^2 = \frac{\partial p}{\partial \rho}$$

- There is no added heat and thus the process adiabatic
- For weak waves the process can also be assumed to be reversible

$$a^2 = \left.\frac{\partial p}{\partial \rho}\right|_s$$

Speed of Sound

$$a^2 = \left.\frac{\partial p}{\partial \rho}\right|_s$$

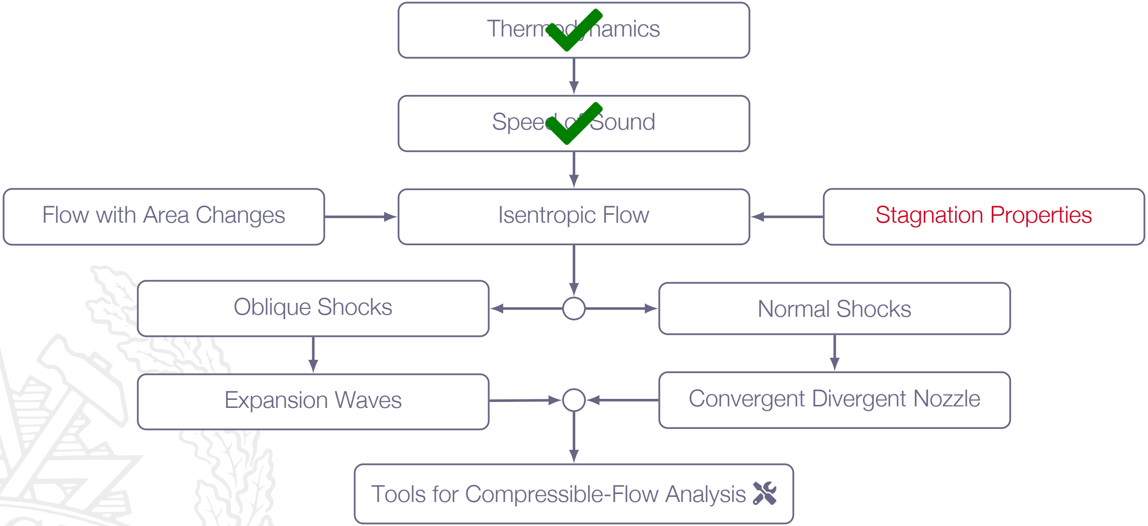
The isentropic relation gives

$$p = \rho^\gamma \Rightarrow \frac{\partial p}{\partial \rho} = \gamma \rho^{\gamma - 1} = \gamma \frac{p}{\rho} = \gamma RT$$

and thus

$$\boxed{a = \sqrt{\gamma RT}}$$

Roadmap - Compressible Flow



Stagnation Enthalpy

Consider high-speed gas flow past an insulated wall

$$h_1 + \frac{1}{2}V_1^2 + gz_1 = h_2 + \frac{1}{2}V_2^2 + gz_2 - q + w_\nu$$

- differences in potential energy extremely small
- outside of the boundary layer, heat transfer and viscous work are zero

$$h_1 + \frac{1}{2}V_1^2 = h_2 + \frac{1}{2}V_2^2 = const$$

Stagnation Enthalpy

$$h + \frac{1}{2}V^2 = h_o$$

”The maximum enthalpy that the fluid would achieve if brought to rest adiabatically”

Stagnation Temperature

For a calorically perfect gas $h = C_p T$

$$h + \frac{1}{2}V^2 = h_o$$

$$C_p T + \frac{1}{2}V^2 = C_p T_o$$

Where T_o is the stagnation temperature

Mach Number Relations

$$C_pT + \frac{1}{2}V^2 = C_pT_o \Rightarrow 1 + \frac{V^2}{2C_pT} = \frac{T_o}{T}$$

$$C_pT = \frac{\gamma R}{\gamma - 1}T = \frac{\gamma RT}{\gamma - 1} = \frac{a^2}{\gamma - 1}$$

$$\boxed{\frac{T_o}{T} = 1 + \left(\frac{\gamma - 1}{2}\right)M^2}$$

Mach Number Relations

Since $a \propto T^{1/2}$ we get

$$\frac{a_o}{a} = \left(\frac{T_o}{T}\right)^{1/2} = \left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right]^{1/2}$$

Mach Number Relations

If the flow is adiabatic and reversible (isentropic), we may use the isentropic relations

$$\frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{\gamma/(\gamma-1)} = \left[1 + \left(\frac{\gamma-1}{2}\right)M^2\right]^{\gamma/(\gamma-1)}$$

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{1/(\gamma-1)} = \left[1 + \left(\frac{\gamma-1}{2}\right)M^2\right]^{1/(\gamma-1)}$$

Stagnation Properties

- p_o and ρ_o - the pressure and density that the flow would achieve if brought to rest isentropically
- All stagnation properties are constants in an isentropic flow
- h_o, T_o, and a_o are constants in an adiabatic flow but not necessarily p_o and ρ_o
- p_o and ρ_o will vary throughout an adiabatic flow as the entropy changes due to friction or shocks

Critical Properties

Another useful set of reference variables is the critical properties (sonic conditions)

$$\frac{T_o}{T} = 1 + \left(\frac{\gamma - 1}{2}\right) M^2 = \{M = 1.0\} = 1 + \left(\frac{\gamma - 1}{2}\right) = \left(\frac{2 + \gamma - 1}{2}\right) = \left(\frac{\gamma + 1}{2}\right)$$

Critical Properties

$$\frac{T^*}{T_o} = \left(\frac{2}{\gamma+1}\right)$$

$$\frac{a^*}{a_o} = \left(\frac{2}{\gamma+1}\right)^{1/2}$$

$$\frac{p^*}{p_o} = \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}$$

$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)}$$

Critical Properties

Air $\gamma = 1.4$

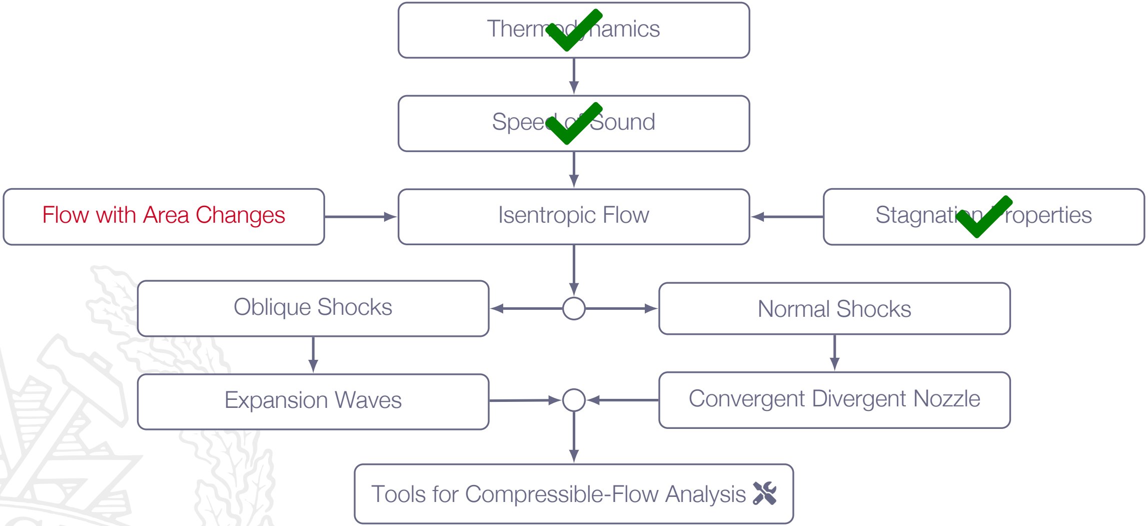
$$\frac{T^*}{T_o} = \left(\frac{2}{\gamma+1}\right) = 0.8333$$

$$\frac{a^*}{a_o} = \left(\frac{2}{\gamma+1}\right)^{1/2} = 0.9129$$

$$\frac{p^*}{p_o} = \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} = 0.5283$$

$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)} = 0.6339$$

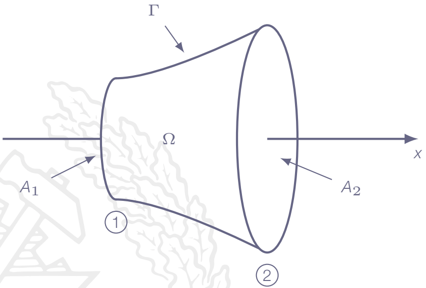
Roadmap - Compressible Flow



Isentropic Quasi-1D Flow

Quasi-1D:

- Flow properties varies in one direction only (x)
- The flow area is a smooth function $A = A(x)$
- Steady-state, inviscid and isentropic flow



The Area-Velocity Relation

Continuity:

$$\rho(x)V(x)A(x) = const \Rightarrow d(\rho VA) = 0 \Rightarrow AVd\rho + \rho AdV + \rho VdA = 0$$

divide by ρVA gives

$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0$$

The Area-Velocity Relation

In the following, isentropic flow is assumed

Stagnation enthalpy:

$$h_o = h + \frac{1}{2}V^2 = const \Rightarrow dh + VdV = 0$$

The first and second law of thermodynamics:

$$Tds = 0 = dh - \frac{dp}{\rho} \Rightarrow dh = \frac{dp}{\rho}$$

and thus

$$\frac{dp}{\rho} + VdV = 0$$

The Area-Velocity Relation

$$\frac{dp}{\rho} + VdV = 0$$

From the definition of the **speed of sound**

$$dp = a^2 d\rho \Rightarrow a^2 \frac{d\rho}{\rho} + VdV = 0 \Rightarrow \frac{d\rho}{\rho} = -\frac{1}{a^2} VdV$$

The Area-Velocity Relation

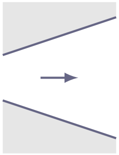
$$\frac{d\rho}{\rho}+\frac{dV}{V}+\frac{dA}{A}=\frac{dV}{V}-\frac{1}{a^2}VdV+\frac{dA}{A}=0$$

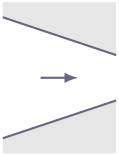
$$\frac{dV}{V}\left(\frac{V^2}{a^2}-1\right)=\frac{dA}{A}$$

$$\boxed{\frac{dV}{V}=\frac{1}{M^2-1}\frac{dA}{A}=-\frac{dp}{\rho V^2}}$$

The Area-Velocity Relation

$$\frac{dV}{V}(M^2 - 1) = \frac{dA}{A}$$





	Subsonic $M < 1$	Supersonic $M > 1$
$dA > 0$	subsonic diffuser $dV < 0$ $dp > 0$	supersonic nozzle $dV > 0$ $dp < 0$
$dA < 0$	subsonic nozzle $dV > 0$ $dp < 0$	supersonic diffuser $dV < 0$ $dp > 0$

The Area-Velocity Relation

$$\frac{dV}{V}(M^2 - 1) = \frac{dA}{A}$$

What happens when $M = 1$?

The Area-Velocity Relation

$$\frac{dV}{V}(M^2 - 1) = \frac{dA}{A}$$

What happens when $M = 1$?

$M = 1$ when $dA = 0$

The Area-Velocity Relation

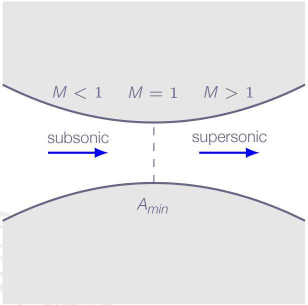
$$\frac{dV}{V}(M^2-1)=\frac{dA}{A}$$

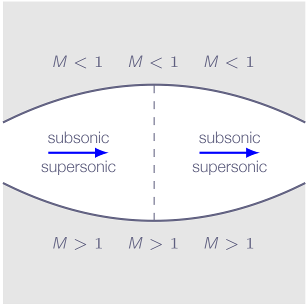
What happens when $M=1$?

$M=1$ when $dA=0$

maximum or minimum area

The Area-Velocity Relation





The Area-Mach-Number Relation

$$\rho A V = \rho^* A^* V^* \Rightarrow \frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{V^*}{V}$$

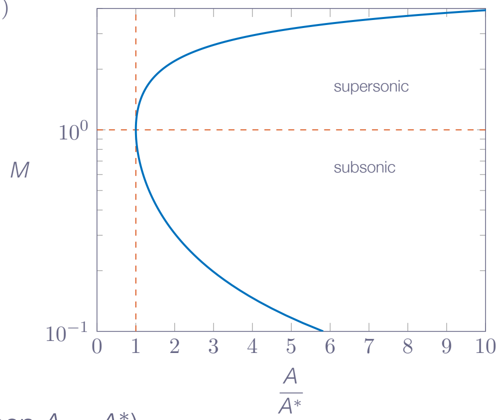
$$\frac{\rho^*}{\rho} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho} = \left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{1/(\gamma-1)}$$

$$\frac{V^*}{V} = \frac{(\gamma R T^*)^{1/2}}{V} = \frac{(\gamma R T)^{1/2}}{V}\left(\frac{T^*}{T_o}\right)^{1/2}\left(\frac{T_o}{T}\right)^{1/2} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{1/2}$$

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{(\gamma+1)/(\gamma-1)}$$

The Area-Mach-Number Relation

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{(\gamma+1)/(\gamma-1)}$$



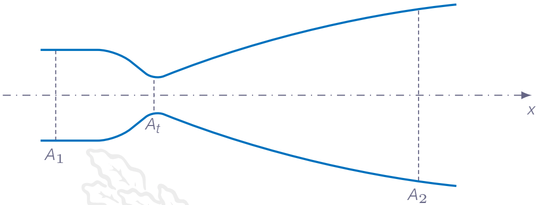
Note!

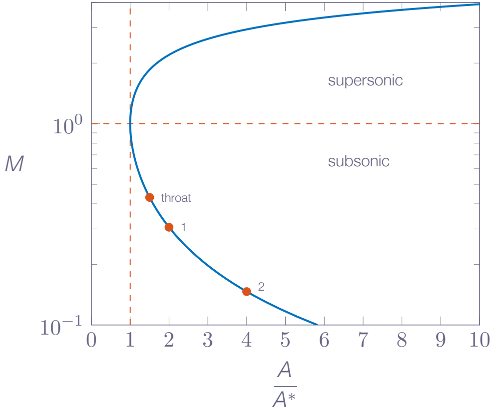
Two possible solutions for each value of $\frac{A}{A^*}$:

one subsonic and one supersonic (except when $A = A^*$)

The Area-Mach-Number Relation

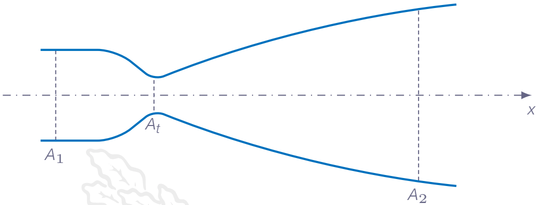
Sub-critical (non-choked) nozzle flow

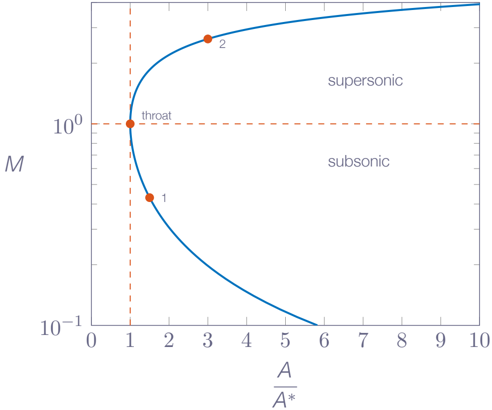




The Area-Mach-Number Relation

Critical (choked) nozzle flow



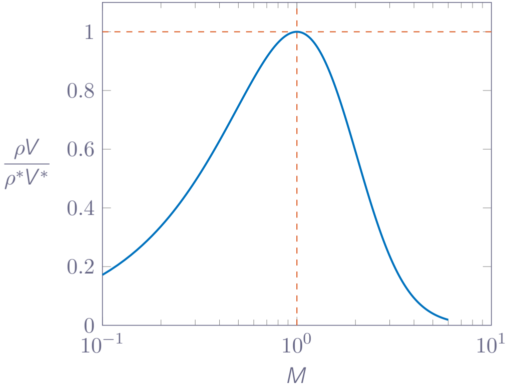


Choking

$$\rho V A = \rho^* A^* V^* \Rightarrow \frac{A^*}{A} = \frac{\rho V}{\rho^* V^*}$$

From the area-Mach-number relation

$$\frac{A^*}{A} = \begin{cases} < 1 & \text{if} \quad M < 1 \\ 1 & \text{if} \quad M = 1 \\ < 1 & \text{if} \quad M > 1 \end{cases}$$



The maximum possible mass flow through a duct is achieved when its throat reaches sonic conditions

Choking

$$\dot{m}_{max} = \rho^* A^* V^*$$

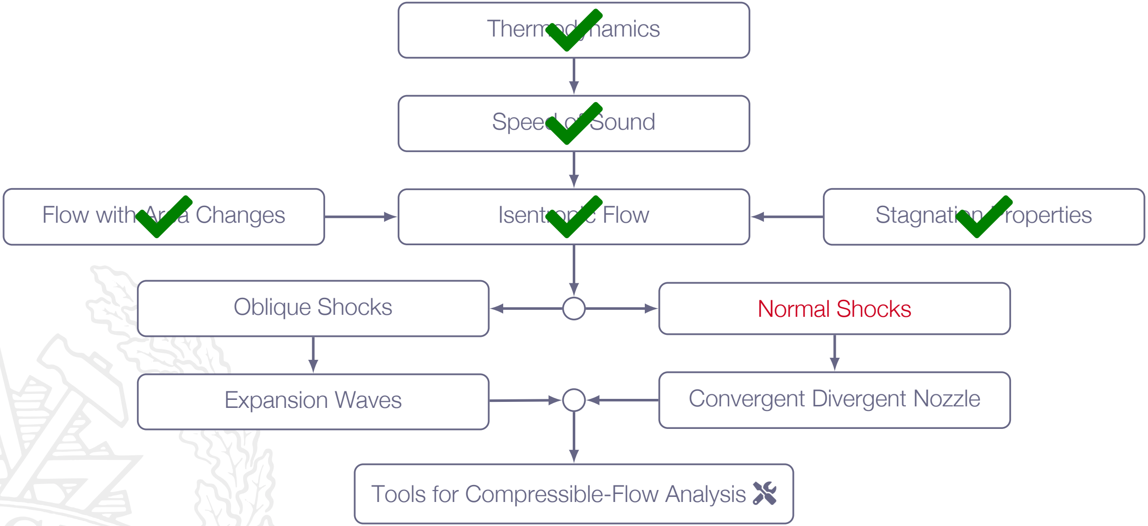
$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma + 1}\right)^{1/(\gamma-1)}$$

$$V^* = \sqrt{\gamma R T^*}$$

$$\frac{T^*}{T_o} = \frac{2}{\gamma + 1}$$

$$\boxed{\dot{m}_{max} = \frac{A^* p_o}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R} \left(\frac{2}{\gamma + 1}\right)^{\frac{(\gamma+1)}{(\gamma-1)}}}}$$

Roadmap - Compressible Flow



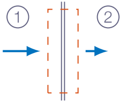
Shock Waves

”Shock waves are nearly discontinuous changes in a supersonic flow”

Reasons for the appearance of shocks in a flow can be for example:

- higher downstream pressure
- sudden changes in flow direction
- blockage by a downstream body
- explosion

Normal Shocks



Continuity:

$$\rho_1 u_1 = \rho_2 u_2$$

Momentum:

$$p_1 - p_2 = \rho_2 u_2^2 - \rho_1 u_1^2$$

Energy:

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2 = h_o$$

The Rankine-Hugoniot relation:

$$h_2 - h_1 = \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right)$$

Normal Shocks

$$h_2 - h_1 = \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right)$$

Note! The Rankine-Hugoniot relation only includes thermodynamic properties (no velocities) and gives a relation between the flow state upstream of the shock and the flow downstream of the shock

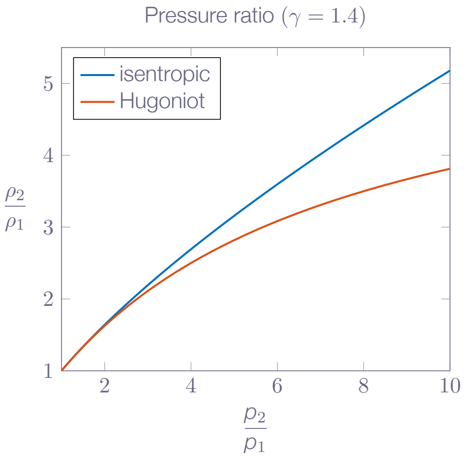
Normal Shocks

The Rankine-Hugoniot relation

$$\frac{\rho_2}{\rho_1} = \frac{1 + \left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right)}{\left(\frac{\gamma+1}{\gamma-1}\right) + \left(\frac{p_2}{p_1}\right)}$$

The isentropic relation

$$\frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{1/\gamma}$$



Normal Shocks

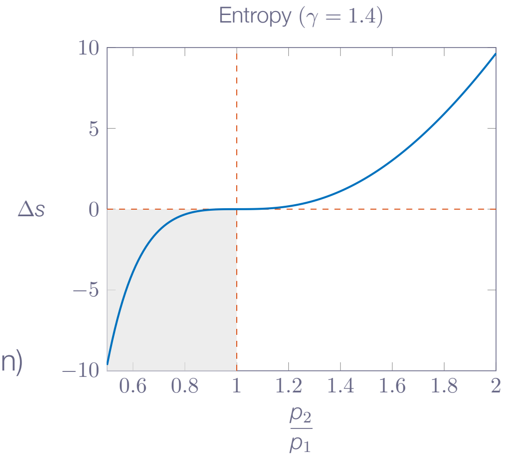
The second law of thermodynamics

$$s_2 - s_1 = C_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

can be rewritten as

$$s_2 - s_1 = C_v \ln \left[\frac{p_2}{p_1} \left(\frac{\rho_1}{\rho_2} \right)^{\gamma} \right]$$

(ρ_2/ρ_1 from the Rankine-Hugoniot relation)



Note! a reduction of entropy is a violation of the second law of thermodynamics

Normal Shocks

For a perfect gas, it is possible to obtain relations for normal shocks that only include upstream variables

Momentum equation: $p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$

$$p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2 = \{\rho_1 u_1 = \rho_2 u_2\} = \rho_1 u_1 (u_1 - u_2) = \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

divide by p_1

$$\frac{p_2}{p_1} = 1 + \frac{\rho_1 u_1^2}{p_1}\left(1 - \frac{u_2}{u_1}\right)$$

$$u_1^2 = M_1^2 a_1^2 = M_1^2 \gamma R T_1 = \gamma M_1^2 \frac{p_1}{\rho_1} \Rightarrow \frac{p_2}{p_1} = 1 + \gamma M_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

Normal Shocks

$$\frac{p_2}{p_1} = 1 + \gamma M_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

Using the energy equation its possible obtain a relation for $\frac{u_2}{u_1}$ (the derivation is quite lengthy though)

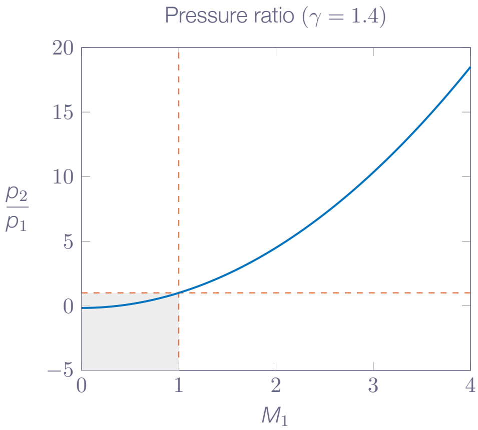
$$\frac{u_2}{u_1} = \frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2}$$

and thus

$$\boxed{\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right)}$$

Normal Shocks

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right)$$



Note! from before we know that p_2/p_1 must be greater than 1.0, which means that M_1 must be greater than 1.0

Normal Shocks

Momentum equation: $p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$

$$M = \frac{u}{a} \Rightarrow p_1 + \rho_1 M_1^2 a_1^2 = p_2 + \rho_2 M_2^2 a_2^2$$

$$a = \sqrt{\gamma RT} = \sqrt{\frac{\gamma p}{\rho}} \Rightarrow p_1 + \rho_1 M_1^2 \frac{\gamma p_1}{\rho_1} = p_2 + \rho_2 M_2^2 \frac{\gamma p_2}{\rho_2}$$

$$p_1 \left(1 + \gamma M_1^2\right) = p_2 \left(1 + \gamma M_2^2\right)$$

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}$$

Normal Shocks

Two ways to calculate the pressure ratio over the shock

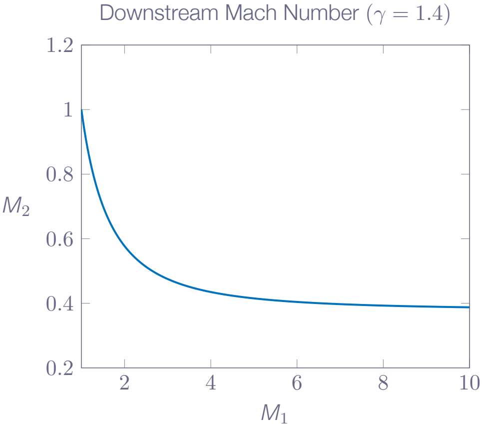
$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right) \qquad \frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}$$

Setting the relations equal gives a relation for the downstream Mach number

$$\boxed{M_2^2 = \frac{(\gamma - 1)M_1^2 + 2}{2\gamma M_1^2 - (\gamma - 1)}}$$

Normal Shocks

$$M_2^2 = \frac{(\gamma - 1)M_1^2 + 2}{2\gamma M_1^2 - (\gamma - 1)}$$

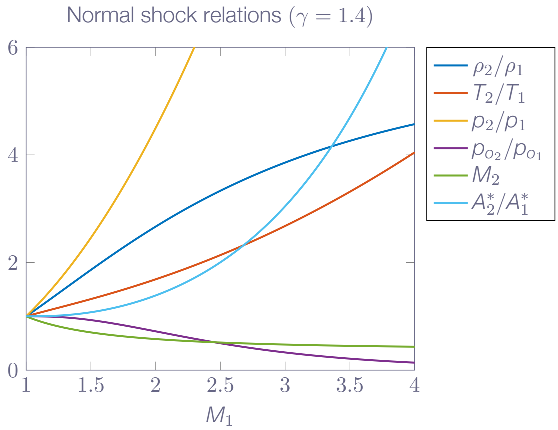


Note! for $\gamma > 1$ and $M_1 > 1$, the downstream Mach number must be less than 1.0, i.e we will always have subsonic flow behind a normal shock

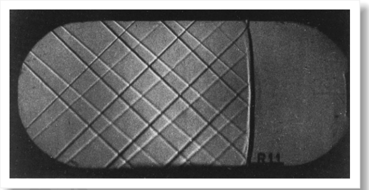
Normal Shocks - Summary

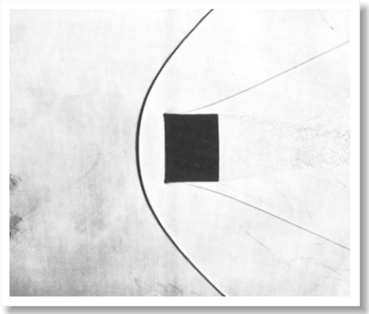
1. Supersonic flow upstream of normal shock
2. Subsonic flow downstream of normal shock
3. Entropy increases over the shock and consequently total pressure decreases
4. Sonic throat area increases
5. Very weak shock waves are nearly isentropic

Normal Shocks



Normal Shocks



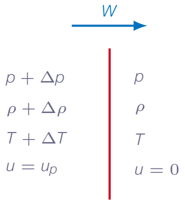


Moving Normal Shocks

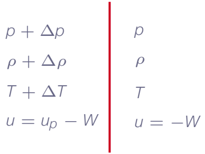
Change frame of reference

- coordinate system moving with the shock
- thermodynamic properties does not change

$$h_2 - h_1 = \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

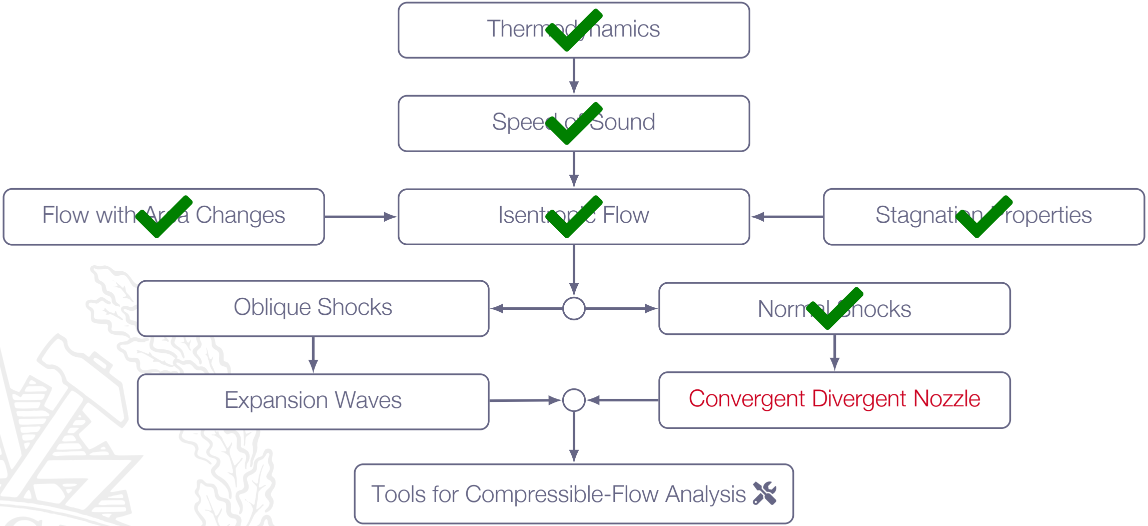


frame of reference:
fixed to the fluid

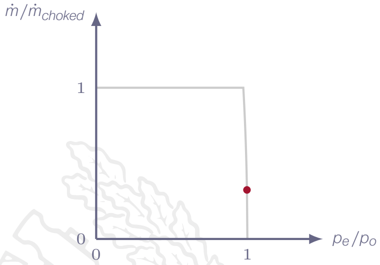


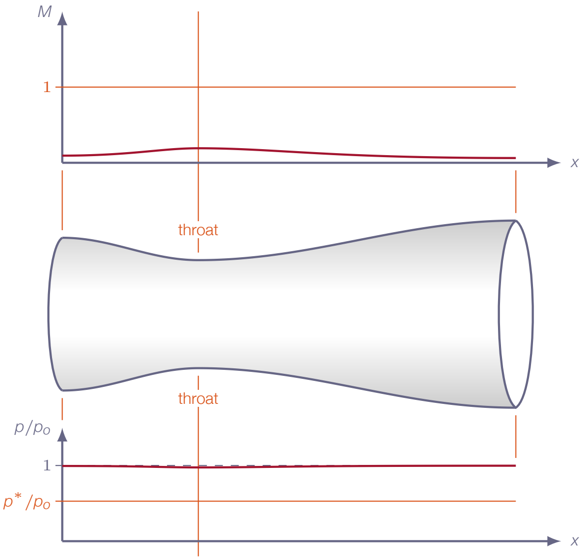
frame of reference:
following the shock

Roadmap - Compressible Flow

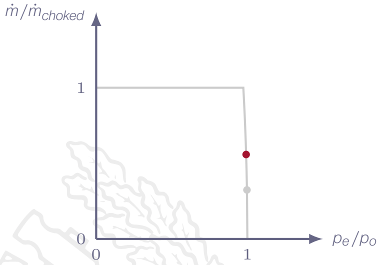


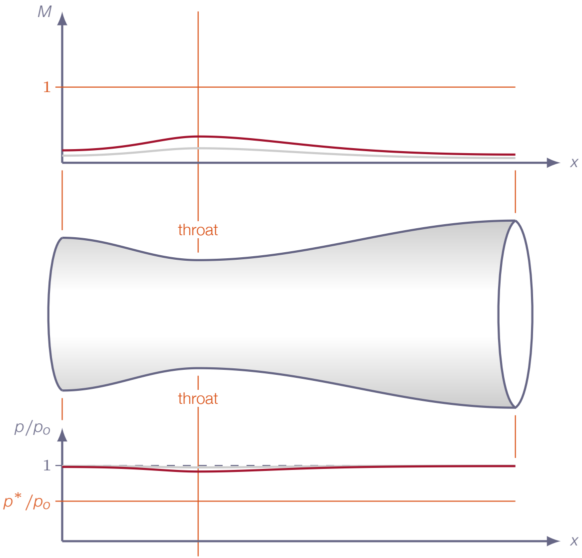
Convergent-Divergent Nozzle





Convergent-Divergent Nozzle





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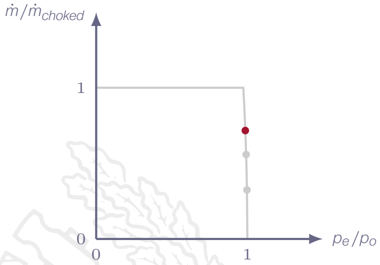

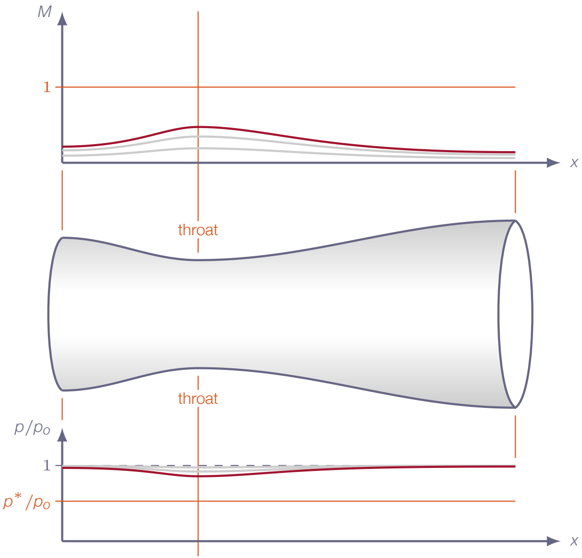

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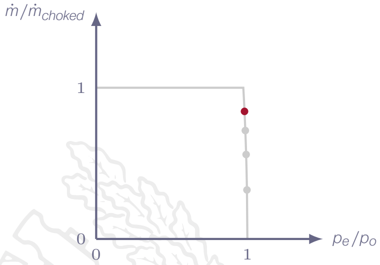

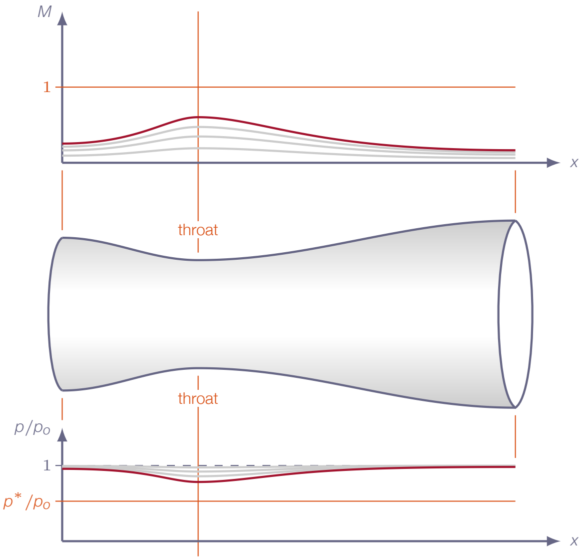

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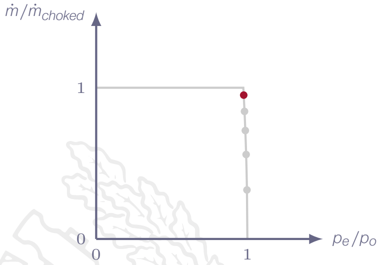

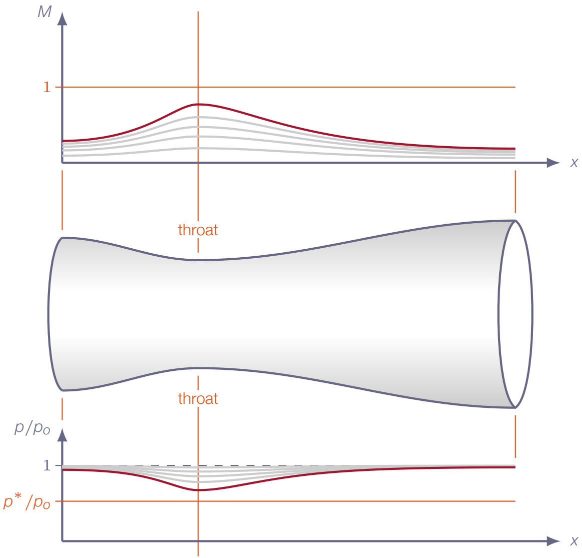

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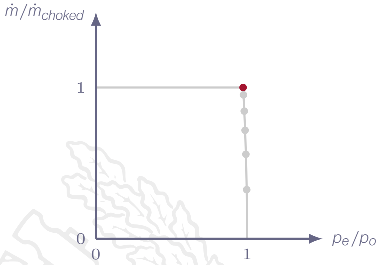

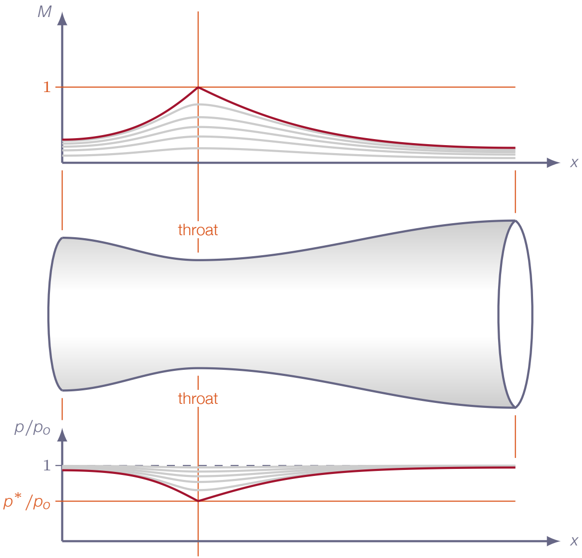

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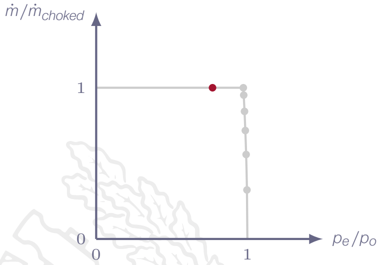

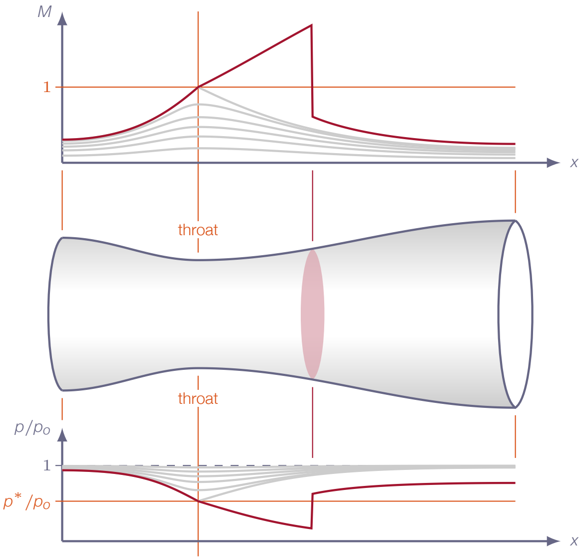

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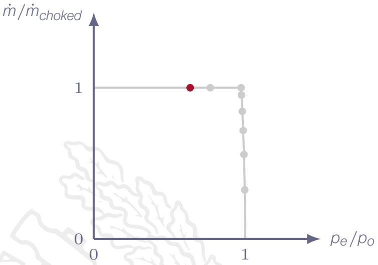

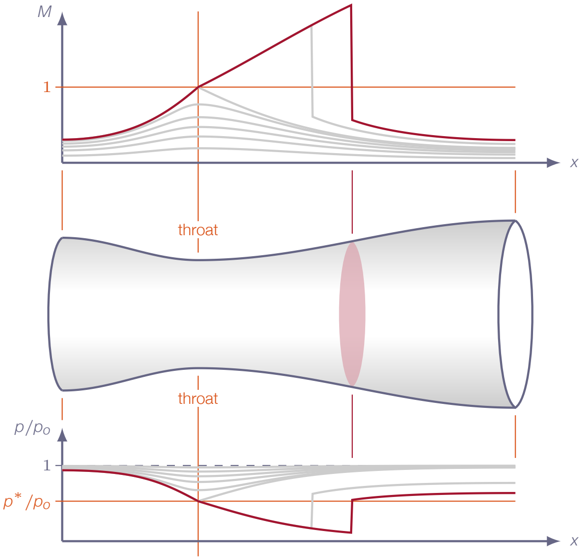

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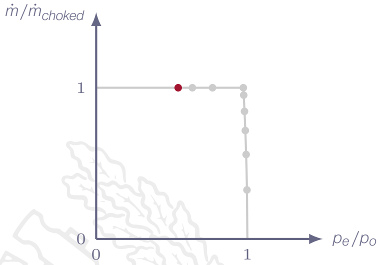
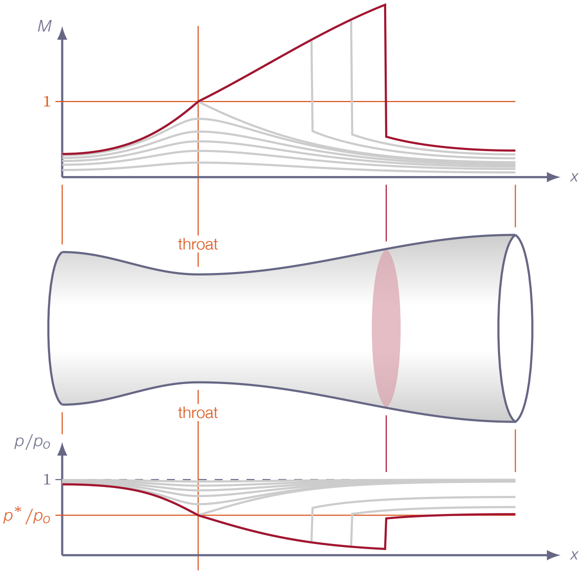
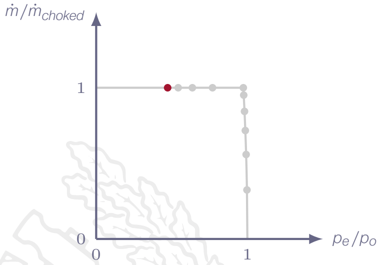
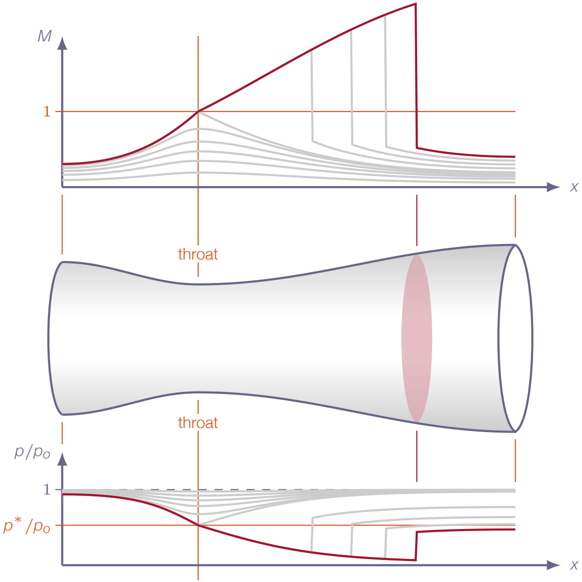




Convergent-Divergent Nozzle





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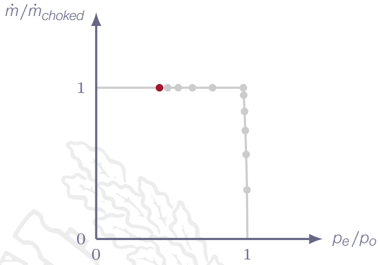
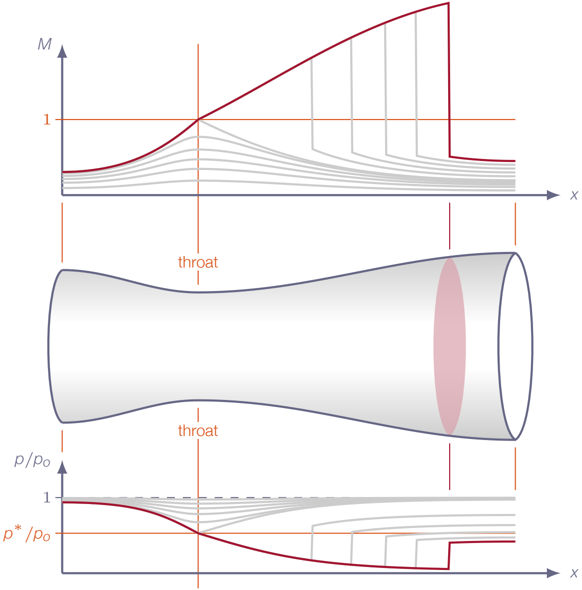
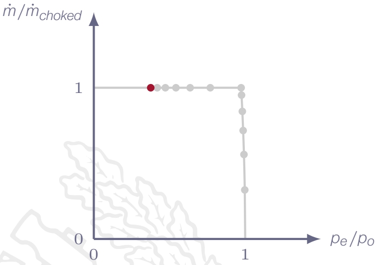
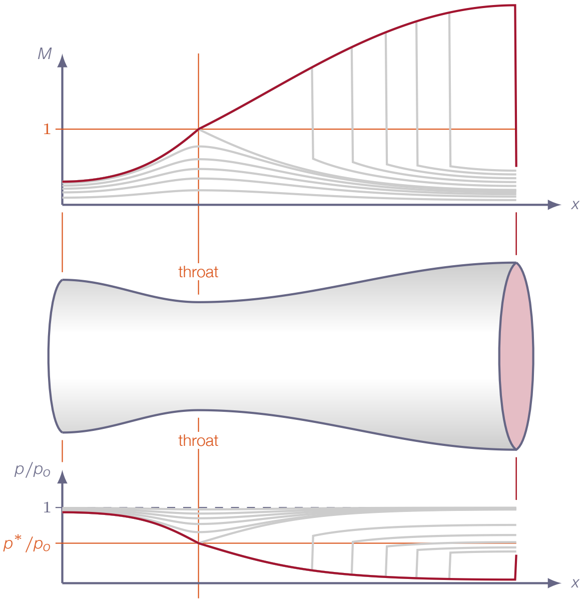


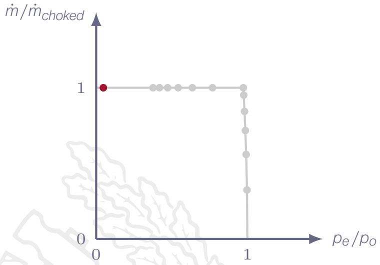


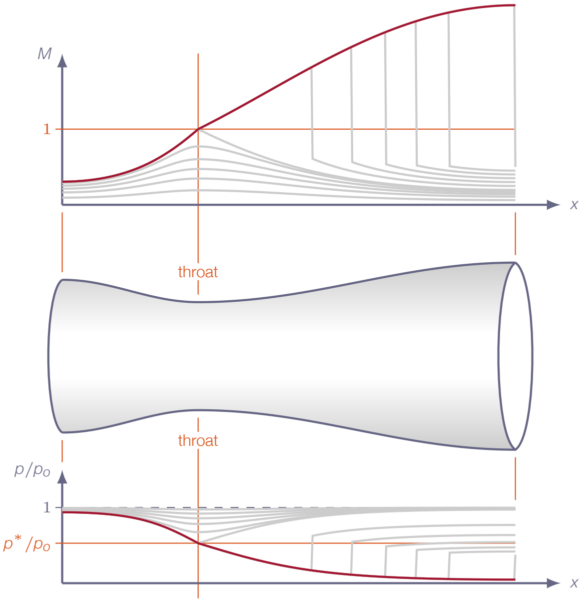
Convergent-Divergent Nozzle



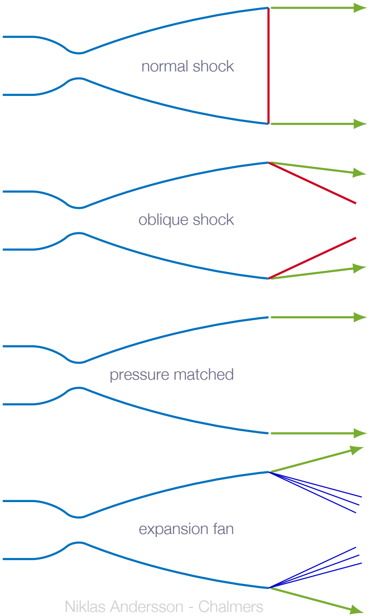


Convergent-Divergent Nozzle





Convergent-Divergent Nozzle



$p_o/p_e = (p_o/p_e)_{ne}$
normal shock at nozzle exit

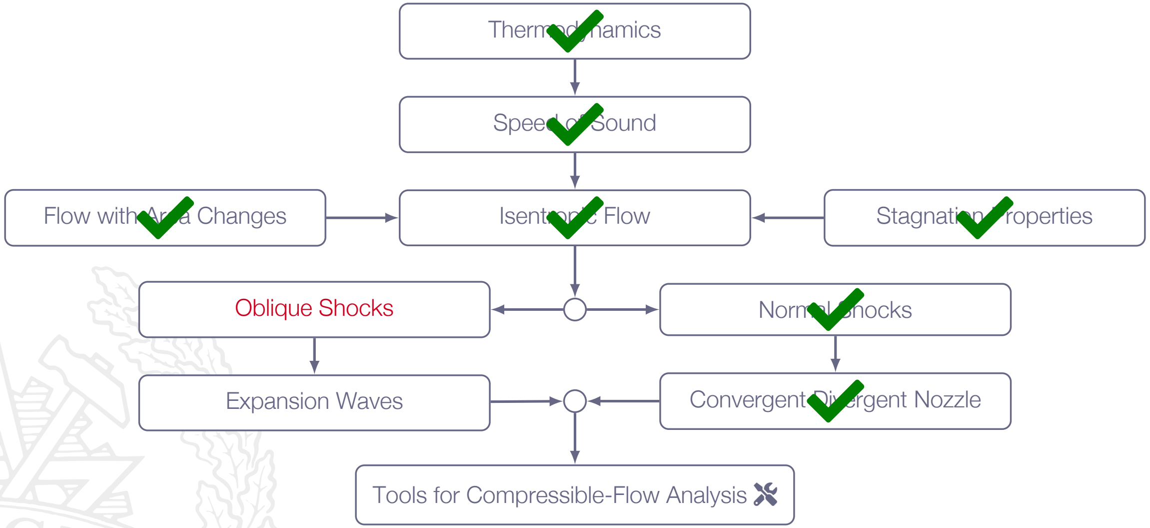
$(p_o/p_e)_{ne} < p_o/p_e < (p_o/p_e)_{sc}$
overexpanded nozzle flow

$p_o/p_e = (p_o/p_e)_{sc}$
pressure matched nozzle flow

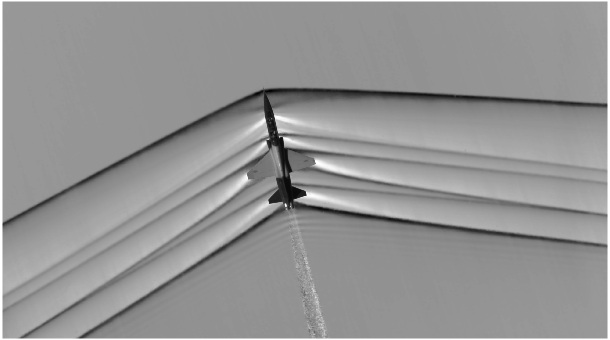
$p_o/p_e > (p_o/p_e)_{sc}$
underexpanded nozzle flow

Convergent-Divergent Nozzle

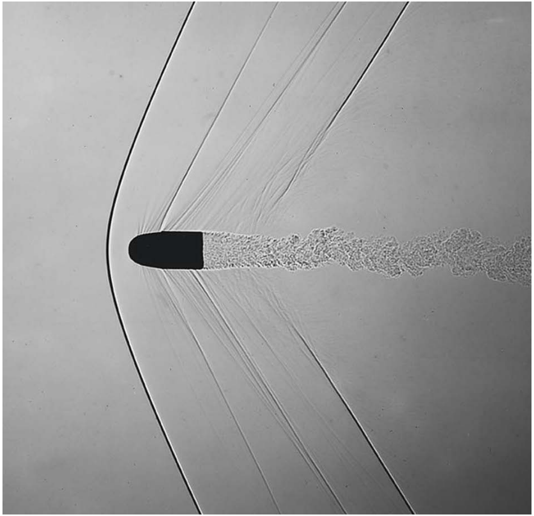
Roadmap - Compressible Flow



Oblique Shocks

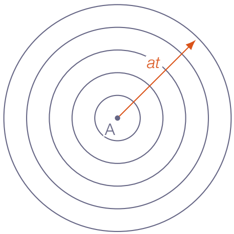


Oblique Shocks

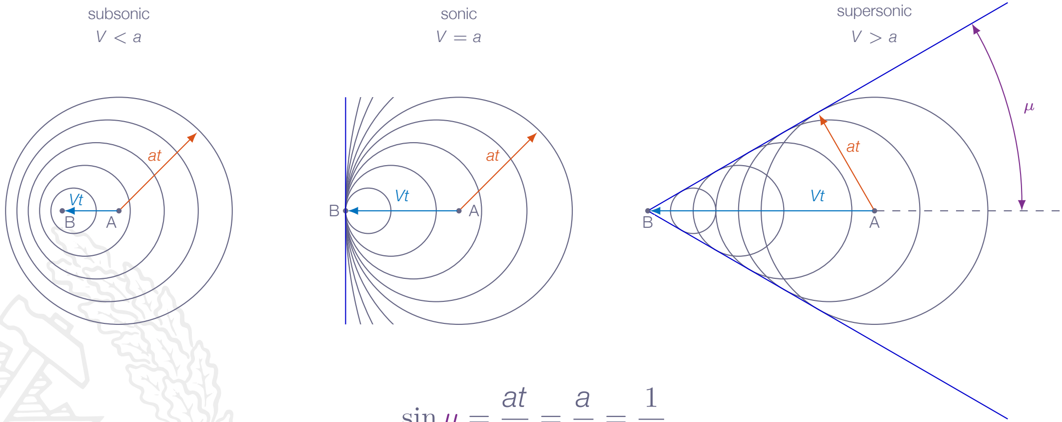


Mach Wave

Sound waves emitted from A (speed of sound a)



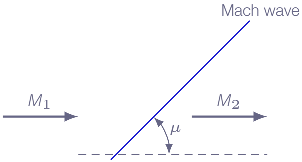
Mach Wave



$$\sin \mu = \frac{at}{Vt} = \frac{a}{V} = \frac{1}{M}$$

Mach Wave

A Mach wave is an infinitely weak oblique shock



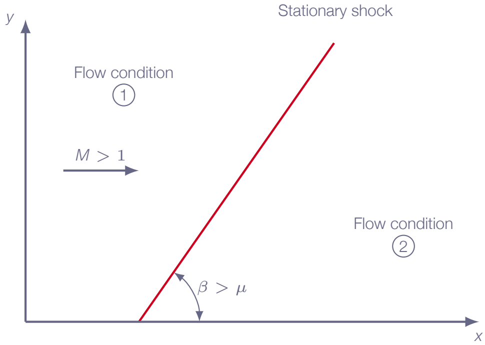
No substantial changes of flow properties over a single Mach wave

$M_1 > 1.0$ and $M_1 \approx M_2$

Isentropic

Oblique Shocks and Mach Waves

Two-dimensional steady-state flow



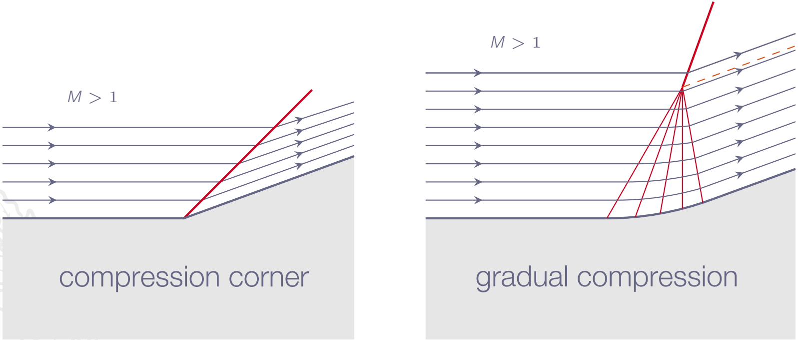
Significant changes of flow properties from 1 to 2

$M_1 > 1.0$, $\beta > \mu$, and $M_1 \neq M_2$

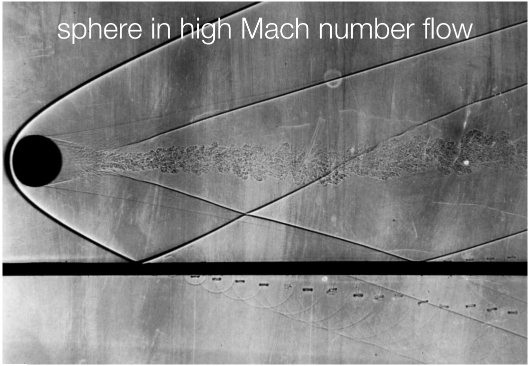
Not isentropic

Oblique Shocks and Mach Waves

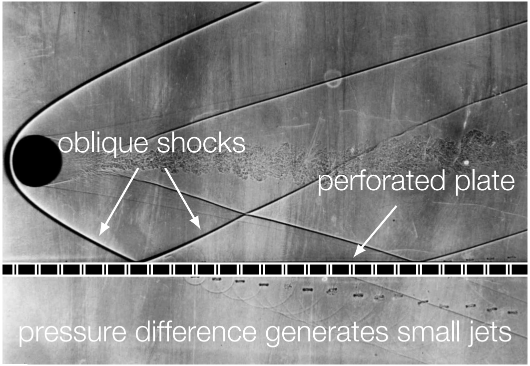
When does an oblique shock appear in a flow?



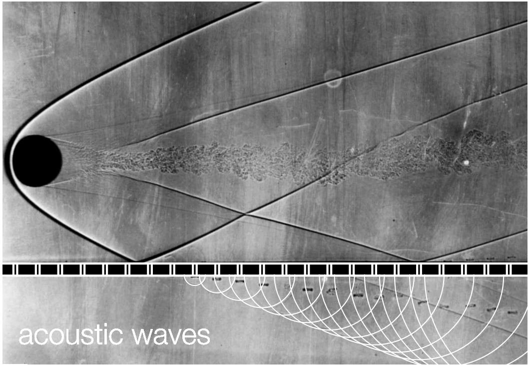
Oblique Shocks and Mach Waves



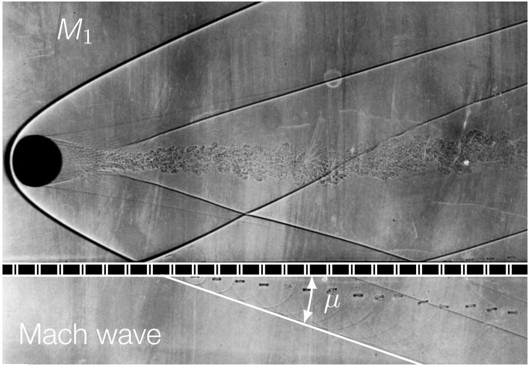
Oblique Shocks and Mach Waves



Oblique Shocks and Mach Waves

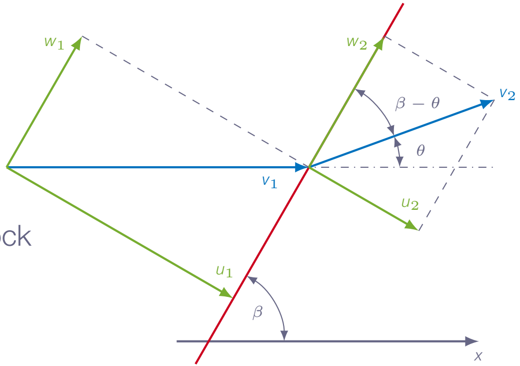


Oblique Shocks and Mach Waves



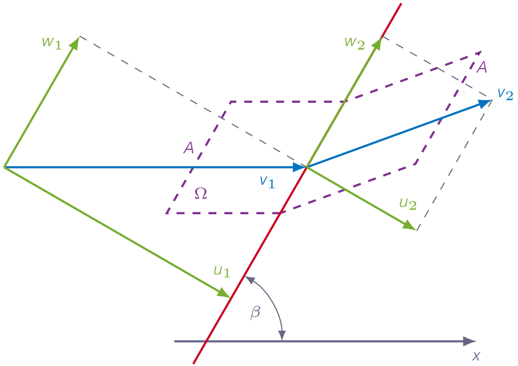
$$\mu = 19^\circ \Rightarrow M_1 = \frac{1}{\sin\mu} \approx 3.1$$

Oblique Shocks



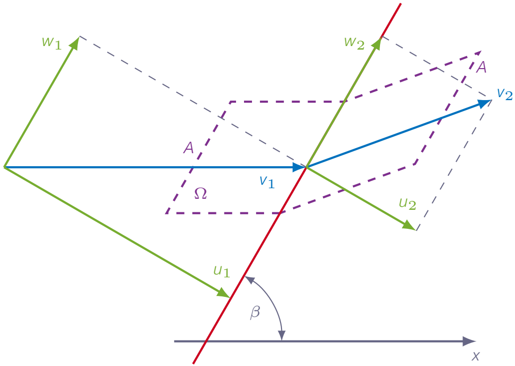
Stationary oblique shock

Oblique Shock Relations



- Two-dimensional steady-state flow
- Control volume aligned with flow stream lines

Oblique Shock Relations

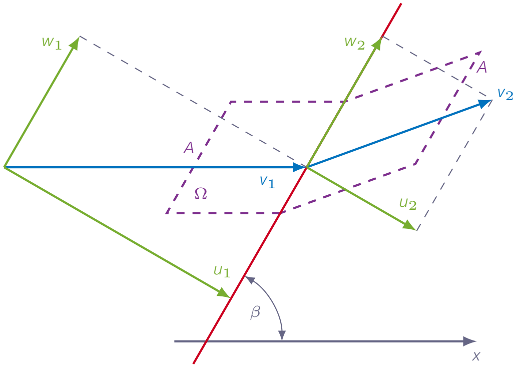


Velocity notations:

$$M_{n_1} = \frac{u_1}{a_1} = M_1 \sin(\beta) \qquad M_1 = \frac{v_1}{a_1}$$

$$M_{n_2} = \frac{u_2}{a_2} = M_2 \sin(\beta - \theta) \qquad M_2 = \frac{v_2}{a_2}$$

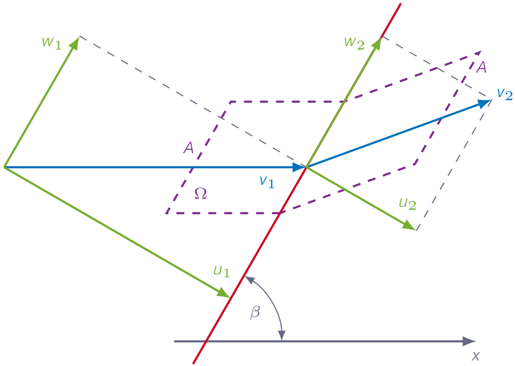
Oblique Shock Relations



Conservation of mass:

$$-\rho_1 u_1 A + \rho_2 u_2 A = 0 \Rightarrow \rho_1 u_1 = \rho_2 u_2$$

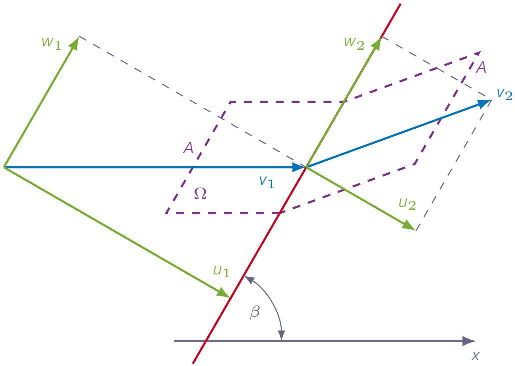
Oblique Shock Relations



Conservation of momentum (shock-normal direction):

$$-(\rho_1 u_1^2 + p_1)A + (\rho_2 u_2^2 + p_2)A = 0 \Rightarrow \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

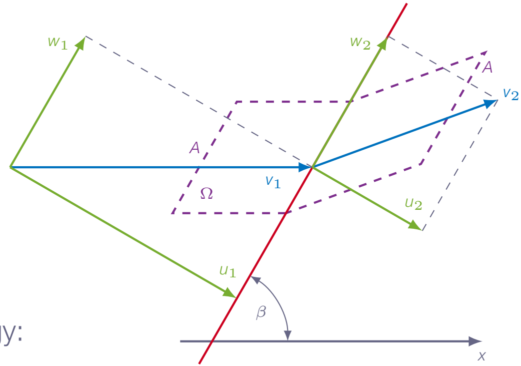
Oblique Shock Relations



Conservation of momentum (shock-tangential direction):

$$-\rho_1 u_1 w_1 A + \rho_2 u_2 w_2 A = 0 \Rightarrow w_1 = w_2$$

Oblique Shock Relations



Conservation of energy:

$$-\rho_1 u_1 [h_1 + \frac{1}{2}(u_1^2 + w_1^2)]A + \rho_2 u_2 [h_2 + \frac{1}{2}(u_2^2 + w_2^2)]A = 0 \Rightarrow h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

Oblique Shock Relations

We can use the same equations as for normal shocks if we replace M_1 with M_{n_1} and M_2 with M_{n_2}

$$M_{n_2}^2 = \frac{M_{n_1}^2 + [2/(\gamma - 1)]}{[2\gamma/(\gamma - 1)]M_{n_1}^2 - 1}$$

Ratios such as ρ_2/ρ_1, p_2/p_1, and T_2/T_1 can be calculated using the relations for normal shocks with M_1 replaced by M_{n_1}

Oblique Shock Relations

What about ratios involving stagnation flow properties, can we use the ones previously derived for normal shocks?

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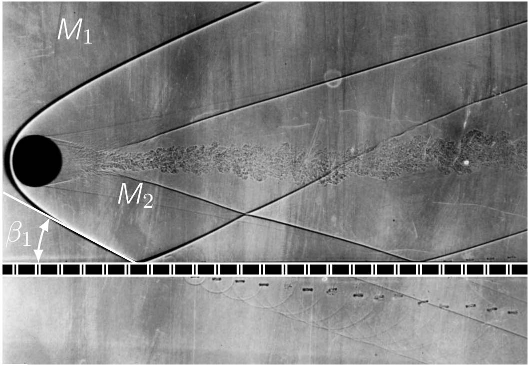
Note! Do not not use ratios involving total quantities, *e.g.* p_{o_2}/p_{o_1}, ρ_{o_2}/ρ_{o_1}, obtained from formulas or tables for normal shock

Oblique Shocks and Mach Waves

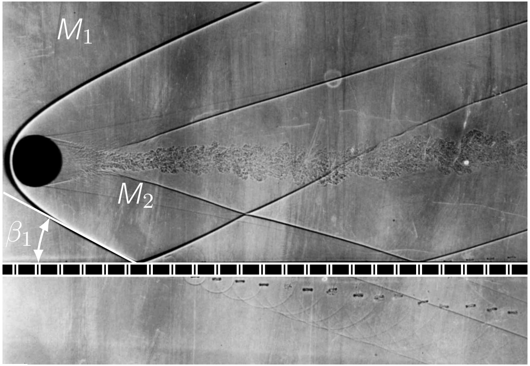


$M_1 > M_2$

$M_2 > 1.0$

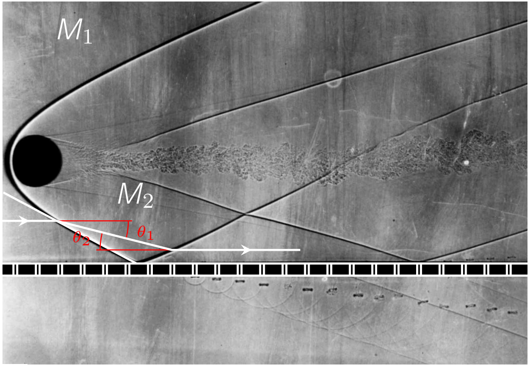
$$\theta_1 = f(M_1, \beta_1), \quad M_2 = f(M_1, \theta_1, \beta_1)$$

Oblique Shocks and Mach Waves



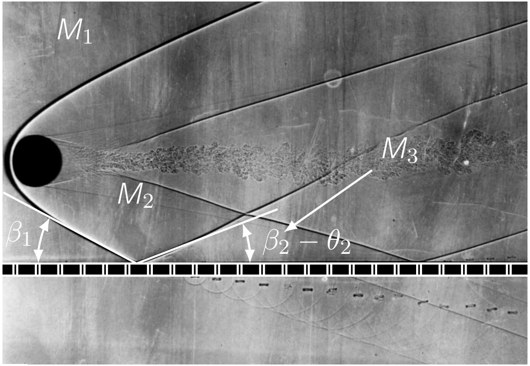
$$\left.\begin{array}{r}\beta_1 = 28^\circ \\ M_1 = 3.1\end{array}\right\} \Rightarrow \theta_1 \approx 11.2^\circ, \quad M_2 \approx 2.5$$

Oblique Shocks and Mach Waves



$$\theta_1 = \theta_2$$

Oblique Shocks and Mach Waves



$$M_1 > M_2 > M_3$$

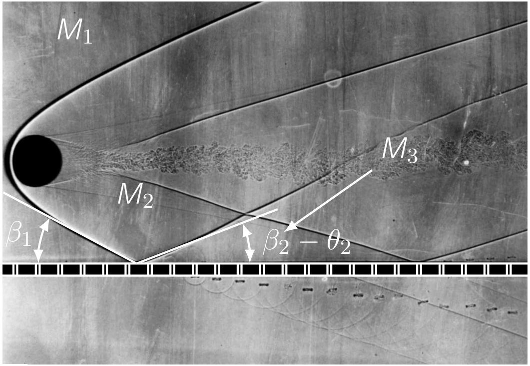
$$M_3 > 1.0$$

$$\beta_2 > \beta_1$$

$$\beta_2 = f(M_2, \theta_2), \quad M_3 = f(M_2, \theta_2, \beta_2)$$

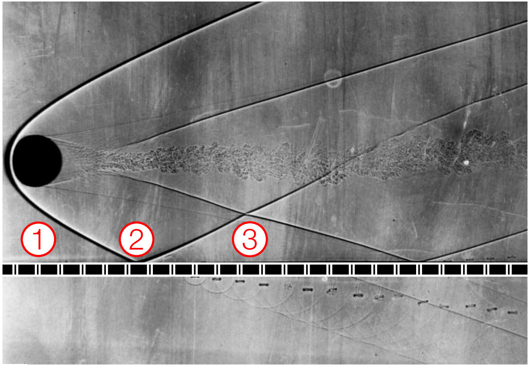
Note! Shock wave reflection at solid wall is **not** specular

Oblique Shocks and Mach Waves



$$\left.\begin{aligned} \theta_2 &= 11.2^\circ \\ M_2 &= 2.5 \end{aligned}\right\} \Rightarrow \beta_2 \approx 33^\circ, \quad M_3 \approx 2.0$$

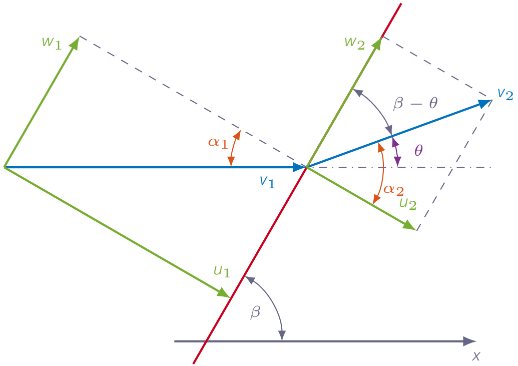
Oblique Shocks and Mach Waves



$$\frac{p_3}{p_1} = \frac{p_2}{p_1}\frac{p_3}{p_2} \approx 4.52$$

$$\frac{T_3}{T_1} = \frac{T_2}{T_1}\frac{T_3}{T_2} \approx 1.57$$

Deflection Angle (*for the interested*)



$$\theta = \alpha_2 - \alpha_1 = \tan^{-1}\left(\frac{w}{u_2}\right) - \tan^{-1}\left(\frac{w}{u_1}\right) \Rightarrow \frac{\partial \theta}{\partial w} = \frac{u_2}{w^2 + u_2^2} - \frac{u_1}{w^2 + u_1^2}$$

Deflection Angle (*for the interested*)

$$\frac{\partial\theta}{\partial w}=\frac{u_2}{w^2+u_2^2}-\frac{u_1}{w^2+u_1^2}=0\Rightarrow$$

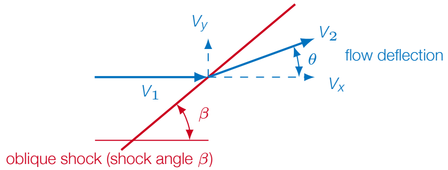
$$\frac{u_2(w^2+u_1^2)-u_1(w^2+u_2^2)}{(w^2+u_2^2)(w^2+u_1^2)}=0\Rightarrow\frac{(u_2-u_1)(w^2-u_1u_2)}{(w^2+u_2^2)(w^2+u_1^2)}=0$$

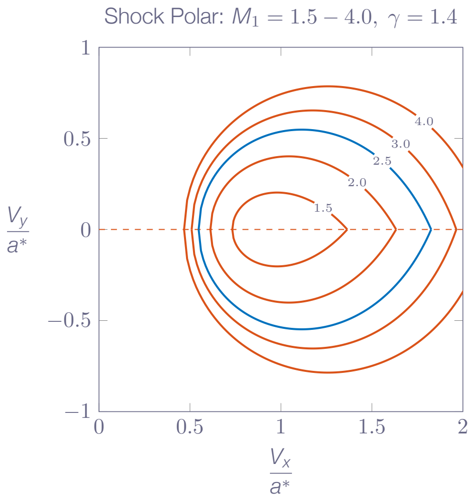
Two solutions:

1. $u_2=u_1$ (no deflection)
2. $w^2=u_1u_2$ (max deflection)

Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number



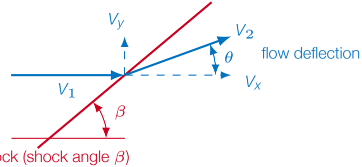


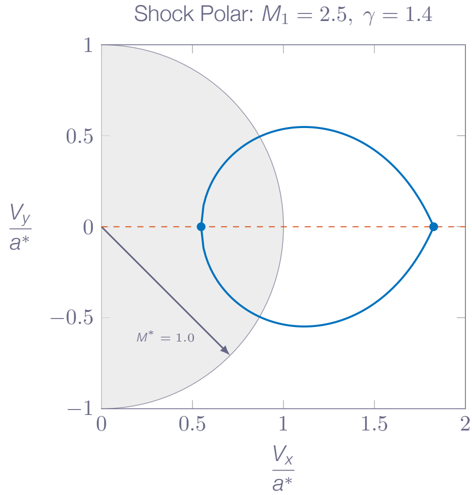
No deflection cases:

- normal shock
  (reduced shock-normal velocity)
- Mach wave
  (unchanged wave-normal velocity)

Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number



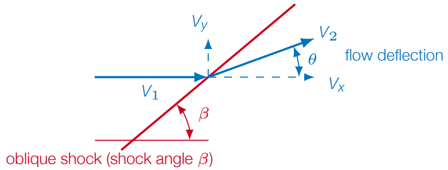


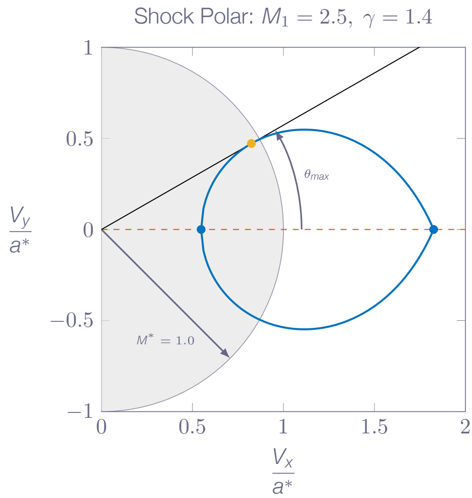
$$M^* = \frac{\sqrt{V_x^2 + V_y^2}}{a^*}$$

Solutions to the left of the sonic line are subsonic

Shock Polar - Flow Deflection - θ_{max}

Graphical representation of all possible deflection angles for a specific Mach number



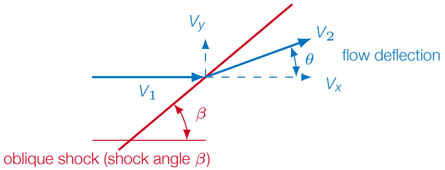


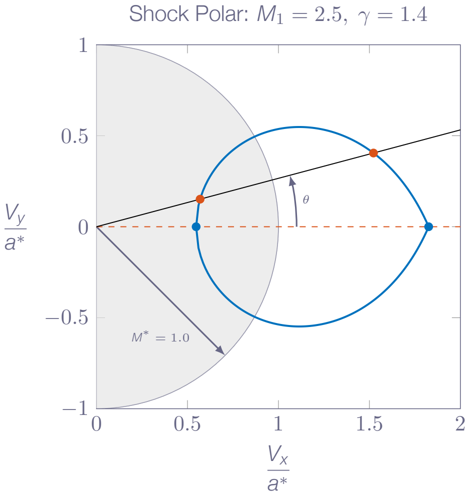
$$\tan\theta = \frac{V_y}{V_x}$$

It is not possible to deflect the flow more than θ_{max}

Shock Polar - Flow Deflection

Graphical representation of all possible deflection angles for a specific Mach number





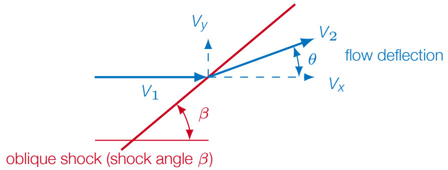
For each deflection angle $\theta < \theta_{max}$, there are two solutions

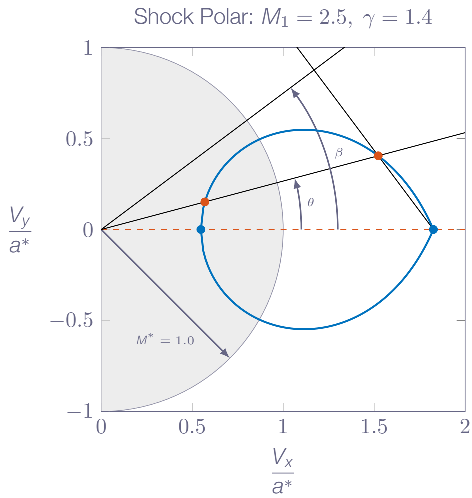
- strong shock solution
- weak shock solution

Weak shocks give lower losses and therefore the preferred solution

Shock Polar - Weak Solution

Graphical representation of all possible deflection angles for a specific Mach number

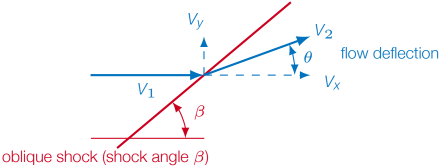


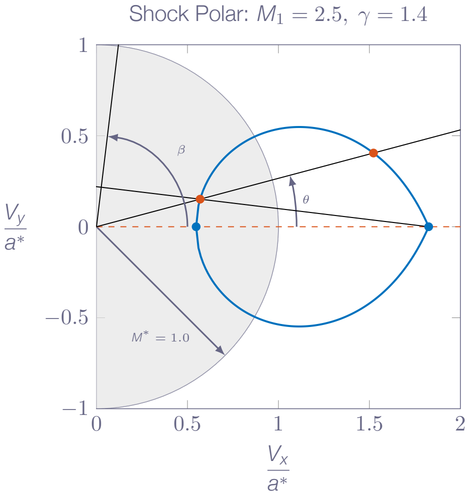


The shock polar can be used to calculate the shock angle β for a given deflection angle θ

Shock Polar - Strong Solution

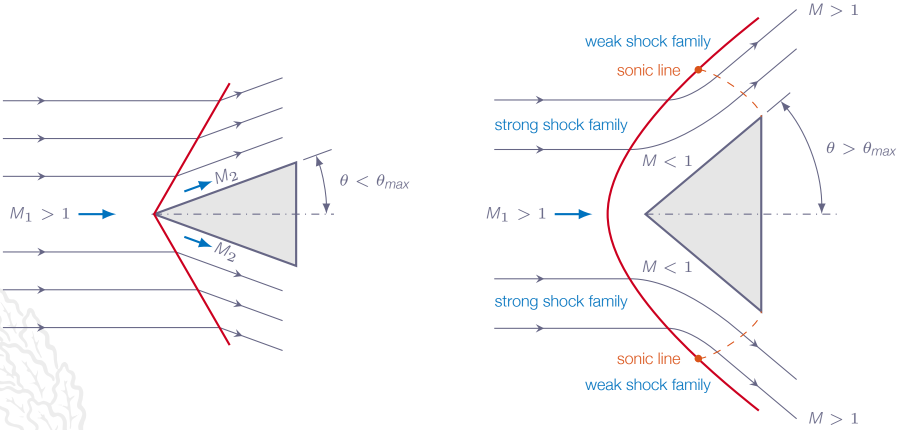
Graphical representation of all possible deflection angles for a specific Mach number





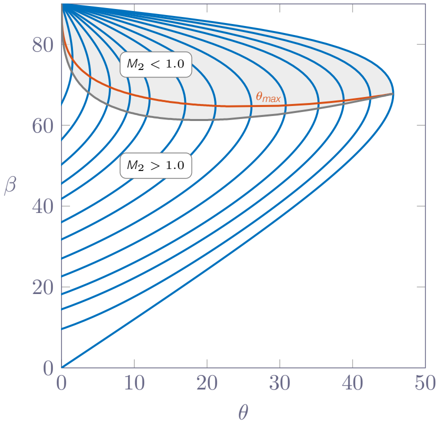
The shock polar can be used to calculate the shock angle β for a given deflection angle θ

Flow Deflection



The θ-β-Mach Relation

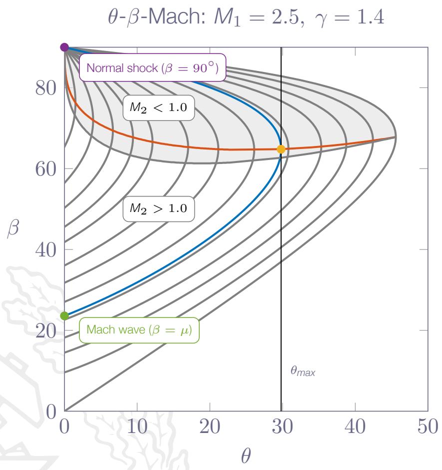
$$\tan(\theta) = \frac{2\cot(\beta)(M_1^2\sin^2(\beta) - 1)}{M_1^2(\gamma + \cos(2\beta)) + 2}$$

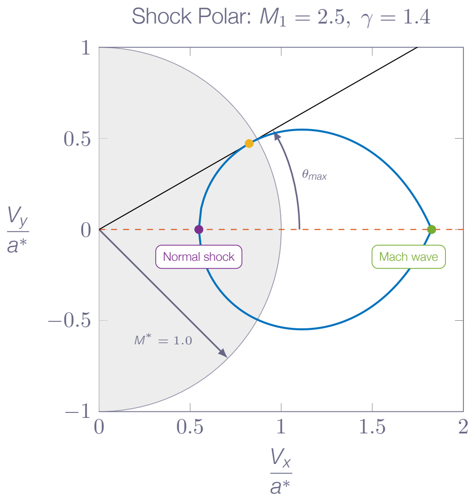


A relation between:

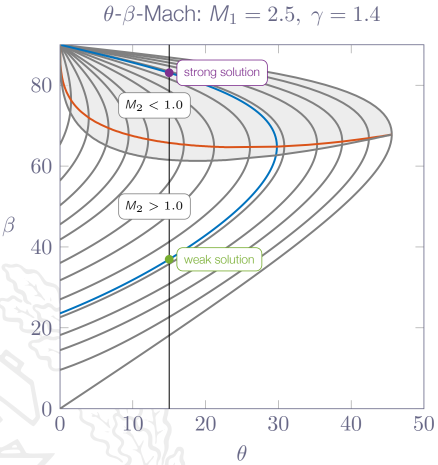
1. flow deflection angle θ
2. shock angle β
3. upstream flow Mach number M_1

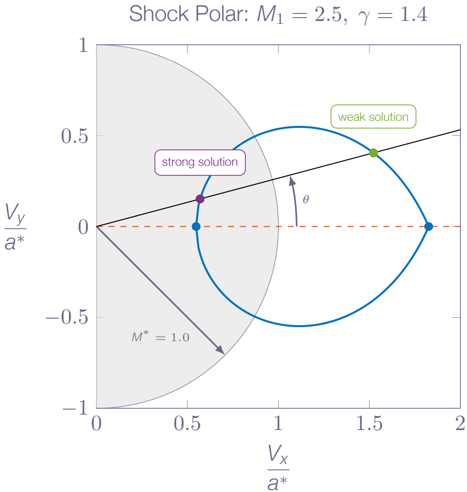
The θ-β-Mach Relation vs. Shock Polar





The θ-β-Mach Relation vs. Shock Polar

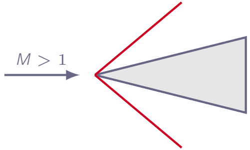




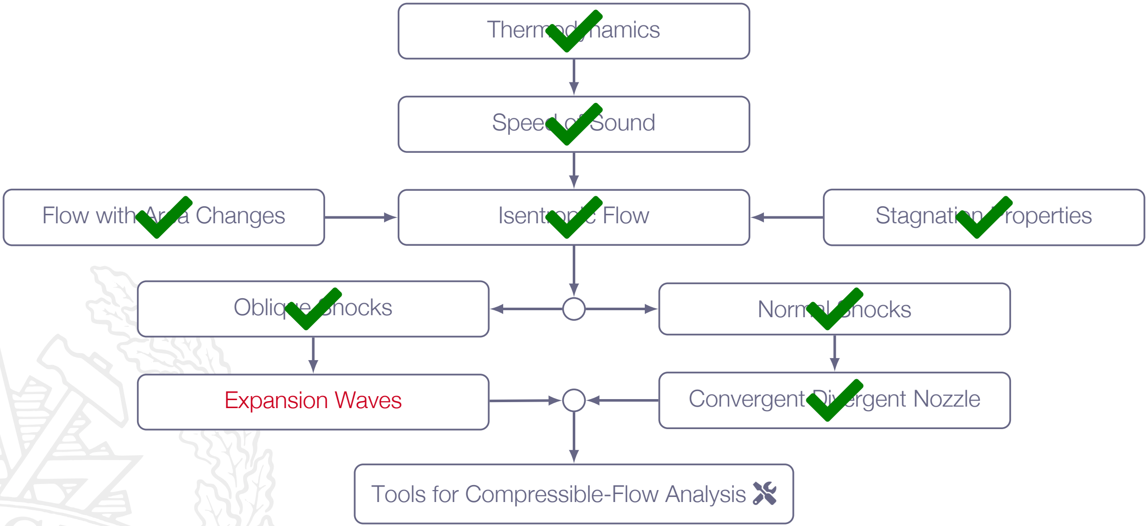
The θ-β-Mach Relation - Wedge Flow

Wedge flow oblique shock analysis:

1. θ-β-M relation $\Rightarrow$ β for given M_1 and θ
2. β gives M_{n_1} according to: $M_{n_1} = M_1 \sin(\beta)$
3. normal shock formula with M_{n_1} instead of M_1 $\Rightarrow$ M_{n_2} (instead of M_2)
4. M_2 given by $M_2 = M_{n_2} / \sin(\beta - \theta)$
5. normal shock formula with M_{n_1} instead of M_1 $\Rightarrow$ ρ_2/ρ_1, p_2/p_1, etc
6. upstream conditions + ρ_2/ρ_1, p_2/p_1, etc $\Rightarrow$ downstream conditions



Roadmap - Compressible Flow

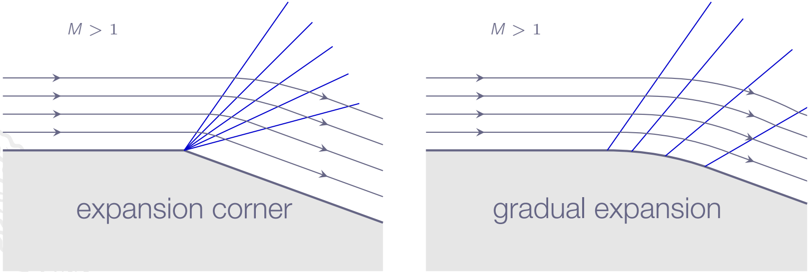


Expansion Waves

- Gradual change of flow angle
- Increasing flow area
- Increasing Mach number
- Accumulation of infinitesimal flow deflections - isentropic

Expansion Waves

What is an expansion wave or expansion region?



The Prandtl-Meyer Function

- The change of flow properties over an expansion region can be calculated using the Prandtl-Meyer function
- The Prandtl-Meyer function derivation is based on the fact that each expansion wave gives an infinitesimal change in flow angle and flow properties

Prandtl-Meyer Function Derivation (*for the interested*)

For small deflection angles, linearization of the θ-β-Mach relation gives

$$\frac{dp}{p} \approx \frac{\gamma M^2}{(M^2-1)^{1/2}} d\theta$$

The momentum equation for inviscid flows gives

$$dp = -d(\rho V^2) = -\rho V dV - V \underbrace{d(\rho V)}_{=0} = -\rho V dV = -\rho V^2 \frac{dV}{V} = -\rho a^2 M^2 \frac{dV}{V} \Rightarrow$$

$$\Rightarrow \{\rho a^2 = \rho\gamma RT = \gamma p\} \Rightarrow \frac{dp}{p} = -\gamma M^2 \frac{dV}{V}$$

Prandtl-Meyer Function Derivation (*for the interested*)

Now, setting the two expressions for dp/p equal

$$-\gamma M^2 \frac{dV}{V} = \frac{\gamma M^2}{(M^2-1)^{1/2}} d\theta \Rightarrow d\theta = -(M^2-1)^{1/2}\frac{dV}{V}$$

$$V = Ma \Rightarrow dV = adM + Mda \Rightarrow \frac{dV}{V} = \frac{dM}{M} + \frac{da}{a}$$

$$d\theta = -(M^2-1)^{1/2}\left(\frac{dM}{M} + \frac{da}{a}\right)$$

Prandtl-Meyer Function Derivation (*for the interested*)

$$d\theta = -(M^2-1)^{1/2}\left(\frac{dM}{M}+\frac{da}{a}\right)$$

$$\frac{a_o}{a} = \left(1+\frac{\gamma-1}{2}M^2\right)^{1/2}$$

$$da = \left(1+\frac{\gamma-1}{2}M^2\right)^{-1/2} da_o + a_o d\left[\left(1+\frac{\gamma-1}{2}M^2\right)^{-1/2}\right]$$

$$\text{isentropic} \Rightarrow \; da_o = 0$$

$$\frac{da}{a} = \frac{d\left[\left(1+\frac{\gamma-1}{2}M^2\right)^{-1/2}\right]}{\left(1+\frac{\gamma-1}{2}M^2\right)^{-1/2}} = \frac{-\frac{1}{2}\left(1+\frac{\gamma-1}{2}M^2\right)^{-3/2}(\gamma-1)MdM}{\left(1+\frac{\gamma-1}{2}M^2\right)^{-1/2}}$$

Prandtl-Meyer Function Derivation (*for the interested*)

$$d\theta = -(M^2-1)^{1/2}\left(\frac{dM}{M}+\frac{da}{a}\right)$$

$$\frac{da}{a} = \frac{-\frac{1}{2}(\gamma-1)MdM}{1+\frac{\gamma-1}{2}M^2} \Rightarrow d\theta = -\frac{2(M^2-1)^{1/2}}{2+(\gamma-1)M^2}\frac{dM}{M}$$

Introducing ω defined such that: $d\omega = -d\theta$, $\omega = 0$ when $M = 1$

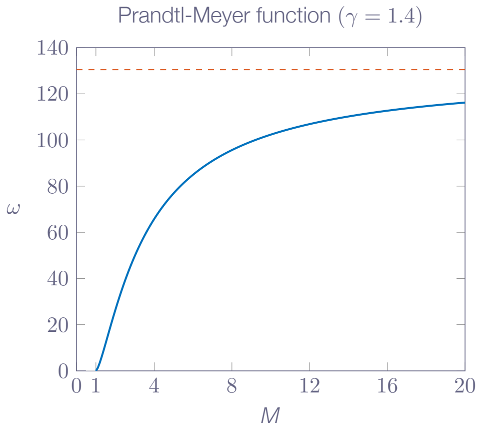
$$\int_0^{\omega} d\omega = \int_1^{M} \frac{2(M^2-1)^{1/2}}{2+(\gamma-1)M^2}\frac{dM}{M}$$

$$\omega(M) = \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}\tan^{-1}\left(\frac{M^2-1}{(\gamma+1)/(\gamma-1)}\right)^{1/2} - \tan^{-1}(M^2-1)^{1/2}$$

The Prandtl-Meyer Function

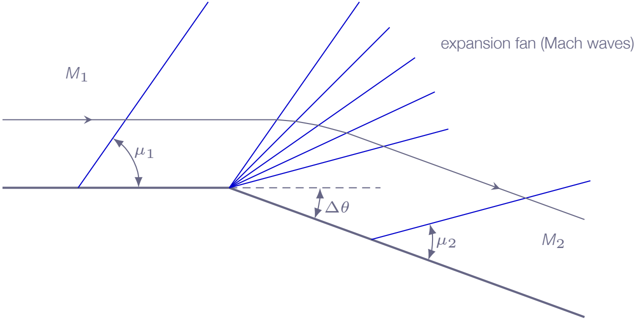
$$\omega(M) = \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2} \tan^{-1}\left(\frac{M^2-1}{(\gamma+1)/(\gamma-1)}\right)^{1/2} - \tan^{-1}(M^2-1)^{1/2}$$

$$\omega(M)|_{M\to\infty} = 130.45^\circ$$



Prandtl-Meyer Expansion Waves

Example:



1. $\theta_1 = 0$, $M_1 > 1$ is given
2. θ_2 is given
3. find M_2 such that $\Delta\theta = \theta_2 - \theta_1 = \omega(M_2) - \omega(M_1)$

Prandtl-Meyer Expansion Waves

Since the flow is isentropic, the isentropic relations apply:

(T_o and p_o are constant)

Calorically perfect gas:

$$\frac{T_o}{T} = \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]$$

$$\frac{p_o}{p} = \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{\frac{\gamma}{\gamma-1}}$$

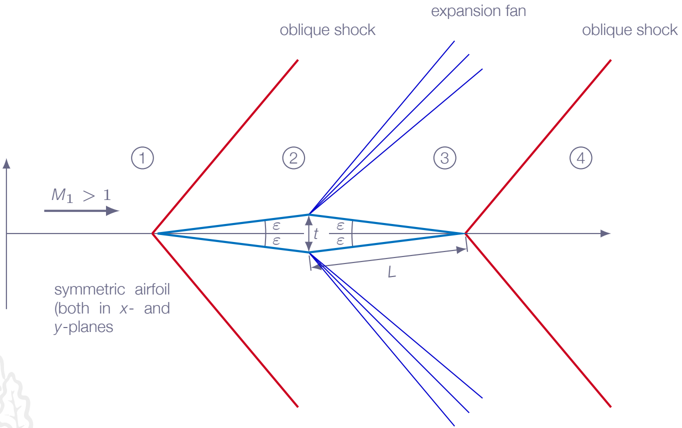
Prandtl-Meyer Expansion Waves

since $T_{o_1} = T_{o_2}$ and $p_{o_1} = p_{o_2}$

$$\frac{T_1}{T_2} = \frac{T_{o_2}}{T_{o_1}}\frac{T_1}{T_2} = \left(\frac{T_{o_2}}{T_2}\right) \Big/ \left(\frac{T_{o_1}}{T_1}\right) = \left[\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right]$$

$$\frac{p_1}{p_2} = \frac{p_{o_2}}{p_{o_1}}\frac{p_1}{p_2} = \left(\frac{p_{o_2}}{p_2}\right) \Big/ \left(\frac{p_{o_1}}{p_1}\right) = \left[\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right]^{\frac{\gamma}{\gamma-1}}$$

Diamond-Wedge Airfoil



Note! symmetric airfoil at zero incidence ⇒ zero lift but what about drag?

Diamond-Wedge Airfoil

1-2 standard oblique shock calculation for flow deflection angle ε and upstream Mach number M_1

2-3 Prandtl-Meyer expansion for flow deflection angle 2ε and upstream Mach number M_2

3-4 standard oblique shock calculation for flow deflection angle ε and upstream Mach number M_3

Diamond-Wedge Airfoil - Wave Drag

Since conditions 2 and 3 are constant in their respective regions, we obtain:

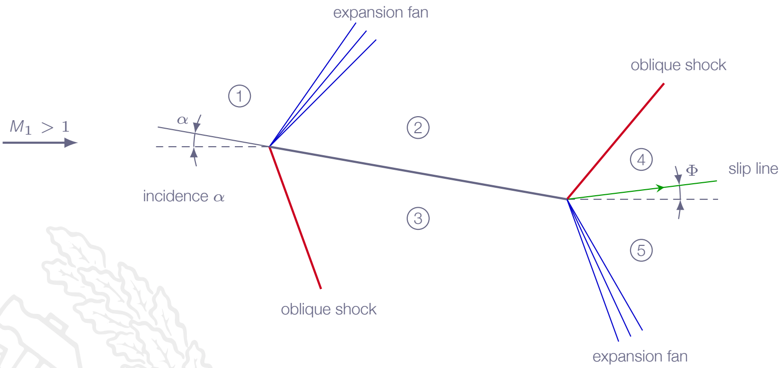
$$D = 2\left[p_2 L \sin(\varepsilon) - p_3 L \sin(\varepsilon)\right] = 2(p_2 - p_3)\frac{t}{2} = (p_2 - p_3)t$$

For supersonic free stream ($M_1 > 1$), with shocks and expansion fans according to figure we will always find that $p_2 > p_3$

which implies $D > 0$

Wave drag (drag due to flow loss at compression shocks)

Flat-Plate Airfoil



Flat-Plate Airfoil

It seems that the angle of the flow downstream of the flat plate would be different than the angle of the flow upstream of the plate. Can that really be correct?!

Flat-Plate Airfoil

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For the flow in the vicinity of the plate this is the correct picture. Further out from the plate, shock and expansion waves will interact and eventually sort the mismatch of flow angles out

Flat-Plate Airfoil

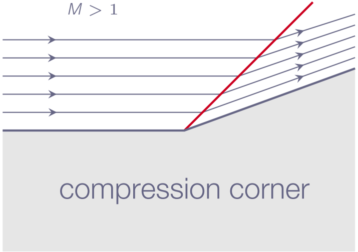
- Flow states 4 and 5 must satisfy:
  - $p_4 = p_5$
  - flow direction 4 equals flow direction 5 (Φ)

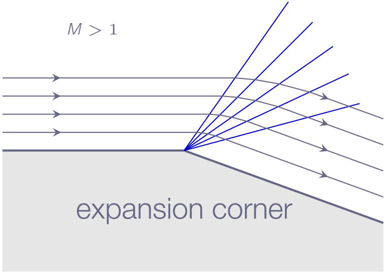
- Shock between 2 and 4 as well as expansion fan between 3 and 5 will adjust themselves to comply with the requirements

- For calculation of lift and drag only states 2 and 3 are needed

- States 2 and 3 can be obtained using standard oblique shock formulas and Prandtl-Meyer expansion

Oblique Shocks and Expansion Waves

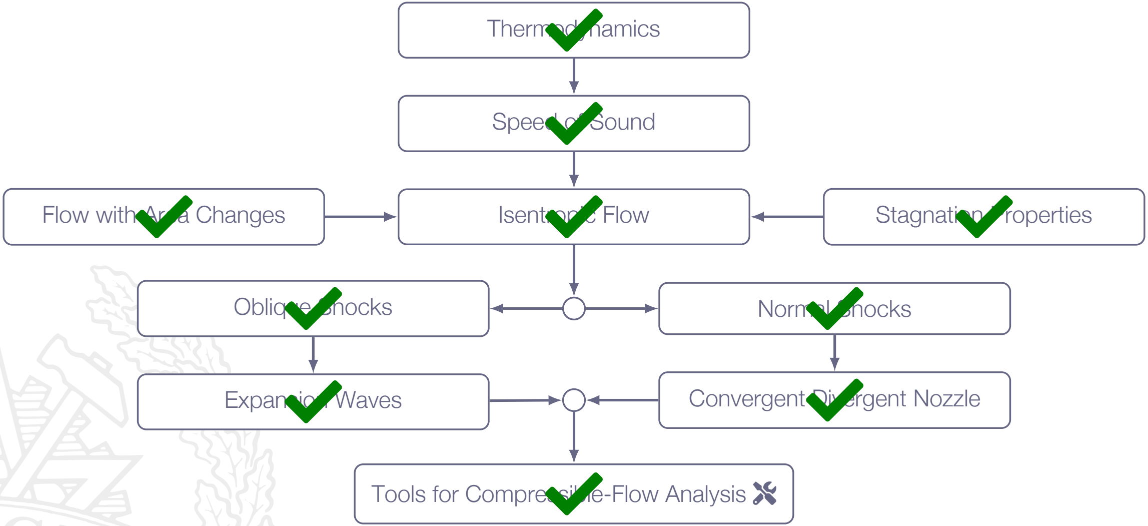




M	decrease
V	decrease
p	increase
ρ	increase
T	increase

M	increase
V	increase
p	decrease
ρ	decrease
T	decrease

Roadmap - Compressible Flow



Supersonic Stereo

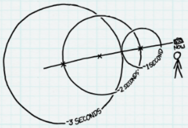
What if you somehow managed to make a stereo travel at twice the speed of sound, would it sound backwards to someone who was just casually sitting somewhere as it flies by?

—Tim Currie

Yes.

Technically, anyway. It would be pretty hard to hear.

The basic idea is pretty straightforward. The stereo is going faster than its own sound, so it will reach you first, followed by the sound it emitted one second ago, followed by the sound it emitted two seconds ago, and so forth.



The problem is that the stereo is moving at Mach 2, which means that two seconds ago, it was over a kilometer away. It's hard to hear music from that distance, particularly when your ears were just hit by (a) a sonic boom, and (b) pieces of a rapidly disintegrating stereo.

Wind speeds of Mach 2 would messily disassemble most consumer electronics. The force of the wind on the body of the stereo is roughly comparable to that of a dozen people standing on it: