Fluid Mechanics MTF053

Turbulence

Based on B. Sundén 1998 *Kompendium i värmeöverföring*

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1 Introduction

Most flows in nature and engineering applications are turbulent such as for example

- the atmospheric boundary layer
- the flow over aircraft wings
- water flow in rivers, pipes and channels
- wake flows behind ships, cars and airplanes

In the following turbulence will be discussed and the governing equations for turbulent flows will be derived.

1.1 Turbulence Characteristics

It is impossible to give an exact definition of turbulence.

1. Turbulent flows are **unsteady** (time dependent)
2. Fluid motion is **irregular**, and the flow structures shows random variations in time and space, which makes it impossible to describe the motion in detail as a function of the space coordinates (x, y, z) and time t. Fortunately, the turbulent motion is irregular in such a way that statistical methods can be applied. It is possible to defined averages of flow properties such as velocity, pressure, and temperature.
3. Turbulent flows are **three dimensional** and **rotational**.
4. Turbulent flow has **high rotational intensity** and unsteady vortical structures in a wide range of sizes occurs in the flow at all times
5. Turbulent flows appear at **high Reynolds numbers**. Small instabilities in a laminar flow develops to turbulence if the Reynolds number is high enough.
6. The turbulence is a **continuous** phenomenon, i.e., the smallest turbulence motions are significantly larger than the mean free path of the molecules
7. Turbulence is **diffusive** which means that velocity fluctuations are spread in the surrounding fluid and thus there will be an increase of heat transfer, increased exchange of momentum, and higher resistance to separation in a turbulent flow than in a laminar flow.
8. Turbulent flows are always **dissipative**. Friction in the flow converts turbulence kinetic energy into internal energy of the fluid. Therefore, the turbulent motion needs continuous addition of energy to compensate for the viscous losses. If no energy is added, the turbulence motions will decay.
9. **Turbulence is not a property of the fluid, it is a property of the flow**

1.2 Methods for Analyzing Turbulent Flows

Turbulent flows have been studied for more than hundred years but no general solution to the turbulence problems exists. The flow equations have been studied in detail, but it is still hard, if not impossible, to do a quantitative analysis of a turbulent flow without using empiric data. Statistical methods applied to the flow equations always leads to the introduction of new unknowns. With the number of equations kept the same this leads to the greatest obstacle of turbulence flow analysis – **the closure problem**. In order to end up with a system of equations where the number of unknowns equals the number of equations, a number of assumptions must be made. We will get back to this later.

A useful method is the so-called dimensional analysis. In many cases it is possible to stipulate that structures in a turbulent flow only depends on a few variables or parameters. In such cases, dimensional analysis often results in a relationship between dependent and independent variables, which gives a solution that is almost known. The only thing needed is a value for a numeric constant. An example is the so-called energy spectra for the turbulence kinetic energy.

Another method is to try to find asymptotic characteristics of the turbulent flow. Turbulent flow is characterized by very high Reynolds numbers. It is therefore reasonable to require that a description of the turbulence should behave in as specific way when the Reynolds number is increased to infinity. This is a requirement that enables specific results. On example is the theory for turbulent boundary layers. Turbulent flows tend to be almost independent of the fluid viscosity (with exception for the smallest scales). The asymptotic behavior leads to so called **Reynolds number similarity** (asymptotic invariance).

The idea of **self-preservation** or local invariance is related to, but not the same, as the above-mentioned asymptotic invariance. For simple flow geometries, the turbulent motions in a specific location (space and time) is a to a high degree a function of the immediate surroundings. It is thus plausible to assume the turbulence to be dynamically similar everywhere if scaled with local length and time scales.

1.3 Length Scales in Turbulent Flows

In turbulent flows there are a wide spectrum of length scales present. The largest eddies are in the same order of magnitude as the dimension of the flowfield.

The size of the smallest eddies is set by viscous effects. The smaller the eddies, the higher the velocity gradients inside of the eddies. Since dissipation of turbulence kinetic energy is proportional to the viscosity and the velocity gradient, it can be assumed that the smallest eddies are responsible of the dissipation of turbulence kinetic energy.
In the same way as there are different length scales, there will be a wide range of time scales in a turbulent flow.

1.4 Transition to Turbulence

Laminar flows becomes unstable at high Reynolds numbers and eventually transition to turbulence occurs. Pipe flows becomes turbulent at a Reynolds number of

$$Re_D = \frac{U_{av} D}{\nu} \approx 2300$$

and a boundary layer over a flat plate becomes turbulent at a Reynolds number of

$$Re_x = \frac{U_\infty x}{\nu} \approx 3 \times 10^5 - 3 \times 10^6$$

Turbulence cannot be sustained by itself but needs to be provided with energy from the surroundings in order to survive. A common source of energy is presence of shear in the mean flow, i.e., that there are gradients in the mean flow, such as for example $\partial U/\partial y$, that leads to production of turbulence kinetic energy. Turbulence may also appear in the interface between two fluids with different velocities. Examples are jet flows and wakes.

2 Flow Equations

2.1 Reynolds Averaging

Figure 1 shows schematically how velocity and pressure varies over time in a specific point in a turbulent flow. The instantaneous flow properties can be perceived as the superposition of a time average and a fluctuating component.

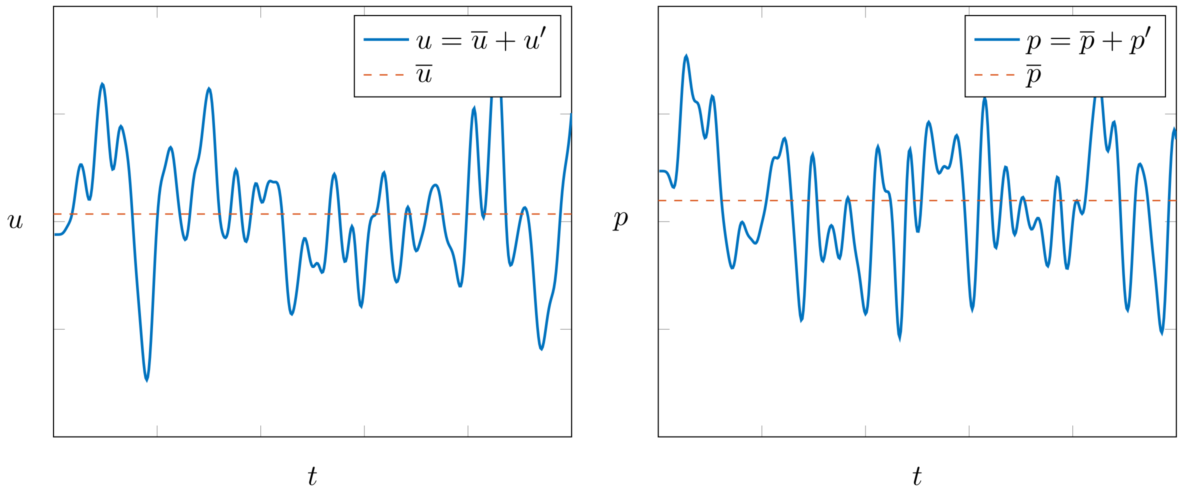

Figure 1: Variations of velocity and pressure in a turbulent flow

The velocity components in the x, y, and z directions and the pressure p can thus be expressed as

$$u = \overline{u} + u' \tag{1}$$

$$v = \overline{v} + v' \tag{2}$$

$$w = \overline{w} + w' \tag{3}$$

$$p = \overline{p} + p' \tag{4}$$

where the time average of the x component of the velocity is obtained as

$$\overline{u} = \frac{1}{T}\int_0^T u dt \tag{5}$$

From Eqns. 1 and 6 we get

$$\overline{u'} = \frac{1}{T}\int_0^T u' dt = 0 \tag{6}$$

Results for the y and z directions are obtained analogously. The time interval T must be large enough to obtain a true time average. The expression of flow properties as the superposition of an average and a fluctuating part is often referred to as Reynolds decomposition (Tennekes and Lumley, 1972).

Since the time averages of the fluctuations are zero, the intensity or strength of the fluctuations are often represented by the RMS value (Root Mean Square)

$$u'_{RMS} = \sqrt{\overline{u'^2}} \tag{7}$$

Figure 2 shows the RMS values of the three velocity components in a flat plate boundary layer.

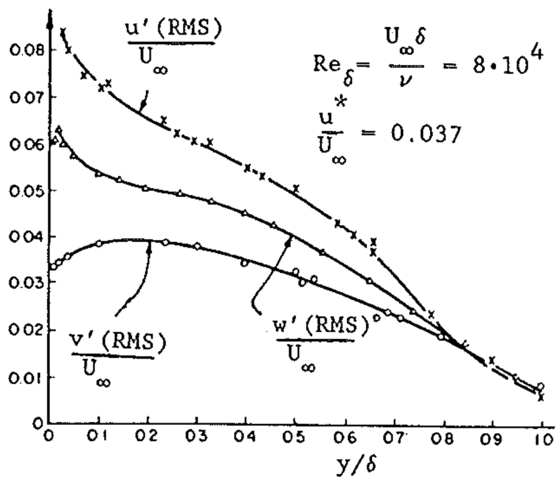

Figure 2: u'_{RMS}, v'_{RMS}, and w'_{RMS} in a flat plate turbulent boundary (from Klebanoff, see Hinze (1975))

2.2 The Continuity Equation

For incompressible flow with constant density, the continuity equation reads

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{8}$$

With the velocities from Eqns. 1 – 3, Eqn. 8 becomes

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

Time averaging the above equation we get

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0 \tag{9}$$

Thus, the continuity equation has the same form as in the laminar case. Moreover, a side result of the above is that

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{10}$$

2.3 Navier-Stokes Equations

The Navier-Stokes equations on their original form applies to both laminar and turbulent flows if the unsteady velocities are used. For incompressible flow without body forces the x component of the equation reads

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{11}$$

Eqns. 1 – 4 in Eqn. 11 gives

$$\begin{aligned} \rho\left(\frac{\partial \overline{u}}{\partial t} + \frac{\partial u'}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{u}\frac{\partial u'}{\partial x} + u'\frac{\partial \overline{u}}{\partial x} + u'\frac{\partial u'}{\partial x} +\right.\\ \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{v}\frac{\partial u'}{\partial y} + v'\frac{\partial \overline{u}}{\partial y} + v'\frac{\partial u'}{\partial y} +\\ \left.\overline{w}\frac{\partial \overline{u}}{\partial z} + \overline{w}\frac{\partial u'}{\partial z} + w'\frac{\partial \overline{u}}{\partial z} + w'\frac{\partial u'}{\partial z}\right) =\\ -\frac{\partial \overline{p}}{\partial x} - \frac{\partial p'}{\partial x} + \mu\left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial^2 \overline{u}}{\partial z^2} + \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2}\right) \end{aligned} \tag{12}$$

Now, let's time average the above equation

$$\rho\left(\frac{\partial\overline{u}}{\partial t}+\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}+\overline{v}\frac{\partial\overline{u}}{\partial z}+\overline{u'\frac{\partial u'}{\partial x}}+\overline{v'\frac{\partial u'}{\partial y}}+\overline{w'\frac{\partial u'}{\partial z}}\right)=-\frac{\partial\overline{p}}{\partial x}+\mu\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}+\frac{\partial^2\overline{u}}{\partial z^2}\right) \tag{13}$$

The last three terms on the left-hand side can be rewritten using Eqn. 10

$$\begin{aligned}\overline{u'\frac{\partial u'}{\partial x}}+\overline{v'\frac{\partial u'}{\partial y}}+\overline{w'\frac{\partial u'}{\partial z}}=\\ \frac{\partial}{\partial x}\overline{u'^2}+\frac{\partial}{\partial y}\overline{u'v'}+\frac{\partial}{\partial z}\overline{u'w'}-\overline{u'\frac{\partial u'}{\partial x}}-\overline{u'\frac{\partial v'}{\partial y}}-\overline{u'\frac{\partial w'}{\partial z}}=\\ \frac{\partial}{\partial x}\overline{u'^2}+\frac{\partial}{\partial y}\overline{u'v'}+\frac{\partial}{\partial z}\overline{u'w'}-\underbrace{\overline{u'\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)}}_{=0}\end{aligned}$$

and thus

$$\overline{u'\frac{\partial u'}{\partial x}}+\overline{v'\frac{\partial u'}{\partial y}}+\overline{w'\frac{\partial u'}{\partial z}}=\frac{\partial}{\partial x}\overline{u'^2}+\frac{\partial}{\partial y}\overline{u'v'}+\frac{\partial}{\partial z}\overline{u'w'} \tag{14}$$

Insert in Eqn. 13

$$\begin{aligned}\frac{\partial\overline{u}}{\partial t}+\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}+\overline{v}\frac{\partial\overline{u}}{\partial z}=\\ -\frac{1}{\rho}\frac{\partial\overline{p}}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}+\frac{\partial^2\overline{u}}{\partial z^2}\right)-\frac{\partial}{\partial x}\overline{u'^2}-\frac{\partial}{\partial y}\overline{u'v'}-\frac{\partial}{\partial z}\overline{u'w'}\end{aligned} \tag{15}$$

The y and z components of the momentum equation are obtained analogously

$$\begin{aligned}\frac{\partial\overline{v}}{\partial t}+\overline{u}\frac{\partial\overline{v}}{\partial x}+\overline{v}\frac{\partial\overline{v}}{\partial y}+\overline{v}\frac{\partial\overline{v}}{\partial z}=\\ -\frac{1}{\rho}\frac{\partial\overline{p}}{\partial y}+\frac{\mu}{\rho}\left(\frac{\partial^2\overline{v}}{\partial x^2}+\frac{\partial^2\overline{v}}{\partial y^2}+\frac{\partial^2\overline{v}}{\partial z^2}\right)-\frac{\partial}{\partial x}\overline{u'v'}-\frac{\partial}{\partial y}\overline{v'^2}-\frac{\partial}{\partial z}\overline{v'w'}\end{aligned} \tag{16}$$

$$\frac{\partial \overline{w}}{\partial t} + \overline{u}\frac{\partial \overline{w}}{\partial x} + \overline{v}\frac{\partial \overline{w}}{\partial y} + \overline{v}\frac{\partial \overline{w}}{\partial z} = \\ -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2 \overline{w}}{\partial x^2} + \frac{\partial^2 \overline{w}}{\partial y^2} + \frac{\partial^2 \overline{w}}{\partial z^2}\right) - \frac{\partial}{\partial x}\overline{u'w'} - \frac{\partial}{\partial y}\overline{v'w'} - \frac{\partial}{\partial z}\overline{w'^2} \tag{17}$$

2.4 Turbulent Stresses

Figure 3 shows a flow where the average velocity is given by $u = u(y)$. The massflow due to the turbulent fluctuation $v\prime$ through the surface element dA in the xz plane is $\dot{m}_y$

$$\dot{m}_y = \rho v' dA \tag{18}$$

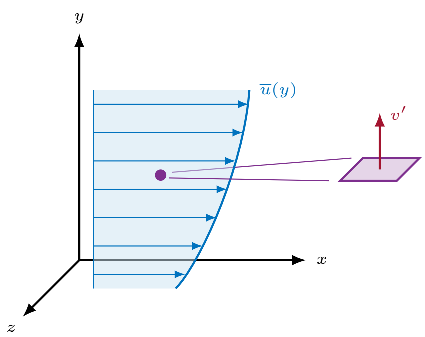

Figure 3: Turbulent stress

According the momentum equation, this will lead to a force in the x direction according to

$$F_x = \dot{m}_y u = \rho dA v' \left(\overline{u} + u'\right) \tag{19}$$

Since the surface element moves in the x direction, the force on the element will be $-F_x$.

Now, let's time average the force and see what happens

$$-\overline{F_x} = \tau dA = -\rho \overline{u'v'} dA \Rightarrow \tau = -\rho \overline{u'v'} \tag{20}$$

Thus we have now found that the term $-\rho\overline{u'v'}$ may be interpreted as a shear stress. It is therefore often referred to as a turbulent shear stress or τ_{turb}. In the same way, it is possible to show that the terms $-\rho\overline{u'^2}$ and $-\rho\overline{u'w'}$ may be interpreted as a normal stress and a shear stress, respectively. The turbulence terms in Eqns. 16 and 17 may also be interpreted as turbulent stresses.

For laminar flow where $\overline{u} = \overline{u}(y)$ we have

$$\tau_{lam} = \mu \frac{\partial \overline{u}}{\partial y} \tag{21}$$

For cases with both laminar and turbulent stresses the total stress is given by $\tau = \tau_{lam} + \tau_{turb}$ or

$$\tau = \mu \frac{\partial \overline{u}}{\partial y} - \rho\overline{u'v'} \tag{22}$$

2.5 Boundary Layer Equations

For two dimensional flow with a steady-state mean flow and constant fluid properties the following boundary layer equations can be derived analogously to the derivation of the boundary layer equations derived for laminar flow (see separate document)

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0 \tag{23}$$

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial P_o}{\partial x} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial}{\partial y}\overline{u'v'} \tag{24}$$

where

$$\frac{\partial P_o}{\partial x} = -\rho U \frac{dU}{dx} \tag{25}$$

2.6 Turbulent Viscosity

The turbulent shear stress was defined in Eqn. 20. In the boundary layer equation (Eqn. 24), the turbulence term is often related to the mean flow. A turbulent viscosity $\rho\varepsilon_m$ is introduced such that

$$\tau_{turb} = -\rho\overline{u'v'} = \rho\varepsilon_m \frac{\partial \overline{u}}{\partial y} \tag{26}$$

As you can see, the turbulent shear stress on the form given by Eqn. 26 is on the same form as the laminar shear stress (Eqn. 21). This assumption is often referred to the Boussinesq assumption named after its inventor (year 1877). With the Boussinesq assumption and Eqn. 22, we get

$$\tau = \rho(\nu + \varepsilon_m)\frac{\partial \overline{u}}{\partial y} \tag{27}$$

3 Turbulent Boundary Layers

In order to be able to estimate the friction of a turbulent pipe flow or of a turbulent flat plate boundary layer flow, the velocity profile must be known. Let's start by investigating a turbulent flat plate boundary layer.

With the total shear stress τ from Eqn.27 in Eqn. 24 becomes

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = \frac{1}{\rho}\frac{\partial \tau}{\partial y} \tag{28}$$

3.1 The Viscous Sublayer

Closest to the flat plate surface $y \to 0$, both $\overline{u}$ and $\overline{v}$ will be very small and thus

$$0 = \frac{1}{\rho}\frac{\partial \tau}{\partial y} \tag{29}$$

which means that close to the wall τ is constant $\tau = \tau_w$, which is the wall shear stress. With Eqn. 27, we get

$$\tau_w = \rho(\nu + \varepsilon_m)\frac{\partial \overline{u}}{\partial y} \tag{30}$$

Now, at the limit $y \to 0$ $\varepsilon_m \to 0$ since both u' and v' vanishes close to the wall and thus

$$\nu\frac{\partial \overline{u}}{\partial y} = \frac{\tau_w}{\rho}$$

Integrating gives

$$\overline{u} = \frac{\tau_w}{\rho\nu}y + c_1$$

where $c_1 = 0$ since $\overline{u} = 0$ when $y = 0$. Introducing the friction velocity u^* such that

$$\tau_w = \rho {u^*}^2 \tag{31}$$

and thus

$$\frac{\overline{u}}{u^*} = \frac{u^* y}{\nu} \text{ or } u^+ = y^+ \tag{32}$$

Eqn. 32 gives the mean flow field velocity in the viscous sublayer where the flow is dominated by friction and is applicable for (Hinze, 1975).

$$0 < \frac{u^* y}{\nu} = y^+ < 5 \tag{33}$$

3.2 The Log Region

Outside of the viscous sublayer but still close enough for Eqn. 30 to be valid, turbulence viscosity dominates the flow $\varepsilon \gg \nu$, which gives

$$\varepsilon_m \frac{\partial \overline{u}}{\partial y} = \frac{\tau_w}{\rho} \tag{34}$$

In order to be able to solve Eqn. 34, Prandtl 1925 introduced the so called mixing length concept. The mixing length, l_m, is analogous to the mean free path in molecular theory. The mixing length is connected to the momentum exchange between different layers and was described by Prandtl as the average path length that a macroscopic identifiable parcel of the turbulent flow moves in space before it, by mixing with the surrounding fluid, loses its identity (see Figure 4).

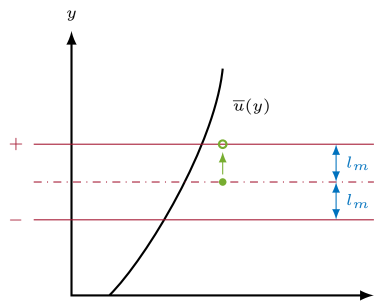

Figure 4: Prandtl's mixing length concept

At a position y in the boundary layer, the mean velocity is $\overline{u}(y)$. Above y at $y+l_m$, the velocity can be approximated as

$$\overline{u}(y+l_m) = \overline{u}(y) + l_m\frac{\partial \overline{u}}{\partial y}$$

and below y at $y - l_m$

$$\overline{u}(y-l_m) = \overline{u}(y) - l_m\frac{\partial \overline{u}}{\partial y}$$

Based on the velocity variation given above, Prandtl postulated that the turbulent velocity fluctuation u' to be approximated as

$$u' \approx l_m\frac{\partial \overline{u}}{\partial y} \tag{35}$$

Prandtl further assumed v' to be of the same order of magnitude as u' and thus the turbulent stress could be expressed as

$$\tau_{turb} = -\rho\overline{u'v'} = \rho l_m^2\left(\frac{\partial \overline{u}}{\partial y}\right)^2 \tag{36}$$

Now, comparing the result with Eqn. 30 we see that

$$\varepsilon_m = l_m^2\frac{\partial \overline{u}}{\partial y} \tag{37}$$

The mixing length l_m varies significantly through the boundary layer. Prandtl's hypothesis was that the mixing length close to the wall is proportional to the wall distance

$$l_m = \kappa y \tag{38}$$

and thus we get

$$\kappa^2 y^2\left(\frac{\partial \overline{u}}{\partial y}\right)^2 = \frac{\tau_w}{\rho}$$

Inserting the friction velocity ($\tau_w = \rho {u^*}^2$)

$$\frac{\partial \overline{u}}{\partial y} = \frac{1}{\kappa}\frac{u^*}{y}$$

Integration gives

$$\overline{u} = \frac{u^*}{\kappa} \ln y + C$$

or

$$\frac{\overline{u}}{u^*} = \frac{1}{\kappa} \ln \frac{u^* y}{\nu} + B \tag{39}$$

Eqn. 39 is valid in the fully turbulent layer ($\varepsilon_m \gg \nu$) or in the so called log layer

$$y^+ = \frac{u^* y}{\nu} > 30 \tag{40}$$

3.3 The Overlap Region

In the y^+ range $5 < y^+ < 30$, we have an overlap region between the viscous sublayer and the fully turbulent log layer. This region of the boundary layer is often referred to as the buffer layer, see Figure 5. Recommended values of the constants in Eqn. 39 are $(1/\kappa) = 2.44$ and $4.9 < B < 5.5$.

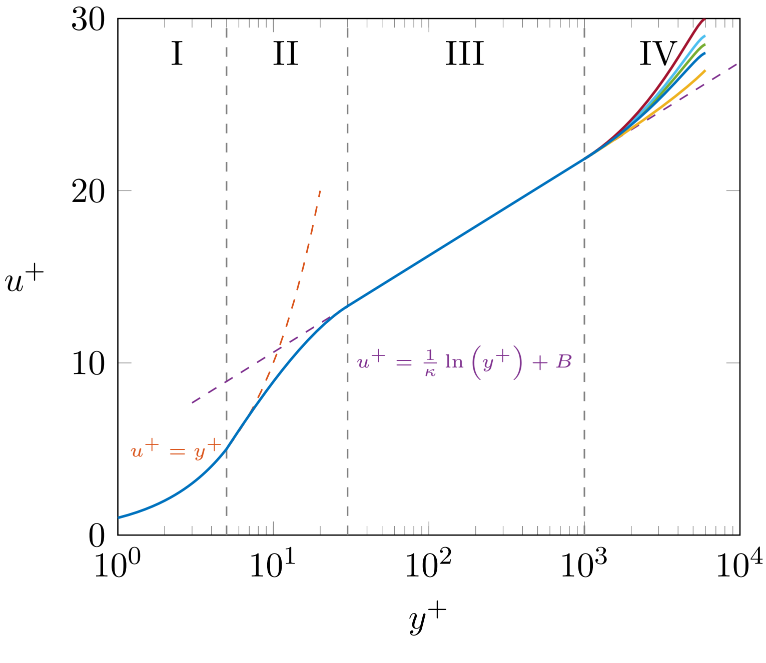

Figure 5: Schematic overview of the velocity distribution in a turbulent boundary layer (I: viscous sublayer, II: buffer layer, III: log-law region, IV: outer layer)

The velocity distribution given by Eqns. 32 and 39 together covers about 20% of the boundary layer, the so called wall layer. In the derivation the wall shear stress was assumed to be constant and experiments have shown that this is in fact the case. The wall region is therefore sometimes referred to as the constant stress layer.

Figures 6 and 7 shows comparisons of theoretical turbulent boundary layer velocity distributions and measured data.

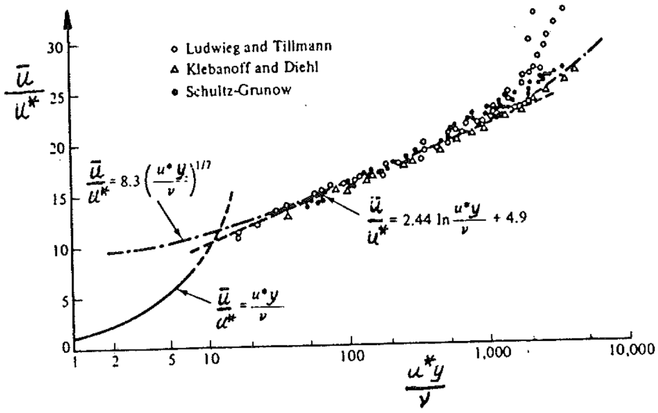

Figure 6: The velocity distribution in a turbulent boundary layer ($\overline{u}/u^*$ as a function of u^*y/ν).

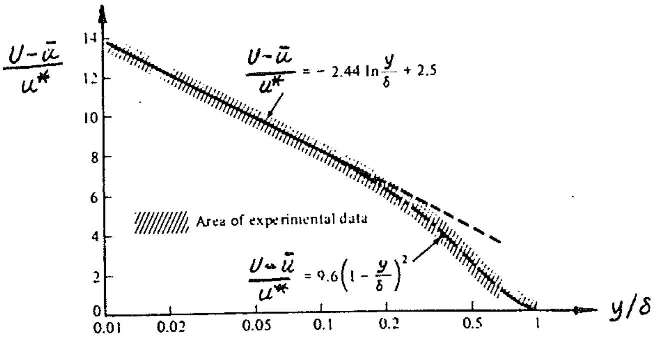

Figure 7: The velocity distribution in a turbulent boundary layer ($(U - \overline{u})/u^*$ as a function of y/δ).

The velocity distributions given by Eqns. 32 and 39 are examples of Reynolds number similarity. For the outer region, $y/\delta > 0.2$, the velocity distribution is often presented as $(U - \overline{u})/u^*$ as a function of y/δ, which is an example of local invarians or self-preservation. For more detail, see Hinze (1975) and Tennekes and Lumley (1972).

The analyze above can also be done for turbulent pipe flow. Figure 8 gives a comparison of theoretical and measured velocity distributions for turbulent pipe flows.

The turbulent velocity distribution is sometimes approximated as

$$\overline{u} = Cy^{1/7} \tag{41}$$

As can be seen from Figure 6, Eqn. 41 is a quite good approximation except in the viscous sublayer.

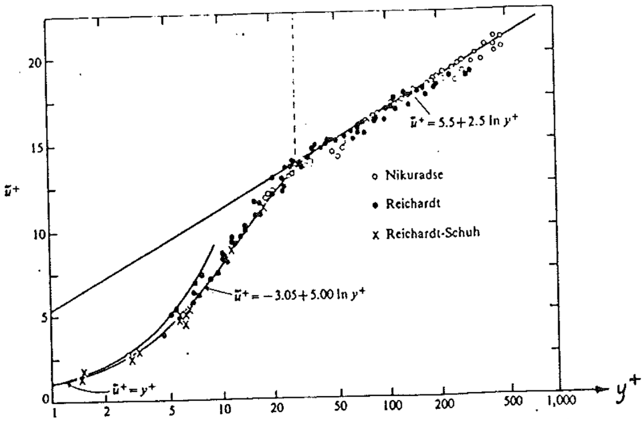

Figure 8: The velocity distribution in the wall layer in turbulent pipe flow.

3.4 The Skin Friction Coefficient for Turbulent Pipe Flow

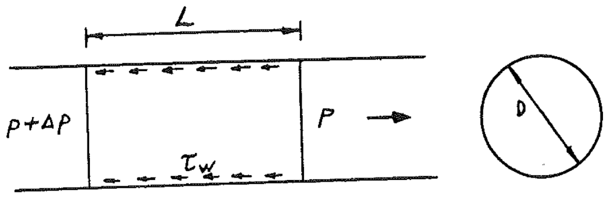

Figure 9: Circular pipe.

For fully developed pipe flow, the pressure drop Δp over a pipe segment (Figure 9) is given by

$$\Delta p = f \frac{L}{D} \frac{\rho V^2}{2} \tag{42}$$

where V is the average pipe velocity. A force balance over a control volume spanning the entire tube (Figure 9) gives that the pressure forces is balanced by the shear force along the pipe perimeter.

$$\Delta p \frac{\pi D^2}{4} = \tau_w \pi D L \Rightarrow \tau_w = \frac{\Delta p D}{4L}$$

or

$$\tau_w = f \frac{\rho V^2}{8} \tag{43}$$

With the skin friction coefficient c_f defined as

$$\tau_w = c_f \frac{\rho V^2}{2}$$

we get

$$c_f = \frac{f}{4} \tag{44}$$

In order to be able to get an estimate of the skin friction coefficient c_f, we will need to obtain the friction factor f. With $\tau_w = \rho {u^*}^2$ and Eqn. 43

$$f = \frac{8{u^*}^2}{V} \tag{45}$$

The average velocity V is obtained as

$$V = \frac{4}{\pi D^2} \int_0^R \overline{u} 2\pi r dr = \frac{8}{D^2} \int_0^R \overline{u} r dr \tag{46}$$

which can be rewritten as

$$\frac{V}{u^*} = \frac{2}{{R^+}^2} \int_0^{R^+} \overline{u}^+ (R^+ - y^+) dy^+ \tag{47}$$

where

$$\overline{u}^+ = \frac{\overline{u}}{u^*},\ y = R - r,\ R^+ = \frac{u^* R}{\nu},\ \text{and}\ y^+ = \frac{u^* y}{\nu}$$

From Eqn. 47 V/u^* can be calculated if $\overline{u}^+(y^+)$ is known. The velocity distribution given by Eqn. 39 can be assumed to apply for the whole pipe cross section and thus

$$\frac{V}{u^*} = \frac{2}{{R^+}^2} \int_0^{R^+} \left(\frac{1}{\kappa} \ln y^+ + B \right) (R^+ - y^+) dy^+ = \cdots = \frac{1}{\kappa} \ln R^+ - \frac{3}{2\kappa} + B$$

With $1/\kappa = 2.5$ and $B = 5.5$ we get

$$\frac{V}{u^*} = 2.5 \ln R^+ + 1.75 \tag{48}$$

Now when we have obtained an expression for V/u^*, the friction factor f can be calculated (Eqn.45) as

$$f = \frac{8}{(2.5 \ln R^+ + 1.75)^2}$$

R^+ can be rewritten as follows

$$R^+ = \frac{u^* R}{\nu} = \frac{u^*}{V} \frac{VD}{2\nu} = \frac{1}{2} Re_D \sqrt{\frac{f}{8}}$$

and thus

$$f = \frac{8}{\left(2.5 \ln \left(\frac{1}{2} Re_D \sqrt{\frac{f}{8}}\right) + 1.75\right)^2}$$

or

$$\frac{1}{\sqrt{f}} = 0.884 \ln \left(\sqrt{f} Re_D\right) - 0.91$$

changing from natural logarithm to ten-base logarithm gives

$$\frac{1}{\sqrt{f}} = 2.03 \log_{10} \left(\sqrt{f} Re_D\right) - 0.91 \tag{49}$$

The above relation is Prandtl's friction law, which makes it possible to calculate the friction factor f if the Reynolds number Re_D is known and with a value of f it is possible to calculate the skin friction coefficient c_f. It has, however, been shown that Eqn. 49 gives better agreement with measurements with slightly changed values of the numerical constants (Özisik, 1977). Figure 10 shows a comparison of the two relations for f and measured data.

$$\frac{1}{\sqrt{f}} = 2.0 \log_{10} \left(\sqrt{f} Re_D\right) - 0.8 \tag{50}$$

Eqn. 50 needs to be solved using iterative methods and a simpler expression is obtained if instead of using the log law to describe the velocity distribution, the $1/7^{th}$-rule is used (Eckert, 1963). This gives the following relation for f

$$f = \frac{0.316}{Re_D^{1/4}} \tag{51}$$

which is often referred to as the Blasius relation. Using Blasius relation f can be calculated directly for a given Reynolds number Re_D.

3.5 Surface Roughness

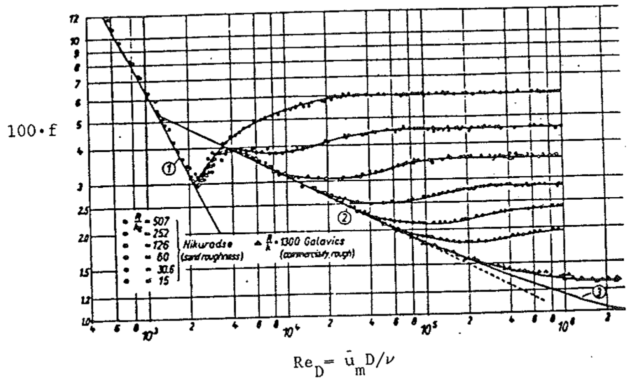

Figure 10: Friction factor for smooth and rough surfaces (1: laminar flow $f = 64/Re_D$, 2: Eqn. 49, and 3: Eqn. 50).

Relations for calculating the friction factor similar to those presented above have been obtained for rough surfaces (Schlichting, 1979). Figure 10 gives shows f for both smooth and rough surfaces. For the rough surfaces, the parameter R/k_s is used to decide which of the curves to use where R is the pipe radius and k_s is the average height of a deviation from the smooth surface. The results presented in the figure are based on measurements made by Nikuradse (Schlichting, 1979) where the the rough surfaces were generated by gluing sand with different grain size to the surface of the tested pipes. For commercially available pipes, the surface roughness and thus the results deviate from those used by Nikuradse. Figure 11 shows friction factors obtained from measurements made by Moody (1944).

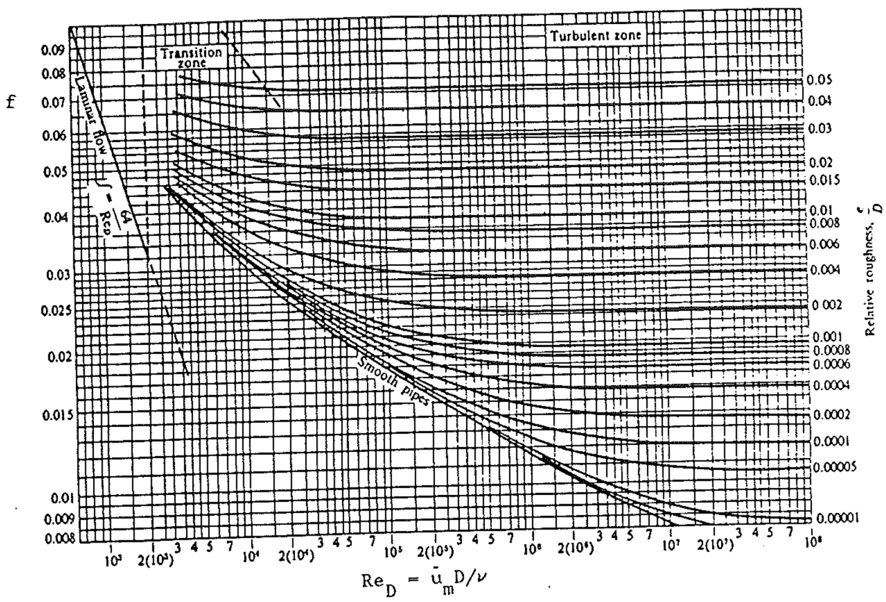

Figure 11: Friction factor f for pipe flow according to Moody (1944).

For a known surface roughness and Reynolds number, the friction factor can be obtained from the Moody chart and then we can calculate the skin friction coefficient.

3.6 The Skin Friction Coefficient for Turbulent Flat Plate Boundary Layer

The same approach as was used to obtain the skin friction for turbulent pipe flow above can be applied to a turbulent flat plate boundary layer with the result

$$c_f = 0.0592 Re_x^{-1/5} \tag{52}$$

where

$$Re_x = \frac{U_\infty x}{\nu}$$

Eqn. 52 can be used in the Reynolds number range $5 \times 10^5 < Re_x < 10^7$.

References

E. R. G. Eckert. *Introduction to Heat and Mass Transfer.* McGraw-Hill, New York, 1963.

J. O. Hinze. *Turbulence.* McGraw-Hill, New York, 2:nd edition edition, 1975.

L. F. Moody. Friction factor for pipe flow. *Trans. ASME*, 66:671–684, 1944.

M. N. Özisik. *Basic Heat Transfer.* McGraw-Hill, Tokyo, 1977.

H. Schlichting. *Boundary Layer Theory.* McGraw-Hill, New York, 7:th edition edition, 1979.

H. Tennekes and J. L. Lumley. *A First Course in Turbulence.* MIT Press, Massachusetts, 1972.