Compressible Flow - TME085

Lecture 11

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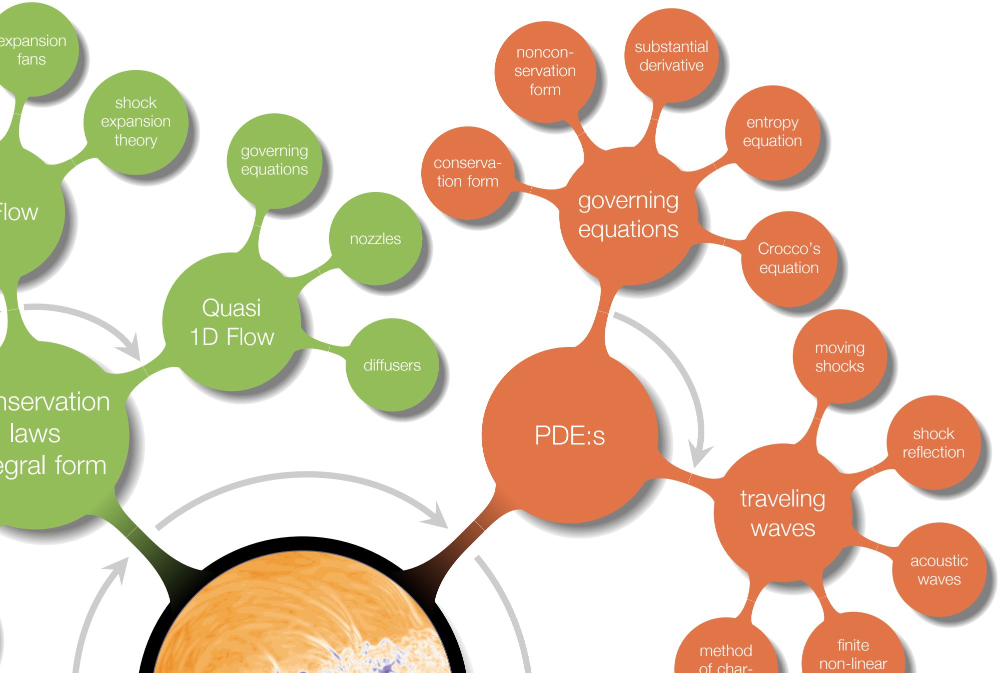
Chapter 7 - Unsteady Wave Motion

Overview



expansion fans
shock expansion theory
flow
governing equations
nozzles
Quasi 1D Flow
diffusers
nservation laws egral form
noncon-servation form
substantial derivative
entropy equation
conserva-tion form
governing equations
Crocco’s equation
moving shocks
PDE:s
shock reflection
traveling waves
acoustic waves
method of char-
finite non-linear

Learning Outcomes

3 **Describe** typical engineering flow situations in which compressibility effects are more or less predominant (e.g. Mach number regimes for steady-state flows)

4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on

8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics

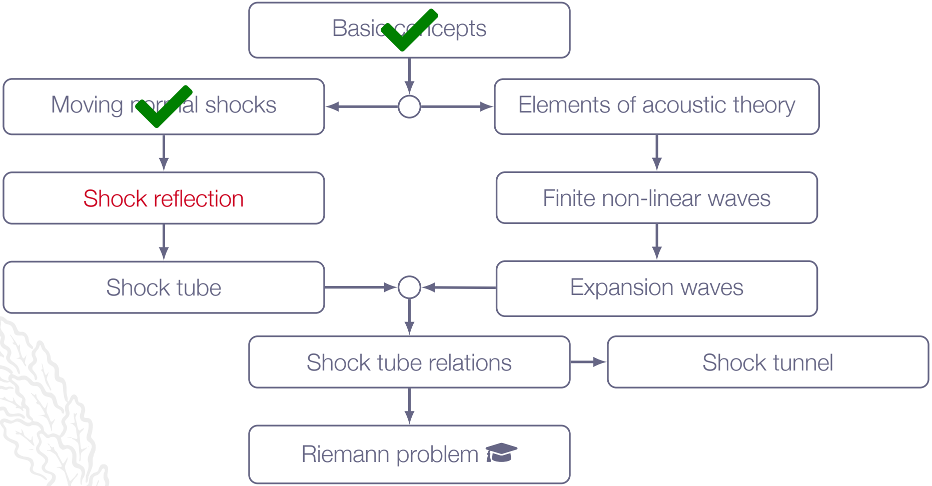
- a 1D isentropic flow*
- b normal shocks*
- j unsteady waves and discontinuities in 1D
- k basic acoustics

9 **Solve** engineering problems involving the above-mentioned phenomena (8a-8k)

11 **Explain** how the equations for aero-acoustics and classical acoustics are derived as limiting cases of the compressible flow equations

moving normal shocks - frame of reference seems to be the key here?!

Roadmap - Unsteady Wave Motion

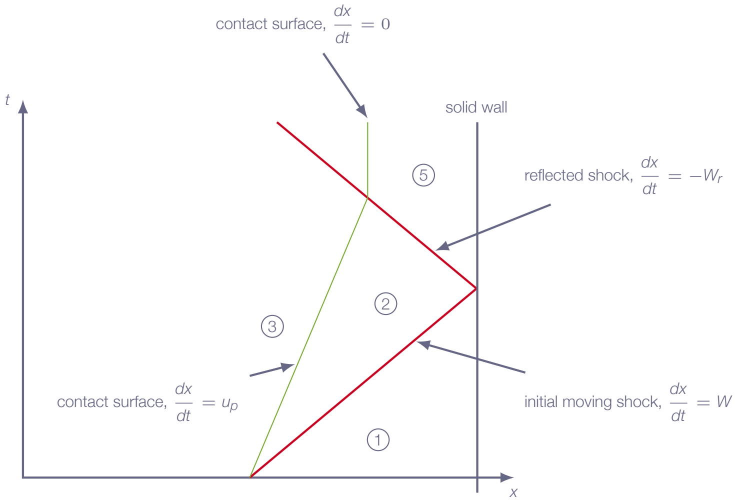


Chapter 7.3
Reflected Shock Wave

One-Dimensional Flow with Friction

what happens when a moving shock approaches a wall?

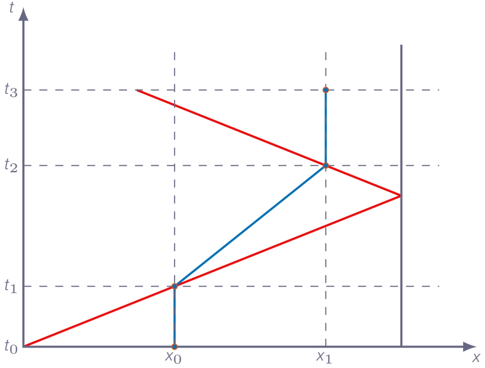
Shock Reflection



Shock Reflection - Particle Path

A fluid particle located at x_0 at time t_0 (a location ahead of the shock) will be affected by the moving shock and follow the blue path

time	location	velocity
t_0	x_0	0
t_1	x_0	u_p
t_2	x_1	u_p
t_3	x_1	0



Shock Reflection Relations

In the frame of reference of the reflected shock we have

velocity ahead of shock: $W_r + u_p$

velocity behind shock: W_r

where W_r is the velocity of the reflected shock and u_p is the induced flow velocity behind the incident shock

Shock Reflection Relations

Continuity:

$$\rho_2(W_r + u_p) = \rho_5 W_r$$

Momentum:

$$p_2 + \rho_2(W_r + u_p)^2 = p_5 + \rho_5 W_r^2$$

Energy:

$$h_2 + \frac{1}{2}(W_r + u_p)^2 = h_5 + \frac{1}{2}W_r^2$$

Shock Reflection Relations

Reflected shock is determined such that $u_5 = 0$

$$\frac{M_r}{M_r^2 - 1} = \frac{M_s}{M_s^2 - 1}\sqrt{1 + \frac{2(\gamma - 1)}{(\gamma + 1)^2}(M_s^2 - 1)\left(\gamma + \frac{1}{M_s^2}\right)}$$

where

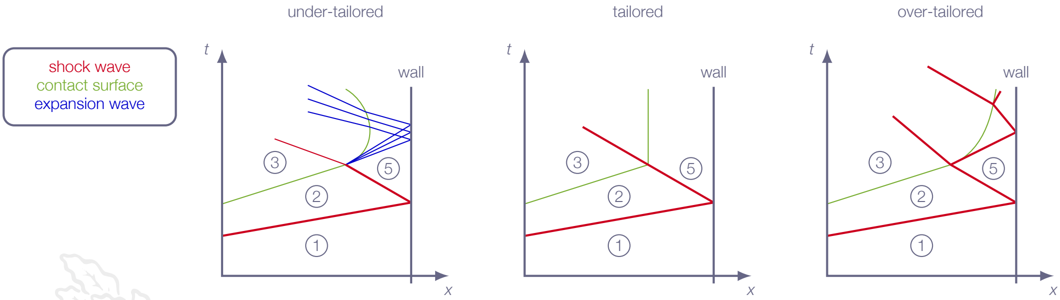
$$M_r = \frac{W_r + u_p}{a_2}$$

Tailored v.s. Non-Tailored Shock Reflection

The time duration of condition 5 is determined by what happens after interaction between reflected shock and contact discontinuity

For special choice of initial conditions (tailored case), this interaction is negligible, thus prolonging the duration of condition 5

Tailored v.s. Non-Tailored Shock Reflection



Under-tailored conditions:

Mach number of incident wave lower than in tailored conditions

Over-tailored conditions:

Mach number of incident wave higher than in tailored conditions

Shock Reflection - Example

Shock reflection in shock tube ($\gamma = 1.4$)
(Example 7.1 in Anderson)

Given data

p_2/p_1	10.0	
T_2/T_1	2.623	
p_1	1.0	bar
T_1	300.0	K

Calculated data

M_s	2.95
M_r	2.09
p_5/p_2	4.978
T_5/T_2	1.77

$$p_5 = \left(\frac{p_5}{p_2}\right)\left(\frac{p_2}{p_1}\right) p_1 = 49.78$$

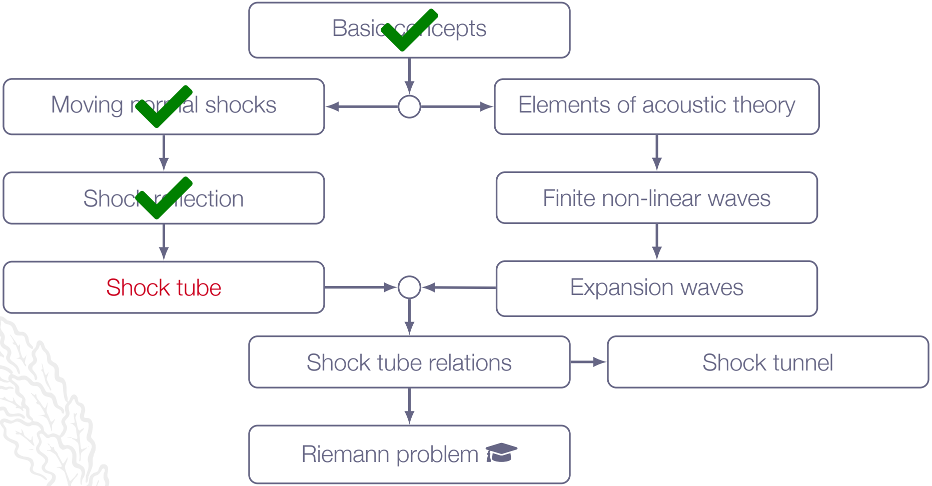
$$T_5 = \left(\frac{T_5}{T_2}\right)\left(\frac{T_2}{T_1}\right) T_1 = 1393$$

Shock Reflection - Shock Tube

Very high pressure and temperature conditions in a specified location with very high precision (p_5, T_5)

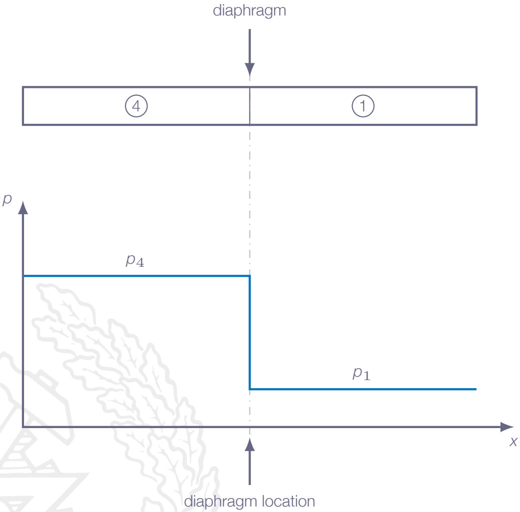
- measurements of thermodynamic properties of various gases at extreme conditions, *e.g.* dissociation energies, molecular relaxation times, etc.
- measurements of chemical reaction properties of various gas mixtures at extreme conditions

Roadmap - Unsteady Wave Motion



The Shock Tube

Shock Tube

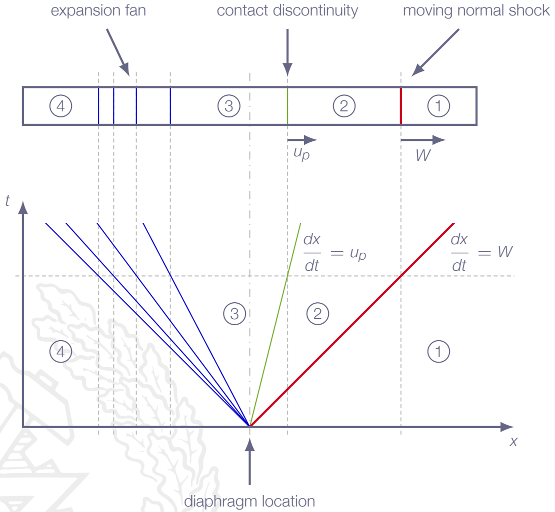


tube with closed ends
diaphragm inside, separating two different constant states
(could also be two different gases)

if diaphragm is removed suddenly (by inducing a breakdown) the two states come into contact and a flow develops

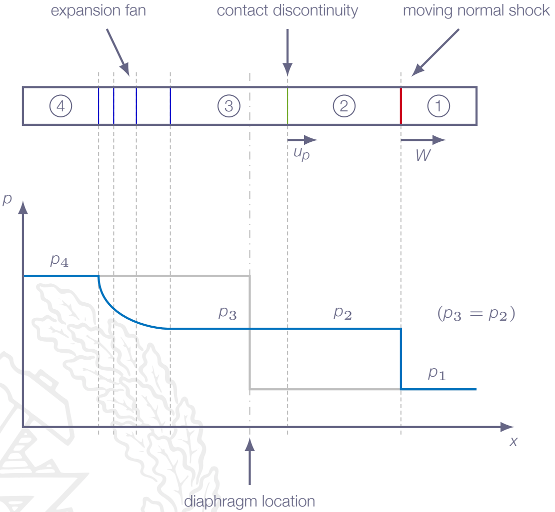
assume that $p_4 > p_1$:
state 4 is "driver" section
state 1 is "driven" section

Shock Tube



flow at some time after diaphragm breakdown

Shock Tube



flow at some time after diaphragm breakdown

Shock Tube - Basic Principles

As the diaphragm is removed, a pressure discontinuity is generated

The only process that can generate a pressure **discontinuity** in the gas is a **shock**

In chapter 3 we learned that the velocity upstream of the shock **must be supersonic**

Since the gas is standing still when the shock tube is started, **the shock must move** in order to establish the required **relative velocity**

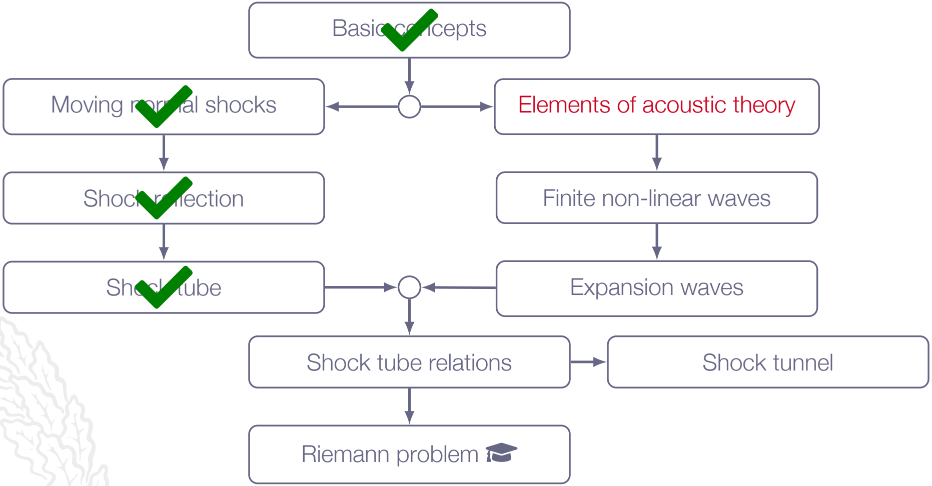
The shock must move in to the gas with the lower pressure

Shock Tube - Basic Principles

By using light gases for the **driver section** (*e.g.* He) and heavier gases for the **driven section** (*e.g.* air) the pressure p_4 required for a specific p_2/p_1 ratio is significantly reduced

If T_4/T_1 is increased, the pressure p_4 required for a specific p_2/p_1 is also reduced

Roadmap - Unsteady Wave Motion



Chapter 7.5
Elements of Acoustic Theory

Sound Waves - Sound Pressure Level

sound wave	L_p [dB]	Δp [Pa]
Weakest audible sound wave	0	2.83×10^{-5}
Loud sound wave	91	1.00×10^{0}
Amplified music	120	2.80×10^{1}
Jet engine @ 30 m	130	9.00×10^{1}
Threshold of pain	140	2.83×10^{2}
Military jet @ 30 m	150	8.90×10^{2}

Example (Loud sound wave):

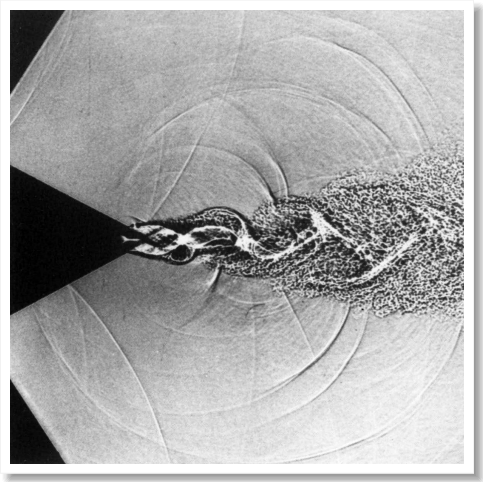
$\Delta p \sim$ 1 Pa (91 dB) gives $\Delta \rho \sim 8.5 \times 10^{-6}$ kg/m^3 and $\Delta u \sim 2.4 \times 10^{-3}$ m/s

Sound Waves - Acoustic Analogy

Schlieren flow visualization of self-sustained oscillation of an under-expanded free jet

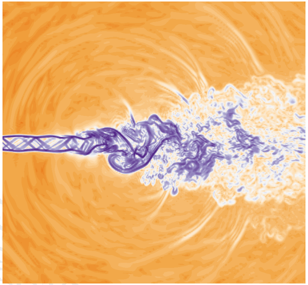
A. Hirschberg

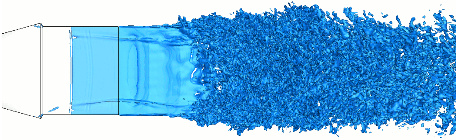
"*Introduction to aero-acoustics of internal flows*", Advances in Aeroacoustics, VKI, 12-16 March 2001



Sound Waves - Acoustic Analogy

Screeching rectangular supersonic jet





Elements of Acoustic Theory

PDE:s for conservation of mass and momentum derived in Chapter 6:

	conservation form	non-conservation form
mass	$\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$	$\dfrac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0$
momentum	$\dfrac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v} + p\mathbf{I}) = 0$	$\rho \dfrac{D\mathbf{v}}{Dt} + \nabla p = 0$

Elements of Acoustic Theory

For adiabatic inviscid flow we also have the entropy equation as

$$\frac{Ds}{Dt} = 0$$

Assume one-dimensional flow

$$\left.\begin{aligned} \rho &= \rho(x,t) \\ \mathbf{v} &= u(x,t)\mathbf{e}_x \\ p &= p(x,t) \\ &\dots \end{aligned}\right\} \Rightarrow$$

continuity $\displaystyle \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$

momentum $\displaystyle \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0$

s=constant

Elements of Acoustic Theory

continuity $$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

momentum $$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0$$

s=constant

More unknowns than equations $\Rightarrow$ the equation system can not be solved

Can $\frac{\partial p}{\partial x}$ be expressed in terms of density?

Leading question; it is possible so let's do just that ...

Elements of Acoustic Theory

From Chapter 1: any thermodynamic state variable is uniquely defined by any two other state variables

$$p = p(\rho, s) \Rightarrow dp = \left(\frac{\partial p}{\partial \rho}\right)_s d\rho + \left(\frac{\partial p}{\partial s}\right)_\rho ds$$

s=constant gives

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_s d\rho = a^2 d\rho$$

$$\Rightarrow \begin{cases} \dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial x} + \rho\dfrac{\partial u}{\partial x} = 0 \\[2ex] \rho\dfrac{\partial u}{\partial t} + \rho u\dfrac{\partial u}{\partial x} + a^2\dfrac{\partial \rho}{\partial x} = 0 \end{cases}$$

Elements of Acoustic Theory

Assume **small perturbations** around stagnant reference condition:

$$\rho = \rho_\infty + \Delta\rho \qquad p = p_\infty + \Delta p \qquad T = T_\infty + \Delta T \qquad u = u_\infty + \Delta u = \{u_\infty = 0\} = \Delta u$$

where ρ_∞, p_∞, and T_∞ are constant

Now, insert $\rho = (\rho_\infty + \Delta\rho)$ and $u = \Delta u$ in the continuity and momentum equations (derivatives of ρ_∞ are zero)

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \Delta u \dfrac{\partial}{\partial x}(\Delta\rho) + (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial x}(\Delta u) = 0 \\[2ex] (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial t}(\Delta u) + (\rho_\infty + \Delta\rho)\Delta u \dfrac{\partial}{\partial x}(\Delta u) + a^2 \dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Elements of Acoustic Theory

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where ρ_∞, p_∞, and T_∞ are constant

Now, insert $\rho = (\rho_\infty + \Delta\rho)$ and $u = \Delta u$ in the continuity and momentum equations (derivatives of ρ_∞ are zero)

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \Delta u \dfrac{\partial}{\partial x}(\Delta\rho) + (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial x}(\Delta u) = 0 \\ (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial t}(\Delta u) + (\rho_\infty + \Delta\rho)\Delta u \dfrac{\partial}{\partial x}(\Delta u) + a^2 \dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Elements of Acoustic Theory

Speed of sound is a thermodynamic state variable $\Rightarrow a^2 = a^2(\rho, s)$. With entropy constant $\Rightarrow a^2 = a^2(\rho)$

Taylor expansion around a_∞ with $(\Delta\rho = \rho - \rho_\infty)$ gives

$$a^2 = a_\infty^2 + \left(\frac{\partial}{\partial\rho}(a^2)\right)_\infty \Delta\rho + \frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2}(a^2)\right)_\infty (\Delta\rho)^2 + \ldots$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \Delta u \dfrac{\partial}{\partial x}(\Delta\rho) + (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial x}(\Delta u) = 0 \\ (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial t}(\Delta u) + (\rho_\infty + \Delta\rho)\Delta u \dfrac{\partial}{\partial x}(\Delta u) + \left[a_\infty^2 + \left(\dfrac{\partial}{\partial\rho}(a^2)\right)_\infty \Delta\rho + \ldots\right]\dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Elements of Acoustic Theory - Acoustic Equations

Since $\Delta\rho$ and Δu are assumed to be small ($\Delta\rho \ll \rho_\infty$, $\Delta u \ll a$)

1. products of perturbations can be neglected
2. higher-order terms in the Taylor expansion can be neglected

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \rho_\infty \dfrac{\partial}{\partial x}(\Delta u) = 0 \\[2ex] \rho_\infty \dfrac{\partial}{\partial t}(\Delta u) + a_\infty^2 \dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Note! The assumption is only valid for small perturbations (sound waves)

This type of derivation is based on linearization, *i.e.* the acoustic equations are **linear**

Elements of Acoustic Theory - Acoustic Equations

Acoustic equations:

”... describe the motion of gas induced by the passage of a sound wave ...”

Elements of Acoustic Theory - Wave Equation

Combining linearized continuity and the momentum equations we get

$$\boxed{\frac{\partial^2}{\partial t^2}(\Delta\rho) = a_\infty^2 \frac{\partial^2}{\partial x^2}(\Delta\rho)}$$

(combine the time derivative of the continuity eqn. and the divergence of the momentum eqn.)

General solution:

$$\Delta\rho(x,t) = \underbrace{F(x - a_\infty t)}_{\text{wave traveling in positive } x\text{-direction with speed } a_\infty} + \underbrace{G(x + a_\infty t)}_{\text{wave traveling in negative } x\text{-direction with speed } a_\infty}$$

F and G may be arbitrary functions

Wave shape is determined by functions F and G

Elements of Acoustic Theory - Wave Equation

Spatial and temporal derivatives of F are obtained according to

$$\begin{cases} \dfrac{\partial F}{\partial t} = \dfrac{\partial F}{\partial (x - a_\infty t)} \dfrac{\partial (x - a_\infty t)}{\partial t} = -a_\infty F' \\[2ex] \dfrac{\partial F}{\partial x} = \dfrac{\partial F}{\partial (x - a_\infty t)} \dfrac{\partial (x - a_\infty t)}{\partial x} = F' \end{cases}$$

spatial and temporal derivatives of G can of course be obtained in the same way...

Elements of Acoustic Theory - Wave Equation

with $\Delta\rho(x,t) = F(x - a_\infty t) + G(x + a_\infty t)$ and the derivatives of F and G we get

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = a_\infty^2 F'' + a_\infty^2 G''$$

and

$$\frac{\partial^2}{\partial x^2}(\Delta\rho) = F'' + G''$$

which gives

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) - a_\infty^2 \frac{\partial^2}{\partial x^2}(\Delta\rho) = 0$$

i.e., the proposed solution fulfils the wave equation

Elements of Acoustic Theory - Wave Equation

F and G may be arbitrary functions, assume $G = 0$

$$\Delta\rho(x, t) = F(x - a_\infty t)$$

If $\Delta\rho$ is constant (constant wave amplitude), $(x - a_\infty t)$ must be a constant which implies

$$x = a_\infty t + c$$

where c is a constant

$$\frac{dx}{dt} = a_\infty$$

Elements of Acoustic Theory - Wave Equation

Let's try to find a relation between $\Delta\rho$ and Δu

$\Delta\rho(x,t) = F(x - a_\infty t)$ (wave in positive x direction) gives:

$$\frac{\partial}{\partial t}(\Delta\rho) = -a_\infty F'$$

and

$$\frac{\partial}{\partial x}(\Delta\rho) = F'$$

$$\underbrace{\frac{\partial}{\partial t}(\Delta\rho)}_{-a_\infty F'} + a_\infty \underbrace{\frac{\partial}{\partial x}(\Delta\rho)}_{F'} = 0$$

or

$$\frac{\partial}{\partial x}(\Delta\rho) = -\frac{1}{a_\infty}\frac{\partial}{\partial t}(\Delta\rho)$$

Elements of Acoustic Theory - Wave Equation

Linearized momentum equation:

$$\rho_\infty \frac{\partial}{\partial t}(\Delta u) = -a_\infty^2 \frac{\partial}{\partial x}(\Delta \rho) \Rightarrow$$

$$\frac{\partial}{\partial t}(\Delta u) = -\frac{a_\infty^2}{\rho_\infty}\frac{\partial}{\partial x}(\Delta \rho) = \left\{\frac{\partial}{\partial x}(\Delta \rho) = -\frac{1}{a_\infty}\frac{\partial}{\partial t}(\Delta \rho)\right\} = \frac{a_\infty}{\rho_\infty}\frac{\partial}{\partial t}(\Delta \rho)$$

$$\frac{\partial}{\partial t}\left(\Delta u - \frac{a_\infty}{\rho_\infty}\Delta \rho\right) = 0 \Rightarrow \Delta u - \frac{a_\infty}{\rho_\infty}\Delta \rho = const$$

In undisturbed gas $\Delta u = \Delta \rho = 0$ which implies that the constant must be zero and thus

$$\boxed{\Delta u = \frac{a_\infty}{\rho_\infty}\Delta \rho}$$

Elements of Acoustic Theory - Wave Equation

Similarly, for $\Delta\rho(x,t) = G(x + a_\infty t)$ (wave in negative x direction) we obtain:

$$\boxed{\Delta u = -\frac{a_\infty}{\rho_\infty}\Delta\rho}$$

Also, since $\Delta p = a_\infty^2 \Delta\rho$ we get:

Right going wave (+x direction) $\Delta u = \dfrac{a_\infty}{\rho_\infty}\Delta\rho = \dfrac{1}{a_\infty\rho_\infty}\Delta p$

Left going wave (-x direction) $\Delta u = -\dfrac{a_\infty}{\rho_\infty}\Delta\rho = -\dfrac{1}{a_\infty\rho_\infty}\Delta p$

Elements of Acoustic Theory - Wave Equation

Δu denotes **induced mass motion** and is positive in the positive x-direction

$$\Delta u = \pm\frac{a_\infty \Delta\rho}{\rho_\infty} = \pm\frac{\Delta p}{a_\infty \rho_\infty}$$

condensation (the part of the sound wave where $\Delta\rho > 0$):
Δu is always in the **same** direction as the wave motion

rarefaction (the part of the sound wave where $\Delta\rho < 0$):
Δu is always in the direction **opposite** to the wave motion

Elements of Acoustic Theory - Wave Equation *Summary*

Combining **linearized** continuity and the momentum equations we get

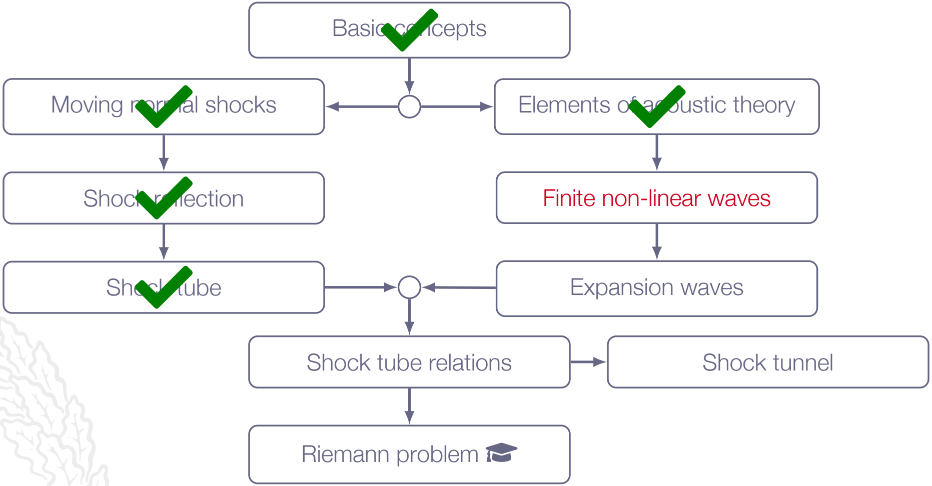
$$\boxed{\frac{\partial^2}{\partial t^2}(\Delta\rho) = a_\infty^2 \frac{\partial^2}{\partial x^2}(\Delta\rho)}$$

Due to the assumptions made, the **equation is not exact**

More and more accurate as the perturbations becomes smaller and smaller

So, how should we describe waves with larger amplitudes?

Roadmap - Unsteady Wave Motion



Chapter 7.6
Finite (Non-Linear) Waves

Finite (Non-Linear) Waves

When $\Delta\rho,\ \Delta u,\ \Delta p,$ ... Become large, the **linearized acoustic equations become poor approximations**

Non-linear equations must be used

One-dimensional non-linear continuity and momentum equations:

$$\boxed{\begin{aligned} \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} &= 0 \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} &= 0 \end{aligned}}$$

Finite (Non-Linear) Waves

We still assume isentropic flow, $ds = 0$

$$\frac{\partial \rho}{\partial t} = \left(\frac{\partial \rho}{\partial p}\right)_s \frac{\partial p}{\partial t} = \frac{1}{a^2}\frac{\partial p}{\partial t} \qquad \frac{\partial \rho}{\partial x} = \left(\frac{\partial \rho}{\partial p}\right)_s \frac{\partial p}{\partial x} = \frac{1}{a^2}\frac{\partial p}{\partial x}$$

Inserted in the continuity equation this gives:

$$\boxed{\begin{aligned} \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho a^2 \frac{\partial u}{\partial x} &= 0 \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} &= 0 \end{aligned}}$$

Finite (Non-Linear) Waves

Add $1/(\rho a)$ times the continuity equation to the momentum equation:

$$\left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right] + \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right] = 0$$

If we instead **subtract** $1/(\rho a)$ times the continuity equation from the momentum equation, we get:

$$\left[\frac{\partial u}{\partial t} + (u-a)\frac{\partial u}{\partial x}\right] - \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u-a)\frac{\partial p}{\partial x}\right] = 0$$

Finite (Non-Linear) Waves

Since $u = u(x,t)$, we have:

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}\frac{dx}{dt}dt$$

Let $\frac{dx}{dt} = u + a$ gives

$$du = \left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right]dt$$

Interpretation: change of u in the direction of line $\frac{dx}{dt} = u + a$

Finite (Non-Linear) Waves

In the same way we get:

$$dp = \frac{\partial p}{\partial t}dt + \frac{\partial p}{\partial x}\frac{dx}{dt}dt$$

and thus

$$dp = \left[\frac{\partial p}{\partial t} + (u + a)\frac{\partial p}{\partial x}\right]dt$$

Interpretation: change of p in the direction of line $\dfrac{dx}{dt} = u + a$

Finite (Non-Linear) Waves

Now, if we combine

$$\left[\frac{\partial u}{\partial t}+(u+a)\frac{\partial u}{\partial x}\right]+\frac{1}{\rho a}\left[\frac{\partial p}{\partial t}+(u+a)\frac{\partial p}{\partial x}\right]=0$$

$$du=\left[\frac{\partial u}{\partial t}+(u+a)\frac{\partial u}{\partial x}\right]dt$$

$$dp=\left[\frac{\partial p}{\partial t}+(u+a)\frac{\partial p}{\partial x}\right]dt$$

we get

$$\boxed{\frac{du}{dt}+\frac{1}{\rho a}\frac{dp}{dt}=0}$$

Characteristic Lines

Thus, along a line $dx = (u + a)dt$ we have

$$\boxed{du + \frac{dp}{\rho a} = 0}$$

In the same way we get along a line where $dx = (u - a)dt$

$$\boxed{du - \frac{dp}{\rho a} = 0}$$

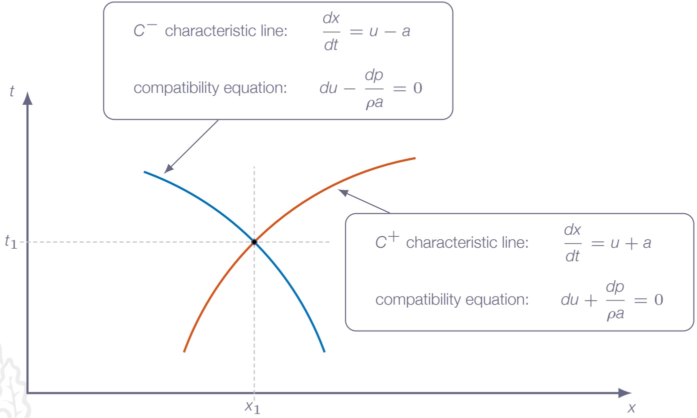
Characteristic Lines

We have found a path through a point (x, t) along which the governing partial differential equations reduces to ordinary differential equations

These paths or lines are called **characteristic lines**

The C^+ and C^- characteristic lines are physically the paths of **right- and left-running acoustic waves** in the xt-plane

Characteristic Lines



Characteristic Lines - Summary

$$\frac{du}{dt} + \frac{1}{\rho a}\frac{dp}{dt} = 0 \quad \text{along } C^+ \text{ characteristic}$$

$$\frac{du}{dt} - \frac{1}{\rho a}\frac{dp}{dt} = 0 \quad \text{along } C^- \text{ characteristic}$$

$$du + \frac{dp}{\rho a} = 0 \quad \text{along } C^+ \text{ characteristic}$$

$$du - \frac{dp}{\rho a} = 0 \quad \text{along } C^- \text{ characteristic}$$

Riemann Invariants

Integration gives:

$$\boxed{\begin{aligned} J^+ &= u + \int \frac{dp}{\rho a} = \text{constant along } C^+ \text{ characteristic} \\ J^- &= u - \int \frac{dp}{\rho a} = \text{constant along } C^- \text{ characteristic} \end{aligned}}$$

We need to rewrite $\dfrac{dp}{\rho a}$ to be able to perform the integrations

Riemann Invariants

For an isentropic processes the **isentropic relations** give:

$$p = c_1 T^{\gamma/(\gamma-1)} = c_2 a^{2\gamma/(\gamma-1)}$$

where c_1 and c_2 are constants and thus

$$dp = c_2 \left(\frac{2\gamma}{\gamma - 1} \right) a^{[2\gamma/(\gamma-1)-1]} da$$

Assume **calorically perfect gas**: $a^2 = \dfrac{\gamma p}{\rho} \Rightarrow \rho = \dfrac{\gamma p}{a^2}$

with $p = c_2 a^{2\gamma/(\gamma-1)}$ we get $\rho = c_2 \gamma a^{[2\gamma/(\gamma-1)-2]}$

Riemann Invariants

$$J^+ = u + \int \frac{dp}{\rho a} = u + \int \frac{c_2 \left(\frac{2\gamma}{\gamma-1}\right) a^{[2\gamma/(\gamma-1)-1]}}{c_2 \gamma a^{[2\gamma/(\gamma-1)-1]}} da = u + \int \frac{2da}{\gamma - 1}$$

$$\boxed{\begin{aligned} J^+ &= u + \frac{2a}{\gamma - 1} \\ J^- &= u - \frac{2a}{\gamma - 1} \end{aligned}}$$

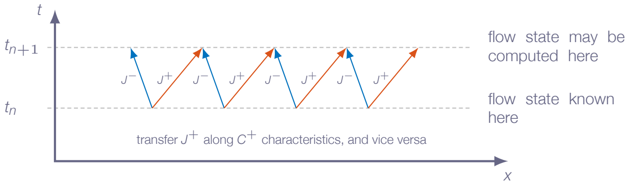
Riemann Invariants

If J^+ and J^- are known at some point $(x,\ t)$, then

$$\left\{\begin{aligned} J^+ + J^- &= 2u \\ J^+ - J^- &= \frac{4a}{\gamma - 1} \end{aligned}\right. \Rightarrow \left\{\begin{aligned} u &= \frac{1}{2}(J^+ + J^-) \\ a &= \frac{\gamma - 1}{4}(J^+ - J^-) \end{aligned}\right.$$

With the Riemann invariants known, the flow state is uniquely defined!

Method of Characteristics



Summary

Acoustic waves

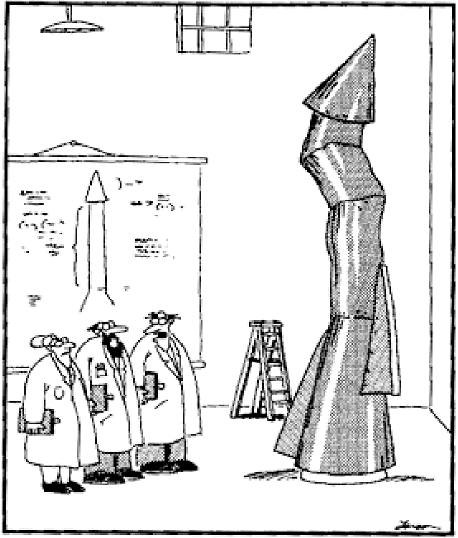
1. $\Delta\rho$, Δu, etc - **very small**
2. All parts of the wave propagate with the same **velocity** a_∞
3. The **wave shape** stays the **same**
4. The flow is governed by **linear relations**

Finite (non-linear) waves

1. $\Delta\rho$, Δu, etc - can be **large**
2. Each local part of the wave propagates at the **local velocity** $(u + a)$
3. The wave **shape changes** with time
4. The flow is governed by **non-linear relations**

Summary

the method of characteristics is a central element in classic compressible flow theory



"It's time we face reality, my friend. ... We're not exactly rocket scientists."