Compressible Flow - TME085

Lecture 7

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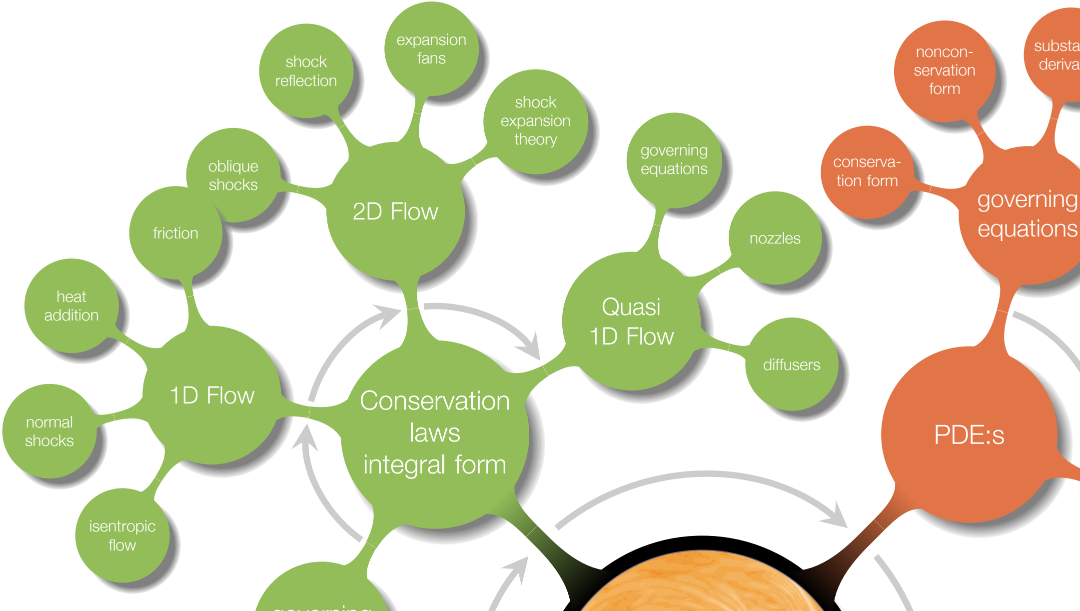
Chapter 5 - Quasi-One-Dimensional Flow

Overview


shock reflection
expansion fans
shock expansion theory
oblique shocks
2D Flow
friction
governing equations
nozzles
Quasi 1D Flow
diffusers
heat addition
1D Flow
normal shocks
isentropic flow
Conservation laws integral form
noncon-servation form
conserva-tion form
governing equations
PDE:s

Learning Outcomes

4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on

6 **Define** the special cases of calorically perfect gas, thermally perfect gas and real gas and **explain** the implication of each of these special cases

7 **Explain** why entropy is important for flow discontinuities

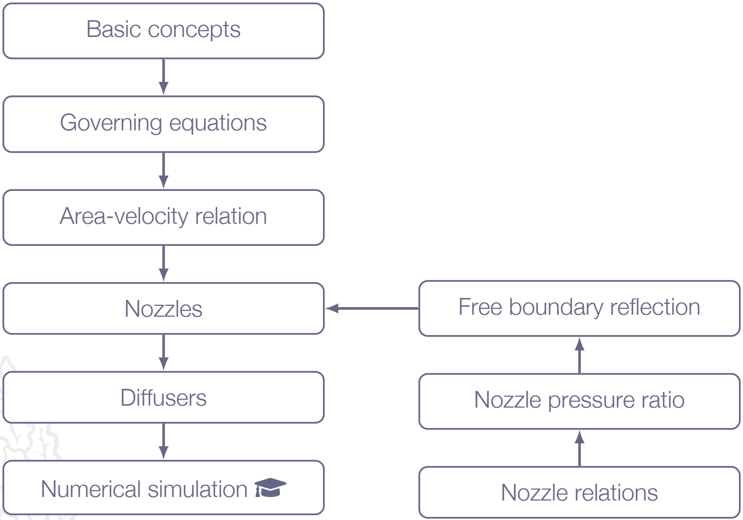
8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics

- a 1D isentropic flow*
- b normal shocks*
- i detached blunt body shocks, nozzle flows

9 **Solve** engineering problems involving the above-mentioned phenomena (8a-8k)

what does quasi-1D mean? either the flow is 1D or not, or?

Roadmap - Quasi-One-Dimensional Flow

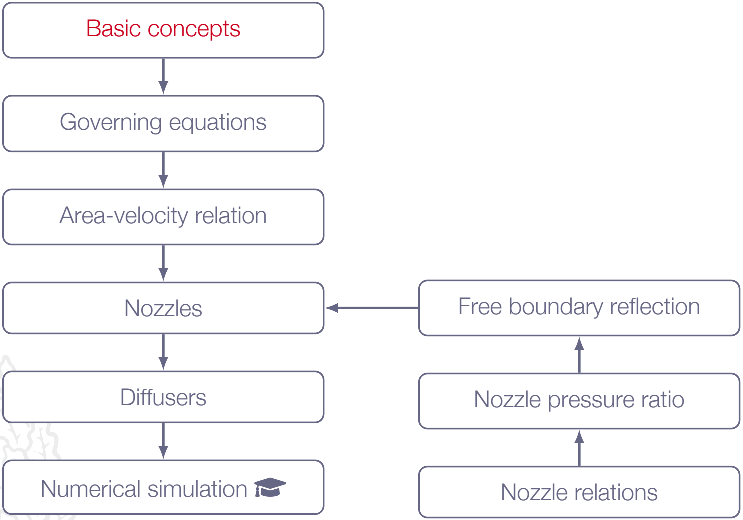


Motivation

By extending the one-dimensional theory to quasi-one-dimensional, we can study important applications such as nozzles and diffusers

Even though the flow in nozzles and diffusers are in essence three dimensional we will be able to establish important relations using the quasi-one-dimensional approach

Roadmap - Quasi-One-Dimensional Flow



Quasi-One-Dimensional Flow

Chapter 3

overall assumption

one-dimensional flow
steady state
constant cross-section area

applications

normal shock
1D flow with heat addition
1D flow with friction

Chapter 4

overall assumption

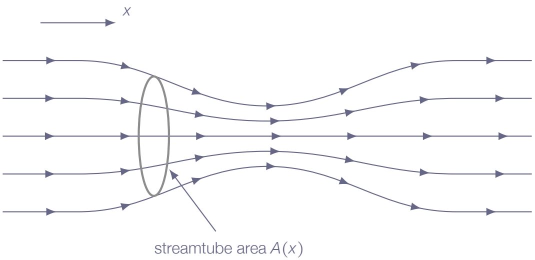
two-dimensional flow
steady state
uniform freestream

applications

oblique shocks
expansion fans
shock-expansion theory

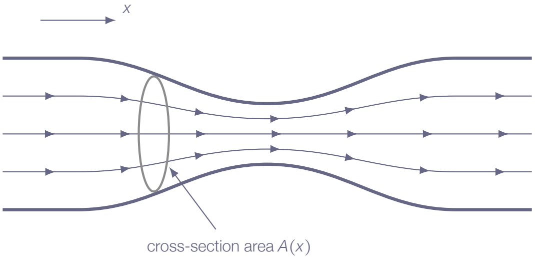
Quasi-One-Dimensional Flow

Extension of one-dimensional flow to allow **variations in streamtube area** (steady-state flow assumption still applied)



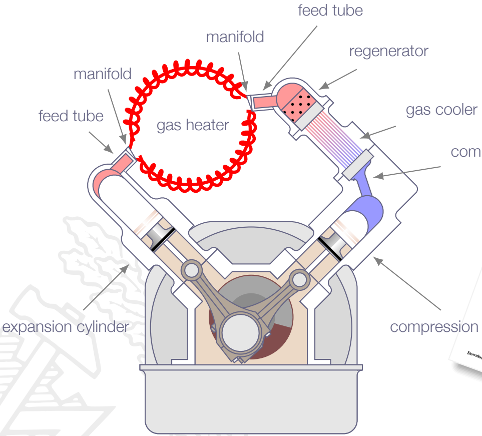
Quasi-One-Dimensional Flow

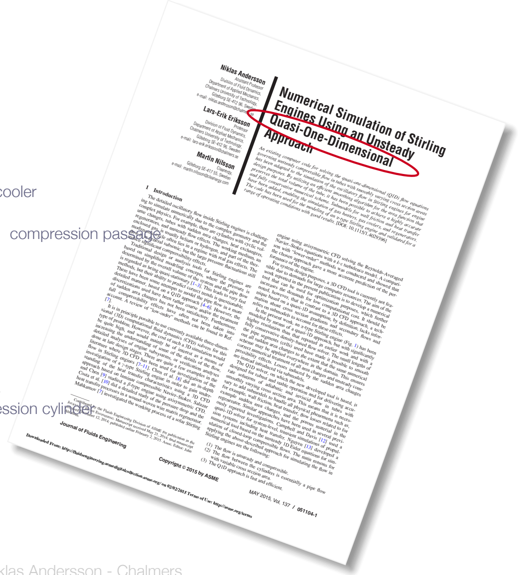
Example: tube with variable cross-section area



Quasi-One-Dimensional Flow - Nozzle Flow

Quasi-One-Dimensional Flow - Stirling Engine





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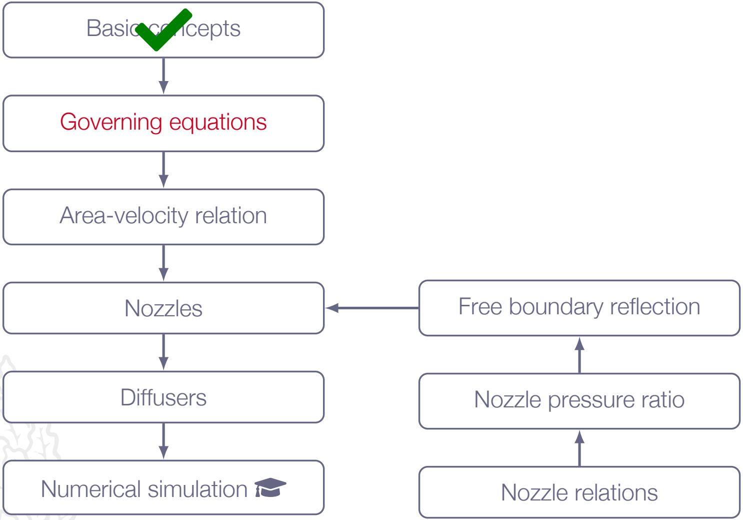
Numerical Simulation of Stirling Engines Using an Unsteady Quasi-One-Dimensional Approach

An existing computer code for solving the quasi-one-dimensional (Q1D) flow equations governing unsteady compressible flow in tubes with smoothly varying cross section areas has been adapted to the simulation of the oscillatory flow in Stirling engines for engine design purposes. By utilizing an efficient smoothing algorithm for the area function that preserves the total volume of the tube, it has been possible to achieve a highly accurate and fully conservative numerical scheme. Submodels for wall friction and heat transfer have been added, enabling the simulation of gas heaters, gas coolers, and regenerators. The code has been used for the modeling of an α-type Stirling engine and validated for a range of operating conditions with good results. [DOI: 10.1115/1.4029396]

1 Introduction

Roadmap - Quasi-One-Dimensional Flow

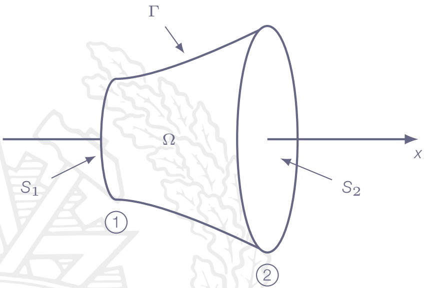


Chapter 5.2
Governing Equations

Governing Equations

Introduce **cross-section-averaged flow quantities** $\Rightarrow$
all quantities depend on x only

$$A = A(x),\ \rho = \rho(x),\ u = u(x),\ p = p(x),\ \ldots$$



Ω	control volume
S_1	left boundary (area A_1)
S_2	right boundary (area A_2)
Γ	perimeter boundary

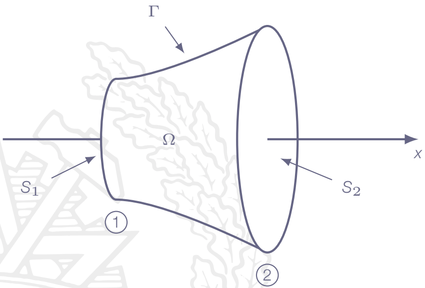
$$\partial\Omega = S_1 \cup \Gamma \cup S_2$$

Governing Equations - Assumptions

1. Inviscid flow (no boundary layers)

2. Steady-state flow (no unsteady effects)

3. No flow through Γ (control volume aligned with streamlines)



Governing Equations - Conservation of Mass

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega} \rho d\mathscr{V}}_{=0} + \underbrace{\oiint\limits_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS}_{-\rho_1 u_1 A_1 + \rho_2 u_2 A_2} = 0$$

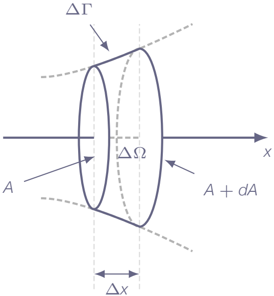
$$\boxed{\rho_1 u_1 A_1 = \rho_2 u_2 A_2}$$

Governing Equations - Conservation of Momentum

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho\mathbf{v}d\mathscr{V}}_{=0} + \oiint\limits_{\partial\Omega}\left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v} + p\mathbf{n}\right]dS = 0$$

$$\oiint\limits_{\partial\Omega}\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}dS = -\rho_1 u_1^2 A_1 + \rho_2 u_2^2 A_2$$

$$\oiint\limits_{\partial\Omega}p\mathbf{n}dS = -p_1 A_1 + p_2 A_2 - \int_{A_1}^{A_2} p dA$$



$$(\rho_1 u_1^2 + p_1)A_1 + \int_{A_1}^{A_2} p dA = (\rho_2 u_2^2 + p_2)A_2$$

Governing Equations - Conservation of Energy

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega} \rho e_o d\mathscr{V}}_{=0} + \oiint\limits_{\partial\Omega} [\rho h_o(\mathbf{v}\cdot\mathbf{n})]\, dS = 0$$

which gives

$$\rho_1 u_1 A_1 h_{o_1} = \rho_2 u_2 A_2 h_{o_2}$$

from continuity we have that $\rho_1 u_1 A_1 = \rho_2 u_2 A_2 \Rightarrow$

$$\boxed{h_{o_1} = h_{o_2}}$$

Governing Equations - Summary

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

$$(\rho_1 u_1^2 + p_1)A_1 + \int_{A_1}^{A_2} p dA = (\rho_2 u_2^2 + p_2)A_2$$

$$h_{o_1} = h_{o_2}$$

Governing Equations - Differential Form

Continuity equation:

$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$ or $\rho u A = c$

where c is a constant $\Rightarrow$

$$d(\rho u A) = 0$$

Governing Equations - Differential Form

Momentum equation:

$$(\rho_1 u_1^2 + p_1)A_1 + \int_{A_1}^{A_2} p dA = (\rho_2 u_2^2 + p_2)A_2 \Rightarrow$$

$$d\left[(\rho u^2 + p)A\right] = pdA \Rightarrow$$

$$d(\rho u^2 A) + d(pA) = pdA \Rightarrow$$

$$u \underbrace{d(\rho u A)}_{=0} + \rho u A du + A dp + pdA = pdA \Rightarrow$$

$$\rho u A du + A dp = 0 \Rightarrow$$

$$\boxed{dp = -\rho u du} \quad \text{(Euler's equation)}$$

Governing Equations - Differential Form

Energy equation:

$$h_{o_1} = h_{o_2} \Rightarrow dh_o = 0$$

$$h_o = h + \frac{1}{2}u^2 \Rightarrow$$

$$\boxed{dh + udu = 0}$$

Governing Equations - Differential Form

Summary (valid for all gases):

$$d(\rho uA) = 0$$

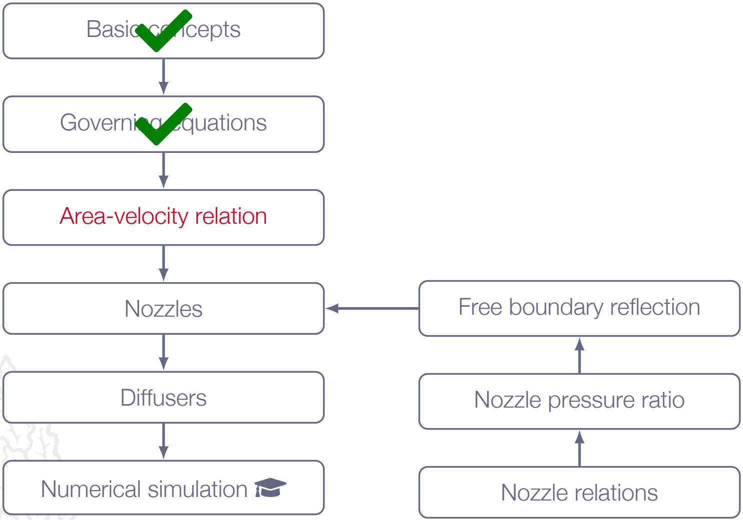
$$dp = -\rho u du$$

$$dh + u du = 0$$

Assumptions:

1. quasi-one-dimensional flow
2. inviscid flow
3. steady-state flow

Roadmap - Quasi-One-Dimensional Flow



Chapter 5.3
Area-Velocity Relation

Area-Velocity Relation

$$d(\rho u A) = 0 \Rightarrow u A d\rho + \rho A du + \rho u dA = 0$$

divide by $\rho u A$ gives

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

Euler's equation:

$$dp = -\rho u du \Rightarrow \frac{dp}{\rho} = \frac{dp}{d\rho}\frac{d\rho}{\rho} = -u du$$

Assuming adiabatic, reversible (isentropic) process and the definition of speed of sound gives

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s = a^2 \Rightarrow a^2 \frac{d\rho}{\rho} = -u du \Rightarrow \frac{d\rho}{\rho} = -M^2 \frac{du}{u}$$

Area-Velocity Relation

Now, inserting the expression for $\frac{d\rho}{\rho}$ in the rewritten continuity equation gives

$$(1 - M^2)\frac{du}{u} + \frac{dA}{A} = 0$$

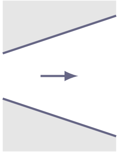
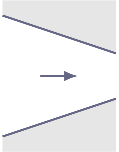
or

$$\frac{dA}{A} = (M^2 - 1)\frac{du}{u}$$

which is the **area-velocity relation**

The Area-Velocity Relation

$$\frac{dA}{A} = \frac{du}{u}(M^2 - 1)$$

		Subsonic $M < 1$	Supersonic $M > 1$
	$dA > 0$	subsonic diffuser $du < 0$ $dp > 0$	supersonic nozzle $du > 0$ $dp < 0$
	$dA < 0$	subsonic nozzle $du > 0$ $dp < 0$	supersonic diffuser $du < 0$ $dp > 0$

The Area-Velocity Relation

$$\frac{du}{u}(M^2 - 1) = \frac{dA}{A}$$

What happens when $M = 1$?

The Area-Velocity Relation

$$\frac{du}{u}(M^2 - 1) = \frac{dA}{A}$$

What happens when $M = 1$?

$M = 1$ when $dA = 0$

The Area-Velocity Relation

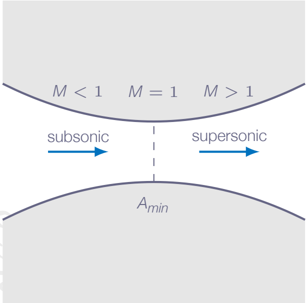
$$\frac{du}{u}(M^2-1)=\frac{dA}{A}$$

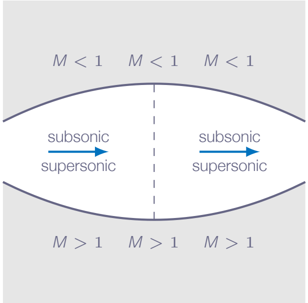
What happens when $M = 1$?

$M = 1$ when $dA = 0$

maximum or minimum area

The Area-Velocity Relation





The Area-Velocity Relation

A converging-diverging nozzle is the **only possibility** to obtain supersonic flow!

A supersonic flow entering a convergent-divergent nozzle will slow down and, if the conditions are right, become sonic at the throat - hard to obtain a shock-free flow in this case

Area-Velocity Relation

$$M \rightarrow 0 \Rightarrow \frac{dA}{A} = -\frac{du}{u}$$

$$\frac{dA}{A} + \frac{du}{u} = 0 \Rightarrow$$

$$\frac{1}{Au}\left[udA + Adu\right] = 0 \Rightarrow$$

$$d(uA) = 0 \Rightarrow Au = c$$

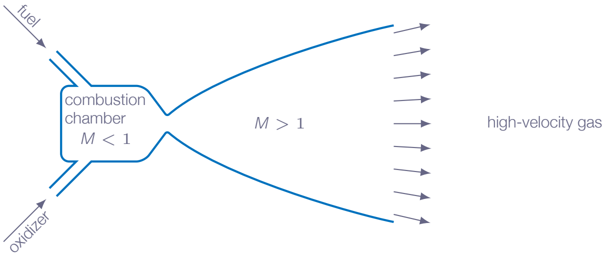
where c is a constant

Area-Velocity Relation

Note 1 The area-velocity relation is only valid for isentropic flow
not valid across a compression shock (due to entropy increase)

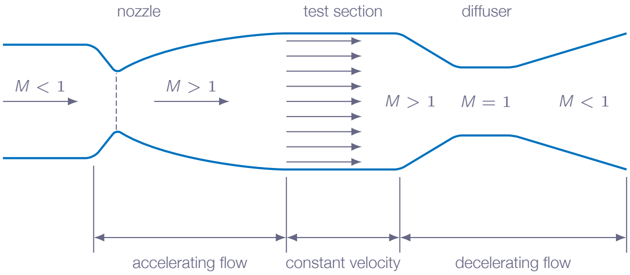
Note 2 The area-velocity relation is valid for all gases

Area-Velocity Relation Examples - Rocket Engine

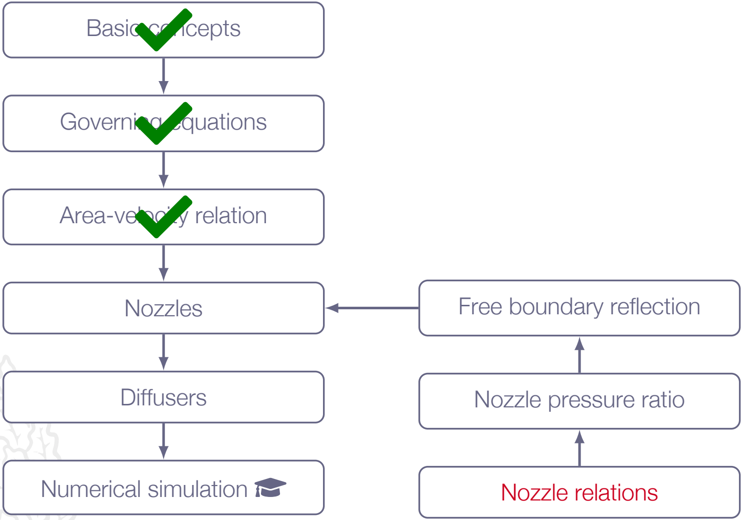


High-temperature, high-pressure gas in combustion chamber expand through the nozzle to very high velocities. Typical figures for a LH^2/LOx rocket engine: $p_o \sim 120$ [bar], $T_o \sim 3600$ [K], exit velocity ~ 4000 [m/s]

Area-Velocity Relation Examples - Wind Tunnel



Roadmap - Quasi-One-Dimensional Flow



Chapter 5.4
Nozzles

Nozzle Flow with Varying Pressure Ratio

time for rocket science!

Nozzle Flow - Relations

Calorically perfect gas assumed:

From Chapter 3:

$$\frac{T_o}{T} = \left(\frac{a_o}{a}\right)^2 = 1 + \frac{1}{2}(\gamma - 1)M^2$$

$$\frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{\frac{1}{\gamma-1}}$$

Nozzle Flow - Relations

Critical conditions:

$$\frac{T_o}{T^*} = \left(\frac{a_o}{a^*}\right)^2 = \frac{1}{2}(\gamma + 1)$$

$$\frac{p_o}{p^*} = \left(\frac{T_o}{T^*}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_o}{\rho^*} = \left(\frac{T_o}{T^*}\right)^{\frac{1}{\gamma-1}}$$

Nozzle Flow - Relations

$$M^{*2} = \frac{u^2}{a^{*2}} = \frac{u^2}{a^2}\frac{a^2}{a^{*2}} = \frac{u^2}{a^2}\frac{a^2}{a_o^2}\frac{a_o^2}{a^{*2}} \Rightarrow$$

$$\left.\begin{aligned} &\frac{u^2}{a^2} = M^2 \\ &\frac{a^2}{a_o^2} = \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-1} \\ &\frac{a_o^2}{a^{*2}} = \frac{1}{2}(\gamma + 1) \end{aligned}\right\} \Rightarrow M^{*2} = M^2 \frac{\frac{1}{2}(\gamma + 1)}{1 + \frac{1}{2}(\gamma - 1)M^2}$$

Nozzle Flow - Relations

For nozzle flow we have

$$\rho u A = c$$

where c is a constant and therefore

$$\rho^* u^* A^* = \rho u A$$

or, since at critical conditions $u^* = a^*$

$$\rho^* a^* A^* = \rho u A$$

which gives

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{a^*}{u} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{a^*}{u}$$

Nozzle Flow - Relations

$$\frac{A}{A^*}=\frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{a^*}{u}$$

$$\left.\begin{aligned}\frac{\rho^*}{\rho_o}&=\left(\frac{T_o}{T^*}\right)^{\frac{-1}{\gamma-1}}\\ \frac{\rho_o}{\rho}&=\left(\frac{T_o}{T}\right)^{\frac{1}{\gamma-1}}\\ \frac{a^*}{u}&=\frac{1}{M^*}\end{aligned}\right\}\Rightarrow\frac{A}{A^*}=\frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{1}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{1}{\gamma-1}}M^*}$$

Nozzle Flow - Relations

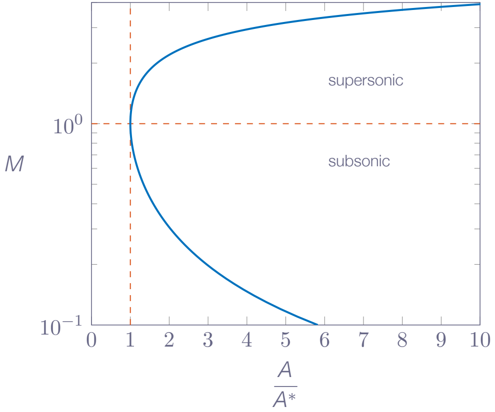
$$\left.\begin{aligned}\left(\frac{A}{A^*}\right)^2 &= \frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{2}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{2}{\gamma-1}} {M^*}^2} \\ {M^*}^2 &= M^2\frac{\frac{1}{2}(\gamma+1)}{1+\frac{1}{2}(\gamma-1)M^2}\end{aligned}\right\} \Rightarrow$$

$$\left(\frac{A}{A^*}\right)^2 = \frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{\gamma+1}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{\gamma+1}{\gamma-1}} M^2}$$

which is the **area-Mach-number relation**

The Area-Mach-Number Relation

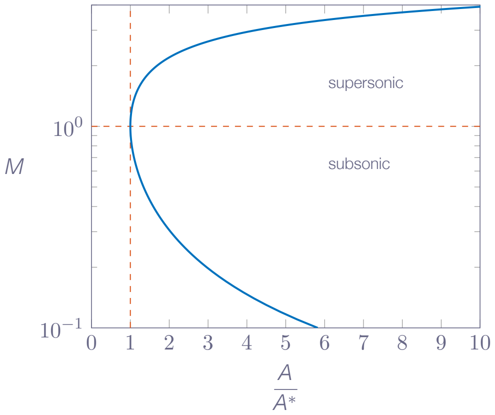
$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{(\gamma+1)/(\gamma-1)}$$



The Area-Mach-Number Relation

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{(\gamma+1)/(\gamma-1)}$$

Note! $\dfrac{A}{A^*} = \dfrac{\rho^* u^*}{\rho u}$



Area-Mach-Number Relation

Note 1 Critical conditions used here are those corresponding to **isentropic flow**. Do not confuse these with the conditions in the cases of one-dimensional flow with heat addition and friction

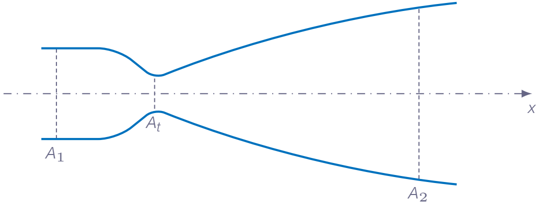
Note 2 For quasi-one-dimensional flow, assuming inviscid steady-state flow, both **total and critical conditions are constant along streamlines** unless shocks are present (then the flow is no longer isentropic)

Note 3 The derived area-Mach-number relation is **only valid for calorically perfect gas and for isentropic flow**. It is not valid across a compression shock

Nozzle Flow

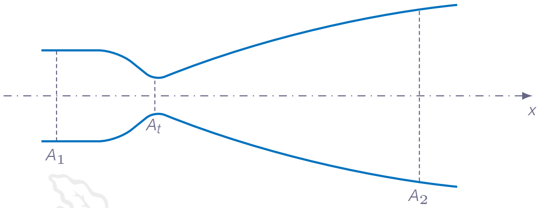
Assumptions:

1. inviscid
2. steady-state
3. quasi-one-dimensional
4. calorically perfect gas



The Area-Mach-Number Relation

Sub-critical (non-choked) nozzle flow

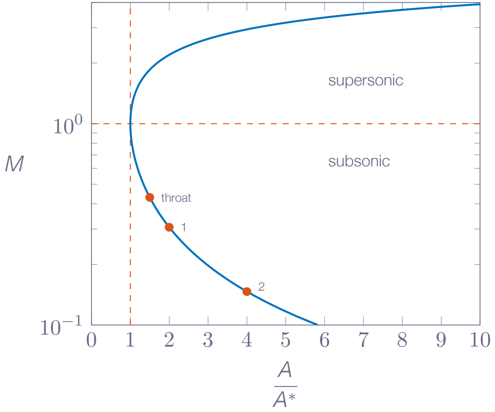


$M < 1$ at nozzle throat

$A_t > A^*$

$M_1 < 1$

$M_2 < 1$



The Area-Mach-Number Relation

Subcritical nozzle flow (non-choked and subsonic $\Rightarrow$ isentropic):

A^* is constant throughout the nozzle ($A^* < A_t$)

M_1 given by the subsonic solution of

$$\left(\frac{A_1}{A^*}\right)^2 = \frac{1}{M_1^2}\left[\frac{2}{\gamma+1}(1+\frac{1}{2}(\gamma-1)M_1^2)\right]^{\frac{\gamma+1}{\gamma-1}}$$

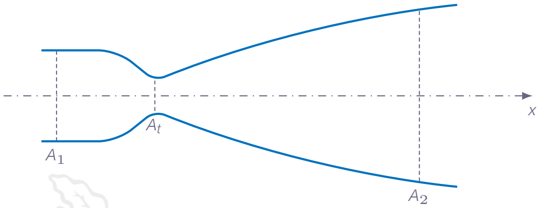
M_2 given by the subsonic solution of

$$\left(\frac{A_2}{A^*}\right)^2 = \frac{1}{M_2^2}\left[\frac{2}{\gamma+1}(1+\frac{1}{2}(\gamma-1)M_2^2)\right]^{\frac{\gamma+1}{\gamma-1}}$$

M is uniquely determined everywhere in the nozzle, with subsonic flow both upstream and downstream of the throat

The Area-Mach-Number Relation

Critical (choked) nozzle flow

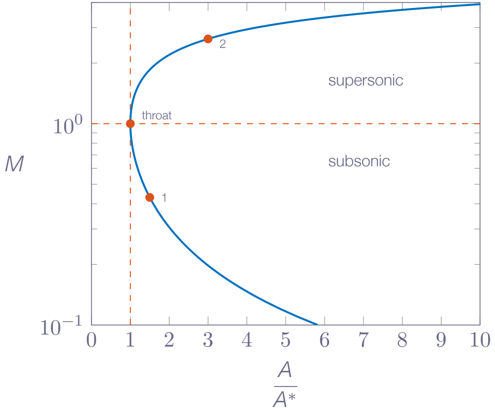


$M = 1$ at nozzle throat

$A_t = A^*$

$M_1 < 1$

$M_2 > 1$



The Area-Mach-Number Relation

Supercritical nozzle flow (choked flow without shocks $\Rightarrow$ isentropic):

A^* is constant throughout the nozzle ($A^* = A_t$)

M_1 given by the subsonic solution of

$$\left(\frac{A_1}{A^*}\right)^2 = \left(\frac{A_1}{A_t}\right)^2 = \frac{1}{M_1^2}\left[\frac{2}{\gamma+1}(1+\frac{1}{2}(\gamma-1)M_1^2)\right]^{\frac{\gamma+1}{\gamma-1}}$$

M_2 given by the supersonic solution of

$$\left(\frac{A_2}{A^*}\right)^2 = \left(\frac{A_2}{A_t}\right)^2 = \frac{1}{M_2^2}\left[\frac{2}{\gamma+1}(1+\frac{1}{2}(\gamma-1)M_2^2)\right]^{\frac{\gamma+1}{\gamma-1}}$$

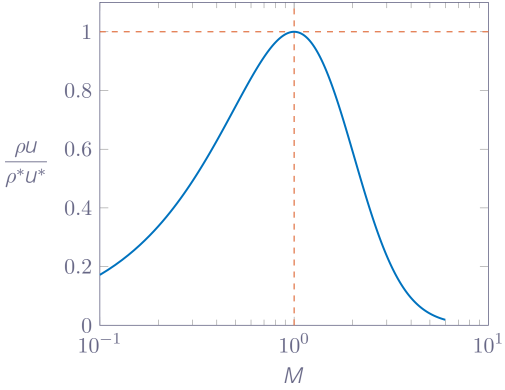
M is uniquely determined everywhere in the nozzle, with subsonic flow upstream of the throat and supersonic flow downstream of the throat

Nozzle Mass Flow

$$\rho u A = \rho^* A^* u^* \Rightarrow \frac{A^*}{A} = \frac{\rho u}{\rho^* u^*}$$

From the area-Mach-number relation

$$\frac{A^*}{A} = \begin{cases} < 1 & \text{if} \quad M < 1 \\ 1 & \text{if} \quad M = 1 \\ < 1 & \text{if} \quad M > 1 \end{cases}$$



The maximum possible massflow through a duct is achieved when its throat reaches sonic conditions

Nozzle Mass Flow

For a choked nozzle:

$$\dot{m} = \rho_1 u_1 A_1 = \rho^* u^* A^* = \rho_2 u_2 A_2$$

$$\left.\begin{aligned} \rho^* &= \frac{\rho^*}{\rho_o}\rho_o = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \frac{p_o}{RT_o} \\ a^* &= \frac{a^*}{a_o}a_o = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}} \sqrt{\gamma RT_o} \end{aligned}\right\} \Rightarrow$$

$$\boxed{\dot{m} = \frac{p_o A_t}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}}$$

Nozzle Mass Flow

$$\dot{m} = \frac{p_o A_t}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R} \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma+1}{\gamma-1}}}$$

The **maximum mass flow** that can be sustained through the nozzle

Valid for quasi-one-dimensional, inviscid, steady-state flow and calorically perfect gas

Note! The massflow formula is valid even if there are shocks present downstream of throat!

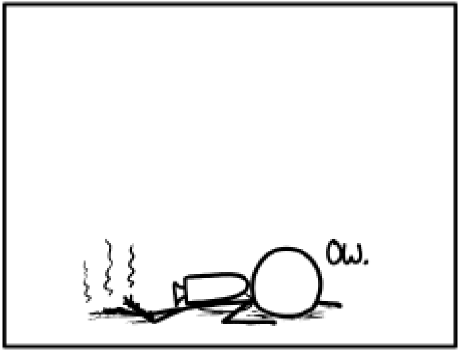
Nozzle Mass Flow

$$\dot{m} = \frac{p_o A_t}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R} \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma+1}{\gamma-1}}}$$

How can we increase mass flow through nozzle?

1. increase p_o
2. decrease T_o
3. increase A_t
4. decrease R
   (increase molecular weight, without changing γ)

ROCKET PACKS ARE EASY.



THE HARD PART IS INVENTING
THE CALF SHIELDS.