Compressible Flow - TME085

Chapter 7

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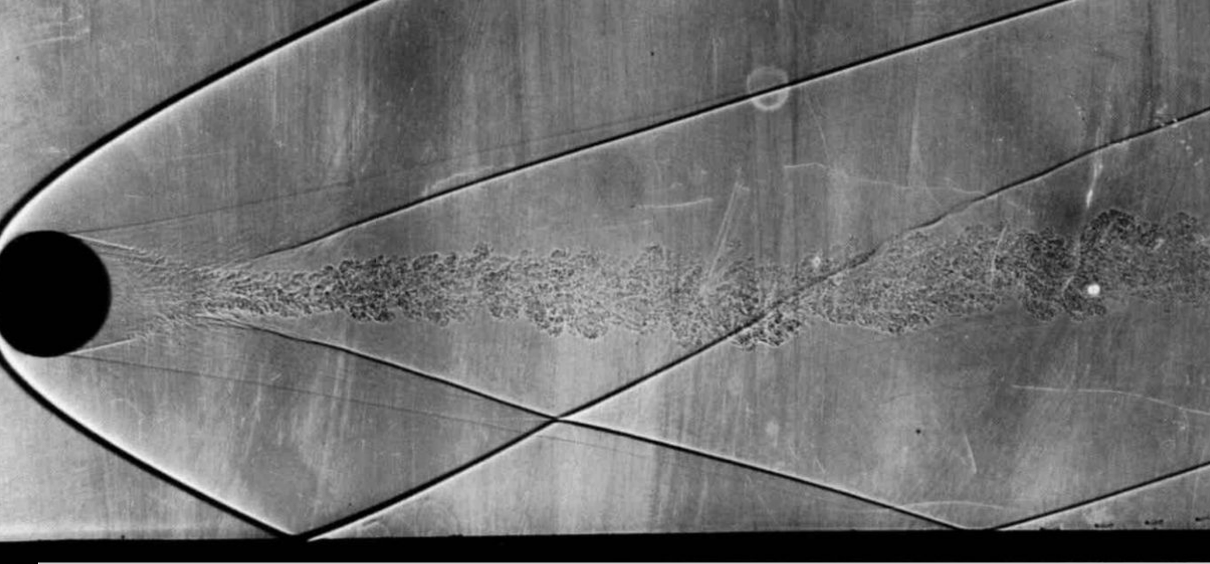
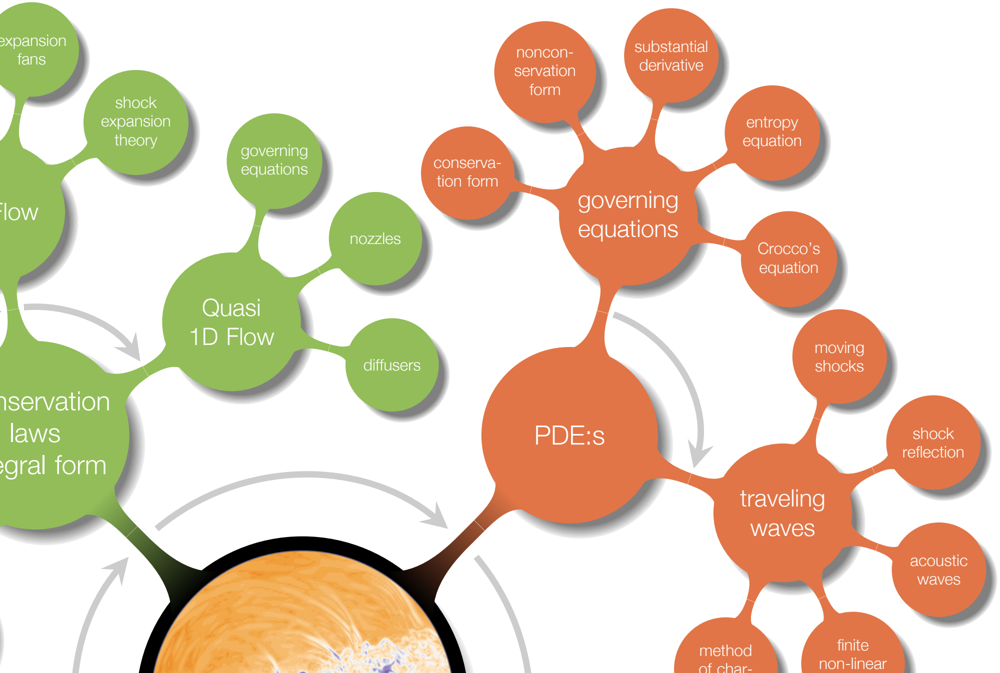


Chapter 7 - Unsteady Wave Motion

Overview



expansion fans
shock expansion theory
Flow
governing equations
nozzles
Quasi 1D Flow
diffusers
nservation laws egral form
noncon-servation form
substantial derivative
entropy equation
conserva-tion form
governing equations
Crocco’s equation
PDE:s
moving shocks
shock reflection
traveling waves
acoustic waves
method of char-
finite non-linear

Learning Outcomes

3 **Describe** typical engineering flow situations in which compressibility effects are more or less predominant (e.g. Mach number regimes for steady-state flows)

4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on

8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics

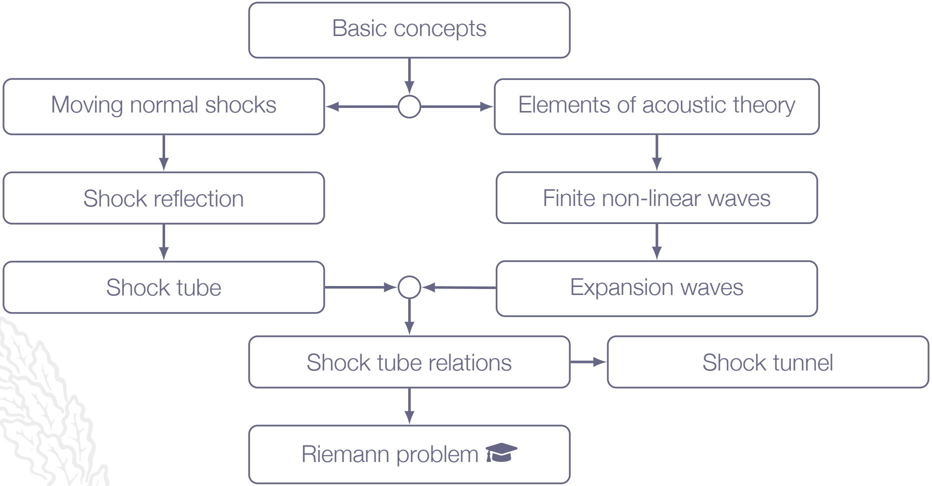
- a 1D isentropic flow*
- b normal shocks*
- j unsteady waves and discontinuities in 1D
- k basic acoustics

9 **Solve** engineering problems involving the above-mentioned phenomena (8a-8k)

11 **Explain** how the equations for aero-acoustics and classical acoustics are derived as limiting cases of the compressible flow equations

moving normal shocks - frame of reference seems to be the key here?!

Roadmap - Unsteady Wave Motion



Motivation

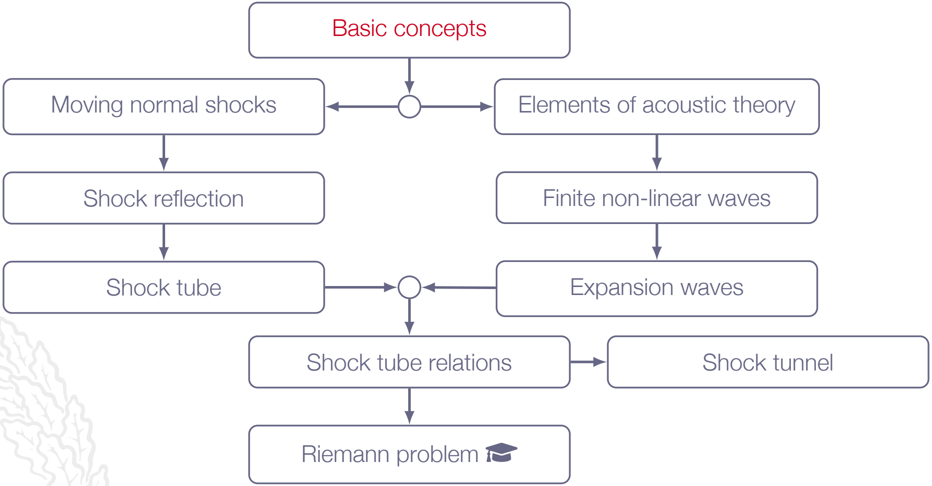
Most practical flows are unsteady

Traveling waves appears in many real-life situations and is an important topic within compressible flows

We will study unsteady flows in one dimension in order to reduce complexity and focus on the physical effects introduced by the unsteadiness

Throughout this section, we will study an application called the shock tube, which is a rather rare application but it lets us study unsteady waves in one dimension and it includes all physical principles introduced in chapter 7

Roadmap - Unsteady Wave Motion



Unsteady Wave Motion - Example #1

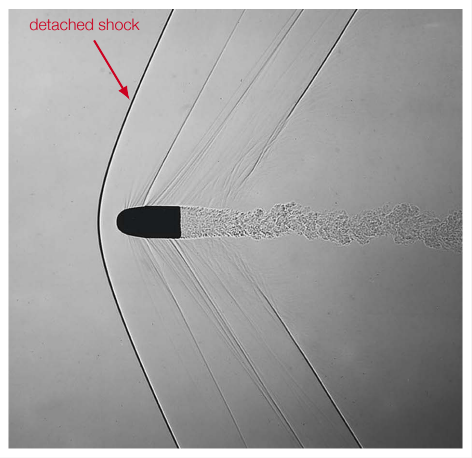
Object moving with supersonic speed through the air

observer moving with the bullet

- steady-state flow
- the detached shock wave is **stationary**

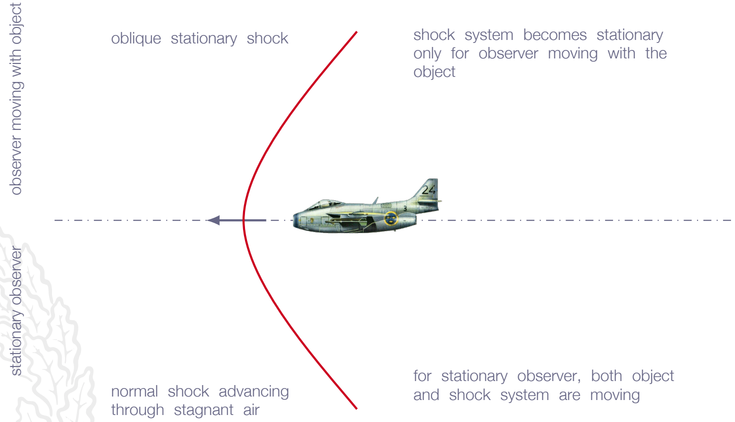
observer at rest

- unsteady flow
- detached **shock wave moves** through the air (to the left)



Unsteady Wave Motion - Example #2

Object moving with supersonic speed through the air



Unsteady Wave Motion - Example #3

Shock wave from explosion

MID-AIR EXPLOSION

GROUND EXPLOSION

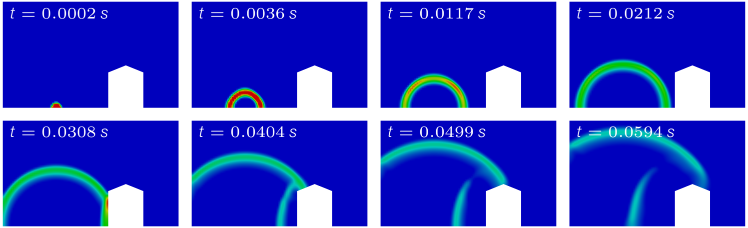
For observer at rest with respect to the surrounding air:

1. the flow is **unsteady**

2. the **shock wave moves** through the air

Unsteady Wave Motion - Example #4

Shock wave from explosion



normal shock moving spherically outwards:

shock **strength decreases** with radius

shock **speed decreases** with radius

Unsteady Wave Motion

inertial frames!

Physical laws are the same for both frame of references

Shock characteristics are the same for both observers (shape, strength, etc)

Recall - the Hugonoit relation does not include velocities, only static thermodynamic quantities that are independent of reference frame

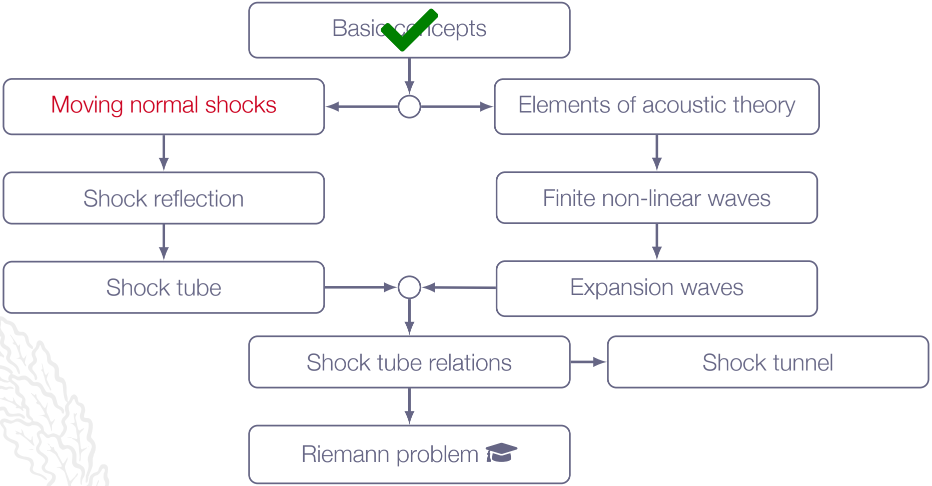
Unsteady Wave Motion

Is there a connection with stationary shock waves?

Answer: Yes!

Locally, in a **moving frame of reference**, the shock may be viewed as a **stationary normal shock**

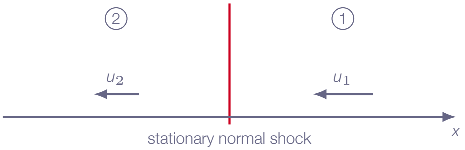
Roadmap - Unsteady Wave Motion



Chapter 7.2
Moving Normal Shock Waves

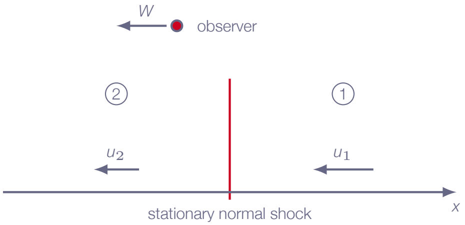
Moving Normal Shock Waves

Chapter 3: stationary normal shock



$u_1 > a_1$	(supersonic flow)
$u_2 < a_2$	(subsonic flow)
$p_2 > p_1$	(sudden compression)
$s_2 > s_1$	(shock loss)

Moving Normal Shock Waves



Introduce observer moving to the left with speed W

- if W is constant the observer is still in an inertial system (all physical laws are unchanged)

The observer sees a normal shock moving to the right with speed W

- gas velocity ahead of shock: $u_1' = W - u_1$
- gas velocity behind shock: $u_2' = W - u_2$

Moving Normal Shock Waves

Now, let $W = u_1 \Rightarrow$

$$u_1' = 0$$

$$u_2' = u_1 - u_2 > 0$$

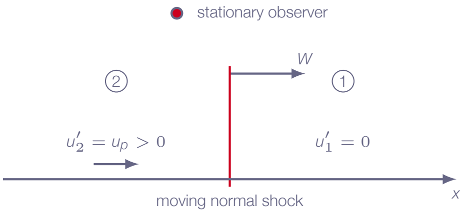
The observer now sees the shock traveling to the right with speed $W = u_1$ into a stagnant gas, leaving a compressed gas ($p_2 > p_1$) with velocity $u_2' > 0$ behind it

Introducing u_p:

$$u_p = u_2' = u_1 - u_2$$

Moving Normal Shock Waves

Analogy:



Case 1

stationary normal shock

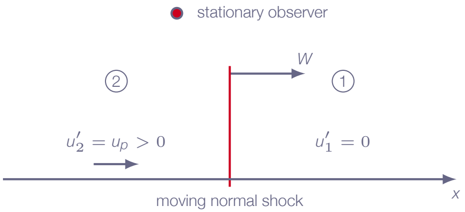
observer moving with velocity W

Case 2

normal shock moving with velocity W

stationary observer

Moving Normal Shock Waves - Governing Equations



For stationary normal shocks we have:

$$\rho_1 u_1 = \rho_2 u_2$$
$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$
$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

With $(u_1 = W)$ and $(u_2 = W - u_p)$ we get:

$$\rho_1 W = \rho_2 (W - u_p)$$
$$\rho_1 W^2 + p_1 = \rho_2 (W - u_p)^2 + p_2$$
$$h_1 + \frac{1}{2}W^2 = h_2 + \frac{1}{2}(W - u_p)^2$$

Moving Normal Shock Waves - Relations

Starting from the governing equations

$$\rho_1 W = \rho_2(W - u_p)$$

$$\rho_1 W^2 + p_1 = \rho_2(W - u_p)^2 + p_2$$

$$h_1 + \frac{1}{2}W^2 = h_2 + \frac{1}{2}(W - u_p)^2$$

and using $h = e + \dfrac{p}{\rho}$

it is possible to show that

$$e_2 - e_1 = \frac{p_1 + p_2}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

Moving Normal Shock Waves - Relations

$$e_2 - e_1 = \frac{p_1 + p_2}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

same Hugoniot equation as for stationary normal shock

This means that we will have same shock strength, *i.e.* same discontinuities in density, velocity, pressure, etc

Moving Normal Shock Waves - Relations

Starting from the Hugoniot equation one can show that

$$\frac{\rho_2}{\rho_1} = \frac{1 + \dfrac{\gamma+1}{\gamma-1}\left(\dfrac{p_2}{p_1}\right)}{\dfrac{\gamma+1}{\gamma-1} + \dfrac{p_2}{p_1}}$$

and

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\left[\frac{\dfrac{\gamma+1}{\gamma-1} + \dfrac{p_2}{p_1}}{1 + \dfrac{\gamma+1}{\gamma-1}\left(\dfrac{p_2}{p_1}\right)}\right]$$

Moving Normal Shock Waves - Relations

For calorically perfect gas and stationary normal shock:

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_s^2 - 1)$$

same as eq. (3.57) in Anderson with $M_1 = M_s$

where

$$M_s = \frac{W}{a_1}$$

M_s is simply the speed of the shock, traveling into the stagnant gas, normalized by the speed of sound in the gas ahead of the shock

Note!

$M_s > 1$, otherwise there is no shock!

shocks always moves faster than sound - no warning before it hits you ☺

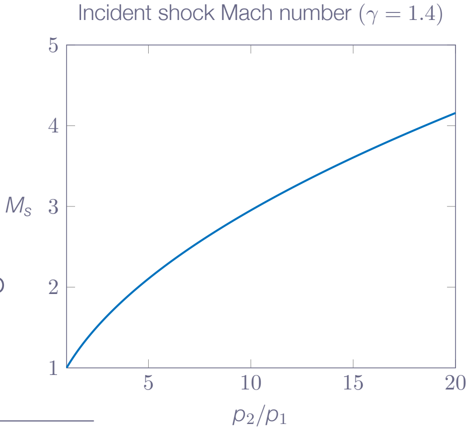
Moving Normal Shock Waves - Relations

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_s^2 - 1)$$

Re-arrange $\Rightarrow$

$$M_s = \sqrt{\frac{\gamma+1}{2\gamma}\left(\frac{p_2}{p_1} - 1\right) + 1}$$

shock speed directly linked to pressure ratio



$$M_s = \frac{W}{a_1} \Rightarrow W = a_1 M_s = a_1\sqrt{\frac{\gamma+1}{2\gamma}\left(\frac{p_2}{p_1} - 1\right) + 1}$$

Moving Normal Shock Waves - Induced Flow Velocity

From the continuity equation we get:

$$u_p = W\left(1 - \frac{\rho_1}{\rho_2}\right) > 0$$

After some derivation we obtain:

$$u_p = \frac{a_1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left[\frac{\frac{2\gamma}{\gamma+1}}{\frac{p_2}{p_1} + \frac{\gamma-1}{\gamma+1}}\right]^{1/2}$$

Moving Normal Shock Waves - Induced Flow Mach Number

$$M_p = \frac{u_p}{a_2} = \frac{u_p}{a_1}\frac{a_1}{a_2} = \frac{u_p}{a_1}\sqrt{\frac{T_1}{T_2}}$$

inserting u_p/a_1 and T_1/T_2 from relations on previous slides we get:

$$M_p = \frac{1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left[\frac{\frac{2\gamma}{\gamma+1}}{\frac{\gamma-1}{\gamma+1} + \frac{p_2}{p_1}}\right]^{1/2}\left[\frac{1 + \left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right)}{\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right) + \left(\frac{p_2}{p_1}\right)^2}\right]^{1/2}$$

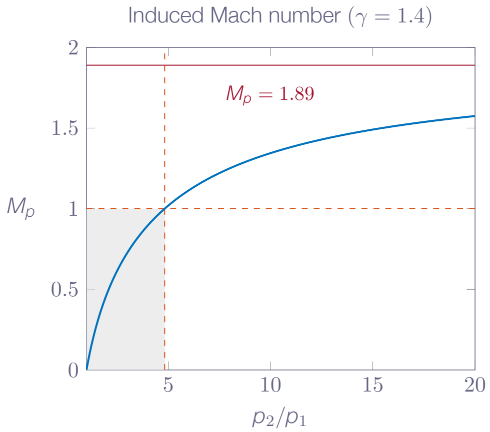
Moving Normal Shock Waves - Induced Flow Mach Number

Note!

$$\lim_{\frac{p_2}{p_1}\to\infty} M_p \to \sqrt{\frac{2}{\gamma(\gamma-1)}}$$

for air ($\gamma = 1.4$)

$$\lim_{\frac{p_2}{p_1}\to\infty} M_p \to 1.89$$



Moving Normal Shock Waves - Example

Moving normal shock with $p_2/p_1 = 10$

($p_1 = 1.0\ bar$, $T_1 = 300\ K$, $\gamma = 1.4$)

$\Rightarrow M_s = 2.95$ and $W = 1024.2\ m/s$

The shock is advancing with almost **three times** the speed of sound!

Behind the shock the induced velocity is $u_p = 756.2\ m/s \Rightarrow$ supersonic flow ($a_2 = 562.1\ m/s$)

May be calculated by formulas 7.13, 7.16, 7.10, 7.11 or by using Table A.2 for stationary normal shock ($u_1 = W$, $u_2 = W - u_p$)

Moving Normal Shock Waves - Total Enthalpy

Note! $h_{o_1} \neq h_{o_2}$

constant total enthalpy is **only valid for stationary shocks!**

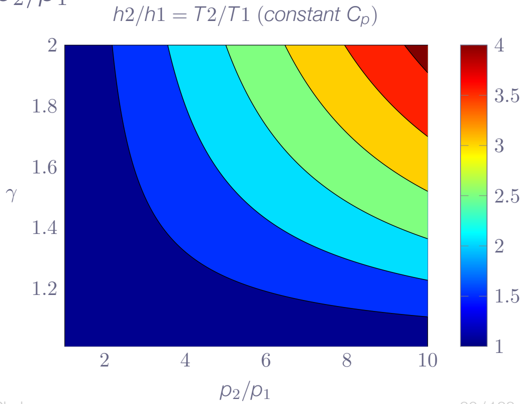
shock is uniquely defined by pressure ratio p_2/p_1

$$u_1 = 0$$

$$h_{o_1} = h_1 + \frac{1}{2}u_1^2 = h_1$$

$$h_{o_2} = h_2 + \frac{1}{2}u_2^2$$

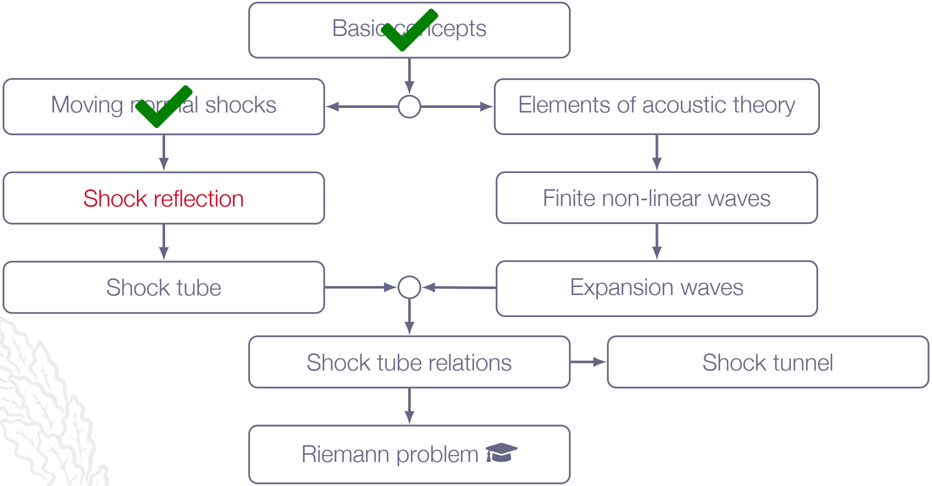
$$h_2 > h_1 \Rightarrow h_{o_2} > h_{o_1}$$



Moving Normal Shock Waves - Total Enthalpy

Gas/Vapor	Ratio of specific heats (γ)	Gas constant R
Acetylene	1.23	319
Air (standard)	1.40	287
Ammonia	1.31	530
Argon	1.67	208
Benzene	1.12	100
Butane	1.09	143
Carbon Dioxide	1.29	189
Carbon Disulphide	1.21	120
Carbon Monoxide	1.40	297
Chlorine	1.34	120
Ethane	1.19	276
Ethylene	1.24	296
Helium	1.67	2080
Hydrogen	1.41	4120
Hydrogen chloride	1.41	230
Methane	1.30	518
Natural Gas (Methane)	1.27	500
Nitric oxide	1.39	277
Nitrogen	1.40	297
Nitrous oxide	1.27	180
Oxygen	1.40	260
Propane	1.13	189
Steam (water)	1.32	462
Sulphur dioxide	1.29	130

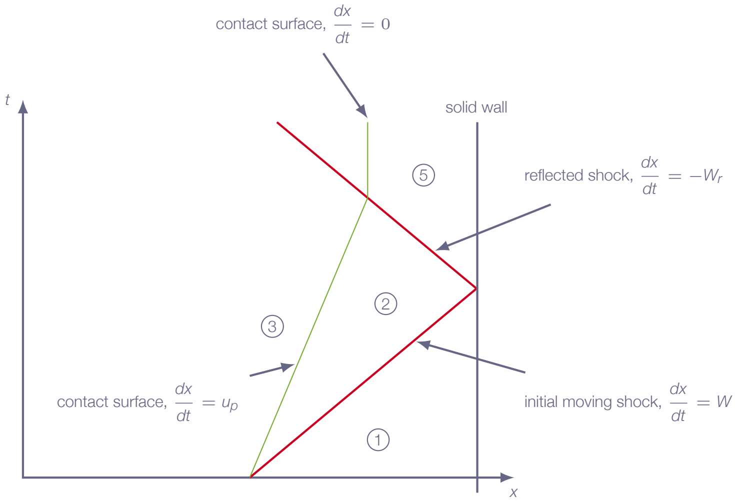
Roadmap - Unsteady Wave Motion



Chapter 7.3
Reflected Shock Wave

what happens when a moving shock approaches a wall?

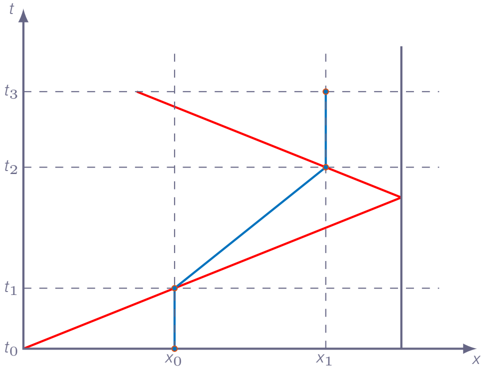
Shock Reflection



Shock Reflection - Particle Path

A fluid particle located at x_0 at time t_0 (a location ahead of the shock) will be affected by the moving shock and follow the blue path

time	location	velocity
t_0	x_0	0
t_1	x_0	u_p
t_2	x_1	u_p
t_3	x_1	0



Shock Reflection Relations

In the frame of reference of the reflected shock we have

velocity ahead of shock: $W_r + u_p$

velocity behind shock: W_r

where W_r is the velocity of the reflected shock and u_p is the induced flow velocity behind the incident shock

Shock Reflection Relations

Continuity:

$$\rho_2(W_r + u_p) = \rho_5 W_r$$

Momentum:

$$p_2 + \rho_2(W_r + u_p)^2 = p_5 + \rho_5 W_r^2$$

Energy:

$$h_2 + \frac{1}{2}(W_r + u_p)^2 = h_5 + \frac{1}{2}W_r^2$$

Shock Reflection Relations

Reflected shock is determined such that $u_5 = 0$

$$\frac{M_r}{M_r^2 - 1} = \frac{M_s}{M_s^2 - 1}\sqrt{1 + \frac{2(\gamma - 1)}{(\gamma + 1)^2}(M_s^2 - 1)\left(\gamma + \frac{1}{M_s^2}\right)}$$

where

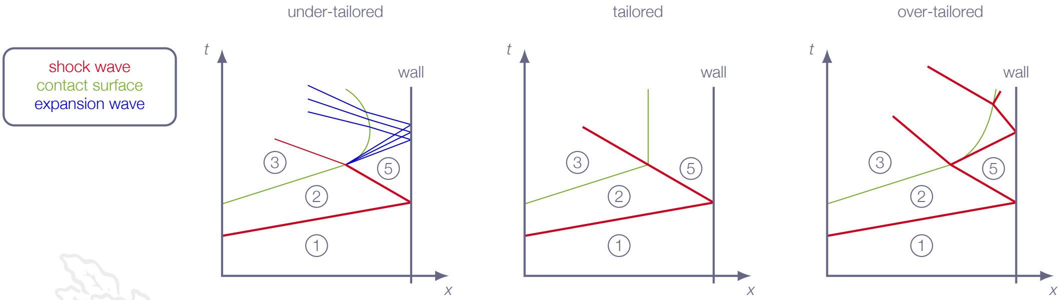
$$M_r = \frac{W_r + u_p}{a_2}$$

Tailored v.s. Non-Tailored Shock Reflection

The time duration of condition 5 is determined by what happens after interaction between reflected shock and contact discontinuity

For special choice of initial conditions (tailored case), this interaction is negligible, thus prolonging the duration of condition 5

Tailored v.s. Non-Tailored Shock Reflection



Under-tailored conditions:

Mach number of incident wave lower than in tailored conditions

Over-tailored conditions:

Mach number of incident wave higher than in tailored conditions

Shock Reflection - Example

Shock reflection in shock tube ($\gamma = 1.4$)
(Example 7.1 in Anderson)

Given data

p_2/p_1	10.0	
T_2/T_1	2.623	
p_1	1.0	bar
T_1	300.0	K

Calculated data

M_s	2.95
M_r	2.09
p_5/p_2	4.978
T_5/T_2	1.77

$$p_5 = \left(\frac{p_5}{p_2}\right)\left(\frac{p_2}{p_1}\right)p_1 = 49.78$$

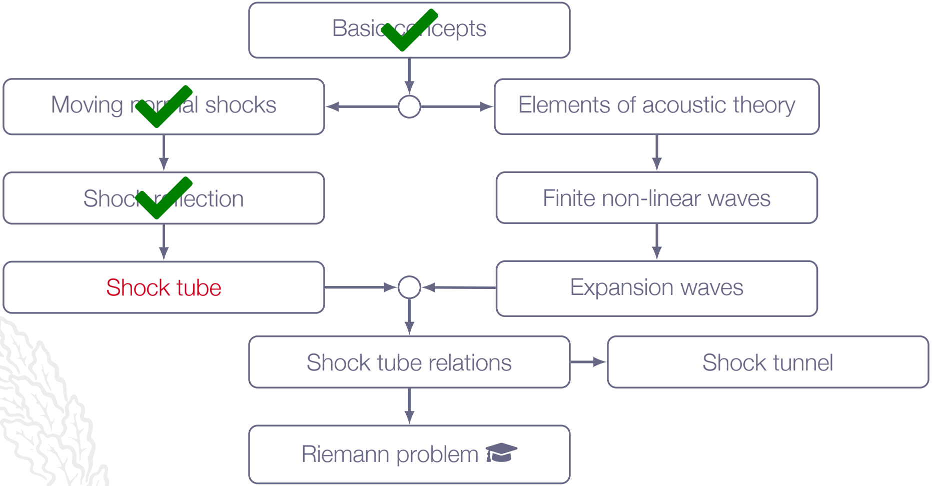
$$T_5 = \left(\frac{T_5}{T_2}\right)\left(\frac{T_2}{T_1}\right)T_1 = 1393$$

Shock Reflection - Shock Tube

Very high pressure and temperature conditions in a specified location with very high precision (p_5, T_5)

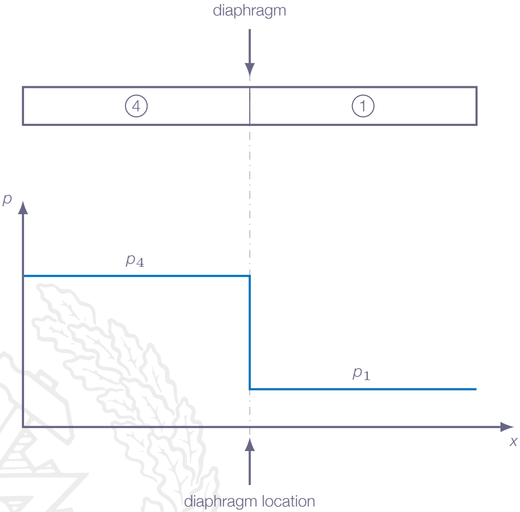
- measurements of thermodynamic properties of various gases at extreme conditions, *e.g.* dissociation energies, molecular relaxation times, etc.
- measurements of chemical reaction properties of various gas mixtures at extreme conditions

Roadmap - Unsteady Wave Motion



The Shock Tube

Shock Tube

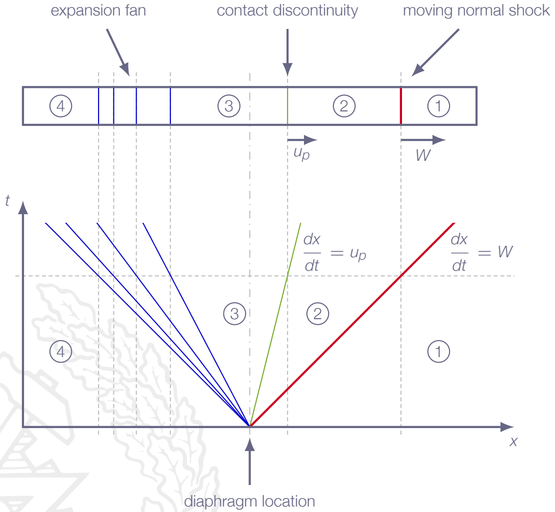


tube with closed ends
diaphragm inside, separating two different constant states
(could also be two different gases)

if diaphragm is removed suddenly (by inducing a breakdown) the two states come into contact and a flow develops

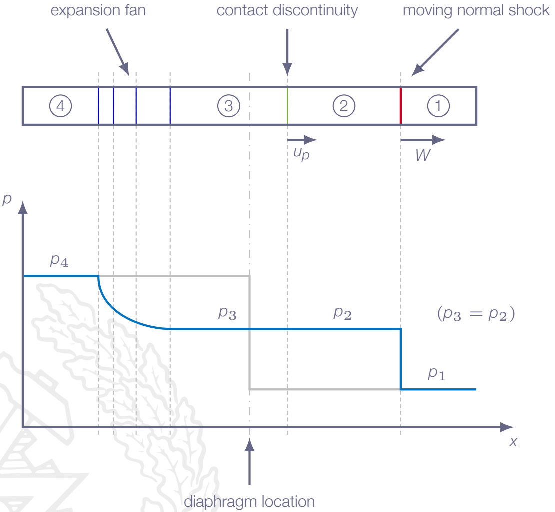
assume that $p_4 > p_1$:
state 4 is "driver" section
state 1 is "driven" section

Shock Tube



flow at some time after diaphragm breakdown

Shock Tube



flow at some time after diaphragm breakdown

Shock Tube - Basic Principles

As the diaphragm is removed, a pressure discontinuity is generated

The only process that can generate a pressure **discontinuity** in the gas is a **shock**

In chapter 3 we learned that the velocity upstream of the shock **must be supersonic**

Since the gas is standing still when the shock tube is started, **the shock must move** in order to establish the required **relative velocity**

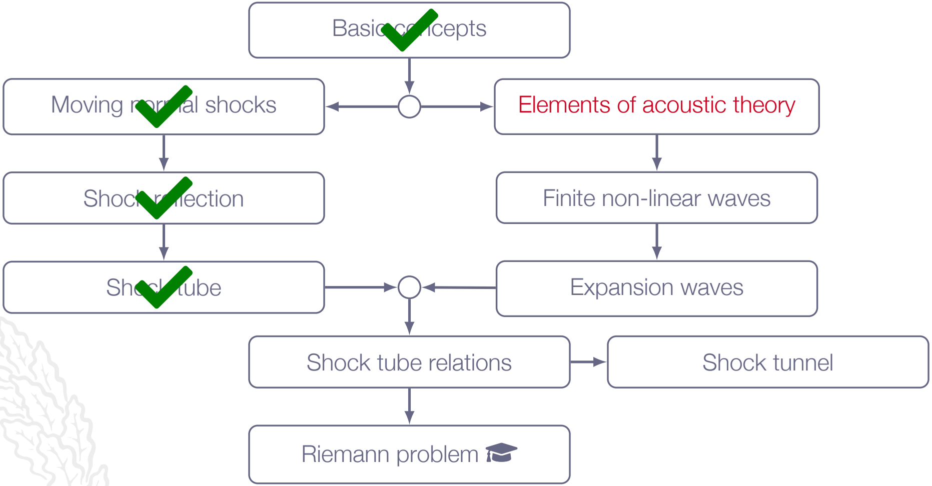
The shock must move in to the gas with the lower pressure

Shock Tube - Basic Principles

By using light gases for the **driver section** (*e.g.* He) and heavier gases for the **driven section** (*e.g.* air) the pressure p_4 required for a specific p_2/p_1 ratio is significantly reduced

If T_4/T_1 is increased, the pressure p_4 required for a specific p_2/p_1 is also reduced

Roadmap - Unsteady Wave Motion



Chapter 7.5
Elements of Acoustic Theory

Sound Waves - Sound Pressure Level

sound wave	L_p [dB]	Δp [Pa]
Weakest audible sound wave	0	2.83×10^{-5}
Loud sound wave	91	1.00×10^{0}
Amplified music	120	2.80×10^{1}
Jet engine @ 30 m	130	9.00×10^{1}
Threshold of pain	140	2.83×10^{2}
Military jet @ 30 m	150	8.90×10^{2}

Example (Loud sound wave):

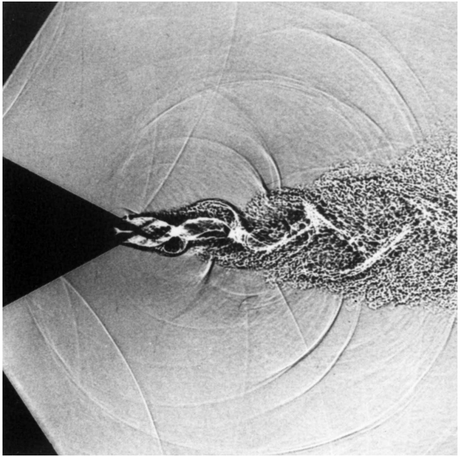
$\Delta p \sim$ 1 Pa (91 dB) gives $\Delta \rho \sim 8.5 \times 10^{-6}$ kg/m^3 and $\Delta u \sim 2.4 \times 10^{-3}$ m/s

Sound Waves - Acoustic Analogy

Schlieren flow visualization of self-sustained oscillation of an under-expanded free jet

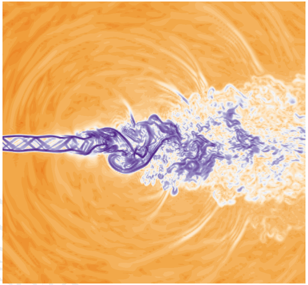
A. Hirschberg

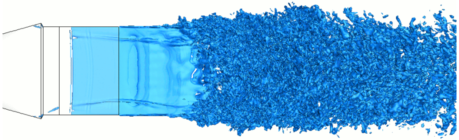
"*Introduction to aero-acoustics of internal flows*", Advances in Aeroacoustics, VKI, 12-16 March 2001



Sound Waves - Acoustic Analogy

Screeching rectangular supersonic jet





PDE:s for conservation of mass and momentum derived in Chapter 6:

	conservation form	non-conservation form
mass	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$	$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0$
momentum	$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v} + p\mathbf{I}) = 0$	$\rho \frac{D\mathbf{v}}{Dt} + \nabla p = 0$

Elements of Acoustic Theory

For adiabatic inviscid flow we also have the entropy equation as

$$\frac{Ds}{Dt} = 0$$

Assume one-dimensional flow

$$\left.\begin{aligned} \rho &= \rho(x,t) \\ \mathbf{v} &= u(x,t)\mathbf{e}_x \\ p &= p(x,t) \\ &\ldots \end{aligned}\right\} \Rightarrow$$

continuity $\dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial x} + \rho\dfrac{\partial u}{\partial x} = 0$

momentum $\rho\dfrac{\partial u}{\partial t} + \rho u\dfrac{\partial u}{\partial x} + \dfrac{\partial p}{\partial x} = 0$

s=constant

Elements of Acoustic Theory

continuity $$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

momentum $$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0$$

s=constant

More unknowns than equations $\Rightarrow$ the equation system can not be solved

Can $\frac{\partial p}{\partial x}$ be expressed in terms of density?

Leading question; it is possible so let's do just that ...

Elements of Acoustic Theory

From Chapter 1: any thermodynamic state variable is uniquely defined by any two other state variables

$$p = p(\rho, s) \Rightarrow dp = \left(\frac{\partial p}{\partial \rho}\right)_s d\rho + \left(\frac{\partial p}{\partial s}\right)_\rho ds$$

s=constant gives

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_s d\rho = a^2 d\rho$$

$$\Rightarrow \begin{cases} \dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial x} + \rho\dfrac{\partial u}{\partial x} = 0 \\[2ex] \rho\dfrac{\partial u}{\partial t} + \rho u\dfrac{\partial u}{\partial x} + a^2\dfrac{\partial \rho}{\partial x} = 0 \end{cases}$$

Elements of Acoustic Theory

Assume **small perturbations** around stagnant reference condition:

$$\rho = \rho_\infty + \Delta\rho \qquad p = p_\infty + \Delta p \qquad T = T_\infty + \Delta T \qquad u = u_\infty + \Delta u = \{u_\infty = 0\} = \Delta u$$

where ρ_∞, p_∞, and T_∞ are constant

Now, insert $\rho = (\rho_\infty + \Delta\rho)$ and $u = \Delta u$ in the continuity and momentum equations (derivatives of ρ_∞ are zero)

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \Delta u \dfrac{\partial}{\partial x}(\Delta\rho) + (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial x}(\Delta u) = 0 \\ (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial t}(\Delta u) + (\rho_\infty + \Delta\rho)\Delta u \dfrac{\partial}{\partial x}(\Delta u) + a^2 \dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Elements of Acoustic Theory

Assume **small perturbations** around stagnant reference condition:

$$\rho = \rho_\infty + \Delta\rho \qquad p = p_\infty + \Delta p \qquad T = T_\infty + \Delta T \qquad u = u_\infty + \Delta u = \{u_\infty = 0\} = \Delta u$$

where ρ_∞, p_∞, and T_∞ are constant

Now, insert $\rho = (\rho_\infty + \Delta\rho)$ and $u = \Delta u$ in the continuity and momentum equations (derivatives of ρ_∞ are zero)

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \Delta u \dfrac{\partial}{\partial x}(\Delta\rho) + (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial x}(\Delta u) = 0 \\ (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial t}(\Delta u) + (\rho_\infty + \Delta\rho)\Delta u \dfrac{\partial}{\partial x}(\Delta u) + a^2 \dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Elements of Acoustic Theory

Speed of sound is a thermodynamic state variable $\Rightarrow a^2 = a^2(\rho, s)$. With entropy constant $\Rightarrow a^2 = a^2(\rho)$

Taylor expansion around a_∞ with $(\Delta\rho = \rho - \rho_\infty)$ gives

$$a^2 = a_\infty^2 + \left(\frac{\partial}{\partial\rho}(a^2)\right)_\infty \Delta\rho + \frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2}(a^2)\right)_\infty (\Delta\rho)^2 + ...$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \Delta u \dfrac{\partial}{\partial x}(\Delta\rho) + (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial x}(\Delta u) = 0 \\ (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial t}(\Delta u) + (\rho_\infty + \Delta\rho)\Delta u \dfrac{\partial}{\partial x}(\Delta u) + \left[a_\infty^2 + \left(\dfrac{\partial}{\partial\rho}(a^2)\right)_\infty \Delta\rho + ...\right]\dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Elements of Acoustic Theory - Acoustic Equations

Since $\Delta\rho$ and Δu are assumed to be small ($\Delta\rho \ll \rho_\infty$, $\Delta u \ll a$)

1. products of perturbations can be neglected
2. higher-order terms in the Taylor expansion can be neglected

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \rho_\infty \dfrac{\partial}{\partial x}(\Delta u) = 0 \\ \rho_\infty \dfrac{\partial}{\partial t}(\Delta u) + a_\infty^2 \dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Note! The assumption is only valid for small perturbations (sound waves)

This type of derivation is based on linearization, *i.e.* the acoustic equations are **linear**

Acoustic equations:

”... describe the motion of gas induced by the passage of a sound wave ...”

Elements of Acoustic Theory - Wave Equation

Combining linearized continuity and the momentum equations we get

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = a_\infty^2 \frac{\partial^2}{\partial x^2}(\Delta\rho)$$

(combine the time derivative of the continuity eqn. and the divergence of the momentum eqn.)

General solution:

$$\Delta\rho(x,t) = F(x - a_\infty t) + G(x + a_\infty t)$$

$F(x - a_\infty t)$: wave traveling in positive x-direction with speed a_∞

$G(x + a_\infty t)$: wave traveling in negative x-direction with speed a_∞

F and G may be arbitrary functions

Wave shape is determined by functions F and G

Elements of Acoustic Theory - Wave Equation

Spatial and temporal derivatives of F are obtained according to

$$\begin{cases} \dfrac{\partial F}{\partial t} = \dfrac{\partial F}{\partial (x - a_\infty t)} \dfrac{\partial (x - a_\infty t)}{\partial t} = -a_\infty F' \\ \\ \dfrac{\partial F}{\partial x} = \dfrac{\partial F}{\partial (x - a_\infty t)} \dfrac{\partial (x - a_\infty t)}{\partial x} = F' \end{cases}$$

spatial and temporal derivatives of G can of course be obtained in the same way...

Elements of Acoustic Theory - Wave Equation

with $\Delta\rho(x,t) = F(x - a_\infty t) + G(x + a_\infty t)$ and the derivatives of F and G we get

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = a_\infty^2 F'' + a_\infty^2 G''$$

and

$$\frac{\partial^2}{\partial x^2}(\Delta\rho) = F'' + G''$$

which gives

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) - a_\infty^2 \frac{\partial^2}{\partial x^2}(\Delta\rho) = 0$$

i.e., the proposed solution fulfils the wave equation

Elements of Acoustic Theory - Wave Equation

F and G may be arbitrary functions, assume $G = 0$

$$\Delta\rho(x,t) = F(x - a_\infty t)$$

If $\Delta\rho$ is constant (constant wave amplitude), $(x - a_\infty t)$ must be a constant which implies

$$x = a_\infty t + c$$

where c is a constant

$$\frac{dx}{dt} = a_\infty$$

Elements of Acoustic Theory - Wave Equation

Let's try to find a relation between $\Delta\rho$ and Δu

$\Delta\rho(x,t) = F(x - a_\infty t)$ (wave in positive x direction) gives:

$$\frac{\partial}{\partial t}(\Delta\rho) = -a_\infty F' \quad \text{and} \quad \frac{\partial}{\partial x}(\Delta\rho) = F'$$

$$\underbrace{\frac{\partial}{\partial t}(\Delta\rho)}_{-a_\infty F'} + a_\infty \underbrace{\frac{\partial}{\partial x}(\Delta\rho)}_{F'} = 0$$

or

$$\frac{\partial}{\partial x}(\Delta\rho) = -\frac{1}{a_\infty}\frac{\partial}{\partial t}(\Delta\rho)$$

Elements of Acoustic Theory - Wave Equation

Linearized momentum equation:

$$\rho_\infty \frac{\partial}{\partial t}(\Delta u) = -a_\infty^2 \frac{\partial}{\partial x}(\Delta \rho) \Rightarrow$$

$$\frac{\partial}{\partial t}(\Delta u) = -\frac{a_\infty^2}{\rho_\infty}\frac{\partial}{\partial x}(\Delta \rho) = \left\{ \frac{\partial}{\partial x}(\Delta \rho) = -\frac{1}{a_\infty}\frac{\partial}{\partial t}(\Delta \rho) \right\} = \frac{a_\infty}{\rho_\infty}\frac{\partial}{\partial t}(\Delta \rho)$$

$$\frac{\partial}{\partial t}\left(\Delta u - \frac{a_\infty}{\rho_\infty}\Delta \rho\right) = 0 \Rightarrow \Delta u - \frac{a_\infty}{\rho_\infty}\Delta \rho = const$$

In undisturbed gas $\Delta u = \Delta \rho = 0$ which implies that the constant must be zero and thus

$$\boxed{\Delta u = \frac{a_\infty}{\rho_\infty}\Delta \rho}$$

Elements of Acoustic Theory - Wave Equation

Similarly, for $\Delta\rho(x,t) = G(x + a_\infty t)$ (wave in negative x direction) we obtain:

$$\Delta u = -\frac{a_\infty}{\rho_\infty}\Delta\rho$$

Also, since $\Delta p = a_\infty^2 \Delta\rho$ we get:

Right going wave (+x direction) $\Delta u = \frac{a_\infty}{\rho_\infty}\Delta\rho = \frac{1}{a_\infty \rho_\infty}\Delta p$

Left going wave (-x direction) $\Delta u = -\frac{a_\infty}{\rho_\infty}\Delta\rho = -\frac{1}{a_\infty \rho_\infty}\Delta p$

Elements of Acoustic Theory - Wave Equation

Δu denotes **induced mass motion** and is positive in the positive x-direction

$$\Delta u = \pm \frac{a_\infty \Delta \rho}{\rho_\infty} = \pm \frac{\Delta p}{a_\infty \rho_\infty}$$

condensation (the part of the sound wave where $\Delta \rho > 0$):
Δu is always in the **same** direction as the wave motion

rarefaction (the part of the sound wave where $\Delta \rho < 0$):
Δu is always in the direction **opposite** to the wave motion

Elements of Acoustic Theory - Wave Equation *Summary*

Combining **linearized** continuity and the momentum equations we get

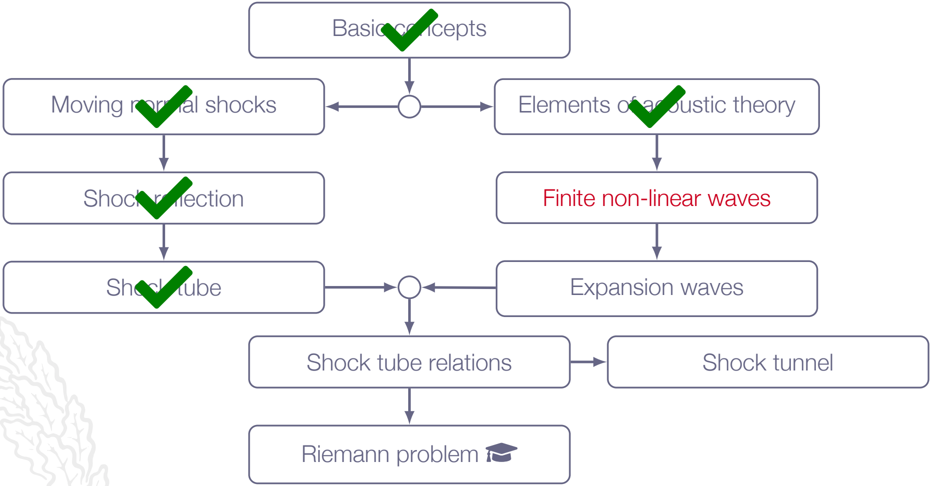
$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = a_\infty^2 \frac{\partial^2}{\partial x^2}(\Delta\rho)$$

Due to the assumptions made, the **equation is not exact**

More and more accurate as the perturbations becomes smaller and smaller

So, how should we describe waves with larger amplitudes?

Roadmap - Unsteady Wave Motion



Chapter 7.6
Finite (Non-Linear) Waves

Finite (Non-Linear) Waves

When $\Delta\rho$, Δu, Δp, ... Become large, the **linearized acoustic equations become poor approximations**

Non-linear equations must be used

One-dimensional non-linear continuity and momentum equations:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

Finite (Non-Linear) Waves

We still assume isentropic flow, $ds = 0$

$$\frac{\partial \rho}{\partial t} = \left(\frac{\partial \rho}{\partial p}\right)_s \frac{\partial p}{\partial t} = \frac{1}{a^2}\frac{\partial p}{\partial t}$$

$$\frac{\partial \rho}{\partial x} = \left(\frac{\partial \rho}{\partial p}\right)_s \frac{\partial p}{\partial x} = \frac{1}{a^2}\frac{\partial p}{\partial x}$$

Inserted in the continuity equation this gives:

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho a^2 \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

Finite (Non-Linear) Waves

Add $1/(\rho a)$ times the continuity equation to the momentum equation:

$$\left[\frac{\partial u}{\partial t}+(u+a)\frac{\partial u}{\partial x}\right]+\frac{1}{\rho a}\left[\frac{\partial p}{\partial t}+(u+a)\frac{\partial p}{\partial x}\right]=0$$

If we instead **subtract** $1/(\rho a)$ times the continuity equation from the momentum equation, we get:

$$\left[\frac{\partial u}{\partial t}+(u-a)\frac{\partial u}{\partial x}\right]-\frac{1}{\rho a}\left[\frac{\partial p}{\partial t}+(u-a)\frac{\partial p}{\partial x}\right]=0$$

Finite (Non-Linear) Waves

Since $u = u(x, t)$, we have:

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}\frac{dx}{dt}dt$$

Let $\frac{dx}{dt} = u + a$ gives

$$du = \left[\frac{\partial u}{\partial t} + (u + a)\frac{\partial u}{\partial x}\right]dt$$

Interpretation: change of u in the direction of line $\frac{dx}{dt} = u + a$

Finite (Non-Linear) Waves

In the same way we get:

$$dp = \frac{\partial p}{\partial t} dt + \frac{\partial p}{\partial x}\frac{dx}{dt} dt$$

and thus

$$dp = \left[\frac{\partial p}{\partial t} + (u + a)\frac{\partial p}{\partial x}\right] dt$$

Interpretation: change of p in the direction of line $\frac{dx}{dt} = u + a$

Finite (Non-Linear) Waves

Now, if we combine

$$\left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right] + \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right] = 0$$

$$du = \left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right] dt$$

$$dp = \left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right] dt$$

we get

$$\boxed{\frac{du}{dt} + \frac{1}{\rho a}\frac{dp}{dt} = 0}$$

Characteristic Lines

Thus, along a line $dx = (u + a)dt$ we have

$$du + \frac{dp}{\rho a} = 0$$

In the same way we get along a line where $dx = (u - a)dt$

$$du - \frac{dp}{\rho a} = 0$$

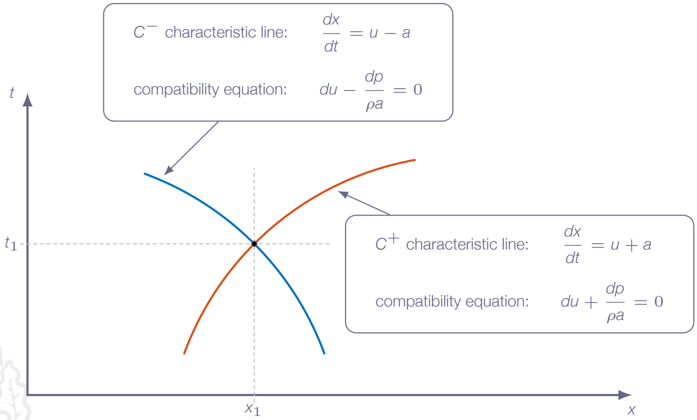
Characteristic Lines

We have found a path through a point (x, t) along which the governing partial differential equations reduces to ordinary differential equations

These paths or lines are called **characteristic lines**

The C^+ and C^- characteristic lines are physically the paths of **right- and left-running acoustic waves** in the xt-plane

Characteristic Lines



Characteristic Lines - Summary

$$\frac{du}{dt} + \frac{1}{\rho a}\frac{dp}{dt} = 0 \quad \text{along } C^+ \text{ characteristic}$$

$$\frac{du}{dt} - \frac{1}{\rho a}\frac{dp}{dt} = 0 \quad \text{along } C^- \text{ characteristic}$$

$$du + \frac{dp}{\rho a} = 0 \quad \text{along } C^+ \text{ characteristic}$$

$$du - \frac{dp}{\rho a} = 0 \quad \text{along } C^- \text{ characteristic}$$

Riemann Invariants

Integration gives:

$$J^{+} = u + \int \frac{dp}{\rho a} = \text{constant along } C^{+} \text{ characteristic}$$

$$J^{-} = u - \int \frac{dp}{\rho a} = \text{constant along } C^{-} \text{ characteristic}$$

We need to rewrite $\frac{dp}{\rho a}$ to be able to perform the integrations

Riemann Invariants

For an isentropic processes the **isentropic relations** give:

$$p = c_1 T^{\gamma/(\gamma-1)} = c_2 a^{2\gamma/(\gamma-1)}$$

where c_1 and c_2 are constants and thus

$$dp = c_2 \left(\frac{2\gamma}{\gamma-1}\right) a^{[2\gamma/(\gamma-1)-1]} da$$

Assume **calorically perfect gas**: $a^2 = \frac{\gamma p}{\rho} \Rightarrow \rho = \frac{\gamma p}{a^2}$

with $p = c_2 a^{2\gamma/(\gamma-1)}$ we get $\rho = c_2 \gamma a^{[2\gamma/(\gamma-1)-2]}$

Riemann Invariants

$$J^+ = u + \int \frac{dp}{\rho a} = u + \int \frac{c_2 \left(\frac{2\gamma}{\gamma-1}\right) a^{[2\gamma/(\gamma-1)-1]}}{c_2 \gamma a^{[2\gamma/(\gamma-1)-1]}} da = u + \int \frac{2da}{\gamma - 1}$$

$$J^+ = u + \frac{2a}{\gamma - 1}$$

$$J^- = u - \frac{2a}{\gamma - 1}$$

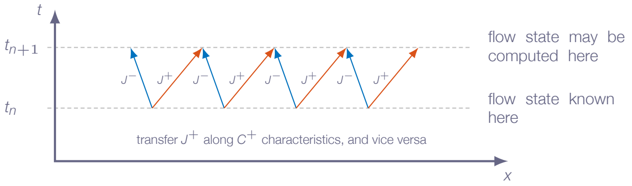
Riemann Invariants

If J^+ and J^- are known at some point $(x,\ t)$, then

$$\left\{\begin{array}{l} J^+ + J^- = 2u \\ \\ J^+ - J^- = \dfrac{4a}{\gamma - 1} \end{array}\right. \Rightarrow \left\{\begin{array}{l} u = \dfrac{1}{2}(J^+ + J^-) \\ \\ a = \dfrac{\gamma - 1}{4}(J^+ - J^-) \end{array}\right.$$

With the Riemann invariants known, the flow state is uniquely defined!

Method of Characteristics



Summary

Acoustic waves

1. $\Delta\rho$, Δu, etc - **very small**
2. All parts of the wave propagate with the same **velocity** a_∞
3. The **wave shape** stays the **same**
4. The flow is governed by **linear relations**

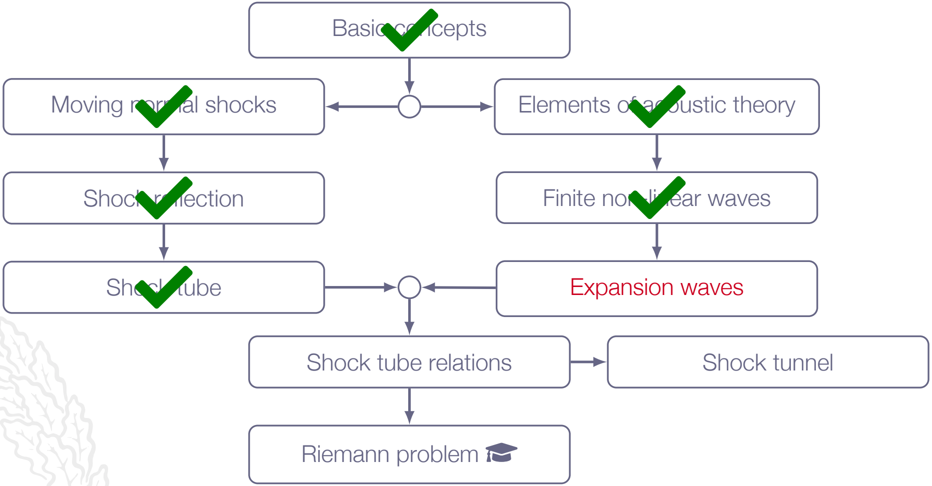
Finite (non-linear) waves

1. $\Delta\rho$, Δu, etc - can be **large**
2. Each local part of the wave propagates at the **local velocity** $(u + a)$
3. The wave **shape changes** with time
4. The flow is governed by **non-linear relations**

Summary

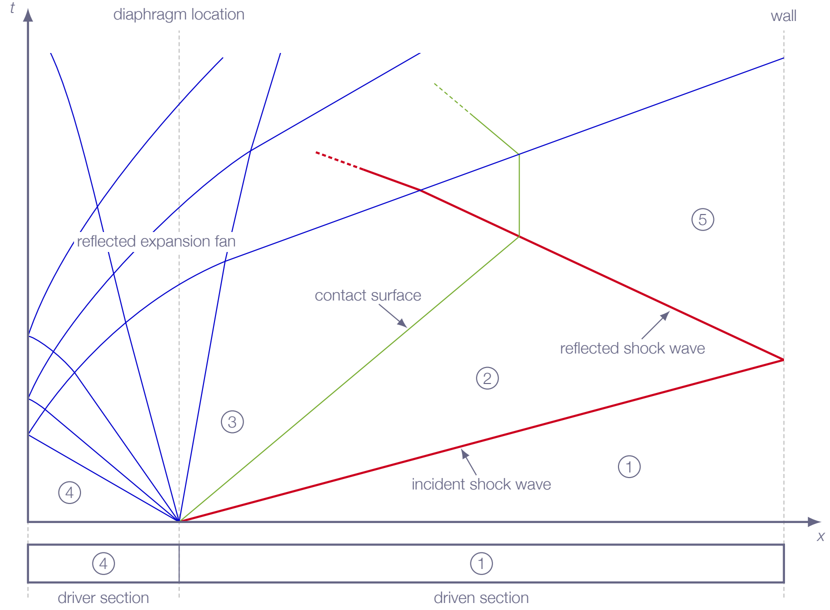
the method of characteristics is a central element in classic compressible flow theory

Roadmap - Unsteady Wave Motion



Chapter 7.7
Incident and Reflected Expansion Waves

Expansion Waves



Expansion Waves

Properties of a left-running expansion wave

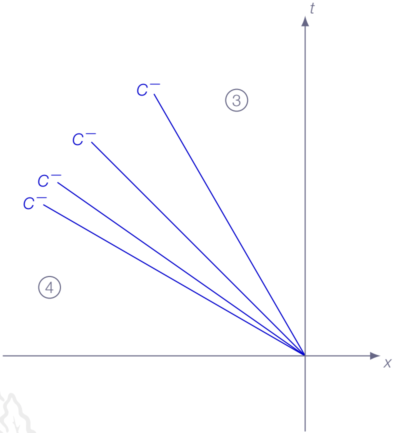
1. All flow properties are constant along C^- characteristics
2. The wave **head** is propagating **into region 4** (high pressure)
3. The wave **tail** defines the **limit of region 3** (lower pressure)
4. Regions 3 and 4 are assumed to be **constant states**

For calorically perfect gas:

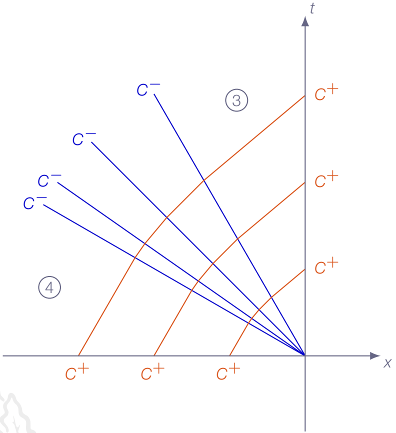
$$J^+ = u + \frac{2a}{\gamma - 1} \quad \text{is constant along } C^+ \text{ lines}$$

$$J^- = u - \frac{2a}{\gamma - 1} \quad \text{is constant along } C^- \text{ lines}$$

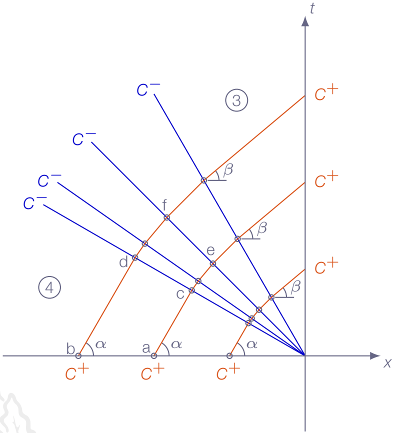
Expansion Waves



Expansion Waves



Expansion Waves



constant flow properties in region 4: $J_a^+ = J_b^+$

J^+ invariants constant along C^+ characteristics:

$$J_a^+ = J_c^+ = J_e^+$$

$$J_b^+ = J_d^+ = J_f^+$$

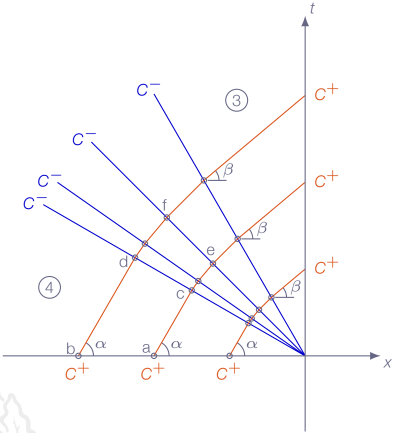
since $J_a^+ = J_b^+$ this also implies $J_e^+ = J_f^+$

J^- invariants constant along C^- characteristics:

$$J_c^- = J_d^-$$

$$J_e^- = J_f^-$$

Expansion Waves



constant flow properties in region 4: $J_a^+ = J_b^+$

J^+ invariants constant along C^+ characteristics:

$$J_a^+ = J_c^+ = J_e^+$$

$$J_b^+ = J_d^+ = J_f^+$$

since $J_a^+ = J_b^+$ this also implies $J_e^+ = J_f^+$

J^- invariants constant along C^- characteristics:

$$J_c^- = J_d^-$$

$$J_e^- = J_f^-$$

$$u_e = \frac{1}{2}(J_e^+ + J_e^-),\ u_f = \frac{1}{2}(J_f^+ + J_f^-),\ \Rightarrow u_e = u_f$$

$$a_e = \frac{\gamma - 1}{4}(J_e^+ - J_e^-),\ a_f = \frac{\gamma - 1}{4}(J_f^+ - J_f^-),\ \Rightarrow a_e = a_f$$

Expansion Waves

Along each C^- line u and a are **constants** which means that

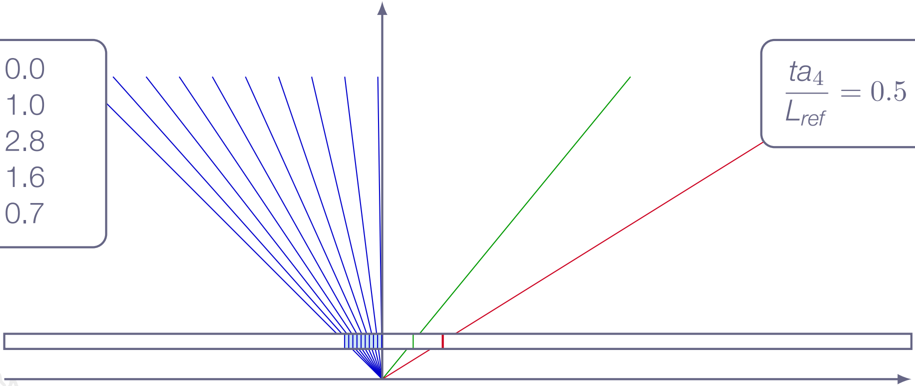
$$\frac{dx}{dt} = u - a = const$$

C^- characteristics are **straight lines** in xt-space

Expansion Waves - Shock Tube

p_4/p_1	10.0
T_4/T_1	1.0
p_2/p_1	2.8
M_s	1.6
M_p	0.7

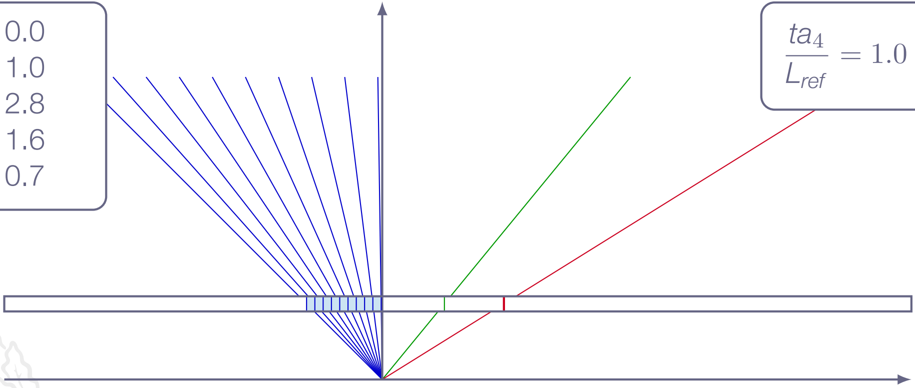
$$\frac{ta_4}{L_{ref}} = 0.5$$



Expansion Waves - Shock Tube

p_4/p_1	10.0
T_4/T_1	1.0
p_2/p_1	2.8
M_s	1.6
M_p	0.7

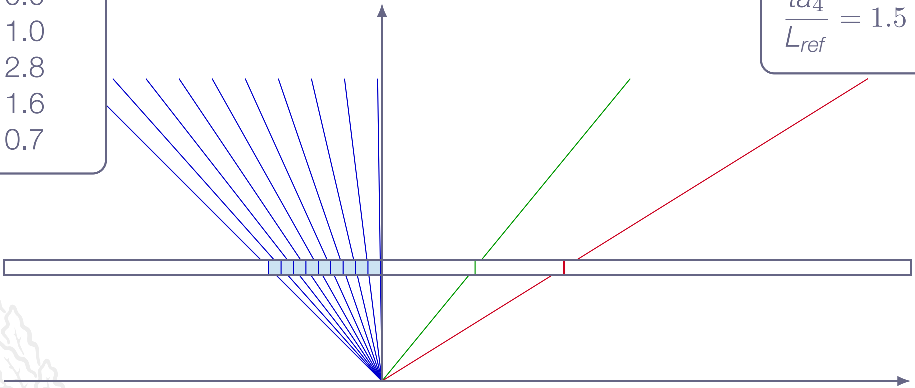
$$\frac{ta_4}{L_{ref}} = 1.0$$



Expansion Waves - Shock Tube

p_4/p_1	10.0
T_4/T_1	1.0
p_2/p_1	2.8
M_s	1.6
M_p	0.7

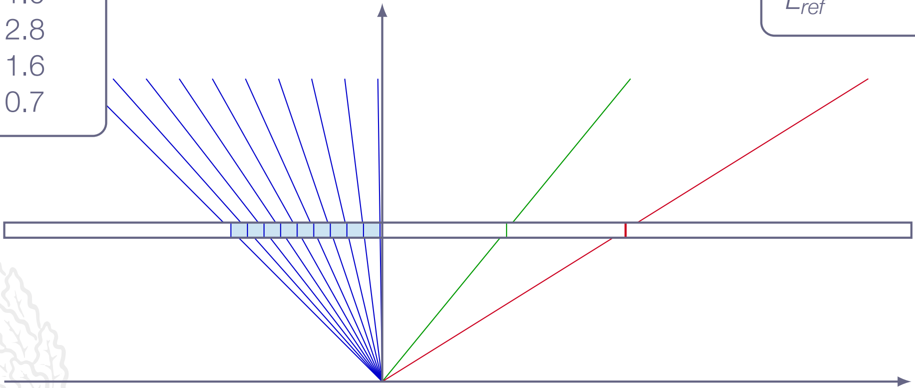
$$\frac{ta_4}{L_{ref}} = 1.5$$



Expansion Waves - Shock Tube

p_4/p_1	10.0
T_4/T_1	1.0
p_2/p_1	2.8
M_s	1.6
M_p	0.7

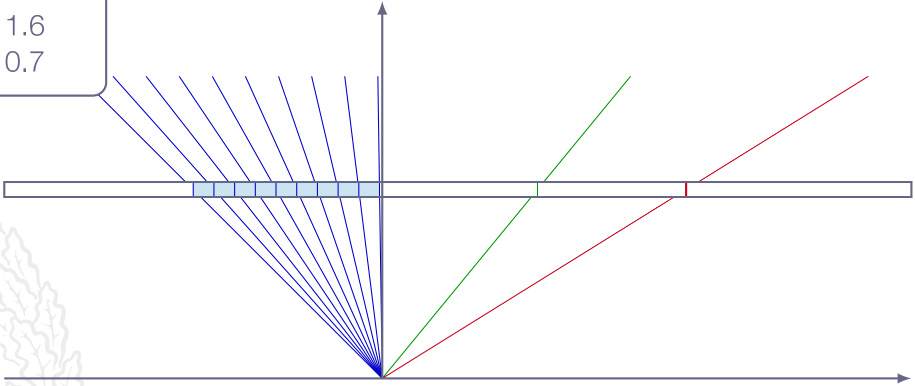
$$\frac{ta_4}{L_{ref}} = 2.0$$



Expansion Waves - Shock Tube

p_4/p_1	10.0
T_4/T_1	1.0
p_2/p_1	2.8
M_s	1.6
M_p	0.7

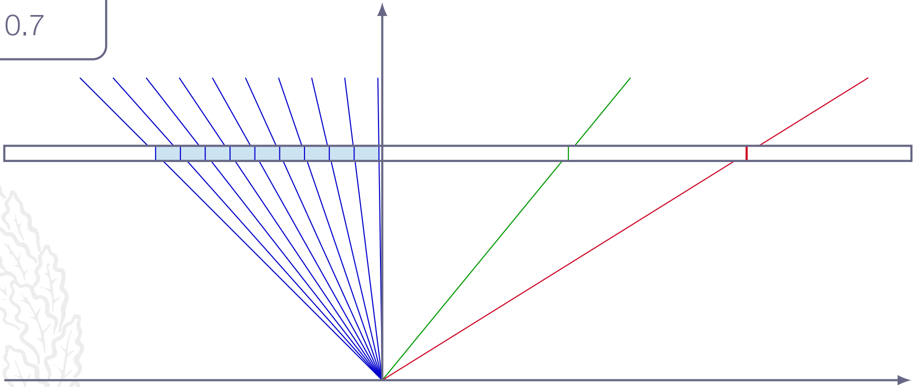
$$\frac{ta_4}{L_{ref}} = 2.5$$



Expansion Waves - Shock Tube

p_4/p_1	10.0
T_4/T_1	1.0
p_2/p_1	2.8
M_s	1.6
M_p	0.7

$$\frac{ta_4}{L_{ref}} = 3.0$$



Shock Tube Expansion Waves - Summary

The start and end conditions are the same for all C^+ lines

J^+ invariants have the same value for all C^+ characteristics

C^- characteristics are straight lines in xt-space

Simple expansion waves centered at $(x, t) = (0, 0)$

Expansion Waves

In a left-running expansion fan:

J^+ is constant throughout expansion fan, which implies:

$$u + \frac{2a}{\gamma - 1} = u_4 + \frac{2a_4}{\gamma - 1} = u_3 + \frac{2a_3}{\gamma - 1}$$

J^- is constant along C^- lines, but varies from one line to the next, which means that

$$u - \frac{2a}{\gamma - 1}$$

is constant along each C^- line

Expansion Waves

Since $u_4 = 0$ we obtain:

$$u + \frac{2a}{\gamma - 1} = u_4 + \frac{2a_4}{\gamma - 1} = \frac{2a_4}{\gamma - 1} \Rightarrow$$

$$\frac{a}{a_4} = 1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}$$

with $a = \sqrt{\gamma R T}$ we get

$$\frac{T}{T_4} = \left[1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}\right]^2$$

Expansion Wave Relations

Isentropic flow $\Rightarrow$ we can use the isentropic relations

complete description in terms of u/a_4

$$\frac{T}{T_4} = \left[1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}\right]^2$$

$$\frac{p}{p_4} = \left[1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}\right]^{\frac{2\gamma}{\gamma-1}}$$

$$\frac{\rho}{\rho_4} = \left[1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}\right]^{\frac{2}{\gamma-1}}$$

Expansion Wave Relations

Since C^{-} characteristics are straight lines, we have:

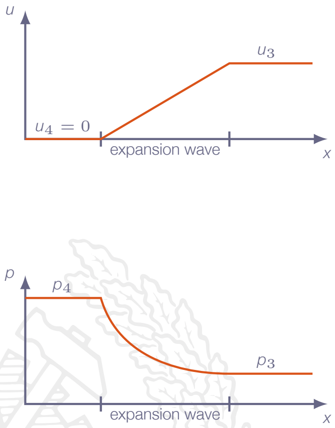
$$\frac{dx}{dt} = u - a \Rightarrow x = (u - a)t$$

$$\frac{a}{a_4} = 1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4} \Rightarrow a = a_4 - \frac{1}{2}(\gamma - 1)u \Rightarrow$$

$$x = \left[u - a_4 + \frac{1}{2}(\gamma - 1)u\right] t = \left[\frac{1}{2}(\gamma - 1)u - a_4\right] t \Rightarrow$$

$$u = \frac{2}{\gamma + 1}\left[a_4 + \frac{x}{t}\right]$$

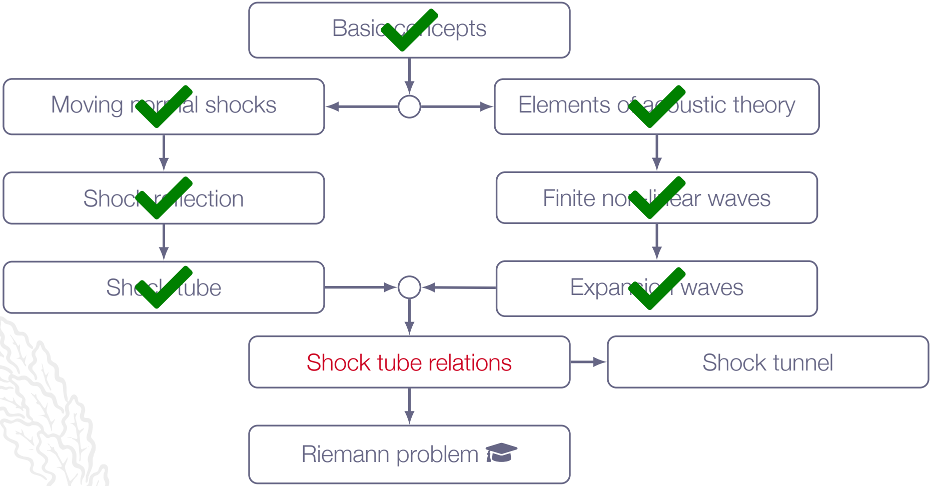
Expansion Wave Relations



Expansion wave head is advancing to the left with speed a_4 into the stagnant gas

Expansion wave tail is advancing with speed $u_3 - a_3$, which may be positive or negative, depending on the initial states

Roadmap - Unsteady Wave Motion



Chapter 7.8
Shock Tube Relations

Shock Tube Relations

$$u_p = u_2 = \frac{a_1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left[\frac{\dfrac{2\gamma_1}{\gamma_1+1}}{\dfrac{p_2}{p_1} + \dfrac{\gamma_1-1}{\gamma_1+1}}\right]^{1/2}$$

$$\frac{p_3}{p_4} = \left[1 - \frac{\gamma_4-1}{2}\left(\frac{u_3}{a_4}\right)\right]^{2\gamma_4/(\gamma_4-1)}$$

solving for u_3 gives

$$u_3 = \frac{2a_4}{\gamma_4-1}\left[1 - \left(\frac{p_3}{p_4}\right)^{(\gamma_4-1)/(2\gamma_4)}\right]$$

Shock Tube Relations

But, $p_3 = p_2$ and $u_3 = u_2$ (no change in velocity and pressure over contact discontinuity)

$$\Rightarrow u_2 = \frac{2a_4}{\gamma_4 - 1}\left[1 - \left(\frac{p_2}{p_4}\right)^{(\gamma_4-1)/(2\gamma_4)}\right]$$

We have now two expressions for u_2 which gives us

$$\frac{a_1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left[\frac{\frac{2\gamma_1}{\gamma_1+1}}{\frac{p_2}{p_1} + \frac{\gamma_1-1}{\gamma_1+1}}\right]^{1/2} = \frac{2a_4}{\gamma_4 - 1}\left[1 - \left(\frac{p_2}{p_4}\right)^{(\gamma_4-1)/(2\gamma_4)}\right]$$

Shock Tube Relations

Rearranging gives:

$$\frac{p_4}{p_1} = \frac{p_2}{p_1}\left\{1 - \frac{(\gamma_4 - 1)(a_1/a_4)(p_2/p_1 - 1)}{\sqrt{2\gamma_1\left[2\gamma_1 + (\gamma_1 + 1)(p_2/p_1 - 1)\right]}}\right\}^{-2\gamma_4/(\gamma_4 - 1)}$$

p_2/p_1 as implicit function of p_4/p_1

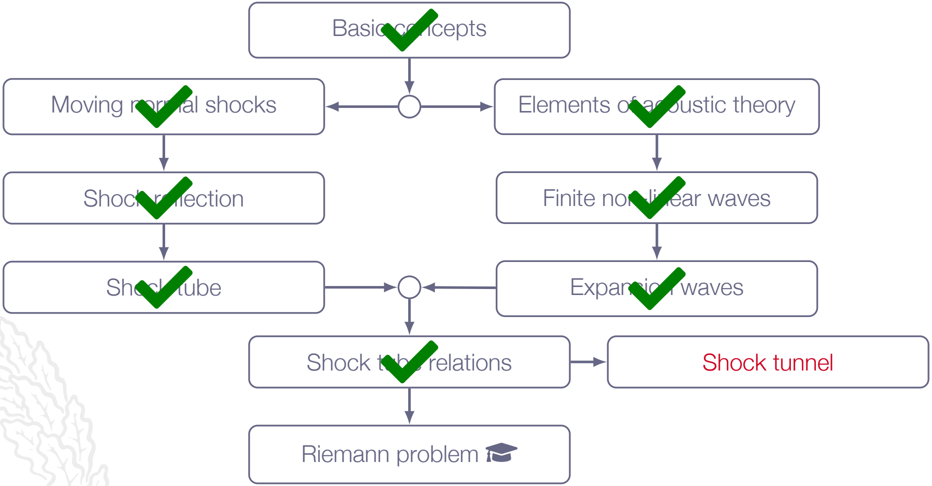
for a given p_4/p_1, p_2/p_1 will increase with decreased a_1/a_4

$$a = \sqrt{\gamma R T} = \sqrt{\gamma (R_u/M) T}$$

the speed of sound in a light gas is higher than in a heavy gas

- driver gas: low molecular weight, high temperature
- driven gas: high molecular weight, low temperature

Roadmap - Unsteady Wave Motion



Shock Tunnel

Addition of a convergent-divergent nozzle to a shock tube configuration

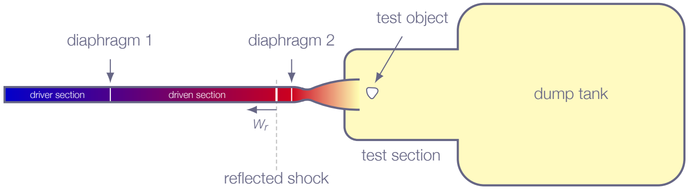
Capable of producing flow conditions which are close to those during the reentry of a space vehicles into the earth's atmosphere

- high-enthalpy, hypersonic flows (short time)
- real gas effects

Example - Aachen TH2:

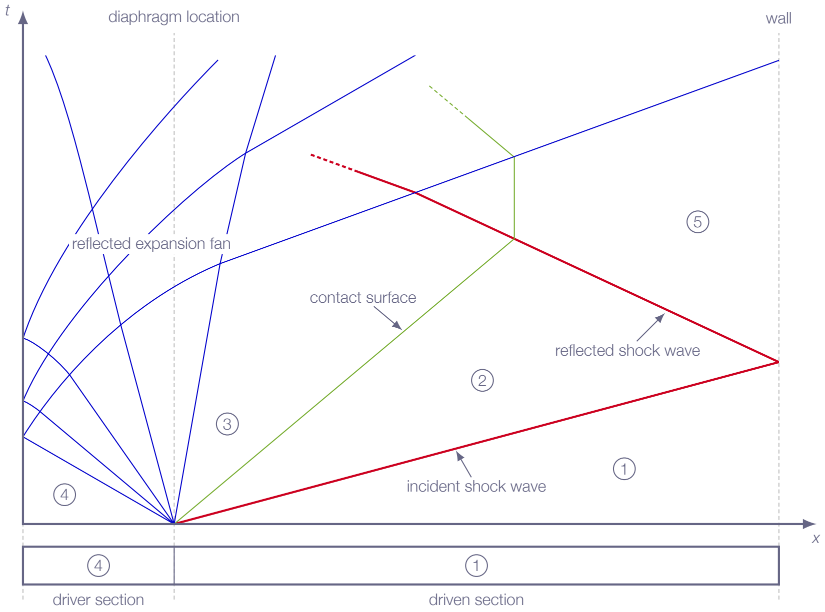
- velocities up to 4 km/s
- stagnation temperatures of several thousand degrees

Shock Tunnel



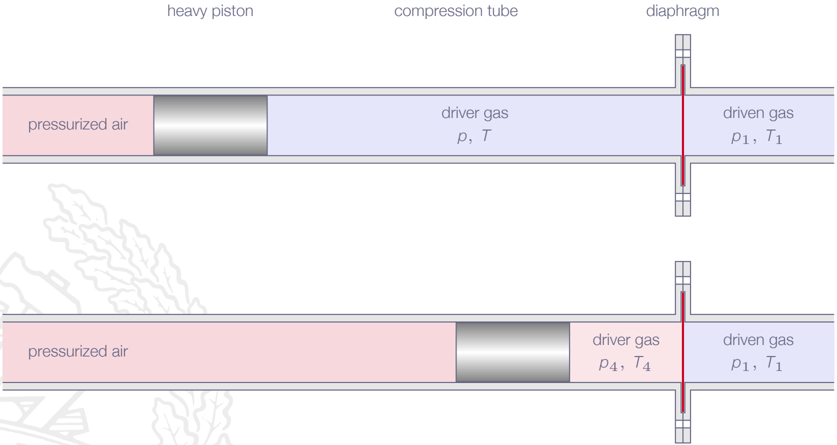
1. High pressure in region 4 (driver section)
   diaphragm 1 burst
   primary shock generated
2. Primary shock reaches end of shock tube
   shock reflection
3. High pressure in region 5
   diaphragm 2 burst
   nozzle flow initiated
   hypersonic flow in test section

Shock Tunnel



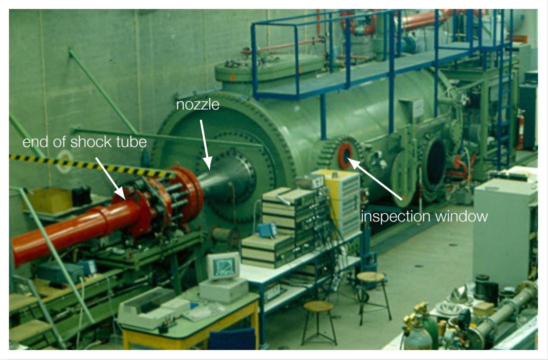
Shock Tunnel

By adding a compression tube to the shock tube a very high p_4 and T_4 may be achieved for any gas in a fairly simple manner



The Aachen Shock Tunnel - TH2

Shock tunnel built 1975



The Aachen Shock Tunnel - TH2

Shock tube specifications:

diameter	140 mm
driver section	6.0 m
driven section	15.4 m
diaphragm 1	10 mm stainless steel
diaphragm 2	copper/brass sheet
max operating (steady) pressure	1500 bar

The Aachen Shock Tunnel - TH2

Driver gas (usually helium):

100 bar $< p_4 <$ 1500 bar

electrical preheating (optional) to 600 K

Driven gas:

0.1 bar $< p_1 <$ 10 bar

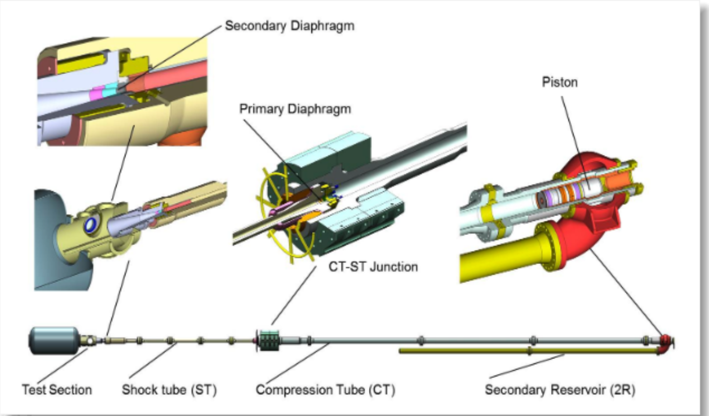
Dump tank evacuated before test

The Aachen Shock Tunnel - TH2

initial conditions			shock		reservoir		free stream			
p_4 [bar]	T_4 [K]	p_1 [bar]	M_s	p_2 [bar]	p_5 [bar]	T_5 [K]	M_∞	T_∞ [K]	u_∞ [m/s]	p_∞ [mbar]
100	293	1.0	3.3	12	65	1500	7.7	125	1740	7.6
370	500	1.0	4.6	26	175	2500	7.4	250	2350	20.0
720	500	0.7	5.6	50	325	3650	6.8	460	3910	42.0
1200	500	0.6	6.8	50	560	4600	6.5	700	3400	73.0
100	293	0.9	3.4	12	65	1500	11.3	60	1780	0.6
450	500	1.2	4.9	29	225	2700	11.3	120	2480	1.5
1300	520	0.7	6.4	46	630	4600	12.1	220	3560	1.2
26	293	0.2	3.4	12	15	1500	11.4	60	1780	0.1
480	500	0.2	6.6	50	210	4600	11.0	270	3630	0.7
100	293	1.0	3.4	12	65	1500	7.7	130	1750	7.3
370	500	1.0	5.1	27	220	2700	7.3	280	2440	26.3

The Caltech Shock Tunnel - T5

Free-piston shock tunnel



The Caltech Shock Tunnel - T5

Compression tube (CT):

length 30 m, diameter 300 mm
free piston (120 kg)
max piston velocity: 300 m/s
driven by compressed air (80 bar - 150 bar)

Shock tube (ST):

length 12 m, diameter 90 mm
driver gas: helium + argon
driven gas: air
diaphragm 1: 7 mm stainless steel
p_4 max 1300 bar

The Caltech Shock Tunnel - T5

Reservoir conditions:

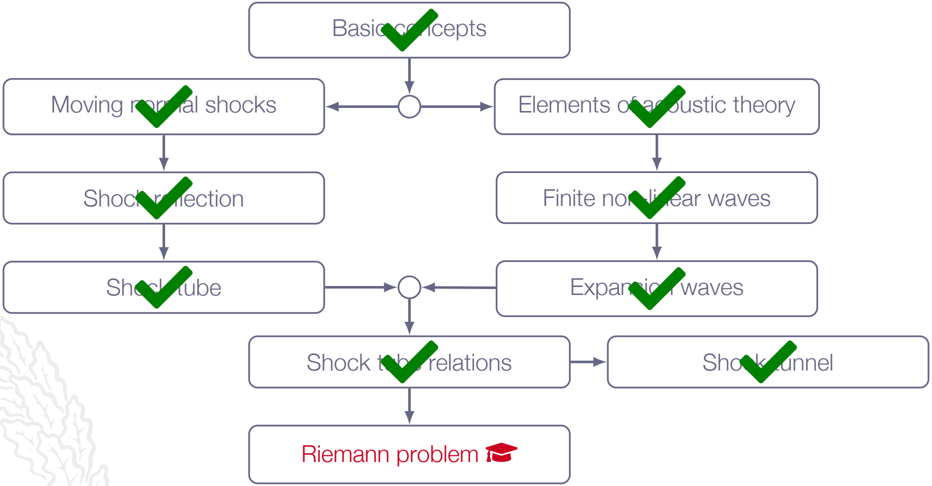
p_5 1000 bar
T_5 10000 K

Freestream conditions (design conditions):

M_∞ 5.2
T_∞ 2000 K
p_∞ 0.3 bar
typical test time 1 ms

Other Examples of Shock Tunnels

Roadmap - Unsteady Wave Motion



Riemann Problem

The shock tube problem is a special case of the general **Riemann Problem**

> *"... A Riemann problem, named after Bernhard Riemann, consists of an initial value problem composed by a conservation equation together with piece-wise constant data having a single discontinuity ..."*

Wikipedia

Riemann Problem

May show that solutions to the shock tube problem have the general form:

$$
\begin{aligned}
p &= p(x/t) \\
\rho &= \rho(x/t) \\
u &= u(x/t) \\
T &= T(x/t) \\
a &= a(x/t)
\end{aligned}
$$

where $x = 0$ denotes the position of the initial jump between states 1 and 4

Riemann Problem - Shock Tube Simulation

Numerical method:

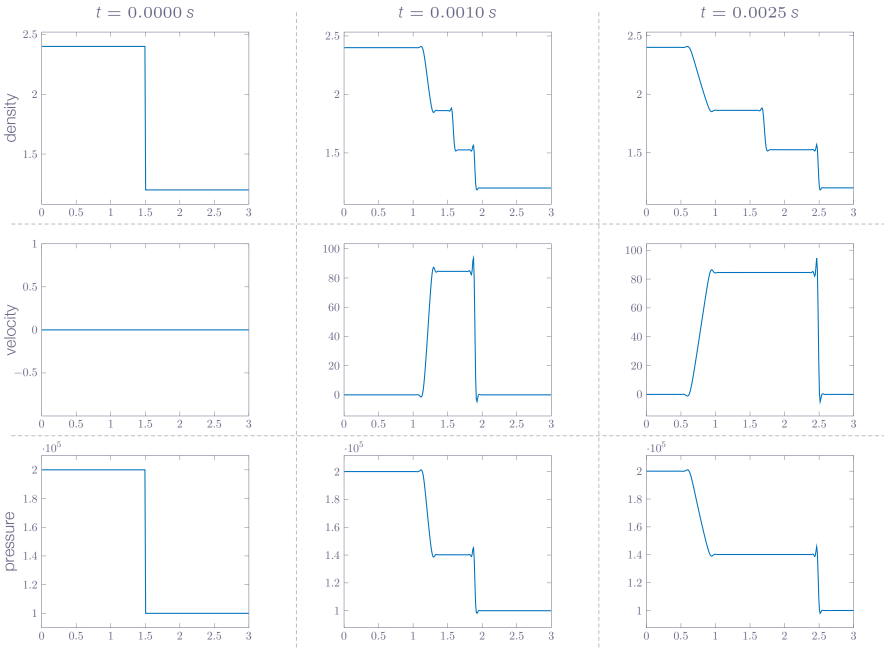
- Finite-Volume Method (FVM) solver
- three-stage Runge-Kutta time stepping
- third-order characteristic upwinding scheme
- local artificial damping

Left side conditions (state 4):

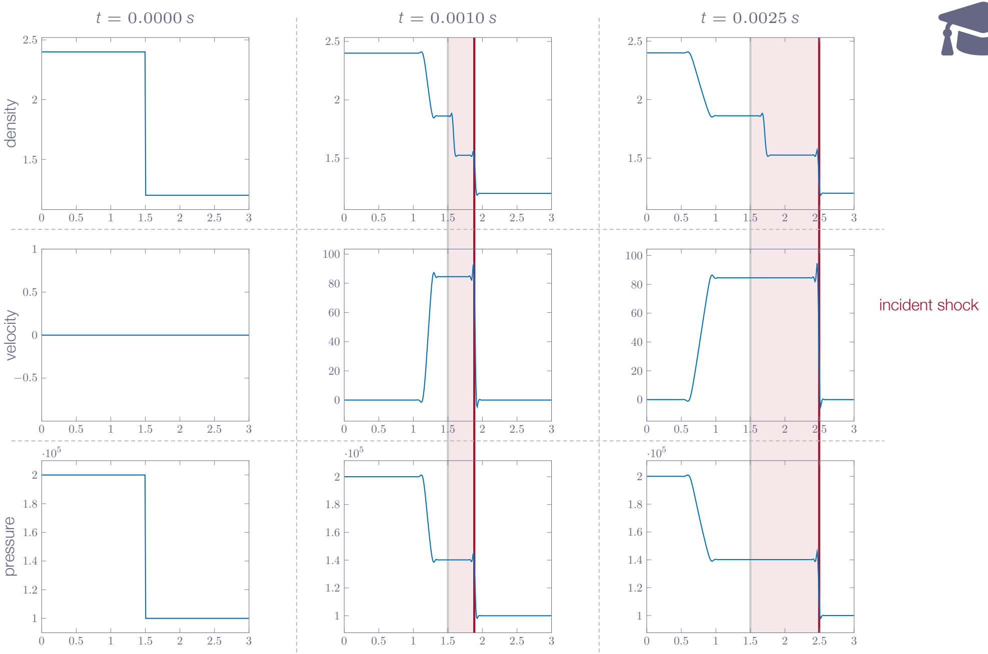
$$\rho = 2.4\ kg/m^3$$
$$u = 0.0\ m/s$$
$$p = 2.0\ bar$$

Right side conditions (state 1):

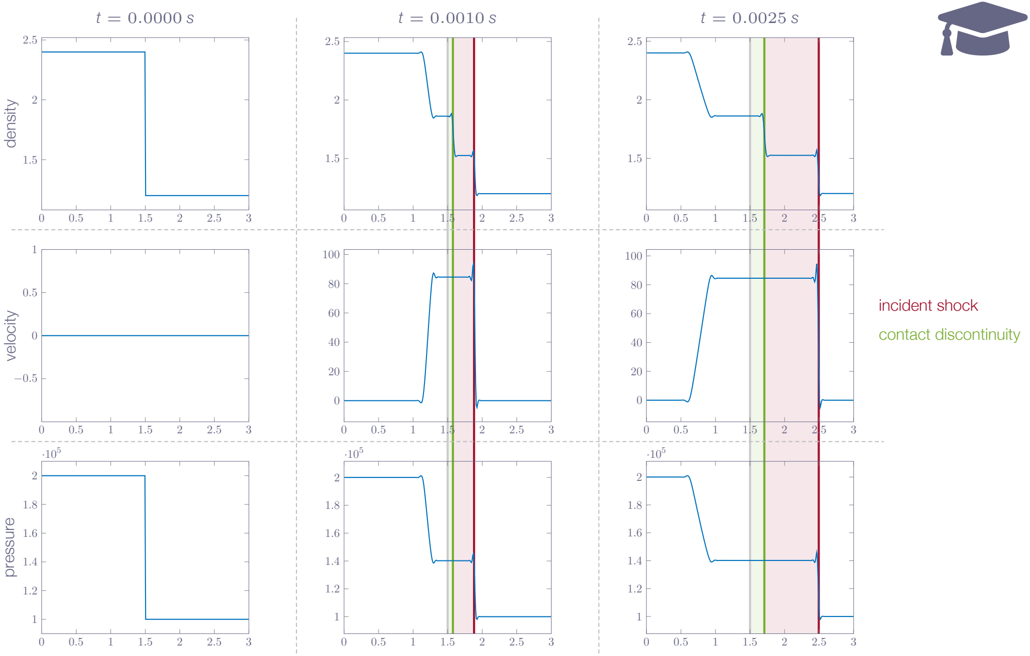
$$\rho = 1.2\ kg/m^3$$
$$u = 0.0\ m/s$$
$$p = 1.0\ bar$$



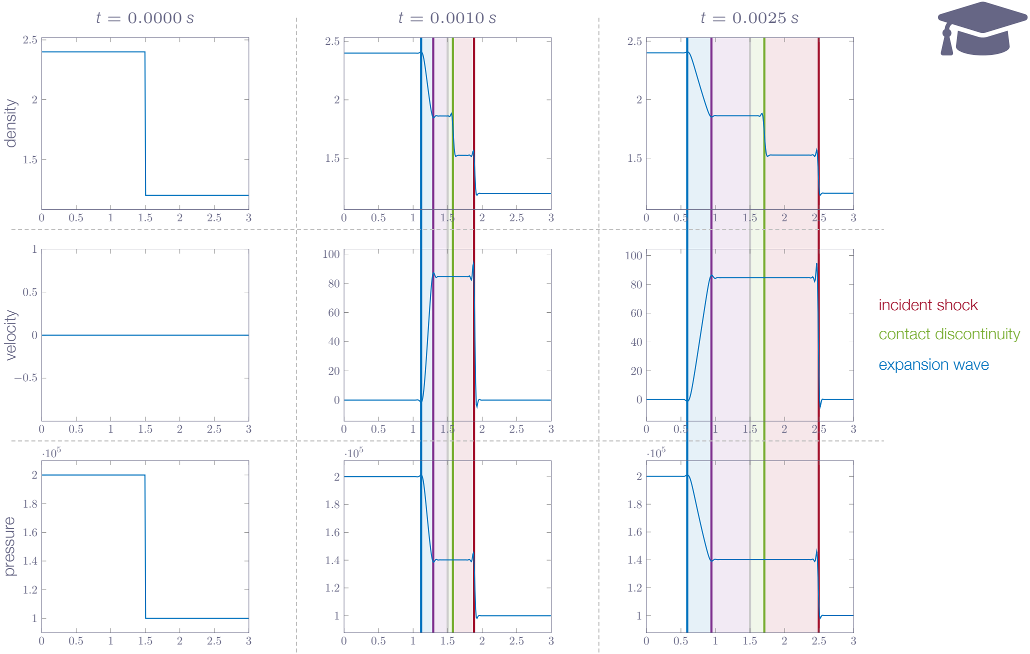
t = 0.0000 s
t = 0.0010 s
t = 0.0025 s
density
velocity
pressure
·10^5



t = 0.0000 s
t = 0.0010 s
t = 0.0025 s
density
velocity
pressure
·10^5
incident shock

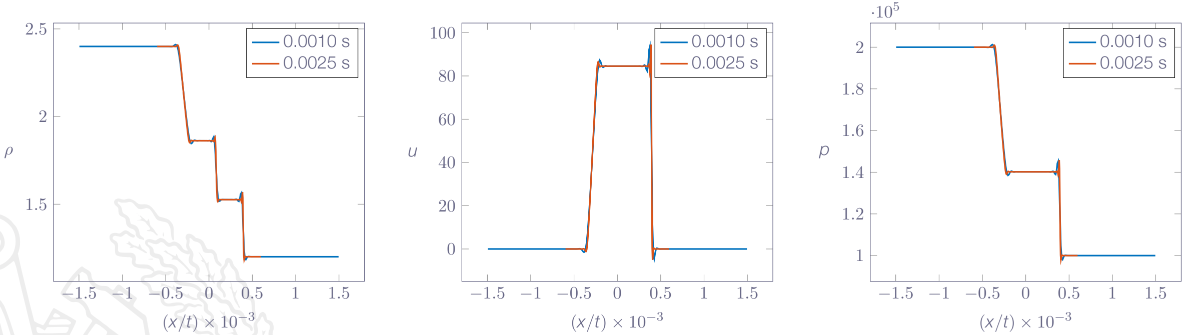


t = 0.0000 s
t = 0.0010 s
t = 0.0025 s
density
velocity
pressure
·10^5
incident shock
contact discontinuity



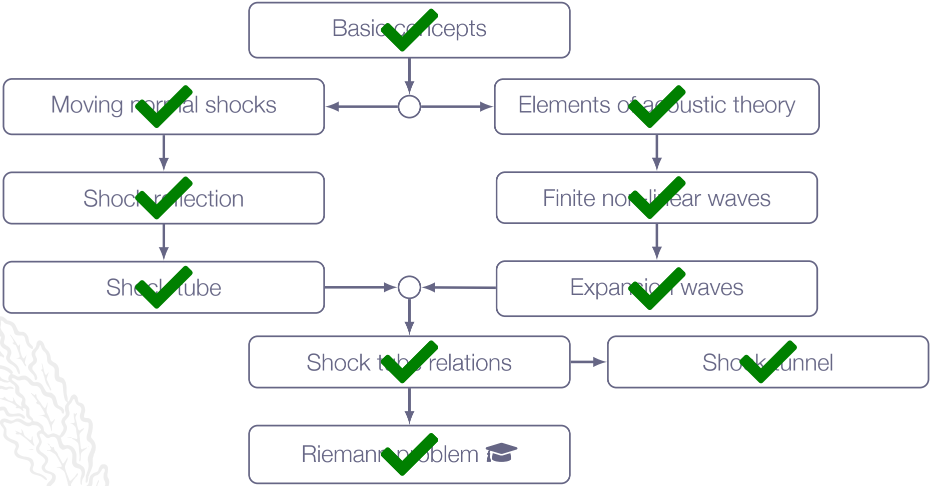
t = 0.0000 s
t = 0.0010 s
t = 0.0025 s
density
velocity
pressure
·10^5
incident shock
contact discontinuity
expansion wave

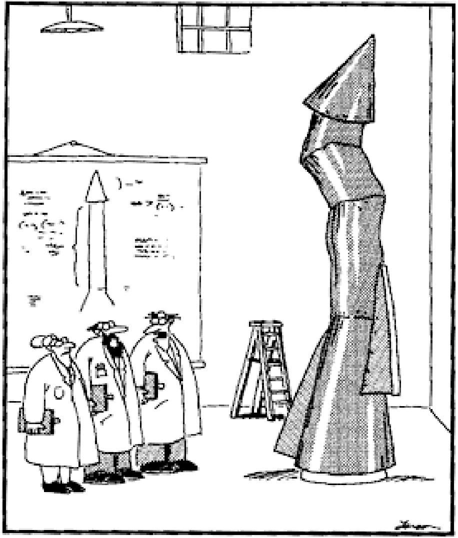
Riemann Problem - Shock Tube Simulation



The solution can be made self similar by plotting the flow field variables as function of x/t

Roadmap - Unsteady Wave Motion





"It's time we face reality, my friend. ... We're not exactly rocket scientists."