Compressible Flow - TME085

Chapter 4

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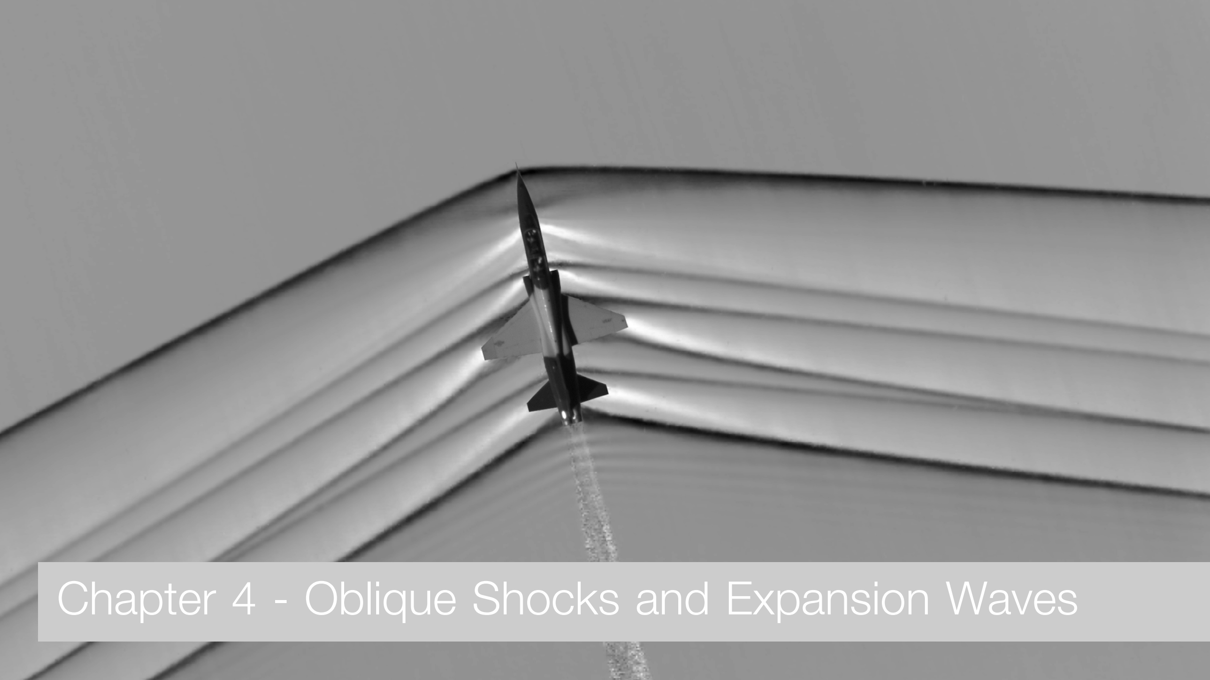
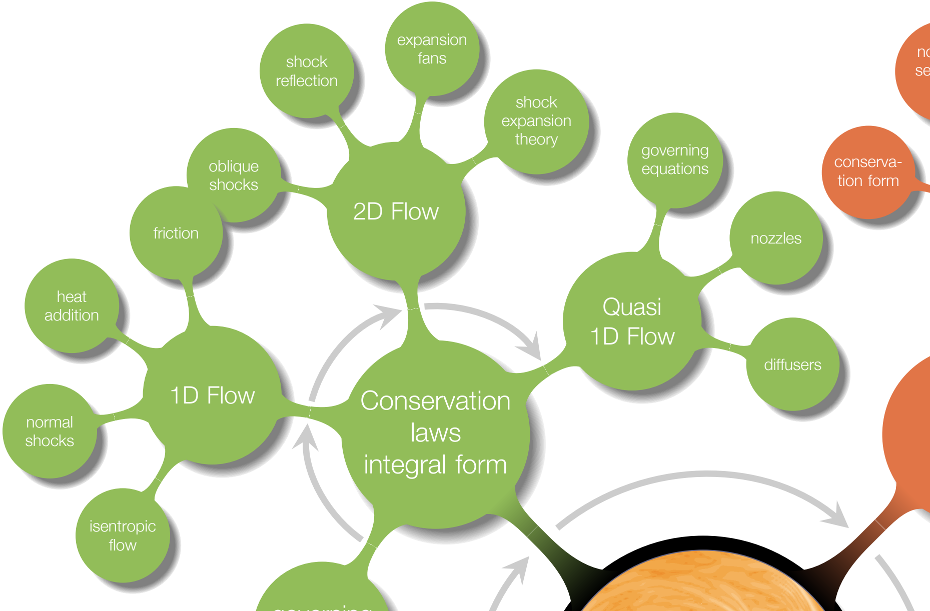


Chapter 4 - Oblique Shocks and Expansion Waves

Overview



shock reflection
expansion fans
shock expansion theory
oblique shocks
2D Flow
governing equations
friction
nozzles
Quasi 1D Flow
heat addition
diffusers
1D Flow
Conservation laws integral form
normal shocks
isentropic flow
conserva-tion form

Learning Outcomes

4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on

7 **Explain** why entropy is important for flow discontinuities

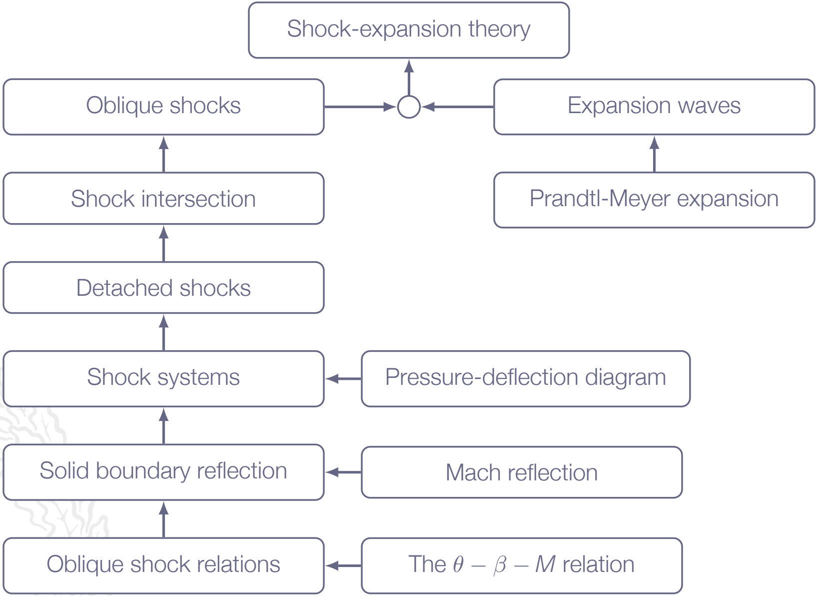
8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics

- b normal shocks*
- e oblique shocks in 2D*
- f shock reflection at solid walls*
- g contact discontinuities
- h Prandtl-Meyer expansion fans in 2D
- i detached blunt body shocks, nozzle flows

9 **Solve** engineering problems involving the above-mentioned phenomena (8a-8k)

why do we get normal shocks in some cases and oblique shocks in other?

Roadmap - Oblique Shocks and Expansion Waves

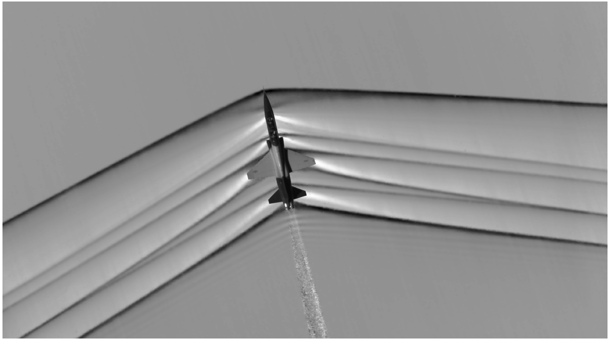


Motivation

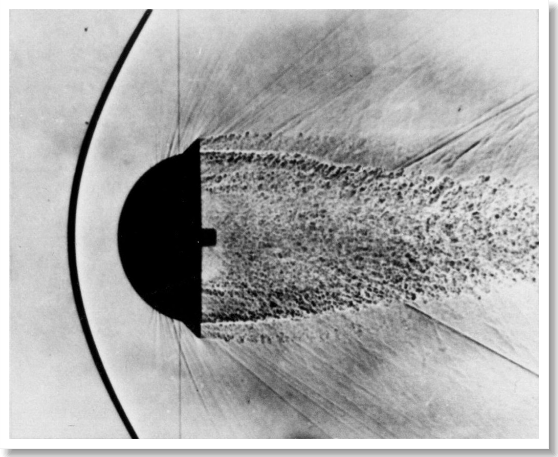
Come on, two-dimensional flow, really?! Why not three-dimensional?

- the normal shocks studied in chapter 3 are a special casees of the more general oblique shock waves that may be studied in two dimensions
- in two dimensions, we can still analyze shock waves using a pen-and-paper approach
- many practical problems or subsets of problems may be analyzed in two-dimensions
- by going from one to two dimensions we will be able to introduce physical processes important for compressible flows

Oblique Shocks and Expansion Waves



Oblique Shocks and Expansion Waves



Oblique Shocks and Expansion Waves - Assumptions

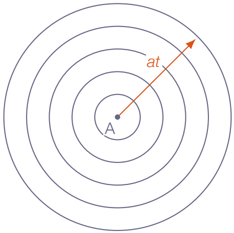
1. Supersonic
2. Steady-state
3. Two-dimensional
4. Inviscid flow (no wall friction)

In real flow, viscosity is non-zero $\Rightarrow$ boundary layers

For high-Reynolds-number flows, boundary layers are thin $\Rightarrow$ inviscid theory still relevant!

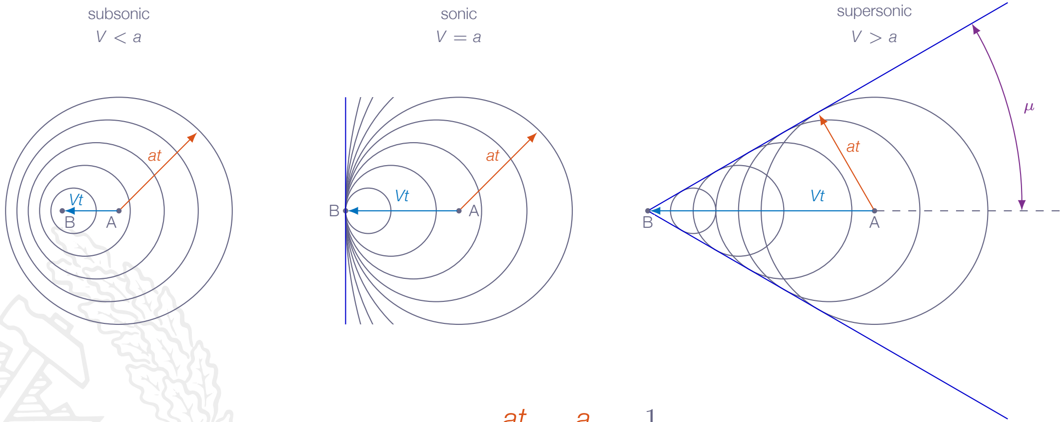
Mach Wave

Sound waves emitted from A (speed of sound *a*)



Mach Waves

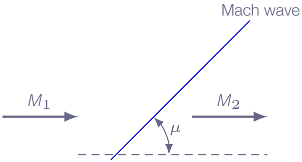
A Mach wave is an infinitely weak oblique shock



$$\sin\mu = \frac{at}{Vt} = \frac{a}{V} = \frac{1}{M}$$

Mach Wave

A Mach wave is an infinitely weak oblique shock

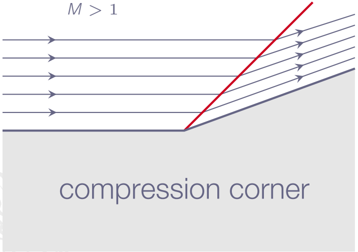


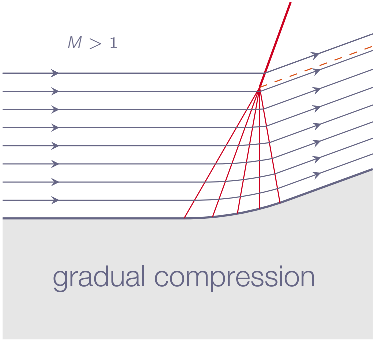
No substantial changes of flow properties over a single Mach wave

$M_1 > 1.0$ and $M_1 \approx M_2$

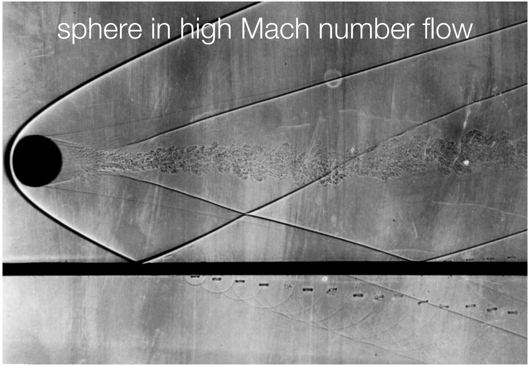
Isentropic

Oblique Shocks

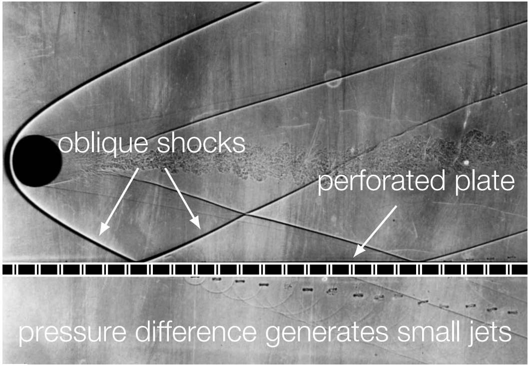




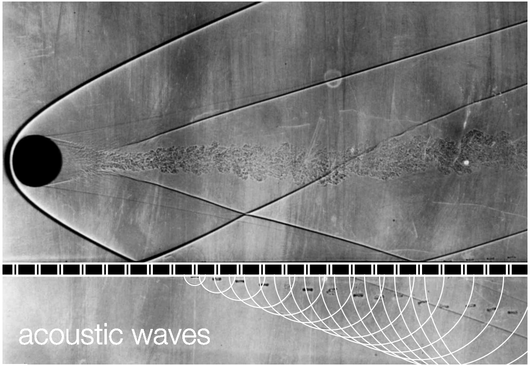
Oblique Shocks and Mach Waves



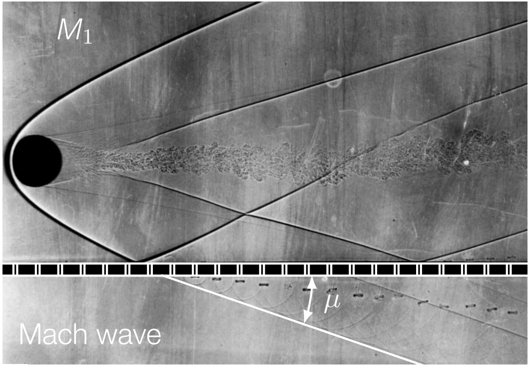
Oblique Shocks and Mach Waves



Oblique Shocks and Mach Waves

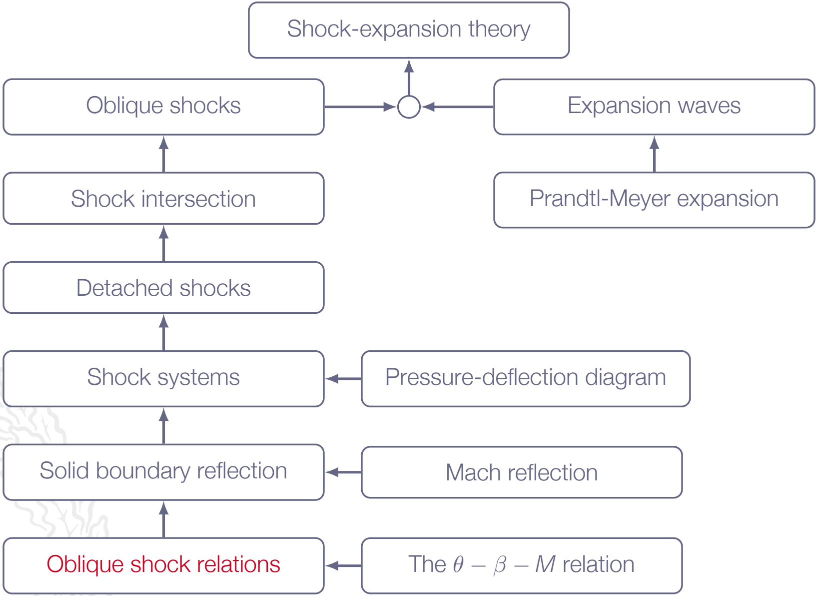


Oblique Shocks and Mach Waves



$$\mu = 19^\circ \Rightarrow M_1 = \frac{1}{\sin\mu} \approx 3.1$$

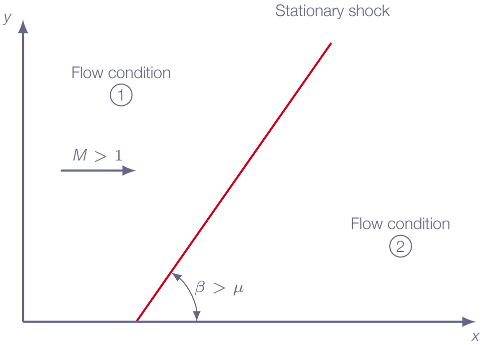
Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.3
Oblique Shock Relations

Oblique Shocks

Two-dimensional steady-state flow



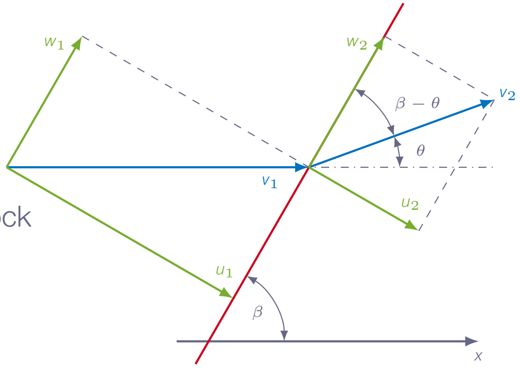
Significant changes of flow properties from 1 to 2

$M_1 > 1.0$, $\beta > \mu$, and $M_1 \neq M_2$

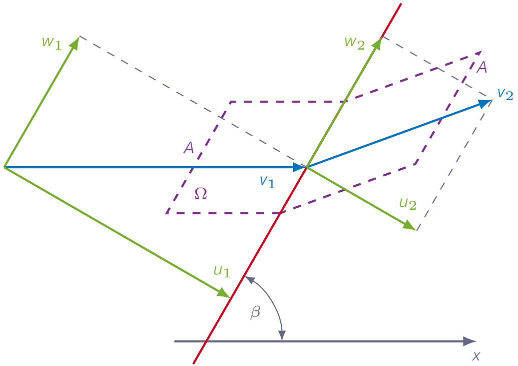
Not isentropic

Oblique Shocks

Stationary oblique shock



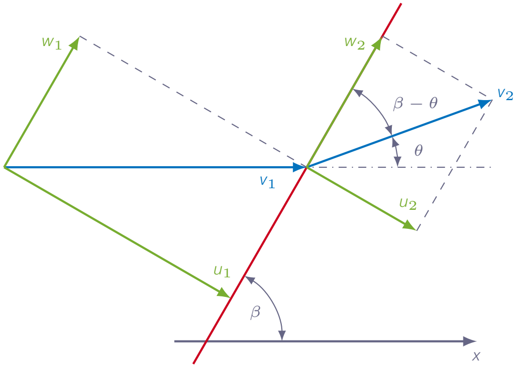
Oblique Shock Relations



Two-dimensional steady-state flow

Control volume aligned with flow stream lines

Oblique Shock Relations



Velocity notations:

$$M_{n_1} = \frac{u_1}{a_1} = M_1 \sin(\beta) \qquad M_1 = \frac{v_1}{a_1}$$

$$M_{n_2} = \frac{u_2}{a_2} = M_2 \sin(\beta - \theta) \qquad M_2 = \frac{v_2}{a_2}$$

Oblique Shock Relations

Conservation of mass:

$$\frac{d}{dt}\iiint_{\Omega} \rho d\mathscr{V} + \oiint_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS = 0$$

Mass conservation for control volume Ω:

$$0 - \rho_1 u_1 A + \rho_2 u_2 A = 0 \Rightarrow$$

$$\boxed{\rho_1 u_1 = \rho_2 u_2}$$

Oblique Shock Relations

Conservation of momentum:

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho\mathbf{v}d\mathscr{V}+\oiint\limits_{\partial\Omega}\left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}+p\mathbf{n}\right]dS=\iiint\limits_{\Omega}\rho\mathbf{f}d\mathscr{V}$$

Momentum in shock-normal direction:

$$0-(\rho_1 u_1^2+p_1)A+(\rho_2 u_2^2+p_2)A=0\Rightarrow$$

$$\boxed{\rho_1 u_1^2+p_1=\rho_2 u_2^2+p_2}$$

Oblique Shock Relations

Momentum in shock-tangential direction:

$$0 - \rho_1 u_1 w_1 A + \rho_2 u_2 w_2 A = 0 \Rightarrow$$

$$\boxed{w_1 = w_2}$$

Oblique Shock Relations

Conservation of energy:

$$\frac{d}{dt}\iiint_{\Omega} \rho e_o d\mathscr{V} + \oiint_{\partial\Omega} [\rho h_o \mathbf{v}\cdot\mathbf{n}]\, dS = \iiint_{\Omega} \rho \mathbf{f}\cdot\mathbf{v} d\mathscr{V}$$

Energy equation applied to the control volume Ω:

$$0 - \rho_1 u_1 [h_1 + \frac{1}{2}(u_1^2 + w_1^2)]A + \rho_2 u_2 [h_2 + \frac{1}{2}(u_2^2 + w_2^2)]A = 0 \Rightarrow$$

$$\boxed{h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2}$$

Oblique Shock Relations

We can use the same equations as for normal shocks if we replace M_1 with M_{n_1} and M_2 with M_{n_2}

$$M_{n_2}^2 = \frac{M_{n_1}^2 + [2/(\gamma - 1)]}{[2\gamma/(\gamma - 1)]M_{n_1}^2 - 1}$$

Ratios such as ρ_2/ρ_1, p_2/p_1, and T_2/T_1 can be calculated using the relations for normal shocks with M_1 replaced by M_{n_1}

Oblique Shock Relations

What about ratios involving stagnation flow properties, can we use the ones previously derived for normal shocks?

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The shock process is adiabatic and thus total temperature is not effected by the shock $\Rightarrow T_{o_2} = T_{o_1}$

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What about the total pressure?

Oblique Shock Relations

What about ratios involving stagnation flow properties, can we use the ones previously derived for normal shocks?

The shock process is adiabatic and thus total temperature is not effected by the shock $\Rightarrow T_{o_2} = T_{o_1}$

What about the total pressure?

$$s_2 - s_1 = C_p \ln\left(\frac{T_{o_2}}{T_{o_1}}\right) - R\ln\left(\frac{p_{o_2}}{p_{o_1}}\right) = \{T_{o_2} = T_{o_1}\} = -R\ln\left(\frac{p_{o_2}}{p_{o_1}}\right)$$

entropy is a thermodynamic flow property and $s_2 - s_1$ is dictated by the shock strength and thus the total pressure ratio is a function of the shock-normal Mach number

Oblique Shock Relations

Note! total pressure is always calculated using the flow Mach number, not the shock-normal Mach number

However, the ratio p_{o_2}/p_{o_1} may be calculated using the shock-normal Mach number

So, be careful when using relations derived for normal shocks for oblique shocks when it comes to total flow conditions...

Oblique Shock Relations

p_{o_2}/p_{o_1} is calculated as: $\frac{p_{o_2}}{p_{o_1}} = \frac{p_{o_2}}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_{o_1}}$

where

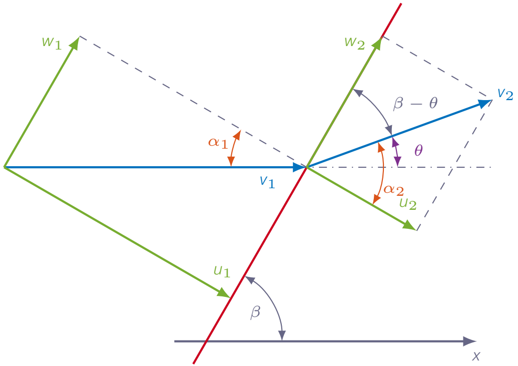
1. $\frac{p_{o_2}}{p_2} = f(M_2)$, $\frac{p_2}{p_1} = f(M_{n_1})$, and $\frac{p_1}{p_{o_1}} = f(M_1)$

or alternatively

2. $\frac{p_{o_2}}{p_2} = f(M_{n_2})$, $\frac{p_2}{p_1} = f(M_{n_1})$, and $\frac{p_1}{p_{o_1}} = f(M_{n_1})$

Note! in the second case the total pressures are **not** the true total pressures of the flow and therefore it is suggested to use the first approach

Deflection Angle (*for the interested*)



$$\theta = \alpha_2 - \alpha_1 = \tan^{-1}\left(\frac{w}{u_2}\right) - \tan^{-1}\left(\frac{w}{u_1}\right)$$

$$\frac{\partial \theta}{\partial w} = \frac{u_2}{w^2 + u_2^2} - \frac{u_1}{w^2 + u_1^2}$$

Deflection Angle (*for the interested*)

$$\frac{\partial\theta}{\partial w} = \frac{u_2}{w^2+u_2^2} - \frac{u_1}{w^2+u_1^2} = 0 \Rightarrow$$

$$\frac{u_2(w^2+u_1^2) - u_1(w^2+u_2^2)}{(w^2+u_2^2)(w^2+u_1^2)} = 0 \Rightarrow \frac{(u_2-u_1)(w^2-u_1u_2)}{(w^2+u_2^2)(w^2+u_1^2)} = 0$$

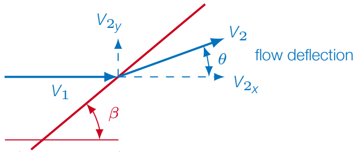
Two solutions:

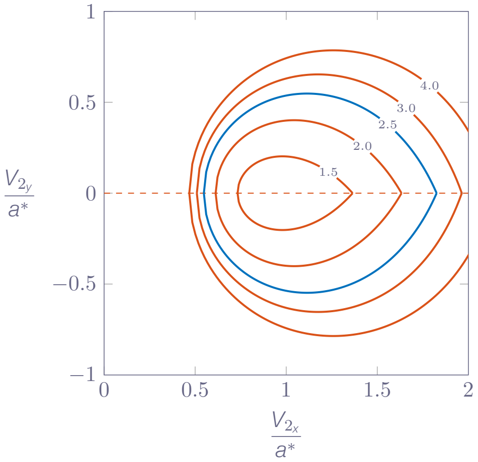
$u_2 = u_1$ (no deflection)

$w^2 = u_1u_2$ (max deflection)

Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number





Note!

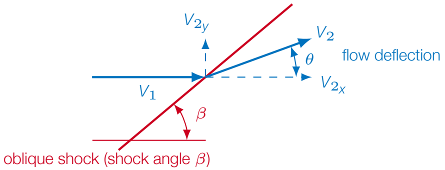
In the shock polar, V_{2_x} and V_{2_y} are normalized by a^*

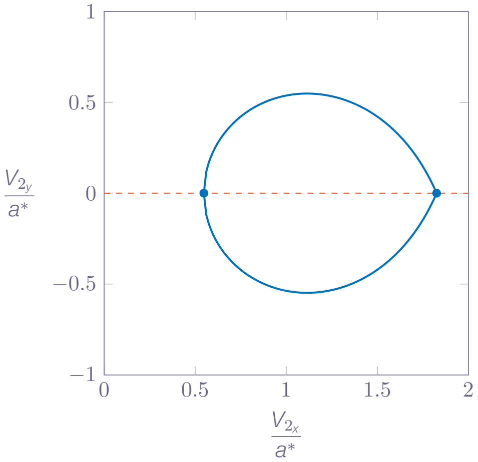
a^* is a constant in a adiabatic flow

a^* is not affected by shocks

Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number





No deflection cases:

normal shock
(reduced shock-normal velocity)

Mach wave
(unchanged shock-normal velocity)

Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

$$M^* = \frac{\sqrt{V_{2_x}^2 + V_{2_y}^2}}{a^*}$$

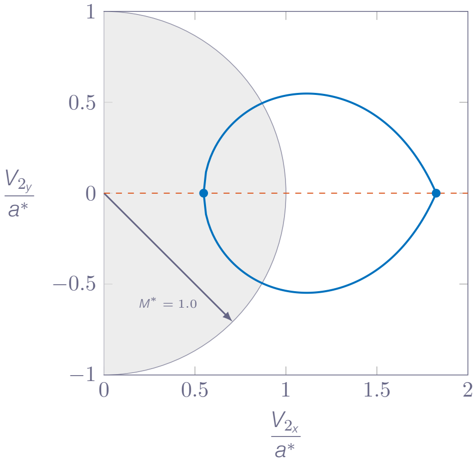
Solutions to the left of the sonic line are subsonic

Recall

$M^* = 1 \Leftrightarrow M = 1$

$M^* < 1 \Leftrightarrow M < 1$

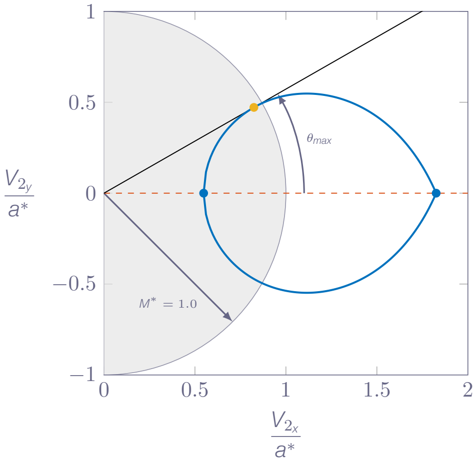
$M^* > 1 \Leftrightarrow M > 1$



Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

It is not possible to deflect the flow more than θ_{max}



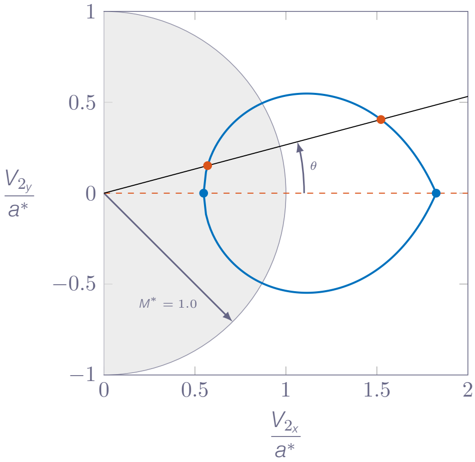
Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

For each deflection angle $\theta < \theta_{max}$, there are two solutions

1. **strong shock** solution
2. **weak shock** solution

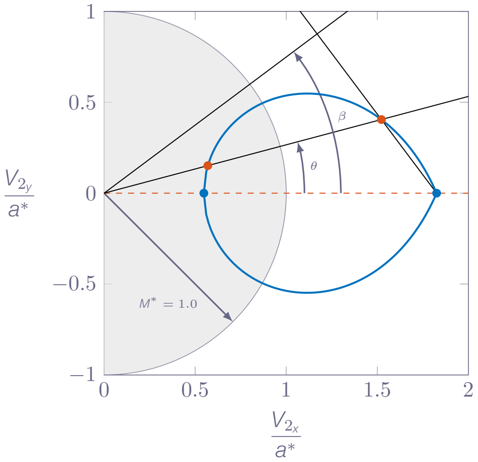
Weak shocks give lower losses and therefore the preferred solution



Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

The shock polar can be used to calculate the shock angle β for a given deflection angle θ



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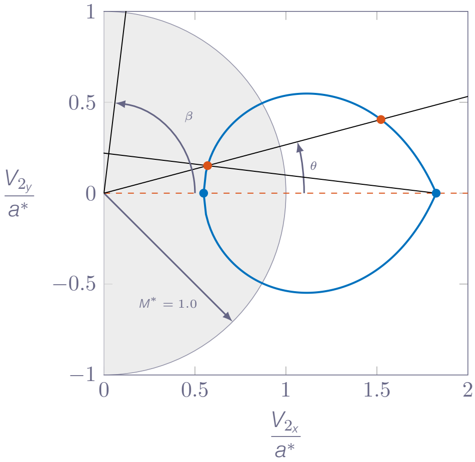
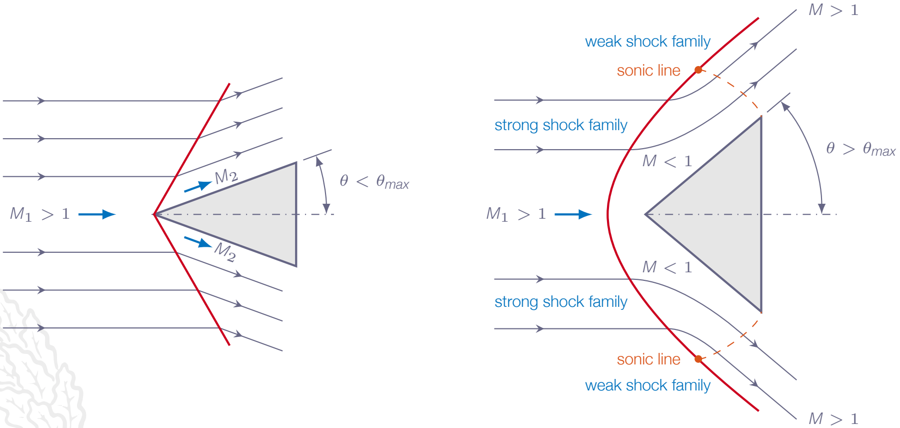
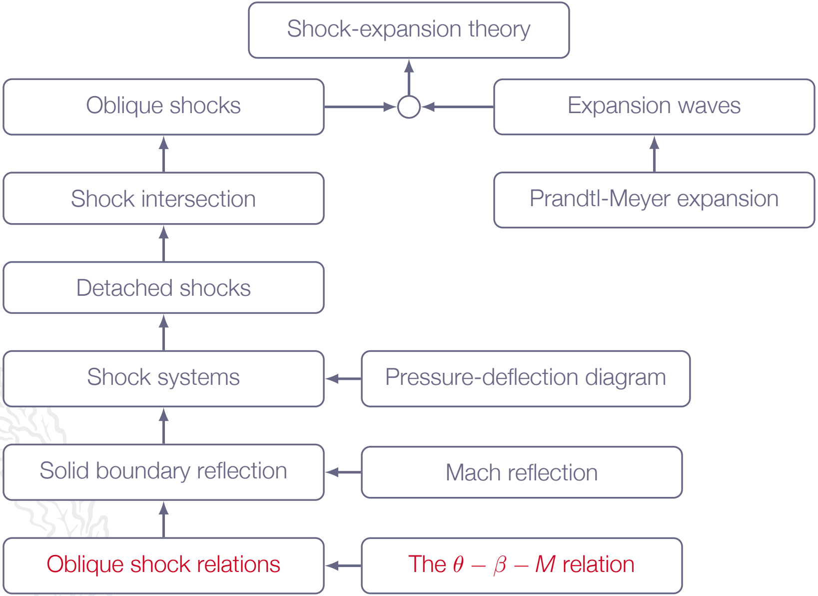


Flow Deflection



Roadmap - Oblique Shocks and Expansion Waves



The θ-β-M Relation

It can be shown that

$$\tan\theta = 2\cot\beta\left(\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}\right)$$

which is the θ-β-M relation

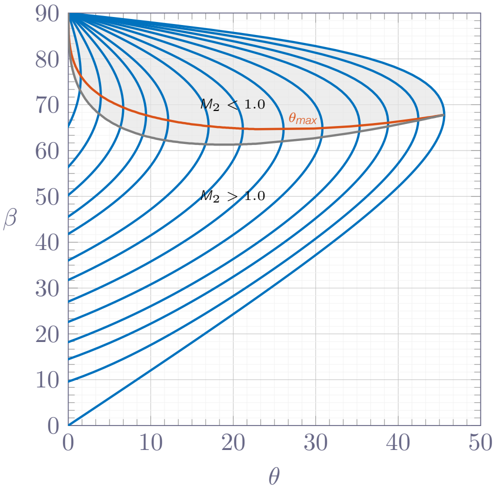
The θ-β-Mach Relation

A relation between:

1. flow deflection angle θ
2. shock angle β
3. upstream flow Mach number M_1

$$\tan(\theta) = 2\cot(\beta)\left(\frac{M_1^2\sin^2(\beta) - 1}{M_1^2(\gamma + \cos(2\beta)) + 2}\right)$$

Note! in general there are two solutions for a given M_1 (or none)



The θ-β-Mach Relation

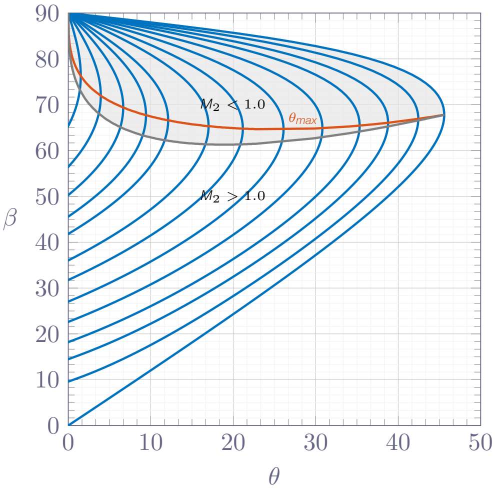
There is a small region where we may find weak shock solutions for which $M_2 < 1$

In most cases weak shock solutions have $M_2 > 1$

Strong shock solutions always have $M_2 < 1$

In practical situations, **weak shock solutions are most common**

Strong shock solution may appear in special situations due to high back pressure, which forces $M_2 < 1$

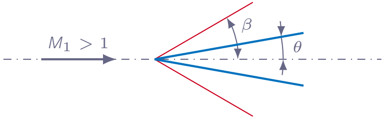


The θ-β-M Relation

Note! In Chapter 3 we learned that the Mach number always reduces to subsonic values behind a shock. This is true for normal shocks (shocks that are normal to the flow direction). It is also true for oblique shocks if looking in the shock-normal direction.

The θ-β-M Relation - Wedge Flow

Wedge flow oblique shock analysis:

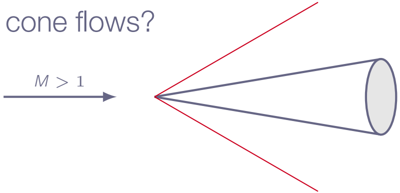


1. θ-β-M relation $\Rightarrow$ β for given M_1 and θ
2. β gives M_{n_1} according to: $M_{n_1} = M_1 \sin(\beta)$
3. normal shock formula with M_{n_1} instead of M_1 $\Rightarrow$ M_{n_2} (instead of M_2)
4. M_2 given by $M_2 = M_{n_2} / \sin(\beta - \theta)$
5. normal shock formula with M_{n_1} instead of M_1 $\Rightarrow$ ρ_2/ρ_1, p_2/p_1, etc
6. upstream conditions + ρ_2/ρ_1, p_2/p_1, etc $\Rightarrow$ downstream conditions

Chapter 4.4
Supersonic Flow over Wedges and Cones

Supersonic Flow over Wedges and Cones

What about cone flows?

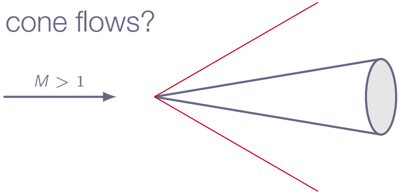


Similar to wedge flow, we do get a constant-strength shock wave, attached at the cone tip (or else a detached curved shock)

The attached shock is also cone-shaped

Supersonic Flow over Wedges and Cones

What about cone flows?

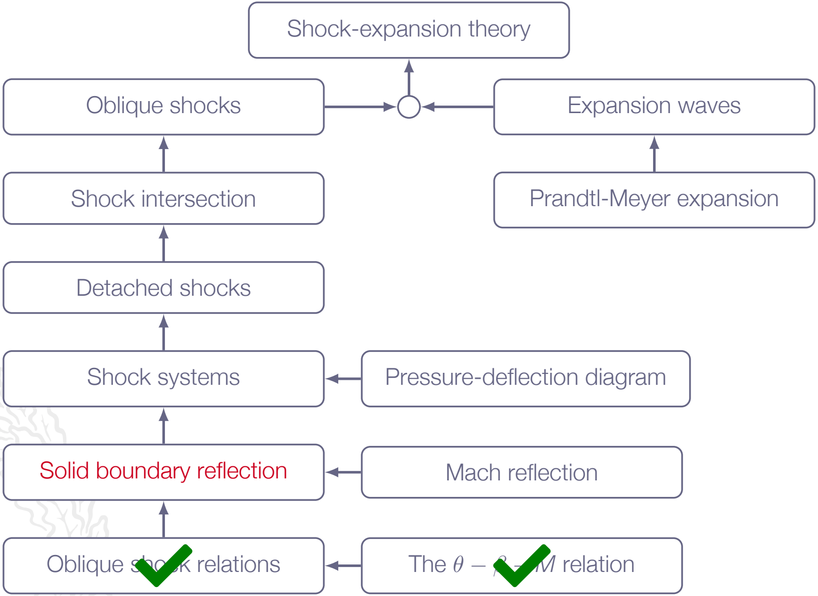


The flow condition immediately downstream of the shock is uniform

However, downstream of the shock the streamlines are curved and the flow varies in a more complex manner (3D relieving effect - as R increases there is more and more space around cone for the flow)

β for cone shock is always smaller than that for wedge shock, if M_1 is the same

Roadmap - Oblique Shocks and Expansion Waves

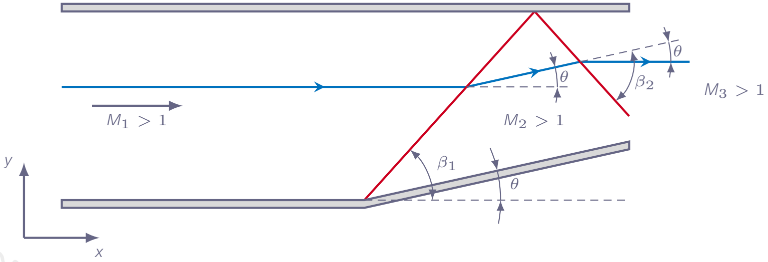


Chapter 4.6
Regular Reflection from a Solid Boundary

Shock Reflection

Regular reflection of oblique shock at solid wall

(see example 4.10)



Assumptions:

- steady-state inviscid flow
- weak shocks

Shock Reflection

first shock

upstream condition

$M_1 > 1$
flow in x-direction

downstream condition

weak shock $\Rightarrow M_2 > 1$
deflection angle θ
shock angle β_1

second shock

upstream condition

downstream of first shock

downstream condition

weak shock $\Rightarrow M_3 > 1$
deflection angle θ
shock angle β_2

Shock Reflection

Solution:

first shock:

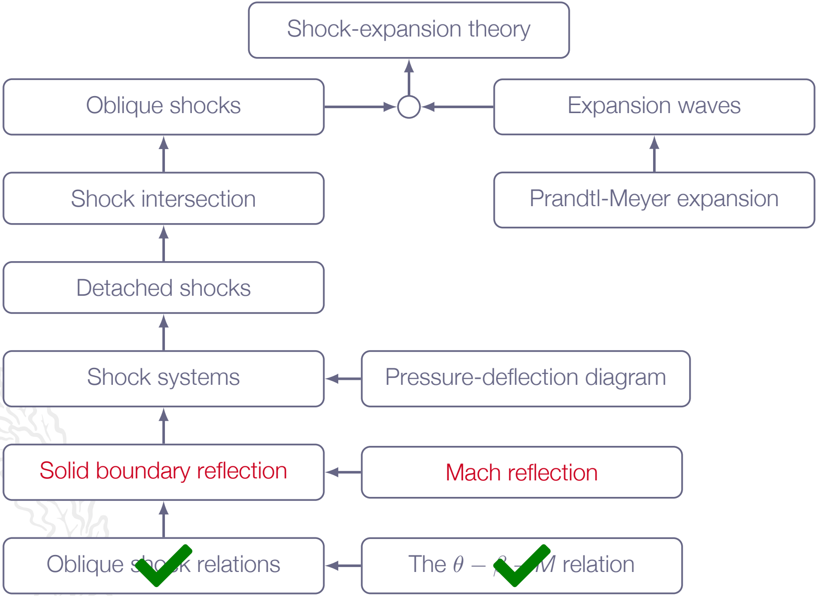
1. β_1 calculated from θ-β-M relation for specified θ and M_1 (*weak solution*)
2. flow condition 2 according to formulas for normal shocks ($M_{n_1} = M_1 \sin(\beta_1)$ and $M_{n_2} = M_2 \sin(\beta_1 - \theta)$)

second shock:

1. β_2 calculated from θ-β-M relation for specified θ and M_2 (*weak solution*)
2. flow condition 3 according to formulas for normal shocks ($M_{n_2} = M_2 \sin(\beta_2)$ and $M_{n_3} = M_3 \sin(\beta_2 - \theta)$)

$\Rightarrow$ complete description of flow and shock waves
(angle between upper wall and second shock: $\Phi = \beta_2 - \theta$)

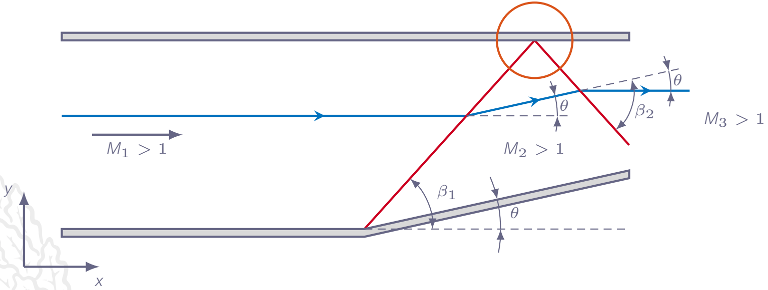
Roadmap - Oblique Shocks and Expansion Waves



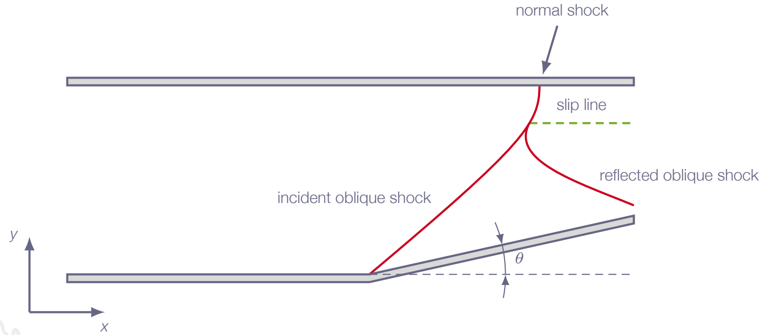
Chapter 4.11
Mach Reflection

Regular Shock Reflection

Regular reflection possible if both primary and reflected shocks are weak (see θ-β-M relation)



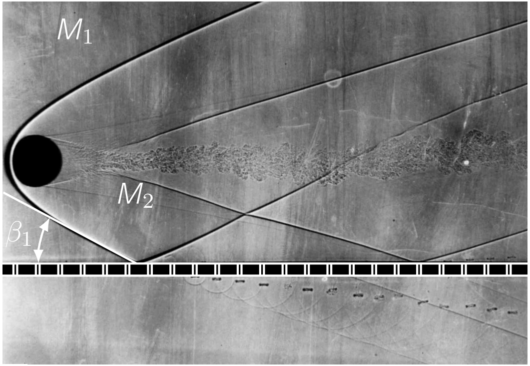
Mach Reflection



Mach reflection:

- appears when regular reflection is not possible
- more complex flow than for a regular reflection
- no analytic solution - numerical solution necessary

Oblique Shocks and Mach Waves

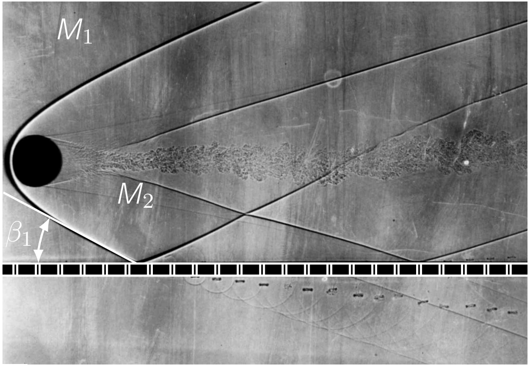


$M_1 > M_2$

$M_2 > 1.0$

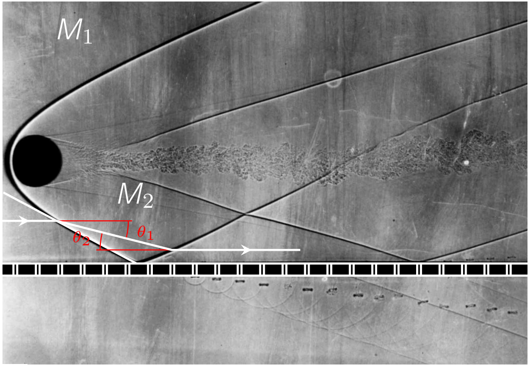
$$\theta_1 = f(M_1, \beta_1), \quad M_2 = f(M_1, \theta_1, \beta_1)$$

Oblique Shocks and Mach Waves



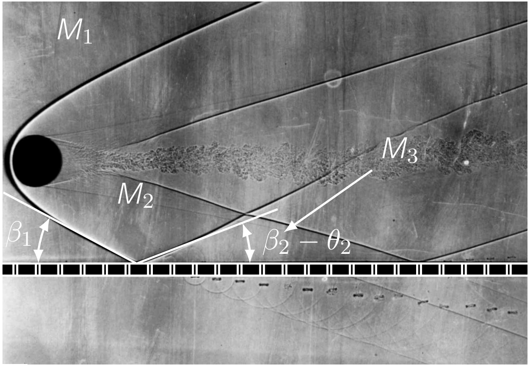
$$\left.\begin{aligned} \beta_1 &= 28^\circ \\ M_1 &= 3.1 \end{aligned}\right\} \Rightarrow \theta_1 \approx 11.2^\circ, \quad M_2 \approx 2.5$$

Oblique Shocks and Mach Waves



$$\theta_1 = \theta_2$$

Oblique Shocks and Mach Waves



$M_1 > M_2 > M_3$

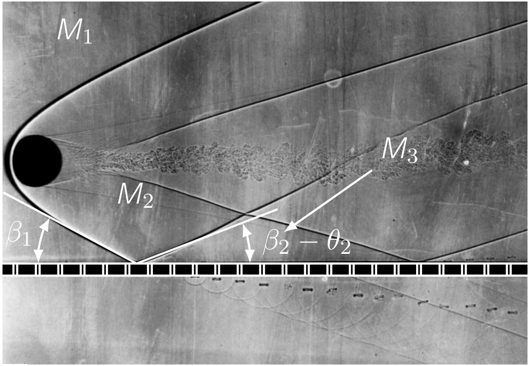
$M_3 > 1.0$

$\beta_2 > \beta_1$

$$\beta_2 = f(M_2, \theta_2), \quad M_3 = f(M_2, \theta_2, \beta_2)$$

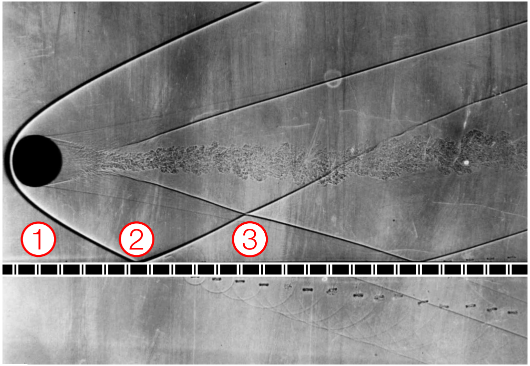
Note! Shock wave reflection at solid wall is **not** specular

Oblique Shocks and Mach Waves



$$\left.\begin{aligned}\theta_2 &= 11.2^\circ \\ M_2 &= 2.5\end{aligned}\right\} \Rightarrow \beta_2 \approx 33^\circ, \quad M_3 \approx 2.0$$

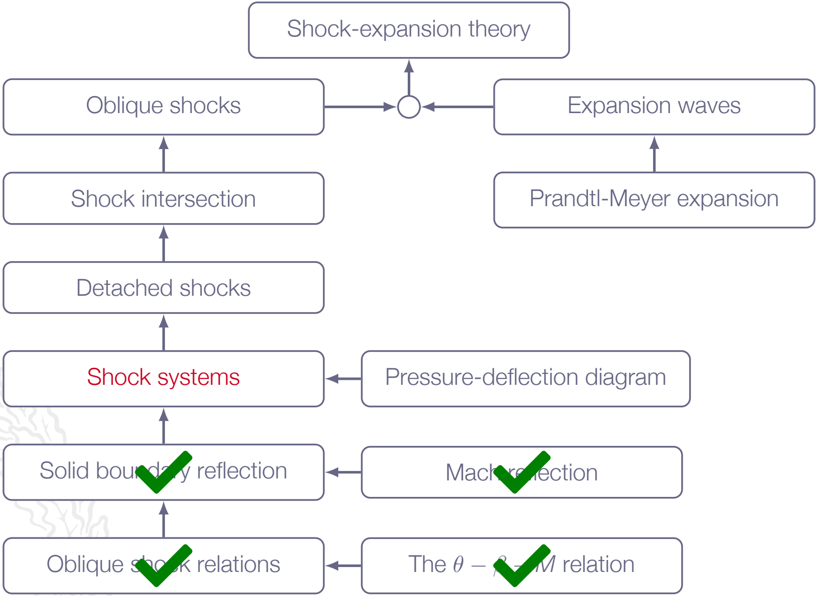
Oblique Shocks and Mach Waves



$$\frac{p_3}{p_1} = \frac{p_2}{p_1}\frac{p_3}{p_2} \approx 4.52$$

$$\frac{T_3}{T_1} = \frac{T_2}{T_1}\frac{T_3}{T_2} \approx 1.57$$

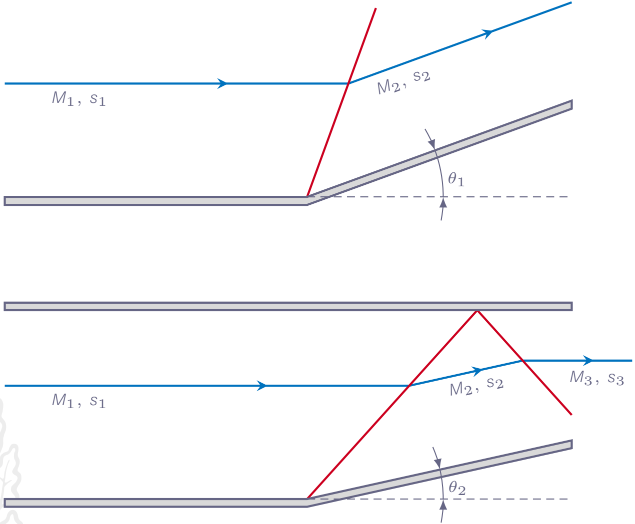
Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.7
Comments on Flow Through Multiple Shock Systems

Flow Through Multiple Shock Systems

Single-shock compression versus multiple-shock compression:



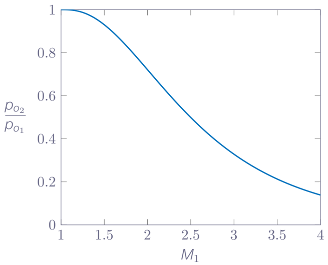
Flow Through Multiple Shock Systems

We may find θ_1 and θ_2 (for same M_1) which gives the same final Mach number

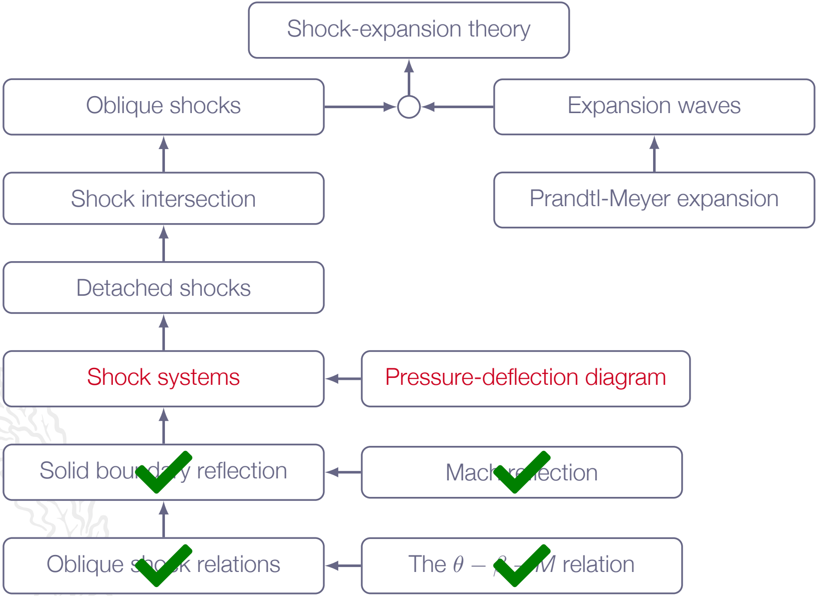
In such cases, the flow with multiple shocks has smaller losses

Explanation: entropy generation at a shock is a very non-linear function of shock strength

Note! the system of multiple shocks might very well result in a larger total flow deflection angle than the single-shock case

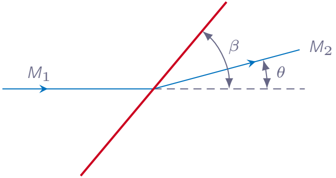


Roadmap - Oblique Shocks and Expansion Waves

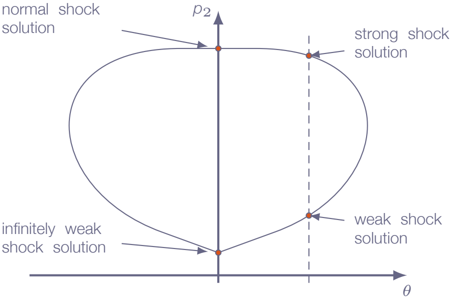


Chapter 4.8
Pressure Deflection Diagrams

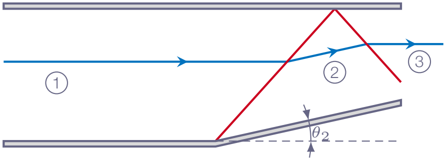
Pressure Deflection Diagrams

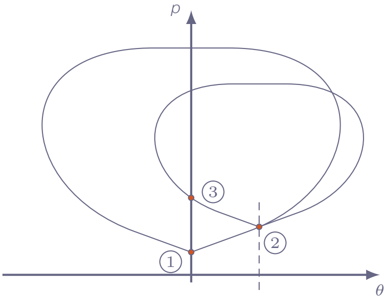


$\Rightarrow$ relation between p_2 and θ

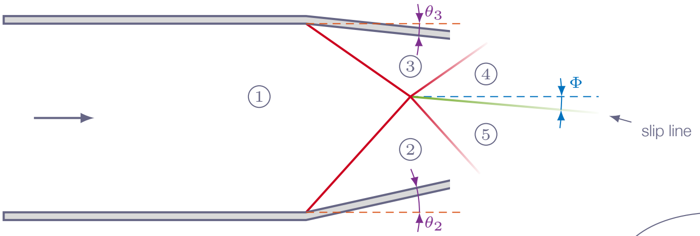


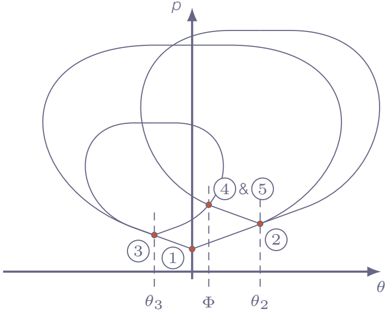
Pressure Deflection Diagrams - Shock Reflection





Pressure Deflection Diagrams - Shock Intersection

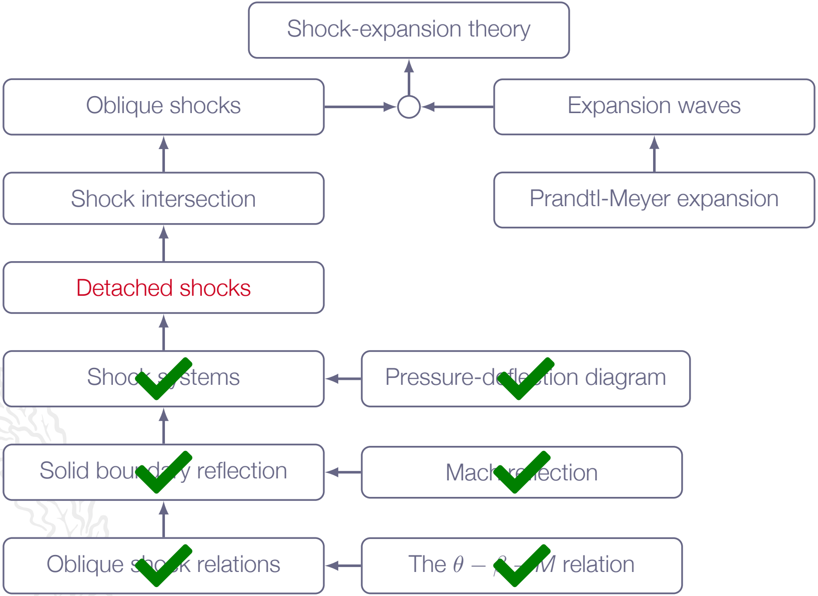




A slip line is a contact discontinuity:

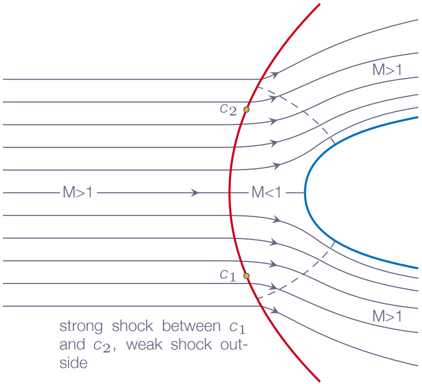
discontinuity in ρ, T, s, v, and M
continuous in p and flow angle

Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.12
Detached Shock Wave in Front of a Blunt Body

Detached Shocks

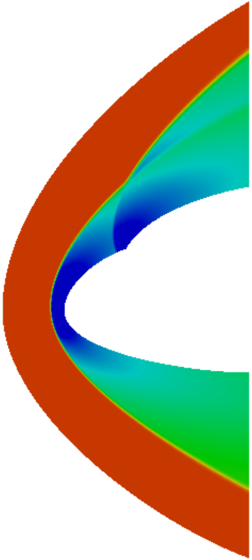


Detached Shocks

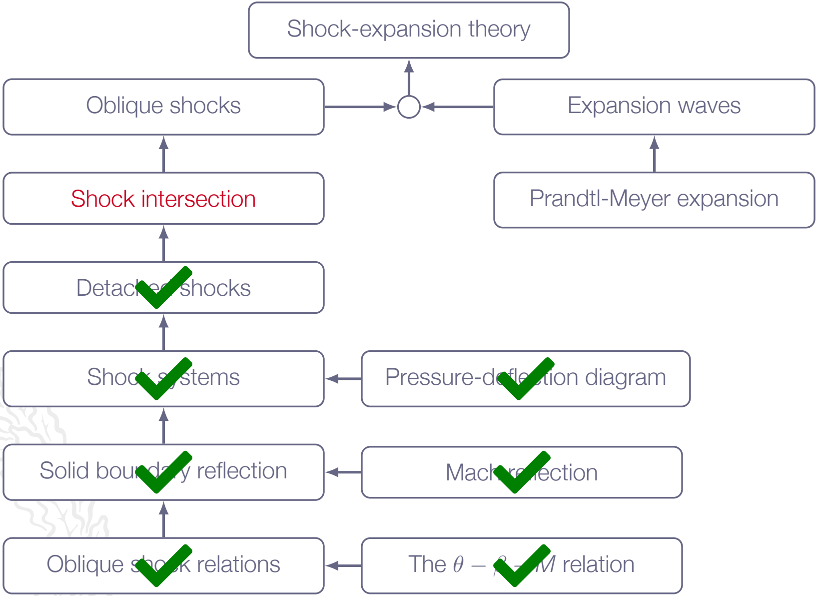
As we move along the detached shock form the centerline, the shock will change in nature as

1. right in front of the body we will have a normal shock
2. strong oblique shock
3. weak oblique shock
4. far away from the body it will approach a **Mach wave**, *i.e.* an infinitely weak oblique shock

Detached Shocks



Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.10
Intersection of Shocks of the Same Family

Mach Waves (*Repetition*)

Oblique shock, angle β, flow deflection θ:

$$M_{n_2}^2 = \frac{M_{n_1}^2 + [2/(\gamma - 1)]}{[2\gamma/(\gamma - 1)]M_{n_1}^2 - 1}$$

where

$$M_{n_1} = M_1 \sin(\beta)$$

and

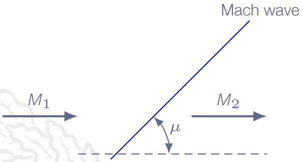
$$M_{n_2} = M_2 \sin(\beta - \theta)$$

Now, let $M_{n_1} \rightarrow 1$ and $M_{n_2} \rightarrow 1 \Rightarrow$ infinitely weak shock!

Such very weak shocks are called **Mach waves**

Mach Waves (*Repetition*)

$$M_{n_1} = 1 \Rightarrow M_1 \sin(\beta) = 1 \Rightarrow \beta = \arcsin(1/M_1)$$

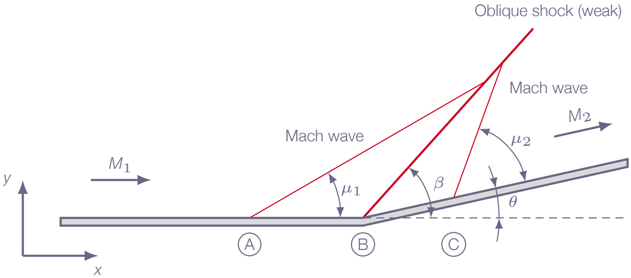


$$M_2 \approx M_1$$

$$\theta \approx 0$$

$$\mu = \arcsin(1/M_1)$$

Mach Waves


Oblique shock (weak)
Mach wave
M_2
μ_2
Mach wave
M_1
y
μ_1
β
θ
A
B
C
x

Mach Waves

1. Mach wave at A: $\sin(\mu_1) = 1/M_1$

2. Mach wave at C: $\sin(\mu_2) = 1/M_2$

3. Oblique shock at B: $M_{n_1} = M_1 \sin(\beta) \Rightarrow \sin(\beta) = M_{n_1}/M_1$

   Existence of shock requires $M_{n_1} > 1 \Rightarrow \beta > \mu_1$

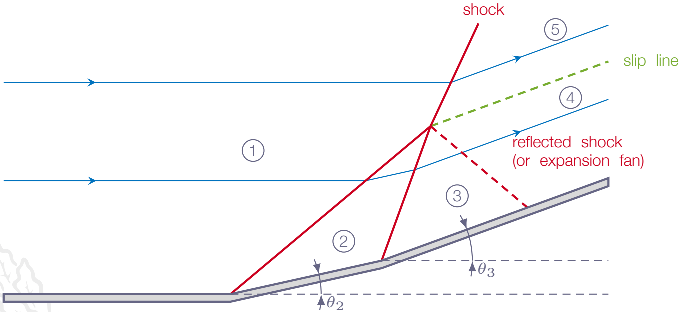
   Mach wave intercepts shock!

4. Also, $M_{n_2} = M_2 \sin(\beta - \theta) \Rightarrow \sin(\beta - \theta) = M_{n_2}/M_2$

   For finite shock strength $M_{n_2} < 1 \Rightarrow (\beta - \theta) < \mu_2$

   Again, Mach wave intercepts shock

Shock Intersection - Same Family



Shock Intersection - Same Family

Case 1: Streamline going through regions 1, 2, 3, and 4 (through two oblique (weak) shocks)

Case 2: Streamline going through regions 1 and 5 (through one oblique (weak) shock)

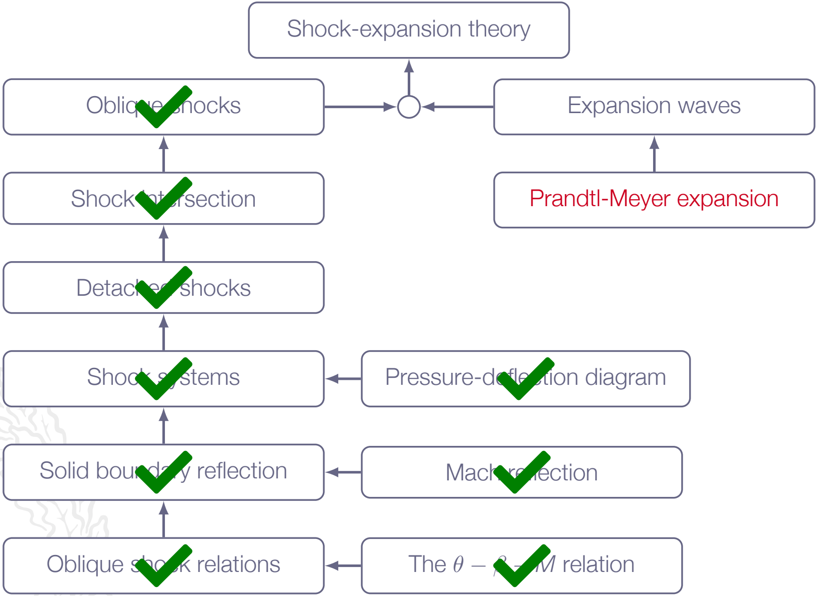
Problem: Find conditions 4 and 5 such that

a. $p_4 = p_5$
b. flow angle in 4 equals flow angle in 5

Solution may give either **reflected shock** or **expansion fan**, depending on actual conditions

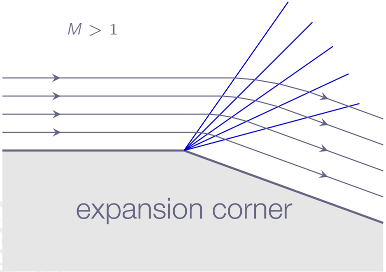
A **slip line** usually appears, across which there is a discontinuity in all variables except p and flow angle

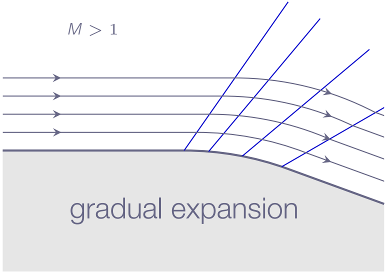
Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.14
Prandtl-Meyer Expansion Waves

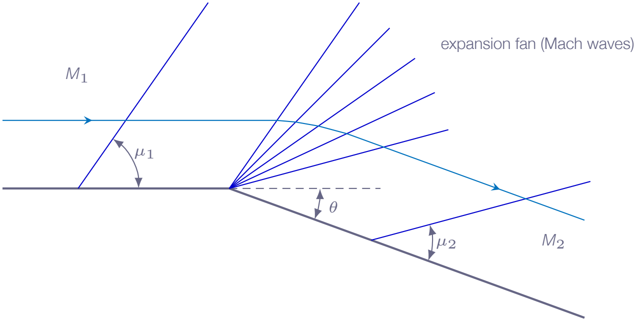
Expansion Waves





Prandtl-Meyer Expansion Waves

An expansion fan is a centered simple wave (also called Prandl-Meyer expansion)



$M_2 > M_1$ (the flow accelerates through the expansion fan)

$p_2 < p_1$, $\rho_2 < \rho_1$, $T_2 < T_1$

Prandtl-Meyer Expansion Waves

Continuous expansion region

Infinite number of weak Mach waves

Streamlines through the expansion wave are smooth curved lines

$ds = 0$ for each Mach wave $\Rightarrow$ the expansion process is **isentropic!**

Prandtl-Meyer Expansion Waves

upstream of expansion $M_1 > 1$, $\sin(\mu_1) = 1/M_1$

flow accelerates as it curves through the expansion fan

downstream of expansion $M_2 > M_1$, $\sin(\mu_2) = 1/M_2$

flow is isentropic $\Rightarrow$ s, p_o, T_o, ρ_o, a_o, ... are constant along streamlines

flow deflection: θ

Prandtl-Meyer Expansion Waves

It can be shown that $d\theta = \sqrt{M^2 - 1}\dfrac{dv}{v}$, where $v = |\mathbf{v}|$ (valid for all gases)

Integration gives

$$\int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \sqrt{M^2 - 1}\frac{dv}{v}$$

the term $\dfrac{dv}{v}$ needs to be expressed in terms of Mach number

$v = Ma \Rightarrow \ln v = \ln M + \ln a \Rightarrow$

$$\frac{dv}{v} = \frac{dM}{M} + \frac{da}{a}$$

Prandtl-Meyer Expansion Waves

Calorically perfect gas and adiabatic flow gives

$$\frac{T_o}{T} = 1 + \frac{1}{2}(\gamma - 1)M^2$$

$$\left\{a = \sqrt{\gamma R T},\ a_o = \sqrt{\gamma R T_o}\right\} \Rightarrow \frac{T_o}{T} = \left(\frac{a_o}{a}\right)^2 \Rightarrow$$

$$\left(\frac{a_o}{a}\right)^2 = 1 + \frac{1}{2}(\gamma - 1)M^2 \Leftrightarrow a = a_o\left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-1/2}$$

Prandtl-Meyer Expansion Waves

Differentiation gives:

$$da = a_o \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-3/2} \left(-\frac{1}{2}\right)(\gamma - 1)MdM$$

or

$$da = a \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-1} \left(-\frac{1}{2}\right)(\gamma - 1)MdM$$

which gives

$$\frac{dv}{v} = \frac{dM}{M} + \frac{da}{a} = \frac{dM}{M} + \frac{-\frac{1}{2}(\gamma - 1)MdM}{1 + \frac{1}{2}(\gamma - 1)M^2} = \frac{1}{1 + \frac{1}{2}(\gamma - 1)M^2}\frac{dM}{M}$$

Prandtl-Meyer Expansion Waves

Thus,

$$\int_{\theta_1}^{\theta_2} d\theta = \theta_2 - \theta_1 = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{1}{2}(\gamma-1)M^2}\frac{dM}{M} = \nu(M_2) - \nu(M_1)$$

where

$$\nu(M) = \int \frac{\sqrt{M^2-1}}{1+\frac{1}{2}(\gamma-1)M^2}\frac{dM}{M}$$

is the so-called **Prandtl-Meyer function**

Prandtl-Meyer Expansion Waves

Performing the integration gives:

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1} \sqrt{M^2-1}$$

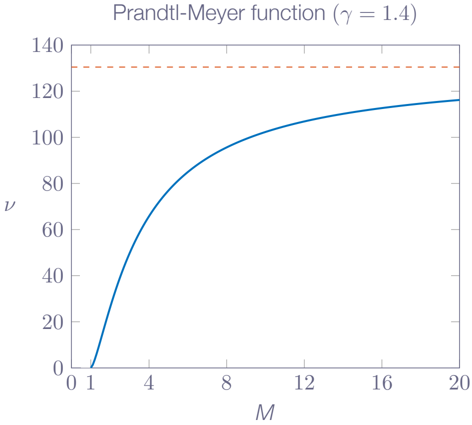
We can now calculate the deflection angle $\Delta\theta$ as:

$$\Delta\theta = \nu(M_2) - \nu(M_1)$$

Prandtl-Meyer Expansion Waves

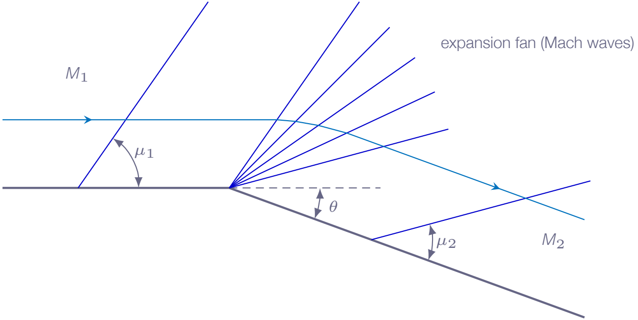
$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1}$$

$$\nu(M)|_{M\to\infty} = 130.45^\circ$$



Prandtl-Meyer Expansion Waves

Example:



1. $\theta_1 = 0$, $M_1 > 1$ is given
2. θ_2 is given
3. problem: find M_2 such that $\theta_2 = \nu(M_2) - \nu(M_1)$
4. $\nu(M)$ for $\gamma = 1.4$ can be found in Table A.5

Prandtl-Meyer Expansion Waves

Since the flow is isentropic, the usual isentropic relations apply:

(p_o and T_o are constant)

Calorically perfect gas:

$$\frac{p_o}{p} = \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_o}{T} = \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]$$

Prandtl-Meyer Expansion Waves

since $p_{o_1} = p_{o_2}$ and $T_{o_1} = T_{o_2}$

$$\frac{p_1}{p_2} = \frac{p_{o_2}}{p_{o_1}}\frac{p_1}{p_2} = \left(\frac{p_{o_2}}{p_2}\right) \Big/ \left(\frac{p_{o_1}}{p_1}\right) = \left[\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right]^{\frac{\gamma}{\gamma-1}}$$

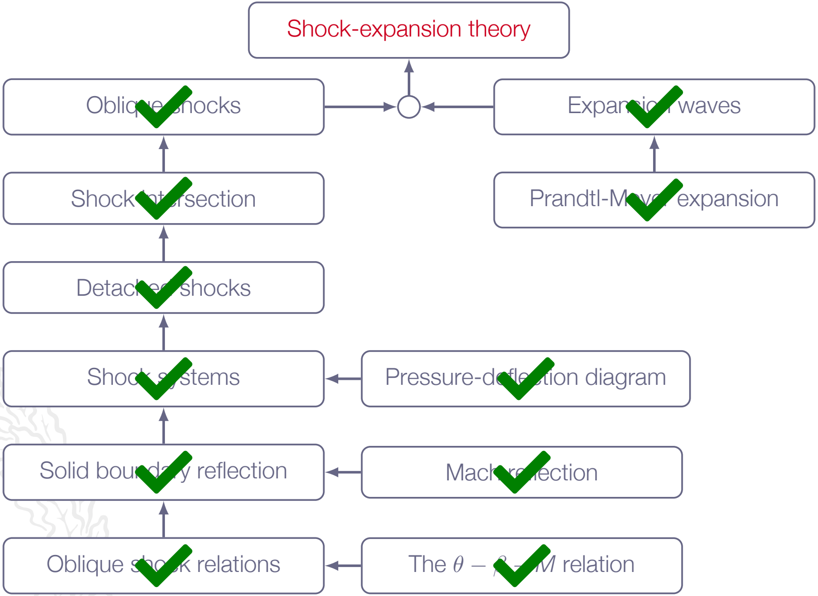
$$\frac{T_1}{T_2} = \frac{T_{o_2}}{T_{o_1}}\frac{T_1}{T_2} = \left(\frac{T_{o_2}}{T_2}\right) \Big/ \left(\frac{T_{o_1}}{T_1}\right) = \left[\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right]$$

Prandtl-Meyer Expansion Waves

Alternative solution:

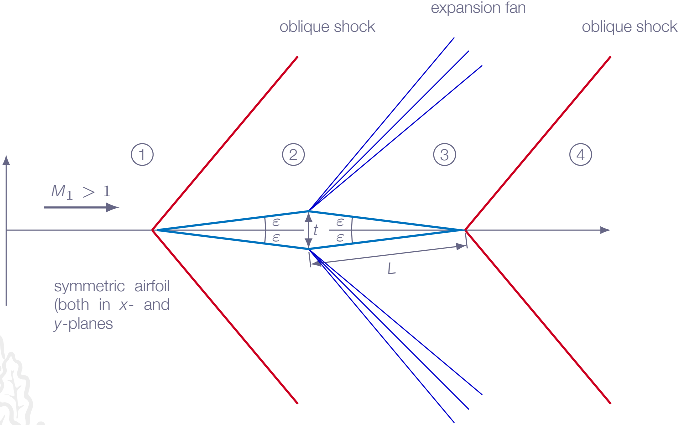
1. determine M_2 from $\theta_2 = \nu(M_2) - \nu(M_1)$
2. compute p_{o_1} and T_{o_1} from p_1, T_1, and M_1 (or use Table A.1)
3. set $p_{o_2} = p_{o_1}$ and $T_{o_2} = T_{o_1}$
4. compute p_2 and T_2 from p_{o_2}, T_{o_2}, and M_2 (or use Table A.1)

Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.15
Shock Expansion Theory

Diamond-Wedge Airfoil



Diamond-Wedge Airfoil

1-2 standard oblique shock calculation for flow deflection angle ε and upstream Mach number M_1

2-3 Prandtl-Meyer expansion for flow deflection angle 2ε and upstream Mach number M_2

3-4 standard oblique shock calculation for flow deflection angle ε and upstream Mach number M_3

Diamond-Wedge Airfoil

symmetric airfoil

zero incidence flow (freestream aligned with flow axis)

gives:

symmetric flow field

zero lift force on airfoil

Diamond-Wedge Airfoil

Drag force:

$$D = -\oiint_{\partial\Omega} p(\mathbf{n} \cdot \mathbf{e}_x) dS$$

$\partial\Omega$	airfoil surface
p	surface pressure
$\mathbf{n}$	outward facing unit normal vector
$\mathbf{e}_x$	unit vector in x-direction

Diamond-Wedge Airfoil

Since conditions 2 and 3 are constant in their respective regions, we obtain:

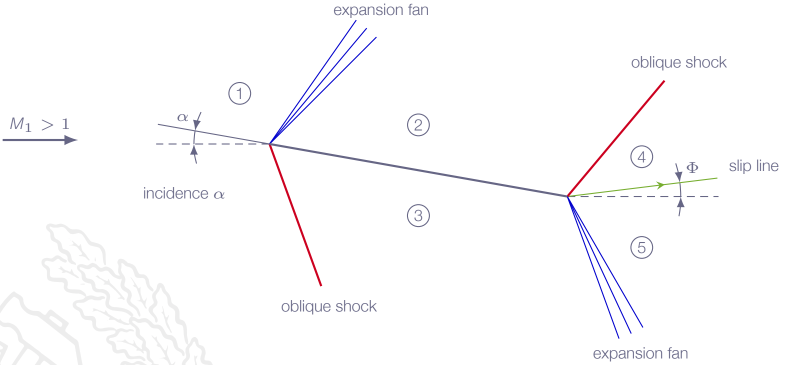
$$D = 2\left[p_2 L \sin(\varepsilon) - p_3 L \sin(\varepsilon)\right] = 2(p_2 - p_3)\frac{t}{2} = (p_2 - p_3)t$$

For supersonic free stream ($M_1 > 1$), with shocks and expansion fans according to figure we will always find that $p_2 > p_3$

which implies $D > 0$

Wave drag (drag due to flow loss at compression shocks)

Flat-Plate Airfoil



Flat-Plate Airfoil

It seems that the angle of the flow downstream of the flat plate would be different than the angle of the flow upstream of the plate. Can that really be correct?!

Flat-Plate Airfoil

It seems that the angle of the flow downstream of the flat plate would be different than the angle of the flow upstream of the plate. Can that really be correct?!

For the flow in the vicinity of the plate this is the correct picture. Further out from the plate, shock and expansion waves will interact and eventually sort the missmatch of flow angles out

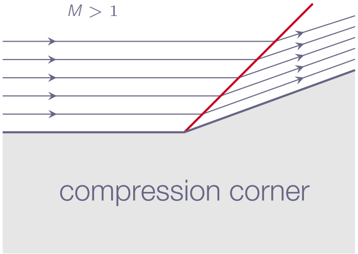
Flat-Plate Airfoil

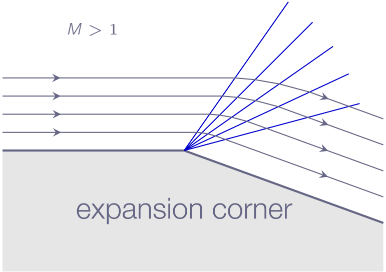
1. Flow states 4 and 5 must satisfy:

   $p_4 = p_5$

   flow direction 4 equals flow direction 5 (Φ)
2. Shock between 2 and 4 as well as expansion fan between 3 and 5 will adjust themselves to comply with the requirements
3. For calculation of lift and drag only states 2 and 3 are needed
4. States 2 and 3 can be obtained using standard oblique shock formulas and Prandtl-Meyer expansion

Oblique Shocks and Expansion Waves

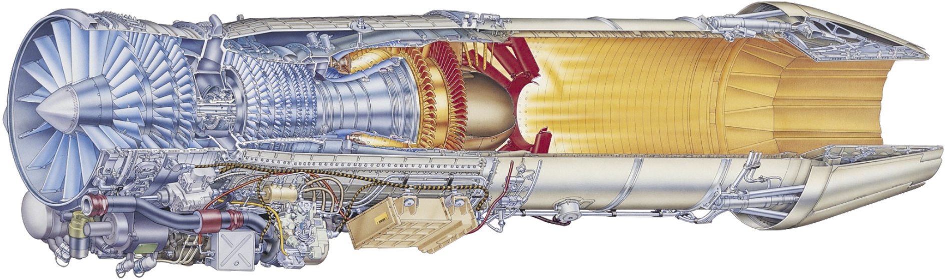




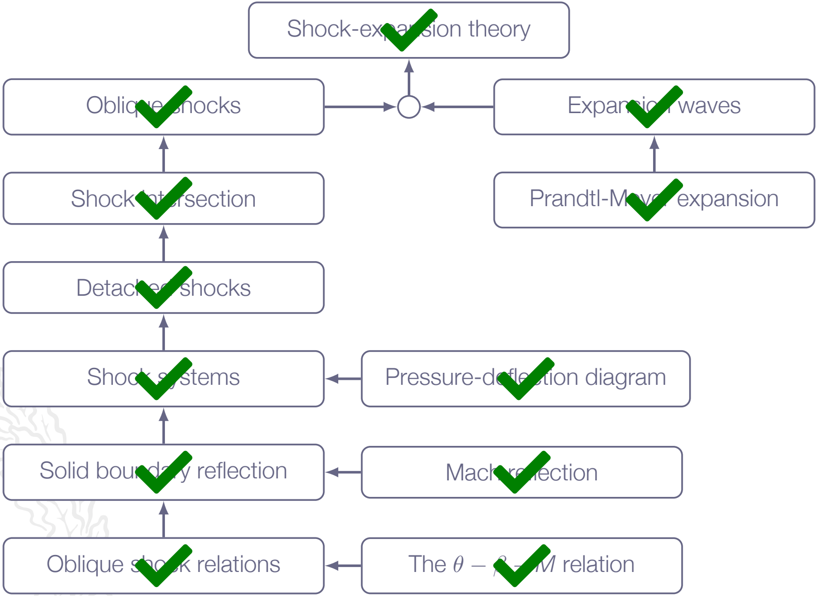
M	decrease
V	decrease
p	increase
ρ	increase
T	increase

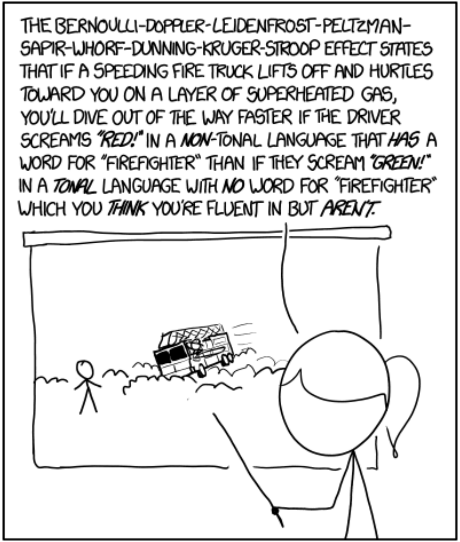
M	increase
V	increase
p	decrease
ρ	decrease
T	decrease

Oblique Shocks and Expansion Waves



Roadmap - Oblique Shocks and Expansion Waves




THE BERNOULLI-DOPPLER-LEIDENFROST-PELTZMAN-SAPIR-WHORF-DUNNING-KRUGER-STROOP EFFECT STATES THAT IF A SPEEDING FIRE TRUCK LIFTS OFF AND HURTLES TOWARD YOU ON A LAYER OF SUPERHEATED GAS, YOU'LL DIVE OUT OF THE WAY FASTER IF THE DRIVER SCREAMS "RED!" IN A NON-TONAL LANGUAGE THAT HAS A WORD FOR "FIREFIGHTER" THAN IF THEY SCREAM "GREEN!" IN A TONAL LANGUAGE WITH NO WORD FOR "FIREFIGHTER" WHICH YOU THINK YOU'RE FLUENT IN BUT AREN'T.