Exercise session 1:

(1.4) The pressure and temperature ratios across a given portion of a shock wave in air are $p_2/p_1 = 4'5$ and $T_2/T_1 = 1'687$, where 1 and 2 denote conditions ahead of and behind the shock wave, respectively. Calculate the change in entropy in J/(kg·K)

Air properties

$\gamma = 1'4$

$R = 287 \ J/(kgK)$

Assuming calorically perfect gas, we can compute c_p as:

(1.22) $$c_p = \frac{\gamma R}{\gamma - 1} = 1004'5 \ \frac{J}{kg \cdot K}$$

Now we can calculate the change in entropy

(1.36) $$s_2 - s_1 = \Delta s = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} = 1004'5 \ln(1'687) - 287 \cdot \ln(4'5)$$

$$\boxed{\Delta s = 93'63 \ \frac{J}{kg \cdot K}}$$

(1.5) Assume that the flow of air through a given duct is isentropic. At one point in the duct, the pressure and temperature are $p_1 = 1800 \ lb/ft^2$ and $T_1 = 500°R$ respectively. At a second point the temperature is 400°R. Calculate the pressure and density at this second point.

First, convert p_1 to Pa and T_1 and T_2 to K

$$p_1 = 1800 \frac{lb}{ft^2} \left(\frac{1 \ ft}{0'3048 m} \right)^2 \frac{4'44822 N}{1 lb} = 86184'43 \ Pa$$

$$T_1 = 500°R \ \frac{5K}{9°R} = 277'78 K$$

$$T_2 = 400°R \ \frac{5K}{9°R} = 222'22 \ K$$

$$\text{Air} \begin{cases} \gamma = 1'4 \\ R = 287 \ J/kg \cdot K \end{cases}$$

Assuming calorically perfect gas, and taking into account that the flow is said to be isentropic inside the duct, we can make use of the isentropic relations to calculate p_2

(1.43) $$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\boxed{p_2 = p_1 \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} = 86184'43 \left(\frac{222'22}{277'78}\right)^{\frac{1'4}{1'4-1}} = 39465'39 \text{ Pa}}$$

Now that we know p_2 and T_2 we can use the equation of state to find ρ_2:

(1.8) $$p_2 v_2 = RT_2 \rightarrow v = \frac{1}{\rho} \rightarrow \boxed{\rho_2 = \frac{p_2}{RT_2} = \frac{39465'39}{287 \cdot 222'22} = 0'6188 \frac{kg}{m^3}}$$

1.7 In the infinitesimal neighborhood surrounding a point in an inviscid flow, the small change in pressure, dp, that corresponds to a small change in velocity, dV, is given by the differential relation $dp = -\rho V dV$. (This equation is called the Euler equation; it is derived in chapter 6)

a) Using this relation, derive a differential relation for the fractional change in density, $d\rho/\rho$, as a function of the fractional change in velocity, dV/V, with the compressibility τ as a coefficient.

b) The velocity at a point in an isentropic flow of air is 10 m/s (a low speed flow), and the density and pressure are $1'23$ kg/m³ and $1'01 \cdot 10^5$ N/m² respectively (corresponding to standard sea level conditions). The fractional change in velocity at the point is 0'01. Calculate the fractional change in density.

c) Repeat part b), except for a local velocity at the point of 1000 m/s (a high-speed flow). Compare this result with b), and comment on the differences.

a) From the definition of compressibility (Eq 1.5)

$$\tau = \frac{1}{\rho}\frac{d\rho}{dp} \Rightarrow dp = \frac{1}{\tau}\frac{d\rho}{\rho}$$

Inserting the given relation $dp = -\rho V dV$

$$-\rho V dV = \frac{1}{\tau}\frac{d\rho}{\rho} \Rightarrow \boxed{\frac{d\rho}{\rho} = -\rho V^2 \tau \frac{dV}{V}}$$

b) In order to use the equation derived in a), we need to calculate τ.

$$(1.5) \qquad \tau = \frac{1}{\rho}\frac{d\rho}{dp} = \frac{1}{\rho}\left(\frac{dp}{d\rho}\right)^{-1}$$

Since the flow is said to be isentropic, so:

$$(1.43) \quad \frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} \Rightarrow p = C\rho^{\gamma}$$

$$\left(\frac{dp}{d\rho}\right)_s = C\gamma\rho^{\gamma-1} = \frac{p\gamma\rho^{\gamma-1}}{\rho^{\gamma}} = \frac{p\gamma}{\rho}$$

Inserting this into 1.5

$$\tau_s = \frac{1}{\rho}\left(\frac{\rho}{p\gamma}\right) = \frac{1}{p\gamma}$$

Inserting this into the relation found in a)

$$\frac{d\rho}{\rho} = -\rho V^2 \frac{1}{p\gamma}\frac{dV}{V} = -\frac{\rho V^2}{p\gamma}\frac{dV}{V}$$

Air $\rightarrow \gamma = 1'4$

$$\boxed{\frac{d\rho}{\rho} = -\frac{1'23 \cdot 10^2}{1'01 \cdot 10^5 \cdot 1'4} \cdot 0'01 = -8'69 \cdot 10^{-6}}$$

c) Just insert the new values

$$\boxed{\frac{d\rho}{\rho} = -\frac{1'23 \cdot 1000^2}{1'01 \cdot 10^5 \cdot 1'4} \cdot 0'01 = -8'69 \cdot 10^{-2}}$$

In order to see why the same 1% change in velocity gives such different changes in density; 0'00087% and 8'7% for the 10m/s case and 1000m/s respectively, we are going to take a look at the Mach numbers of each case. We assume calorically perfect gas. From the equation of state we can calculate temperature

(1.9) $p = \rho R T \Rightarrow \left\{ \text{air}: R = 287 \frac{J}{kg K} \right\} \Rightarrow T = \frac{p}{\rho R} = 286'11 K$

which is the same for both cases. Now the speed of sound can be calculated as

$$a = \sqrt{\gamma R T} = \sqrt{1'4 \cdot 287 \cdot 286'11} = 339'05 \text{ m/s}$$

So we have the following Mach numbers:

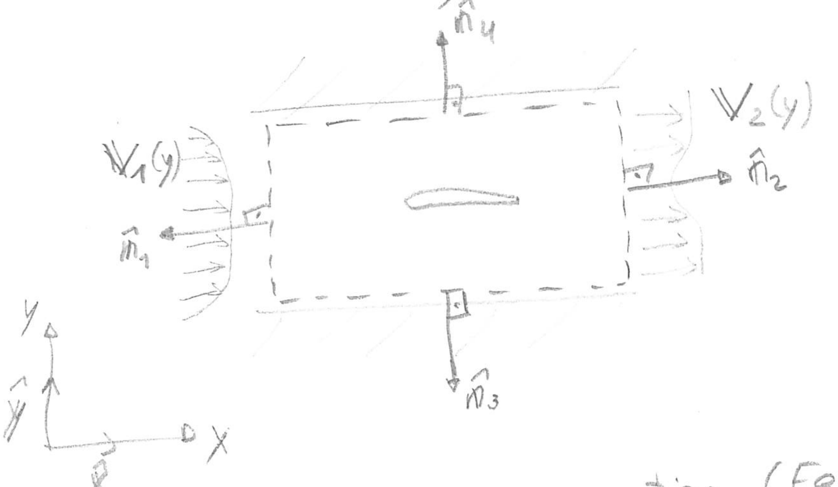
$$M_{low} = \frac{V_{low}}{a} = \frac{10}{339'05} = 0'029$$

$$M_{high} = \frac{V_{high}}{a} = \frac{1000}{339'05} = 2'94$$

As shown in slide 23 of the 1st lecture, the low-speed case has $M < 0'1$ and hence it can be considered incompressible, whereas the high-speed case would be highly compressible as it is somewhere between the supersonic and hypersonic regimes.

(2.1) Wings that spanned the entire test section → 2D flow. Using a control volume approach, derive a formula for the drag per unit span on the models as a function of the integral of the measured velocity distribution. For simplicity assume incompressible flow

A wind tunnel with an airfoil in the test section can be schematically drawn as:



$\mathbb{F} = \{F_x, F_y\}$
$n_1 = -\hat{x}$
$n_2 = \hat{x}$
$n_3 = -\hat{y}$
$n_4 = \hat{y}$
$\mathbb{V} = \{u, v, w\}$

Starting from momentum equation (Eq. 2.11)

$$\cancel{\frac{\partial}{\partial t}\iiint_{\Omega} \rho \mathbb{V}\, dV} + \oiint_{\partial\Omega} \left[\rho (\mathbb{V}\cdot n)\mathbb{V} + p n\right] ds = \cancel{\iiint_{\Omega} \rho f\, dV} + \mathbb{F}$$

steady flow

no external forces

First look at the surface integral

$$\oiint_{\partial\Omega} \left[\rho(\mathbb{V}\cdot n)\mathbb{V} + p n\right] ds = \int_{\Gamma_1} \left[\rho(-u_1(y))\cdot \mathbb{V}_1(y) + p_1(y) n_1\right] dy\, dz +$$

$$\int_{\Gamma_2} \left[\rho(u_2(y))\cdot \mathbb{V}_2(y) + p_2(y) n_2\right] dy\, dz + \int_{\Gamma_3} \left[\rho(-v_3(x))\, \mathbb{V}_3(x) + p_3(x) n_3\right] dx\, dz +$$

$$\int_{\Gamma_4} \left[\rho(v_4(x))\, \mathbb{V}_4(x) + p_4(x) n_4\right] dx\, dz = \mathbb{F}$$

Since we are asked to calculate drag per unit span, $dz = 1$.
For drag we are interested in the x component of the equation.
Thus,

$$\int_{\Gamma_1} -\left[\rho u_1^2(y) + p_1(y)\right] dy + \int_{\Gamma_2} \left[\rho u_2^2(y) + p_2(y)\right] dy = F_x$$

Since we are told to be dealing with incompressible flow we can take density out of the integral

$$-\rho\int_{T_1} u_1^2(y)\,dy + \rho\int_{T_2} u_2^2(y)\,dy - \int_{T_1} P_1(y)\,dy + \int_{T_2} P_2(y)\,dy = F_x$$

For simplicity we assume $p_1(y) = p_2(y)$ and $T_1 = T_2 = T$, so we can rewrite

$$\rho\int_{T}\left[u_2^2(y) - u_1^2(y)\right]dy = F_x$$

Force per unit span exerted on the fluid by the airfoil

Therefore the drag force is equal to $-F_x$

$$\boxed{\text{Drag} = \rho\int_{T}\left[u_1^2(y) - u_2^2(y)\right]dy}$$

2.2 Same problem as before; find equation for lift.

Starting from what we got before, right after we only consider the x component of the momentum equation

$$\int_{T_1}\left[-\rho u_1(y)\mathbb{V}_1(y) + p_1(y)\mathbb{n}_1\right]dy\,dz + \int_{T_2}\left[\rho u_2(y)\mathbb{V}_2(y) + p_2(y)\mathbb{n}_2\right]dy\,dz +$$

$$\int_{T_3}\left[-\rho V_3(x)\mathbb{V}_3(x) + p_3(x)\mathbb{n}_3\right]dx\,dz + \int_{T_4}\left[\rho V_4(x)\mathbb{V}_4(x) + p_4(x)\mathbb{n}_4\right]dx\,dz = \mathbb{F}$$

Since we are assuming the problem to be 2D and we are asked to calculate per unit span, $dz = 1$. Looking at the y component of the equation (constant density due to incompressible assumption)

$$-\rho\int_{T_1} u_1(y)V_1(y)\,dy + \rho\int_{T_2} u_2(y)V_2(y)\,dy - \int_{T_3} P_3(x)\,dx + \int_{T_4} P_4(x)\,dx = F_y$$

Force exerted on the fluid by the airfoil

Therefore the lift is equal to $-F_y$

$$\boxed{\text{Lift} = \rho\int_{T_1} u_1(y)V_1(y)\,dy - \rho\int_{T_2} u_2(y)V_2(y)\,dy + \int_{T_3} p_3(x)\,dx - \int_{T_4} p_4(x)\,dx}$$