# TME085 - Compressible Flow

2018-03-15, 08.30-13.30, M-building

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Approved aids:

- *TME085 Compressible Flow - Formulas, tables and graphs* (provided with exam)
- *Beta - Mathematics Handbook for Science and Engineering*
- Optional calculator/Valfri miniräknare (graph drawing calculators with cleared memory allowed)

Grading:

| number of points on exam | 24-35 | 36-47 | 48-60 |
|---|---|---|---|
| grade | 3 | 4 | 5 |

Note that at least 6 points needs to be in the theoretical part (Part I) and at least 10 points in the problem part (Part II)

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## Part I - Theory Questions (20 p.)

T1. (1 p.)
What conditions must be satisfied for a steady-state compressible flow to be isentropic?

T2. (1 p.)
For a stationary normal shock, describe how the entropy, velocity, pressure and total temperature of a fluid particle is affected as it passes through the shock.

T3. (2 p.)

(a) High temperature effects in compressible flows are found when analyzing for example very strong shocks or nozzle flows with extremely high total pressure and total enthalpy. What is the root cause of these effects and what do we mean by equilibrium gas? What kind of thermodynamic relations are needed to compute the flow of equilibrium gas?

(b) What is the difference between a calorically perfect gas and a thermally perfect gas?

T4. (2 p.)

(a) In shock tubes, unsteady contact discontinuities are sometimes found. Describe in words what they are and under which circumstances they may be formed. Which of the variables $p$, $T$, $\rho$, $u$, $s$ is/are necessarily continuous across such a contact discontinuity?

(b) A stationary normal shock with upstream Mach number $M_1$ ($M_1 > 1$) is compared to a moving normal shock, traveling with Mach number $M_S$ into quiescent (non-moving) air. If $M_1 = M_S$, is there any physical difference between the two shock waves apart from the fact that they have different speeds relative to the observer?

T5. (2 p.)
Derive the special formula for the speed of sound a for a calorically perfect gas:

$$a = \sqrt{\gamma RT}$$

starting from the general formula:

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

T6. (2 p.)
Derive the area-velocity relation in quasi-1D flow starting from the mass conservation relation

$$d(\rho u A) = 0.,$$

Euler's equation

$$dp = -\rho u du,$$

and the definition of the speed of sound

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

What are the implications of the area-velocity relation for quasi-one-dimensional flow?

T7. (2 p.)

(a) Describe the choking of flow that occurs for pipe flow with friction. What happens if the real length $L$ of a pipe is longer than $L^*$ (for either subsonic flow or supersonic flow)?

(b) What is meant by choking the flow in a nozzle? Describe it.

T8. (2 p.)
Derive the continuity equation in non-conservation form from the corresponding conservation form.

T9. (4 p.)

(a) Derive Crocco's relation starting from the momentum equation and the energy equation (the first and second law of thermodynamics)

(b) Describe in words the significance of Crocco's equation

(c) Prove that a steady-state irrotational flow with constant total enthalpy must also be isentropic

(d) What does Crocco's relation say about the flow behind a curved shock

T10. (1 p.)
Describe in words how a finite-volume spatial discretization can be achieved.

T11. (1 p.)
How can we use our knowledge of characteristics (and their speed of propagation) to guide us when determining suitable boundary conditions for compressible flows?

## Part II - Problems (40 p.)


### Problem 1 - SCRAMJET (10 p.)

A SCRAMJET engine is an engine concept for supersonic propulsion without moving parts (see Figure 1). Air is lead into the combustion chamber section of the engine through a sequence of oblique shocks at the engine inlet. After the combustion chamber the air passing through the engine is expanded in a nozzle. The geometry of the engine in the figure can be modified by moving the lower wall in the flow direction such that the oblique shock intersects with the lower wall at the edge, which is crucial for the engine functionality.

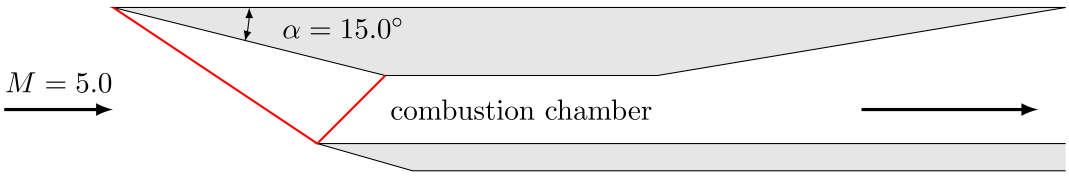


**Figure 1: schematic view of a SCRAMJET engine**

(a) With the geometry given in Figure 1 and the functionality described above, what is the lowest possible Mach for which the engine can be operated? What is the implication of that result? (1 p.)

(b) Calculate the Mach number, temperature and pressure at the combustion chamber entry assuming that the engine operates with a surrounding pressure and temperature of $P_{amb}$=0.1 bar and $T_{amb}$=270 K and a free stream Mach number of M=5.0. (3 p.)

(c) The engine is designed to produce sonic flow (M=1.0) at the end of the combustion chamber (at the axial location where the flow enters the divergent nozzle). Calculate the heat per mass airflow needed to be added in the combustion chamber. The change in gas composition due to the addition of fuel and the consequent reactions do not have to be considered. (3 p.)

(d) Calculate the outlet pressure and Mach number (at the end of the diverging nozzle) given that the outlet flow area is two times the area at the end of the combustion chamber. (3 p.)

**Solution:**

**1a.**

The lowest possible Mach number for which the engine can be operated can be found by examining the $\theta-\beta-M$-relation. For Mach numbers lower than $\sim$1.62 there are no solutions for cases with a deflection angle $\theta = 15°$. The implication of this is that engine can not be started from standing still conditions but needs to be brought to high supersonic velocities before starting up.

**1b.**

First oblique shock (free stream Mach number and ambient conditions given):

$M_1$=5.0

$\theta$=15°

$P_{amb}$=0.1 bar

$T_{amb}$=270 K

Assume air with $\gamma$=1.4

The $\theta - \beta - M$-relation with give Mach number and deflection angle gives the Mach angle of the first shock $\beta_1$=24.3°.

Now, let's calculate move through the first oblique shock and calculate the conditions behind it (in region 2)

The shock-normal Mach number is obtained as (eqn. 4.7)

$M_{n1} = M_1 \sin(\beta_1)$=2.06

Now we can obtain ratios of pressure and temperature over the shock as a function of the shock-normal Mach number $M_{n1}$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_{n1}^2 - 1\right) \tag{4.9}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_{n1}^2}{(\gamma-1)M_{n1}^2 + 2} \tag{4.8}$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} \tag{4.11}$$

The downstream shock-normal Mach number can be obtained using

$$M_{n2}^2 = \frac{M_{n1}^2 + (2/(\gamma-1))}{[2\gamma/(\gamma-1)]M_{n1}^2 - 1} \tag{4.10}$$

Finally, calculate the Mach number in region 2

$$M_2 = M_{n2}/\sin(\beta_1 - \theta) \tag{4.12}$$

which gives $M_2$=3.5

Second oblique shock:

The flow must be deflected back the angle $\theta$ to get axial flow in the combustion chamber $M_2$=3.504, $\theta$=15°

The $\theta-\beta-M$-relation with give Mach number and deflection angle gives the Mach angle of the second shock $\beta_2$=29.2°.

To get through the second oblique shock, we will follow the same procedure as for the first shock

$M_{n21} = M_2 \sin(\beta_2)$=1.7 (eqn. 4.7)

$$\frac{p_3}{p_2} = 1 + \frac{2\gamma}{\gamma+1}\left(M_{n21}^2 - 1\right) \tag{4.9}$$

$$\frac{p_3}{p_1} = \frac{p_2}{p_1}\frac{p_3}{p_2} = \left(1 + \frac{2\gamma}{\gamma+1}\left(M_{n1}^2 - 1\right)\right)\left(1 + \frac{2\gamma}{\gamma+1}\left(M_{n21}^2 - 1\right)\right)$$

$$\frac{\rho_3}{\rho_2} = \frac{(\gamma+1)M_{n21}^2}{(\gamma-1)M_{n21}^2 + 2} \tag{4.8}$$

$$\frac{T_3}{T_2} = \frac{p_3}{p_2}\frac{\rho_2}{\rho_3} \tag{4.11}$$

$$\frac{T_3}{T_1}\frac{T_3}{T_2}\frac{T_2}{T_1} = \left(\frac{p_3}{p_2}\frac{\rho_2}{\rho_3}\right)\left(\frac{p_2}{p_1}\frac{\rho_1}{\rho_2}\right) = \frac{p_3}{p_1}\frac{\rho_1}{\rho_3}$$

As for the first shock, we can calculate the Mach number in region 3 using the downstream shock-normal mach number

$$M_{n22}^2 = \frac{M_{n21}^2 + (2/(\gamma-1))}{[2\gamma/(\gamma-1)]M_{n21}^2 - 1} \tag{4.10}$$

$$M_3 = M_{n22}/\sin(\beta_2 - \theta) \tag{4.12}$$

We now have everything we need in order to be able to calculate the combustion chamber inlet conditions (region 3)

$p_3$=1.547 bar

$T_3$=686.3 K

$M_3$=2.608

**1c.**

In the combustion chamber the initial supersonic flow is brought to sonic speed by adding heat, *i.e.* at the exit of the combustion chamber the $M_4$=1.0

In order to be able to calculate the heat addition needed, we need the total temperatures both at the inlet of the combustion chamber and at the outlet

$$T_{o_3} = T_3\left(1 + \frac{\gamma-1}{2}M_3^2\right) \tag{3.28}$$

*Note that we could also have calculated the total temperatur at the inlet of the combustion chamber using the free stream temperature and Mach number, $T_{o_1}$, since the total temperature is constant through the oblique shocks.*

Using equation 3.88 we get a relation between the total temperature in region 3 (the combustion chamber inlet) and the total temperature at sonic conditions $T_o^*$.

$$\frac{T_{o3}}{T_o^*} = \frac{(\gamma+1)M_3^2}{(1+\gamma M_3^2)}\left(2+(\gamma-1)M_3^2\right) \tag{3.88}$$

Since the exit conditions are sonic $T_{o_4} = T_o^*$

and thus

$$q = C_p(T_o^* - T_{o3}) \tag{3.77}$$

Assuming calorically perfect gas we calculate $C_p$ as

$$C_p = \frac{R\gamma}{\gamma-1} = \{R = 287\} = 1004.5 \tag{1.22}$$

Inserting $C_p$ in the relation above, we get $q = 710.8$ kJ/kg

**1d.**

Calculate exit conditions in the diverging nozzle

The flow enters the nozzle at M=1.0 (sonic conditions), the flow in the nozzle will be supersonic (at least initially). Let's assume that the nozzle flow is free of shocks (isentropic flow) and that we can assume that the gas can be treated as calorically perfect.

The area ratio of the nozzle is given to be 2.0

Using the area-Mach-number relation with $A/A^*$=2.0 gives an exit Mach number of $M_5$=2.2

If the flow is isentropic (no losses), the total pressure will be constant through the nozzle.

To get the total pressure at the exit plane we can thus use the total pressure at the end of the combustion chamber ($p_{o_4}$)

We have the pressure at the inlet of the combustion chamber ($p_3$)

Using the relations for one-dimensional flow with heat addition, we can calculate the pressure at the end of the combustion chamber ($p_4$)

$$\frac{p_4}{p_3} = \frac{1+\gamma M_3^2}{1+\gamma M_4^2} = \{M_4 = 1.0\} = \frac{1+\gamma M_3^2}{1+\gamma} \tag{3.78}$$

From $p_4$ the total pressure $p_{o_4}$ can be calculated as

$$p_{o_4} = p_4\left(1+\frac{\gamma-1}{2}M_4^2\right)^{\gamma/(\gamma-1)} = \{M_4 = 1.0\} = p_4\left(1+\frac{\gamma-1}{2}\right)^{\gamma/(\gamma-1)} \tag{3.30}$$

Under isentropic the total pressure will, as stated above, be constant, *i.e.* $p_{o_5} = p_{o_4}$

Now, we can calculate the static pressure at the exit plane

$$p_5 = p_{o_5}\left(1+\frac{\gamma-1}{2}M_5^2\right)^{\gamma/(1-\gamma)} \tag{3.30}$$

which gives $p_5$=1.2 bar

Now, let's check that our assumptions are valid

What back pressure would result in a shock at exit for the given conditions (the first condition without internal shocks)

$$\frac{p_b}{p_5} = 1 + \frac{2\gamma}{\gamma + 1}(M_5^2 - 1) \tag{3.57}$$

The back pressure matching a normal shock at exit for the calculated Mach number is $p_b$=6.6 bar

The surrounding pressure was given to be $p_{atm}$=0.1 bar which is significantly lower than the calculated back pressure for normal shock at exit and thus it is safe to say that there will be no normal shocks inside the nozzle. There will, however, be shocks in the downstream flow but that is out of scope.

## Problem 2 - Free Piston (10 p.)

A free piston is situated in a cylinder filled with air with a pressure of 1.0 bar and a temperature of 288 K. At time $t = 0.0$, the piston is moved impulsively to the left with the velocity $U_{piston} = 300[m/s]$. The piston motion induces a shock moving ahead of the piston and an expansion region behind the piston as indicated in Figure 2.

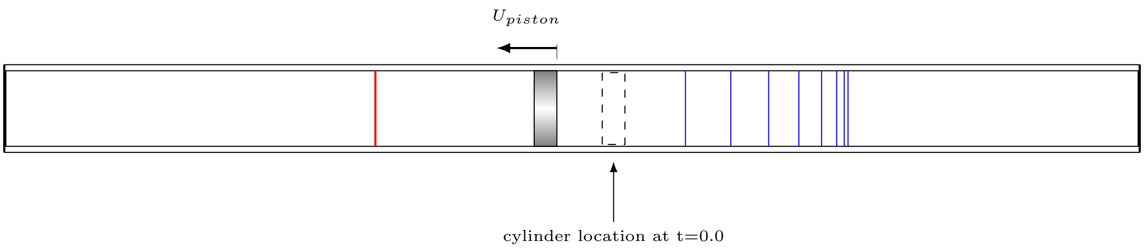


**Figure 2: Schematic view of flow induced by a free piston moving in a cylinder**

(a) Make a schematic drawing of the sequence of events after the release of the free piston at $t = 0.0$ in a $xt$-diagram (2 p.)

(b) Calculate the pressures in the regions just ahead and behind the piston at the situation illustrated in Figure 2 (8 p.)

**Solution:**

**2a.**

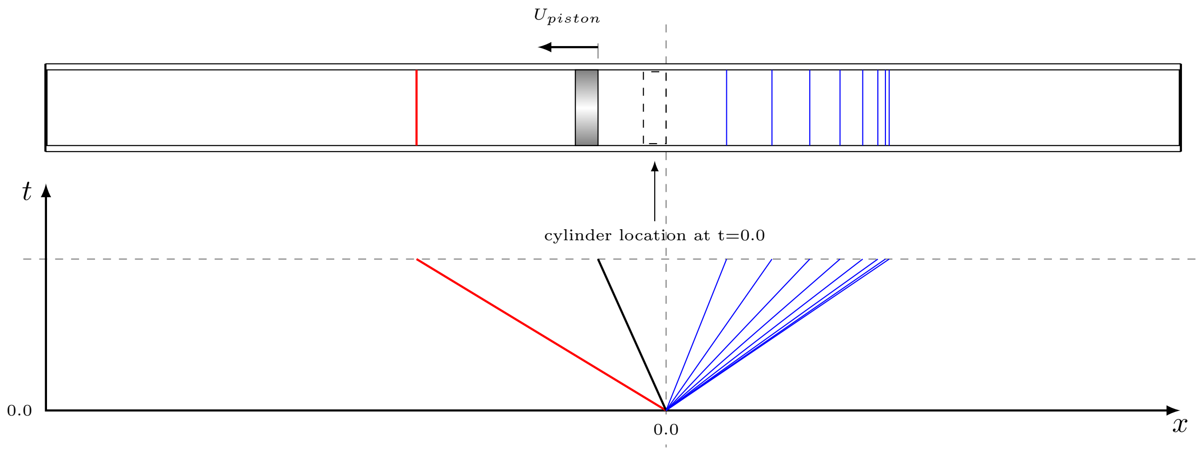


**Figure 3:** *xt***-diagram showing the movement of finite non linear waves in the cylinder. The red line represents the movement of the shock, the black line the piston movement, and the blue lines the expansion region.**

**2b.**

Assume that the gas in the cylinder is air ($\gamma$=1.4) that can be treated as calorically perfect

*ahead of piston*:

The shock wave induced by the piston moves into a region of stagnant air. Behind the moving shock the induced velocity must be the same as the piston speed (air cannot move into the piston and thus the air ahead of it must move with the same speed)

We can use the relations for finite nonlinear waves to get an expression for the induced velocity

$$u_p = \frac{a_1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left[\frac{\frac{2\gamma}{\gamma+1}}{\frac{p_2}{p_1} + \frac{\gamma-1}{\gamma+1}}\right]^{1/2} \tag{7.16}$$

where region 1 is the region ahead of the shock wave (the stagnant fluid into which the shock is moving) and region 2 is the region between the piston head and the shock wave. $p_2/p_1$ is thus the pressure ratio over the shock wave

The induced velocity is the same as the piston speed $u_p$=300.0 m/s which means that we can calculate the pressure ratio over the shock wave by solving the above-given relation iteratively

The speed of sound in region 1 is calculated as $a_1 = \sqrt{\gamma R T_1}$

This gives $p_2/p_1$=3.05 and thus $p_2$=3.05 bar

*behind piston*:

An expansion region is induced behind the piston due to the piston motion. In the expansion region there is a smooth transition from the conditions just behind the piston in region 3 (where, again, the fluid must move with the piston velocity) and the stagnant fluid at the rear end of the piston (region 4)

We can use the expansion wave relations to calculate the pressure behind the piston

$$\frac{p_3}{p_4} = \left(1 - \frac{\gamma - 1}{2}\left(\frac{u_p}{a_4}\right)\right)^{2\gamma/(\gamma-1)} \tag{7.89}$$

which gives $p_3$=0.26 bar

## Problem 3 - Sphere at high speed (10 p.)

The Schlieren image in Figure 4 shows a sphere traveling at supersonic speed over a perforated plate. The image clearly shows the formation of a bow shock in front of the sphere and the shock reflection at the plate below. As the shock wave in front of the bullet passes the holes in the perforated plate, acoustic waves are generated. Viscous losses can be assumed to be small.

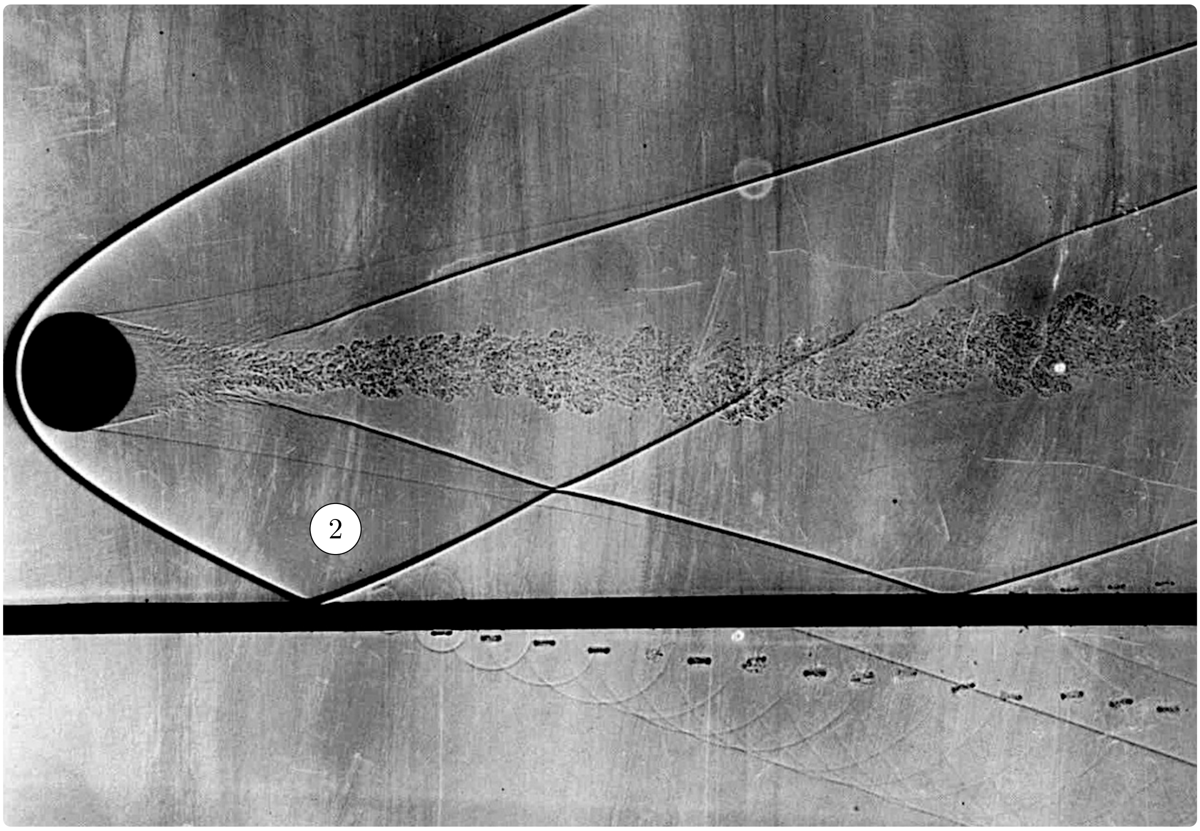


**Figure 4: Schlieren photograph of sphere at supersonic speed**

(a) Estimate the speed (Mach number) at which the sphere is traveling through the air (5 p.)

(b) Estimate the pressure behind the shock wave close to the wall (indicated with a "2" in Figure 4) (5 p.)

**Solution:**

In order to be able to solve this problem we need to make some assumptions. Let's assume that the gas in which the sphere is moving is air ($\gamma$=1.4) at standard atmospheric conditions. Further let's assume that we can treat the gas as calorically perfect.

**3a.**

We are asked to make an estimation of the Mach number of the moving sphere. The sphere moves at supersonic speed over a perforated plate and below the plate acoustic waves are radiated. The fronts of the acoustic waves forms a Mach wave and by estimating the angle of this Mach wave, we can calculate the Mach number

The angle is estimated to be $\mu$=19.2° by measuring in the figure

$$\sin(\mu) = \frac{1}{M} \tag{4.1}$$

inserting the estimated Mach angle gives $M$=3.03

**3b.**

The position where the pressure is to be calculated is downstream of the bow shock that is formed in front of the sphere. Close to the flat plate, the bow shock has turned into an oblique shock and therefore it is possible to use the oblique shock theory to solve this problem. Now, to be able to pressure behind the shock we need the shock angle and again it is possible to estimate this angle by measuring in the provided figure.

Measuring in the figure gives $\beta$=26.6°

The shock-normal Mach number may now be calculated using

$$M_{n1} = M\sin(\beta) \tag{4.7}$$

Using the shock-normal Mach number we can get the pressure ratio over the shock

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_{n1}^2 - 1\right) \tag{3.57}$$

Inserting the $M_{n1}$ we get $p_2/p_1$=1.98 and thus if we assume that pressure in front of the sphere is $p_1$=1.0 bar, the pressure in region 2 will be $p_2$=1.98 bar

## Problem 4 - Nozzle flow (10 p.)

An engineer designs a convergent divergent nozzle that is supposed to produce a supersonic flow. The exit-to-throat area ratio of the nozzle is 4.0 ($A_t = 0.5\ m^2$, $A_e = 2.0\ m^2$). Preliminary calculations using a quasi-1D solver showed that, for the chosen plenum conditions ($T_o$ and $P_o$ in the reservoir upstream of the nozzle), there would not be any shocks inside the nozzle, *i.e.* the flow inside the nozzle can be assumed to be isentropic and the flow in the diverging part of the nozzle was supersonic. Happy with the results, the engineer sent his parameters to the workshop where the nozzle and surrounding equipment was designed according to the specifications. At the first test, measurements showed that the velocity downstream of the nozzle exit was subsonic and there was no signs of shock diamonds or expansion regions downstream of the nozzle exit plane. Puzzled by the result, the engineer went back to his desk trying to figure out what happened.

*The ambient conditions in the test cell are $p_{amb}$=1.0 bar and $T_{amb}$=300.0 K. The gas that flows through the nozzle is air that can be assumed to be calorically perfect.*

(a) What happened, what did the engineer forget from the compressible flow course? (2 p.)

(b) Calculate total pressure $p_o$ and total temperature $T_o$ upstream of the nozzle (plenum chamber) (4 p.)

(c) Calculate the plenum conditions for which the flow would be free of losses both inside the nozzle and downstream of the nozzle exit (neglecting viscous losses) (4 p.)

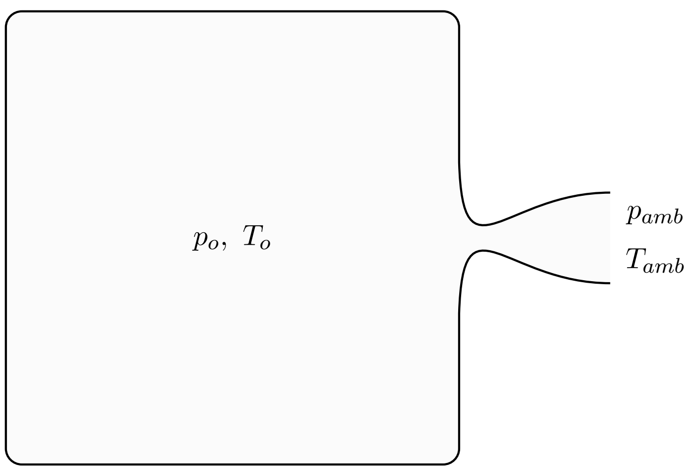


**Figure 5: Schematic overview of plenum chamber and convergent-divergent nozzle**

**Solution:**

**4a.**

The engineer forgot that a quasi-2D solver will only show what happens in the nozzle. The fact that the shock was designed to be free of shocks does not mean that there will not be shocks later. Moreover, the engineer forgot about the possibility of formation of a normal shock at the exit plane. This will not show up in the quasi-1D solver and the fact that no shocks were seen in the test facility indicates that this is what happened.

**4b.**

We are asked to calculated the plenum conditions ($p_o$, $T_o$) that the engineer obtained for this, presumably, first design

Assuming that there are no shocks inside the nozzle (this is what the quasi-1D solver analysis showed), we can assume the flow to be isentropic. Furthermore we know from the problem specifications that the gas flowing in the nozzle is air ($\gamma$=1.4) that can be assumed to be calorically perfect.

We have been given the area reatio $A/A^*$=4.0 (sonic flow at the nozzle throat $\Rightarrow A_t = A^*$)

The area-Mach number relation gives the nozzle exit Mach number $M_e$=2.9

Shock at exit gives:

$$\frac{p_{amb}}{p_e} = 1 + \frac{2\gamma}{\gamma+1}\left(M_e^2 - 1\right) \tag{3.57}$$

$$\frac{T_{amb}}{T_e} = \left[1 + \frac{2\gamma}{\gamma+1}\left(M_e^2 - 1\right)\right]\left[\frac{2+(\gamma-1)M_e^2}{(\gamma+1)M_e^2}\right] \tag{3.59}$$

Using the static conditions at the exit ($p_e$ and $T_e$) we can calculate the total conditions which are constant through the nozzle and thus the same as the sought plenum conditions

$$p_o = p_e(1 + \frac{\gamma-1}{2}M_e^2)^{\gamma/(\gamma-1)} \tag{3.30}$$

$$T_o = T_e(1 + \frac{\gamma-1}{2}M_e^2) \tag{3.28}$$

This gives us

$p_o$=3.386 bar

$T_o$=313.7 K

**4c.**

Now we are asked to calculate the plenum conditions for the case when there are no shocks inside the nozzle or in the downstream flow. This is the so-called supercritical condition and appears when the exit pressure matches the back pressure. Since the nozzle flow is still shock free, the exit Mach number will be unchanged and thus the plenum conditions can be calculated from the downstream conditions as follows

$$p_o = p_b(1 + \frac{\gamma - 1}{2} M_e^2)^{\gamma/(\gamma-1)} \tag{3.30}$$

$$T_o = T_b(1 + \frac{\gamma - 1}{2} M_e^2) \tag{3.28}$$

This gives us

$p_o$=33.6 bar

$T_o$=818.9 K