TME085 - Compressible Flow
2016-08-17, 08.30-13.30, M-building

Approved aids:

- *TME085 Compressible Flow - Formulas, tables and graphs* (provided with exam)

- *Beta - Mathematics Handbook for Science and Engineering*

- Optional calculator/Valfri miniräknare (graph drawing calculators with cleared memory allowed)

Grading:

number of points on exam	24-35	36-47	48-60
grade	3	4	5

Note that at least 6 points needs to be in the theoretical part (Part I) and at least 10 points in the problem part (Part II)

Part I - Theory Questions (20 p.)

T1. (2 p.)

(a) What is the general definition (valid for any gas) of the "*total*" conditions p_o, T_o, ρ_o,... etc at some location in the flow?

(b) For a steady-state adiabatic compressible flow of calorically perfect gas, which of the variables p_o (total pressure) and T_o (total temperature) is/are constant along streamlines? Why?

T2. (1 p.)
Derive the relation

$$\frac{T_o}{T} = 1 + \frac{1}{2}(\gamma - 1)M^2$$

for calorically perfect gas from the energy equation form

$$h_o = h + \frac{1}{2}V^2.$$

T3. (1 p.)
What are the implications of the area-velocity relation for quasi-one-dimensional flow?

T4. (1 p.)
An engineer wants to apply a numerical solution scheme for compressible flow. The flow he is interested in contains shocks. He has to choose between two different solution methods – one which is based on the conservation form of the governing equations and one which is based on the non-conservation form of the governing equations. Which method should he choose and why?

T5. (2 p.)

(a) High temperature effects in compressible flows are found when analyzing for example very strong shocks or nozzle flows with extremely high total pressure and total enthalpy. What is the root cause of these effects and what do we mean by equilibrium gas? What kind of thermodynamic relations are needed to compute the flow of equilibrium gas?

(b) What is the difference between a calorically perfect gas and a thermally perfect gas?

T6. (1 p.)
In shock tubes, unsteady contact discontinuities are sometimes found. Describe in words what they are and under which circumstances they may be formed. Which of the variables p, T, ρ, u, s is/are necessarily continuous across such a contact discontinuity?

T7. (2 p.)

(a) What is meant by an under-expanded or over-expanded nozzle flow?

(b) What is it meant by choking the flow in a nozzle? Describe it.

T8. (1 p.)
Assume a steady-state flow through a pipe with varying cross-section area. If the pipe has negligible heat transfer and wall friction and there are no shocks, then the flow is

(a) adiabatic

(b) isentropic

(c) isobaric (constant pressure)

(d) isenthalpic (constant enthalpy)

Which statements are true and which are false?

T9. (2 p.)
Derive the area-velocity relation in quasi-1D flow starting from the mass conservation relation

$$d(\rho u A) = 0.,$$

Euler's equation

$$dp = -\rho u du,$$

and the definition of the speed of sound

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

What are the implications of the area-velocity relation for quasi-one-dimensional flow?

T10. (2 p.)

(a) When applying a time-marching flow solution scheme the so-called CFL number is an important parameter. Define the CFL number and describe its significance. What is a typical maximum CFL number for an explicit time stepping scheme.

(b) What is meant by the terms "*density-based*" and "*fully coupled*" when discussing CFD codes for compressible flow?

T11. (2 p.)

(a) What is it meant by choking the flow in a nozzle? Describe it.

(b) How does the absolute Mach number change after a *weak* and a *strong* stationary oblique shock, repectively?

T12. (2 p.)
Describe in words the significance of Crocco's equation.

Prove that a steady-state irrotational flow with constant total enthalpy must also be isentropic.

T13. (1 p.)
How can we use our knowledge of characteristics (and their speed of propagation) to guide us when determining suitable boundary conditions for compressible flows?

Part II - Problems (40 p.)

Problem 1 - Ramjet engine (10 p.)

An engineer wants to build and test a very simple ramjet engine for missile propulsion at supersonic speeds. The engine consists of a cylindrical inlet section; followed by a cylindrical heater section, and finally an axisymmetric diverging nozzle (see Figure below). The design speed of the missile is Mach 3 at sea level. The inlet is designed for 'straight inflow', i.e. the flow should go straight into the inlet without any change of direction or velocity. Inside the inlet section a stationary normal shock will be positioned, in order to decelerate the flow to subsonic conditions at station 2. In the heater section heat is added (by combustion) in order to achieve sonic conditions at station 3. In the expansion section the flow is further accelerated (due to the increasing duct area) to supersonic speeds. The expansion is designed to achieve pressure matching at the nozzle exit (station 4), i.e. that the exit pressure equals the ambient pressure. The ambient pressure and temperature are assumed to be 1 bar and 300 K respectively. The tube diameter in the inlet and heater sections is 10 cm. Some simplifications are to be allowed in the calculations; calorically perfect gas is assumed with γ=1.4 and R=287 J/kg/K. All viscous effects are to be neglected.

Calculate

(a) the inflow velocity u_1 and the mass flow through the engine

(b) the exit velocity u_4

(c) the heating power needed for the engine operation

(d) the net thrust of the engine - draw some conclusions about the practical application of the engine
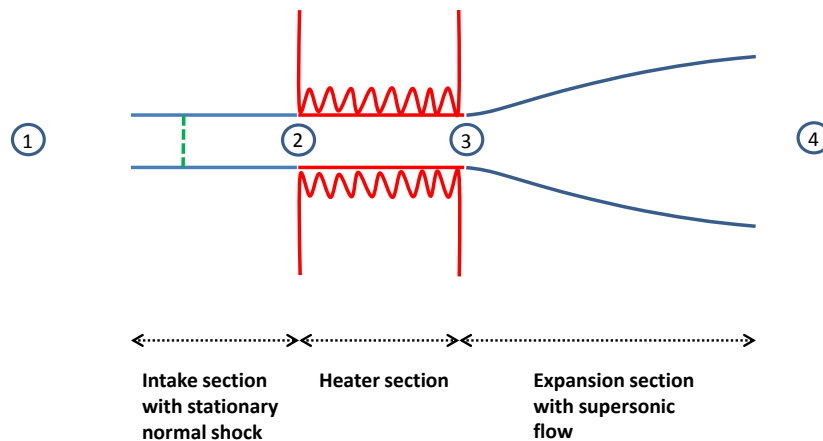
Problem 2 - Convergent-Divergent Nozzle (10 p.)

A converging-diverging nozzle with an exit to throat area ratio, A_e/A_t, of 1.633, is designed to operate with atmospheric pressure at the exit plane, $p_e = p_{atm}$. The converging-diverging nozzle area, A, varies with position, x, as:

$$\frac{A(x)}{A_t} = \left(\frac{A_e}{A_t} - 1\right)\left(2\frac{x}{L} - 1\right)^2 + 1$$
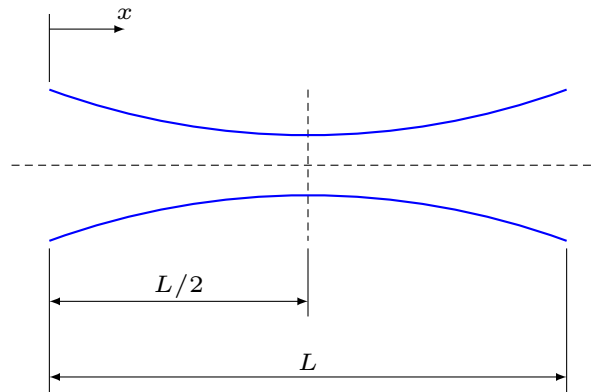
(a) Determine the range(s) of pressure ratios (p_o/p_{atm}) for which the nozzle will be free from normal shocks

(b) Will there be a normal shock in the nozzle if nozzle pressure ratio is p_o/p_{atm}=1.5? If so, at what position (x/L) will the normal shock occur?
Hint: calculating the exit Mach number in case of existence of an internal normal shock is a good starting point

Problem 3 - Pipe flow with friction (10 p.)

A pipe with circular cross-section has a constant inner diameter $D = 1$ cm and a length L=9.45 m. The pipe is insulated to prevent any heat transfer to the flow of air inside. The inflow end of the pipe is connected to a reservoir of air with total pressure p_o and total temperature T_o. The outflow end of the pipe is open to the ambient air, with pressure $p_{amb} = 1$ bar. The static temperature of the flow coming out of the pipe exit is known to be 300 K. Furthermore, the massflow of air in the pipe is known to be 9.5 grams per second. The flow is subsonic.

Compute:

(a) the Mach number at the inflow end of the pipe

(b) the reservoir pressure p_o

The air may be treated as a calorically perfect gas with γ=1.4 and R=287 J/kg/K. The friction coefficient of the pipe is found to be $f = 0.005$.

Problem 4 - Shock-Expansion Theory (10 p.)

A symmetric diamond-shaped airfoil is placed in a supersonic flow of air. The airfoil is oriented with zero angle of attack relative to the freestream flow. The extent of the airfoil on the z-direction is long in comparison to the extent in the x- and y-directions, hence an assumption of two-dimensional flow can be justified. Calculate the drag (*wave drag*) per unit length if the freestream Mach number is M_1=2.0 and the upstream pressure is 101325 Pa.

airfoil parameters			
half angle	ε	5	[degrees]
coord length	c	1.0	[m]
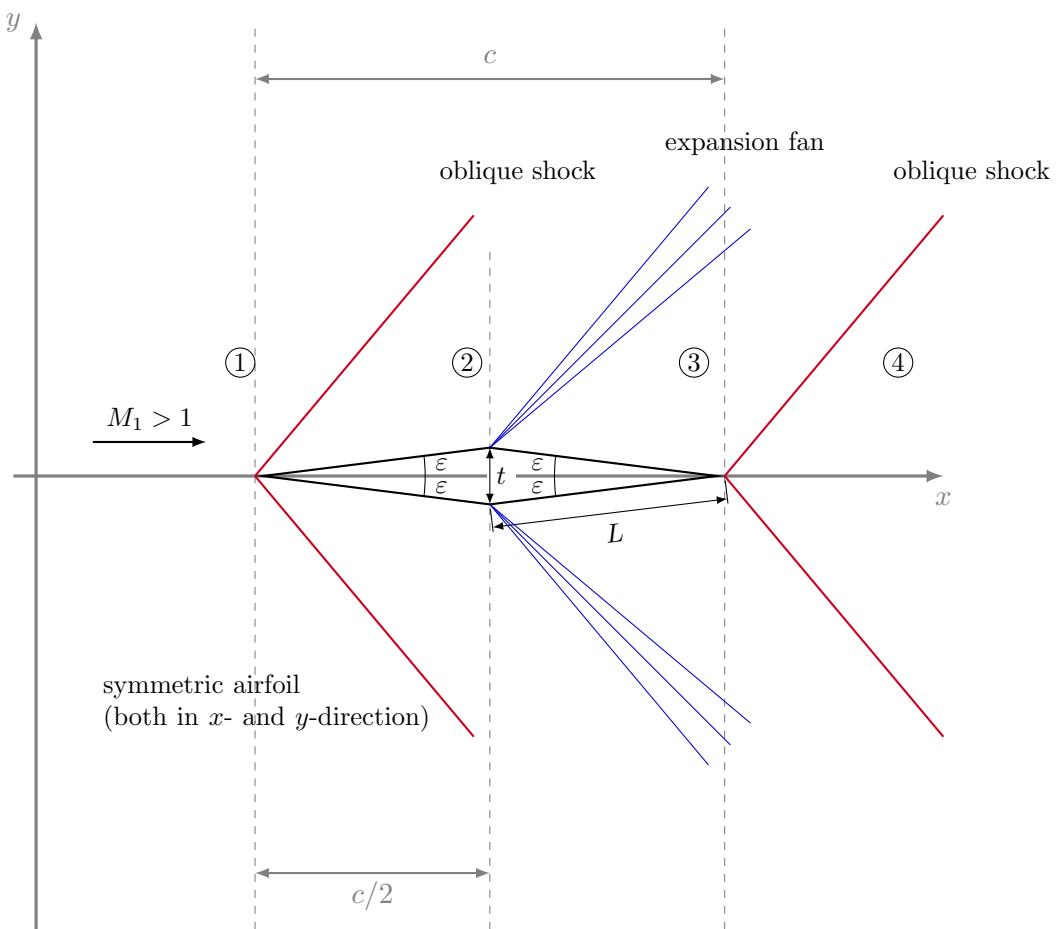
P1   (RAMJET ENGINE)
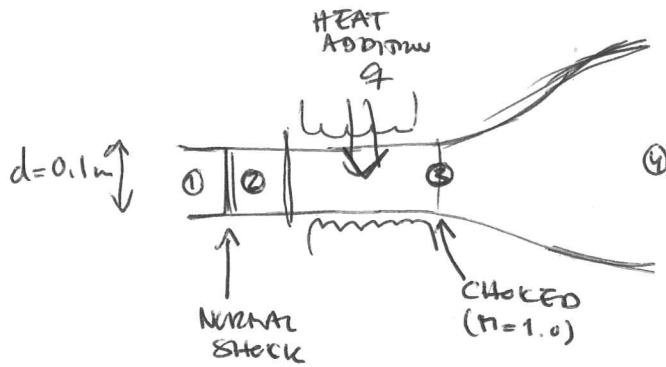
$M_1 = 3.0$
$P_1 = 100.0 \, kPa$
$T_1 = 300.0 \, K$
$P_4 = 100.0 \, kPa$
$M_3 = 1.0$

$u_1 = M_1 a_1 = M_1 \sqrt{\gamma R T_1} = 1041.6 \, m/s$
$\dot{m} = u_1 \rho_1 A_1 = u_1 \frac{P_1}{R T_1} \frac{\pi d^2}{4} = 9.5 \, kg/s$

$1 \rightarrow 2$ NORMAL SHOCK

(3.51) $M_2^2 = \dfrac{1 + ((\gamma+1)/2) M_1^2}{\gamma M_1^2 - (\gamma-1)/2} \Rightarrow M_2 = 0.48$

(3.57) $\dfrac{P_2}{P_1} = 1 + \dfrac{2\gamma}{\gamma+1}(M_1^2 - 1) \Rightarrow P_2 = 1033.33 \, kPa$

(3.59) $\dfrac{T_2}{T_1} = \dfrac{P_2}{P_1}\left(\dfrac{2 + (\gamma-1) M_1^2}{(\gamma+1) M_1^2}\right) \Rightarrow T_2 = 805.7 \, K$

(3.28) $\dfrac{T_0}{T_2} = 1 + \dfrac{\gamma-1}{2} M_2^2 \Rightarrow T_{02} = 840.0 \, K$

(3.30) $\dfrac{P_0}{P_2} = \left(1 + \dfrac{\gamma-1}{2} M_2^2\right)^{\gamma/(\gamma-1)} \Rightarrow P_{02} = 1206.1 \, kPa$

$2 \rightarrow 3$ HEAT ADDITION

$M_3 = 1.0$ (choked) $\Rightarrow T_3 = T^*, \; P_3 = P^*$

(3.85) $\dfrac{P_2}{P^*} = \dfrac{1 + \gamma}{1 + \gamma M_2^2} \Rightarrow P^* = P_3 = 566.7 \, kPa$

(3.86) $\dfrac{T_2}{T^*} = M_2^2 \left(\dfrac{1 + \gamma}{1 + \gamma M_2^2}\right)^2 \Rightarrow T^* = T_3 = 1070.4 \, K$

(3.28) $\dfrac{T_{03}}{T_3} = 1 + \dfrac{\gamma-1}{2}M_3^2 = \dfrac{\gamma+1}{2}$ $\Rightarrow T_{03} = 1289.9\,k$

(3.30) $\dfrac{P_{03}}{P_3} = \left(1 + \dfrac{\gamma-1}{2}M_3^2\right)^{\gamma/(\gamma-1)} = \left(\dfrac{\gamma+1}{2}\right)^{\gamma/(\gamma-1)} \Rightarrow P_{03} = 1072.7\,kPa$

ADDED HEAT: $q_{23} = q_2^* = C_p\left(T_{03} - T_{02}\right) = 446.9\,kJ/kg$

HEAT POWER: $P_{23} = q_{23}\cdot \dot{m} = 4.29\,MW$

3 -> 4   SUPERSONIC EXPANSION

$P_4 = P_1$, $P_{04} = P_{03}$ (ISENTROPIC EXPANSION), $T_{04} = T_{03}$

(3.30) $\dfrac{P_{04}}{P_4} = \left(1 + \dfrac{\gamma-1}{2}M_4^2\right)^{\gamma/(\gamma-1)} \Rightarrow M_4 = 2.2$

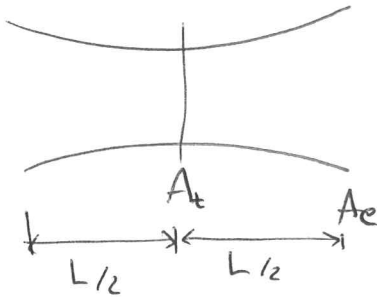(3.28) $\dfrac{T_{04}}{T_4} = \left(1 + \dfrac{\gamma-1}{2}M_4^2\right)^{\gamma/(\gamma-1)} \Rightarrow T_4 = 652.9\,k$

$U_4 = M_4\,a_4 = M_4\sqrt{\gamma R T_4} = 1127.1\,m/s$

$F_{THRUST} = \dot{m}\left(U_4 - U_1\right) = 813.0\,N$

P2   (CONVERGENT - DIVERGENT NOZZLE)

$$A_e / A_t = 1.633$$

$$\frac{A(x)}{A_t} = \left(\frac{A_e}{A_t} - 1\right)\left(2\frac{x}{L} - 1\right)^2 + 1$$

a) DETERMINE THE RANGES OF NOZZLE PRESSURE RATIONS
FOR WHICH THE FLOW IS FREE FROM NORMAL SHOCKS.

FREE FROM NORMAL SHOCKS!

- SUBSONIC FLOW   ⟹ UP TO CHOKED FLOW

- SUPERSONIC FLOW   (AFTER SHOCK AT EXIT)

(5.20) $\left(\dfrac{A_e}{A_t}\right)^2 = \dfrac{1}{M_e^2}\left(\dfrac{2}{\gamma+1}\left(1 + \dfrac{\gamma-1}{2}M_e^2\right)\right)^{(\gamma+1)/(\gamma-1)}$

SUBSONIC SOLUTION: $M_{e_c} = 0.39$

SUPERSONIC SOLUTION: $M_{e_{sc}} = 1.96$

(3.30)

$$NPR_c = \frac{P_{oc}}{P_b} = \left(1 + \frac{\gamma-1}{2}M_{e_c}^2\right)^{\gamma/(\gamma-1)} = 1.11$$

$$NPR_{sc} = \frac{P_{o_{sc}}}{P_b} = \left(1 + \frac{\gamma-1}{2}M_{e_{sc}}^2\right)^{\gamma/(\gamma-1)} = 7.36$$

SHOCK AT EXIT:

$$\frac{P_b}{P_e} = 1 + \frac{2\gamma}{\gamma+1}\left(M_{e_{sc}}^2 - 1\right)$$

$$NPR_{nse} = \frac{P_{ome}}{P_{b\,nse}} = \frac{P_{osc}}{P_e}\frac{P_e}{P_b} = 1.70$$

SHOCK-FREE FLOW:

$$1.0 < NPR < 1.11$$

$$NPR > 1.70$$

b) $NPR = 1.5$ $1.11 < NPR < 1.70$ => NORMAL SHOCK IN NOZZLE

(5.28) $\quad M_e^2 = \dfrac{-1}{\gamma-1} + \sqrt{\dfrac{1}{(\gamma-1)^2} + \left(\dfrac{2}{\gamma-1}\right)\left(\dfrac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}\left(\dfrac{P_{o1}\,A_t}{P_e\cdot A_e}\right)^2}$

=> $M_e = 0.52$

CHOKED FLOW:

(5.21)

$$\dot{m} = \dfrac{P_{o1}A_1^*}{\sqrt{T_0}}\sqrt{\dfrac{\gamma}{R}\left(\dfrac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}}$$

$$\dot{m} = \dfrac{P_{o2}A_2^*}{\sqrt{T_0}}\sqrt{\dfrac{\gamma}{R}\left(\dfrac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}}$$

$\dot{m} = const$ => $\quad P_{o1}A_1^* = P_{o2}A_2^*$

$$\dfrac{P_{o2}}{P_{o1}} = \dfrac{A_1^*}{A_2^*}$$

$$\left(\dfrac{A_e}{A_2^*}\right)^2 = \dfrac{1}{M_e^2}\left(\dfrac{2}{\gamma+1}\left(1 + \dfrac{\gamma-1}{2}M_e^2\right)\right)^{(\gamma+1)/(\gamma-1)}$$

WITHOUT SHOCK: $\quad \dfrac{A_e}{A_1^*} = \dfrac{A_e}{A_t} = 1.633$

FIND $\quad P_{o2}/P_{o1} \quad$ IN THE NOZZLE:

FOR A LOCATION x:

(3.57) $\quad \dfrac{P_y}{P_x} = 1 + \dfrac{2\gamma}{\gamma+1}(M_x^2 - 1)$

(3.51) $\quad M_y^2 = \dfrac{1 + ((\gamma-1)/2)M_x^2}{\gamma M_x^2 - (\gamma-1)/2}$

(3.30) $\quad \dfrac{P_{o1}}{P_x} = \left(1 + \dfrac{\gamma-1}{2}M_x^2\right)^{\gamma/(\gamma-1)}$

(3.30) $\quad \dfrac{P_{o2}}{P_y} = \left(1 + \dfrac{\gamma-1}{2}M_y^2\right)^{\gamma/(\gamma-1)}$
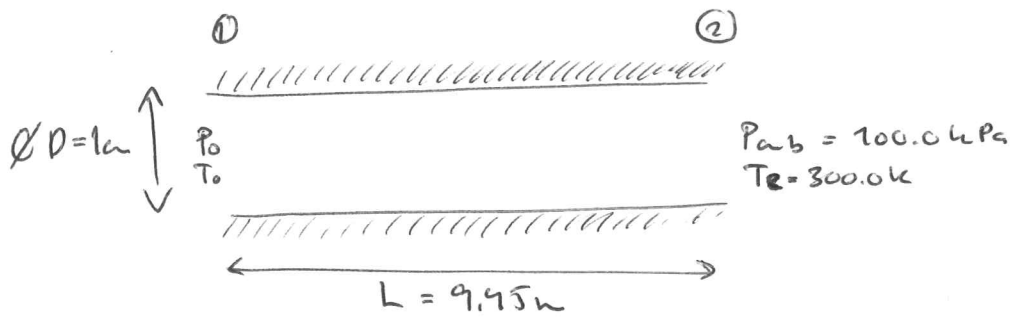
ITERATE TO FIND $M_x = 1.83$
(MACH NUMBER UPSTREAM OF SHOCK)

$$\left(\frac{A_x}{A^*_1}\right)^2 = \frac{1}{\eta_x^2}\left(\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}\eta_x^2\right)\right)^{(\gamma+1)/(\gamma-1)}$$

$$\Rightarrow A_x/A^*_1 = 1.47$$

$$\& \quad \frac{A(x)}{A_t} = \left(\frac{A_c}{A_t}-1\right)\left(2\frac{x}{L}-1\right)^2 + 1 \quad \Rightarrow \quad x/L = 0.93$$

P3   (PIPE FLOW WITH FRICTION)

$$\varnothing D = 1cm \qquad \text{①} \qquad\qquad \text{②}$$

P_0, T_0 $P_{ab} = 100.0 \, kPa$

$T_e = 300.0 \, K$

$$L = 9.45 \, m$$

$$\dot{m} = 9.5 \cdot 10^{-3} \, kg/s$$

$$\bar{f} = 0.005$$

$$\dot{m} = 9.5 \cdot 10^{-3} = \frac{P_e}{R T_e} \, u_2 \, \frac{\pi D^2}{4} = u_2 = 104.1 \, m/s$$

$$M_2 = \frac{u_2}{a_2} = \frac{u_2}{\sqrt{\gamma R T_2}} = 0.3$$

(3.107) $\qquad \dfrac{4 \bar{f} L_2^*}{D} = \dfrac{1 - M_2^2}{\gamma M_2^2} + \dfrac{\gamma+1}{2\gamma} \ln\left(\dfrac{(\gamma+1) M_2^2}{2 + (\gamma-1) M_2^2} \right)$

$$\Rightarrow L_2^* = 2.65 \, m$$

$$L_1^* = L_2^* + L = 12.1 \, m$$

(3.107) $\qquad \dfrac{4 \bar{f} L_1^*}{D} = \dfrac{1 - M_1^2}{\gamma M_1^2} + \dfrac{\gamma+1}{2\gamma} \ln\left(\dfrac{(\gamma+1) M_1^2}{2 + (\gamma-1) M_1^2} \right)$
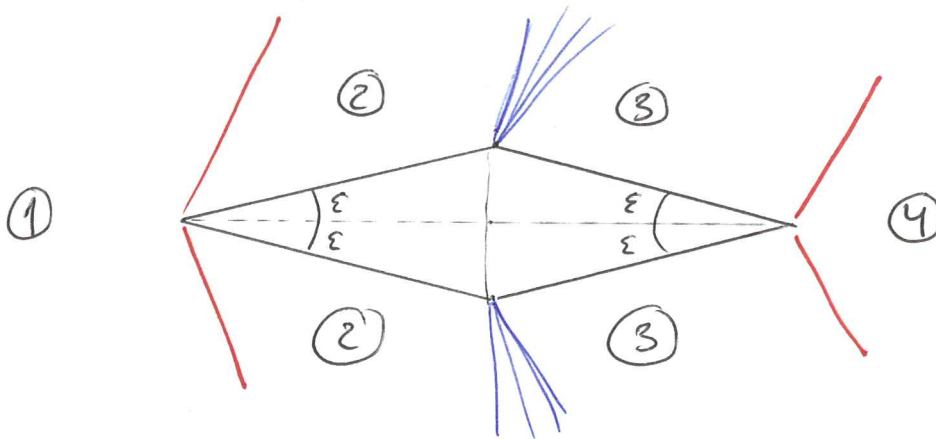
$$\Rightarrow \underline{M_1 = 0.16}$$

(3.104) $\qquad \dfrac{P_2}{P^*} = \dfrac{1}{M_2}\left(\dfrac{\gamma+1}{2+(\gamma-1)M_2^2} \right)^{1/2} \Rightarrow P^* = 27.6 \, kPa$

$\qquad\qquad \dfrac{P_1}{P^*} = \dfrac{1}{M_1}\left(\dfrac{\gamma+1}{2+(\gamma-1)M_1^2} \right)^{1/2} \Rightarrow P_1 = 188.7 \, kPa$

(3.30) $\qquad \dfrac{P_{01}}{P_1} = \left(1 + \dfrac{\gamma-1}{2} M_1^2 \right)^{\gamma/(\gamma-1)} \Rightarrow \underline{P_0 = 192.1 \, kPa}$

P4   (SHOCK-EXPANSION THEORY)

$\varepsilon = 5°$

Symmetric wing   chord $\pm C = 1.0m$

$M_1 = 2.0$

$P_1 = 101385 \, Pa$

OBLIQUE SHOCK   1→2

$\left. \begin{array}{l} \theta = \varepsilon \\ M = M_1 \end{array} \right\}$ $(\theta - \beta - M)$ => $\beta = 34.3°$

(4.7) $M_{n_1} = M_1 \sin\beta$

(4.9) $\dfrac{P_2}{P_1} = 1 + \dfrac{2\gamma}{\gamma+1}(M_{n_1}^2 - 1)$

(4.10) $M_{n_2}^2 = \dfrac{M_{n_1}^2 + (2/(\gamma-1))}{(2\gamma/(\gamma-1))M_{n_1}^2 - 1}$

(4.12) $M_2 = \dfrac{M_{n_2}}{\sin(\beta-\varepsilon)}$

$\left. \vphantom{\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}} \right\}$

$M_2 = 1.82$

$\dfrac{P_2}{P_1} = 1.32$ => $P_2 = 133.3 \, kPa$

EXPANSION   2 → 3

(4.44) $\nu_2 = \sqrt{\dfrac{\gamma+1}{\gamma-1}} \ tan^{-1}\sqrt{\dfrac{\gamma-1}{\gamma+1}(M_2^2-1)} - tan^{-1}\sqrt{M_2^2-1}$

FROM 2→3 , THE FLOW MUST BE TURNED BACK ε
DEGREES AND THEN ANOTHER ε DEGREES TO FOLLOW THE
REAR PART OF THE WING , THUS $\Delta \varepsilon = 2\varepsilon$

$\Rightarrow$ $\nu_3 = \nu_2 + \Delta \varepsilon$ $= 31.84°$

(4.44) $\nu_3 = \sqrt{\dfrac{\gamma+1}{\gamma-1}} \ tan^{-1}\sqrt{\dfrac{\gamma-1}{\gamma+1}(M_3^2-1)} - tan^{-1}\sqrt{M_3^2-1}$

$\Rightarrow M_3 = 2.18$
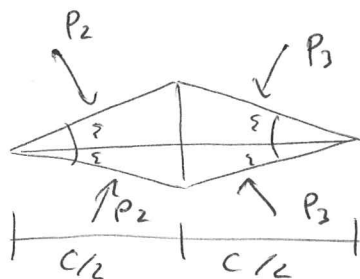
(3.30) $\dfrac{P_{02}}{P_2} = \left(1 + \dfrac{\gamma-1}{2}M_2^2\right)^{\gamma/(\gamma-1)}$

$\dfrac{P_{03}}{P_3} = \left(1 + \dfrac{\gamma-1}{2}M_3^2\right)^{\gamma/(\gamma-1)}$

$P_{02} = P_{03}$ (EXPANSION IS ISENTROPIC)

$\Rightarrow$ $\dfrac{P_3}{P_2} = \left(\dfrac{1 + \frac{(\gamma-1)}{2}M_2^2}{1 + \frac{(\gamma-1)}{2}M_3^2}\right)^{\gamma/(\gamma-1)}$ $= 0.57$

$\dfrac{P_3}{P_1} = \dfrac{P_3}{P_2}\dfrac{P_2}{P_1} = 0.75$ $\Rightarrow$ $P_3 = 75.77 \ kPa$

$F_D = 2\left((P_2 - P_3)\dfrac{c}{2} tan(\varepsilon)\right)$

$= (P_2 - P_3)c \ tan(\varepsilon)$

$= 5.0 \ kN$