Fluid Mechanics MTF053

Equations for Boundary Layer Flows

Based on Appelqvist, Frössling & Loyd. 1998 *Strömning med friktion*

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1 Introduction

In a boundary layer, the Navier-Stokes equations can be simplified by comparing the importance of the different terms, which results in a reduced set of equations often referred to as the boundary layer equations. This reduced set of equations is, however, still a system of non-linear partial differential equations and, although it is easier to handle the reduced set of equations than the full Navier-Stokes equations, there are only a limited set of explicit solutions but of course it is possible to solve the equations numerically. For laminar boundary layers, solutions may be obtained for most geometries whereas for turbulent boundary layers, one often need to use experimental data.

An alternative to using the differential equations is to use relations based on integral equations. These equations are obtained by applying the momentum equation or the energy equation for steady-state flow to a control volume as shown schematically below. The integral relations can be used to obtain approximate solutions. For turbulent flows that, as stated above, requires experimental data to be able to solve the differential equations, the integral relations provide alternative methods. The integral relations can in some cases be as accurate as the boundary layer equations.

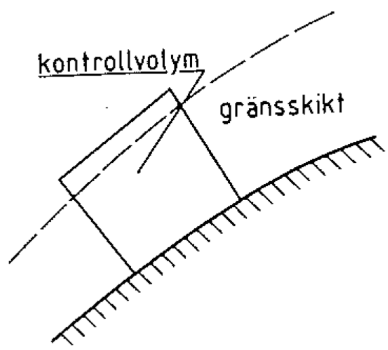

Figure 1: Boundary layer control volume

2 Laminar Boundary Layers

Let's assume two-dimensional, incompressible and steady-state flow with negligible body forces. The governing equations on non-dimensional form reads

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{1}$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) \tag{2}$$

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right) \tag{3}$$

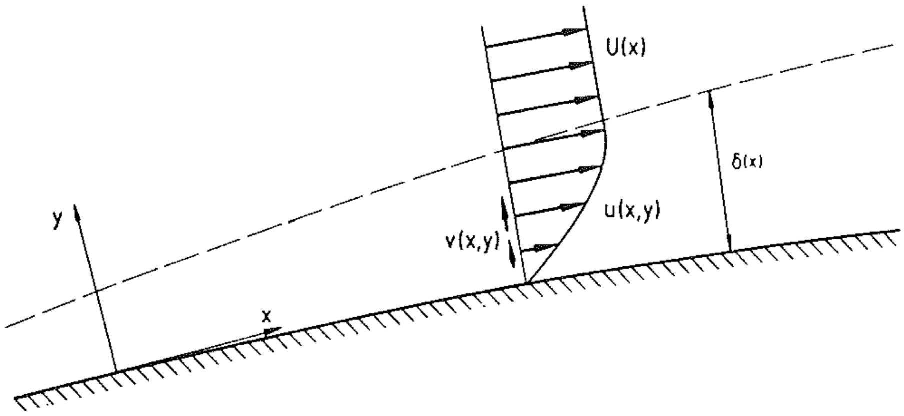

Figure 2: Boundary layer variables and definitions

The non-dimensional variables are obtained by scaling with the freestream velocity $U = U(x)$ and a characteristic length L

$$x^* = \frac{x}{L} \qquad u^* = \frac{u}{U}$$

$$y^* = \frac{y}{L} \qquad v^* = \frac{v}{U}$$

$$p^* = \frac{p}{\rho U^2} \qquad Re = \frac{UL}{\nu}$$

In order to simplify the governing equations, the terms in the non-dimensional equations are compared in order to find out if some terms can be neglected. As part of this comparison, we need an estimate of the size of each of the terms within the boundary layer.

$$u^* = \frac{u}{U} \sim 1$$

$$x^* = \frac{x}{L} \sim 1$$

$$y^* = \frac{y}{L} \sim \frac{\delta}{L} = \delta^*$$

The size of velocity gradient $\partial u^*/\partial y^*$ is in the same order as that of a line starting from zero at the wall and reaching the freestream velocity U at the edge of the boundary layer $y = \delta$ (see figure below)

$$\frac{\partial u^*}{\partial y^*} \sim \frac{1}{\delta^*}$$

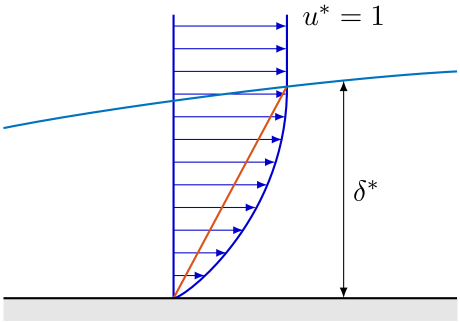

Figure 3: First-order velocity derivative estimate

The value of $\partial u^*/\partial y^*$ changes from $1/\delta^*$ at the wall to zero at the edge of the boundary layer which mean that

$$\frac{\partial^2 u^*}{\partial {y^*}^2} \sim \frac{1}{{\delta^*}^2}$$

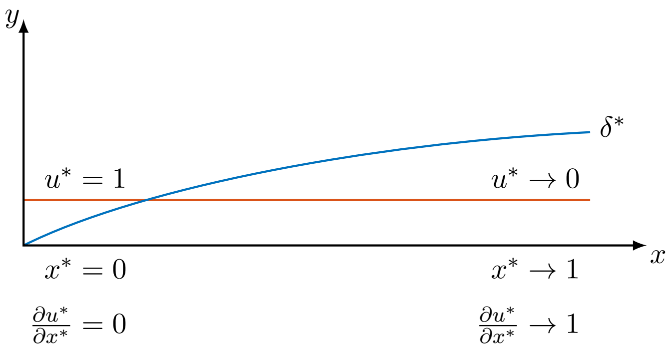

Figure 4: Second-order velocity derivative estimate

With the velocity gradients in the wall-normal direction analyzed, we move on to study the gradients in the flow direction, which gives

$$\frac{\partial u^*}{\partial x^*} \sim \frac{1}{1} = 1$$

$$\frac{\partial^2 u^*}{\partial {x^*}^2} \sim 1$$

If we now have a look at the continuity equation (Eqn. 1), we see that

$$\frac{\partial v^*}{\partial y^*} \sim \frac{\partial u^*}{\partial x^*} \sim 1$$

Since the velocity is zero at the wall, the velocity at the outer edge of the boundary layer (at $y^* = \delta^*$) must be in the order of δ^* and thus for the greater part of the boundary layer we will have

$$v^* \sim \delta^*$$

The derivatives of v^* that remains to be analyzed are obtained in the same way as was done for the derivatives of u^*.

Now, if we assume that the boundary layer is very thin compared to its length (a fair assumption for boundary layers)

$$\delta^* = \frac{\delta}{L} \ll 1$$

The viscous term in the x-component of the momentum equation reads

$$\frac{1}{Re}\left(\frac{\partial^2 u^*}{\partial {x^*}^2} + \frac{\partial^2 u^*}{\partial {y^*}^2}\right)$$

The first term is in the order of 1 and the second term is in the order of $1/{\delta^*}^2$. With the assumption of thin boundary layers in mind this means that

$$\frac{\partial^2 u^*}{\partial {x^*}^2} \ll \frac{\partial^2 u^*}{\partial {y^*}^2}$$

In the same way we see that

$$\frac{\partial^2 v^*}{\partial {x^*}^2} \ll \frac{\partial^2 v^*}{\partial {y^*}^2}$$

Eqn. 2, now reads

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re} \frac{\partial^2 u^*}{\partial {y^*}^2} \tag{4}$$

In a boundary layer friction forces can be assumed to be of the same size as momentum forces. In the left-hand side of Eqn. 4 all terms are in the order of 1 and thus the terms on the right-hand side must be in the order of 1 as well.

$$\frac{1}{Re} \frac{\partial^2 u^*}{\partial {y^*}^2} \sim 1 \Rightarrow Re \sim \frac{1}{{\delta^*}^2}$$

Now, let's take a look at the y-component of the momentum equation

$$\overline{u} \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re} \frac{\partial^2 v^*}{\partial {y^*}^2} \tag{5}$$

With the momentum and friction terms all being in the order of δ^*, we see that the pressure term must be of the same size.

$$\frac{\partial p^*}{\partial y^*} \sim \delta^*$$

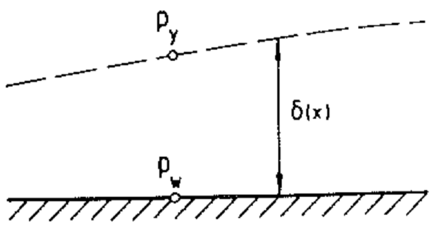

Figure 5: Pressure difference in the boundary layer

This means that if one subtracts the pressure at the wall from the pressure at the edge of the boundary layer one gets a difference in the order of ${\delta^*}^2$.

$$p^*_\delta - p^*_w \sim {\delta^*}^2$$

This means that the pressure is independent of the distance from the wall and thus just a function of x

$$p = p(x)$$

The pressure at the wall, p_w, is in other words equal to the pressure outside of the boundary layer if the boundary layer is thin in relation to its length. This is a result that is used when measuring static pressure.

The analysis of the y-component of the momentum equation gives the result that the pressure is a function of x only and the pressure gradient can be written as a total derivative. The simplified governing equations now back on dimensional form.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{7}$$

Boundary conditions:

$$\begin{aligned} y &= 0 & \qquad u = v &= 0 \\ y &\to \delta & \qquad u &\to U \end{aligned}$$

The pressure term can either be obtained from experiments or calculated from the potential flow velocity using the Bernoulli equation which after differentiation becomes

$$-\frac{1}{\rho}\frac{dp}{dx} = U\frac{dU}{dx} \tag{8}$$

The boundary layer equations can be used for weakly curved surfaces if the coordinate system is set up such that the x-coordinate is aligned with the surface and the y-axis is in the surface normal direction. It is, however, required that the boundary layer thickness, δ, is small in comparison to the surface curvature radius. It is also possible to derive a more generic set of boundary layer equations for three-dimensional unsteady cases.

3 Turbulent Boundary Layers

The boundary layer equations for turbulent flows are obtained starting from the Reynolds-Averaged Navier-Stokes (RANS) equations. In the same way as for the laminar flow equations, the terms in the equations are compared in terms of size using order of magnitude estimates.

For two-dimensional flow we get

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0 \tag{9}$$

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{d\overline{p}}{dx} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial}{\partial y}\overline{u'v'} \tag{10}$$

A term caused by the turbulent shear stress appears in the equation, which makes it impossible to solve the equations directly. The solution to this closure problem is to either express the turbulence shear stress as a function of averaged flow field properties using a valid hypothesis or to use additional equations for the turbulent properties. In both these cases using experimental data is necessary.

Note that over-lined quantities denote ensemble averages.

3.1 Boundary Layer Thickness

For laminar flows it is possible to estimate the boundary layer thickness directly analysis of terms leading up to the boundary layer equations. From before we have that

$$Re \sim \frac{1}{\delta^{*2}} \Rightarrow \delta^* \sim \frac{1}{\sqrt{Re}} \Rightarrow \delta \sim \frac{L}{\sqrt{Re}}$$

The transition from boundary layer to freestream is asymptotic and therefore there are several definitions of boundary layer thickness. Often the boundary layer thickness is defined as the location where the velocity has reached 99% of the freestream velocity, i.e., $u = 0.99U$.

the displacement thickness, δ^*, and momentum thickness, θ, can also be used as measures of the boundary layer thickness.

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy \tag{11}$$

3.2 Integral Estimates

The momentum integral relation is derived by applying the momentum equation for steady-state flow on a control volume aligned with the boundary layer as the control volume $ABCD$ in the figure below.

The extent of the control volume in the z direction is one unit depth.

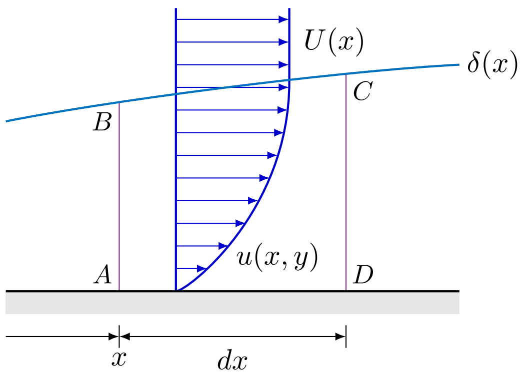

Figure 6: Boundary layer control volume

Massflow balance

ingoing massflow through surface AB:

$$\dot{m}_{AB} = \rho \int_0^\delta u dy$$

outgoing massflow through surface CD:

$$\dot{m}_{CD} = \rho \int_0^\delta u dy + \frac{d}{dx}\left[\rho \int_0^\delta u dy\right] dx = \rho \int_0^\delta u dy + \rho \frac{d}{dx}\left[\int_0^\delta u dy\right] dx$$

Since surface AD is aligned with the wall, there must be a massflow through surface BC that balance the difference $\dot{m}_{CD} - \dot{m}_{AB}$ and thus

$$\dot{m}_{BC} = \rho \frac{d}{dx}\left[\int_0^\delta u dy\right] dx$$

Momentum balance

ingoing momentum through surface AB:

$$\dot{I}_{AB} = \rho \int_0^\delta u^2 dy$$

outgoing momentum through surface CD:

$$\dot{I}_{CD} = \rho \int_0^\delta u^2 dy + \rho \frac{d}{dx} \left[\int_0^\delta u^2 dy \right] dx$$

ingoing momentum through surface BC:

$$\dot{I}_{BC} = U \dot{m}_{BC} = \rho U \frac{d}{dx} \left[\int_0^\delta u dy \right] dx$$

Thus, the net momentum flow out from the control volume is

$$\dot{I}_{CD} - \dot{I}_{AB} - \dot{I}_{BC} = \rho \frac{d}{dx} \left[\int_0^\delta u^2 dy \right] dx - \rho U \frac{d}{dx} \left[\int_0^\delta u dy \right] dx$$

Net force on the control volume

Finally, we need to analyze the forces on the control volume in the x-direction. We will have a friction force along the wall in the negative x-direction. On the other surfaces there will be pressure forces.

$$F_{AD} = -\tau_w dx$$

$$F_{AB} = p\delta$$

$$F_{CD} = -\left(p + \frac{dp}{dx} dx \right) \left(\delta + \frac{d\delta}{dx} dx \right)$$

$$F_{BC} \approx \left(p + \frac{1}{2} \frac{dp}{dx} dx \right) \frac{d\delta}{dx} dx$$

The net force excluding second-order terms

$$dF_x = -\tau_w dx - \delta \frac{dp}{dx} dx$$

We can now write out the x-component of the momentum equation for the control volume

$$\rho U \frac{d}{dx} \left[\int_0^\delta u dy \right] - \rho \frac{d}{dx} \left[\int_0^\delta u^2 dy \right] = \tau_w + \delta \frac{dp}{dx} \tag{12}$$

This equation is often referred to as the momentum equation for boundary layers or the von Kármán integral relation. Eqn. 12 can also be derived from the boundary layer equations Eqn. 7 or Eqn. 10. Eqn. 12 is valid both for laminar and turbulent boundary layers. In the latter, for ensemble-averaged flow variables.

From Eqn. 8 we have

$$-\frac{1}{\rho}\frac{dp}{dx} = U\frac{dU}{dx}$$

which can be used to replace the pressure derivative in Eqn. 12

$$\delta\frac{dp}{dx} = -\delta\rho U\frac{dU}{dx} = -\rho U\frac{dU}{dx}\int_0^\delta dy$$

The first term on the left-hand side of Eqn. 12 can be rewritten using the chain rule of differentiation.

$$\rho U\frac{d}{dx}\left[\int_0^\delta udy\right] = \rho\frac{d}{dx}\left[U\int_0^\delta udy\right] - \rho\frac{dU}{dx}\int_0^\delta udy$$

Insert in Eqn. 12 gives

$$\rho\frac{d}{dx}\left[U\int_0^\delta udy\right] - \rho\frac{dU}{dx}\int_0^\delta udy - \rho\frac{d}{dx}\left[\int_0^\delta u^2dy\right] = \tau_w - \rho U\frac{dU}{dx}\int_0^\delta dy$$

Collect terms

$$\rho\frac{d}{dx}\left[U\int_0^\delta udy - \int_0^\delta u^2dy\right] + \rho\frac{dU}{dx}\left[U\int_0^\delta dy - \int_0^\delta udy\right] = \tau_w$$

Dividing by ρ and rewriting the integral terms gives

$$\frac{d}{dx}\int_0^\delta u(U-u)dy + \frac{dU}{dx}\int_0^\delta (U-u)dy = \frac{\tau_w}{\rho} \qquad (13)$$

Note that along a flat plate $dU/dx = 0$, which removes the second term on the left-hand side.