

# Compressible Flow - TME085

## Lecture 12

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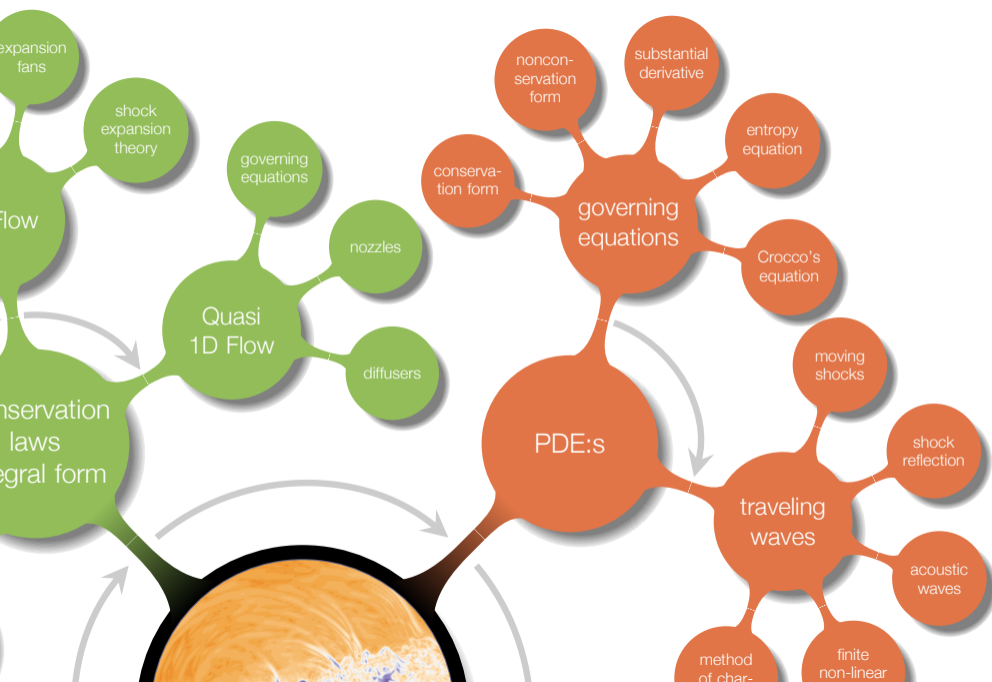
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# Chapter 7

## Unsteady Wave Motion



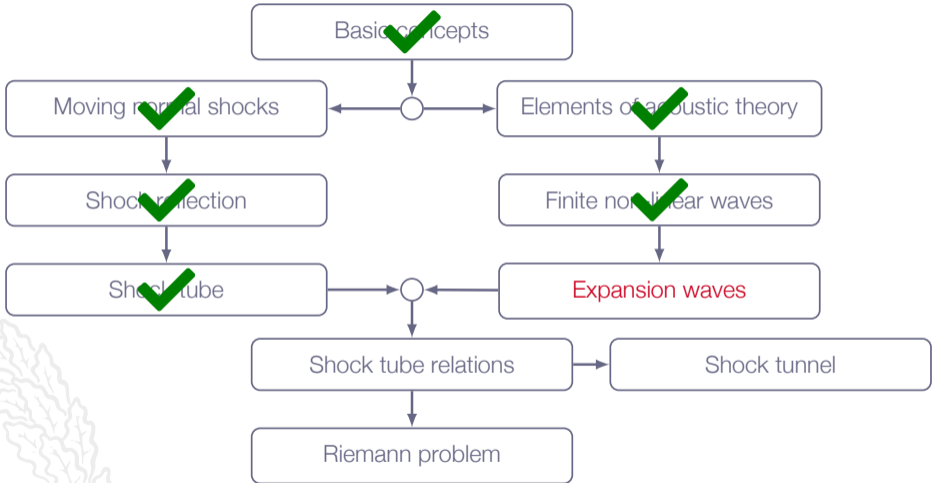


# Learning Outcomes

- 3 **Describe** typical engineering flow situations in which compressibility effects are more or less predominant (e.g. Mach number regimes for steady-state flows)
- 4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on
- 8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics
  - a 1D isentropic flow\*
  - b normal shocks\*
  - j unsteady waves and discontinuities in 1D
  - k basic acoustics
- 9 **Solve** engineering problems involving the above-mentioned phenomena (8a-8k)
- 11 **Explain** how the equations for aero-acoustics and classical acoustics are derived as limiting cases of the compressible flow equations

*moving normal shocks - frame of reference seems to be the key here?!*

# Roadmap - Unsteady Wave Motion

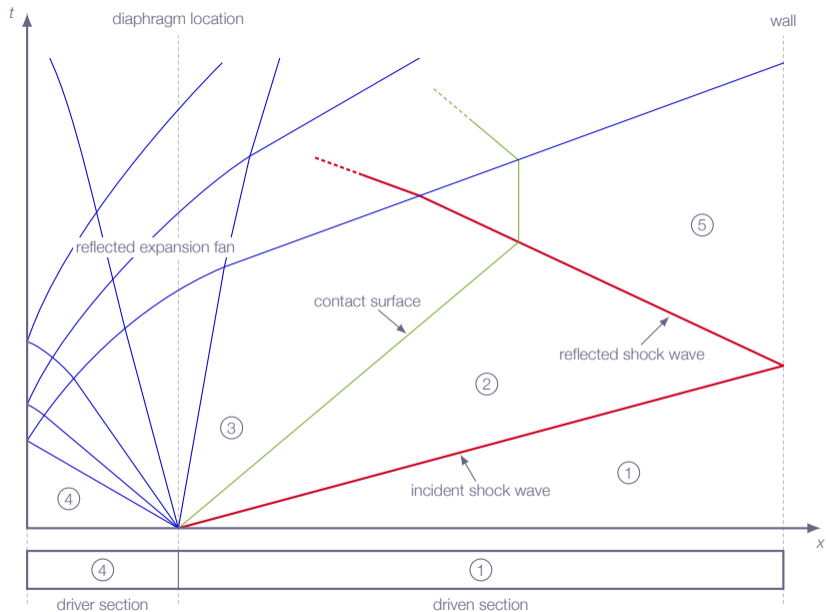


# Chapter 7.7

## Incident and Reflected Expansion Waves



# Expansion Waves



# Expansion Waves

Properties of a left-running expansion wave

1. All flow properties are constant along  $C^-$  characteristics
2. The wave **head** is propagating **into region 4** (high pressure)
3. The wave **tail** defines the **limit of region 3** (lower pressure)
4. Regions 3 and 4 are assumed to be **constant states**

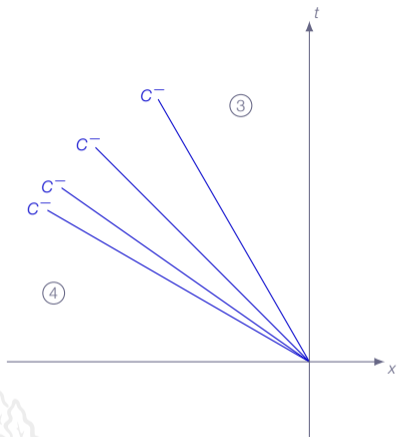
For calorically perfect gas:

$$J^+ = u + \frac{2a}{\gamma - 1} \quad \text{is constant along } C^+ \text{ lines}$$

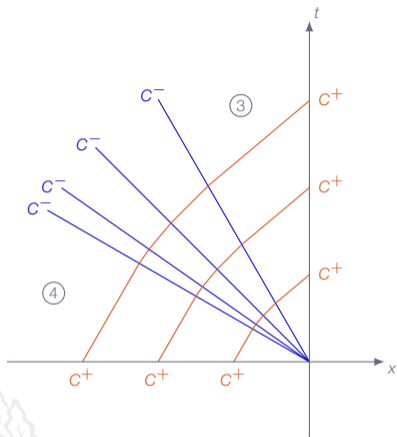
$$J^- = u - \frac{2a}{\gamma - 1} \quad \text{is constant along } C^- \text{ lines}$$



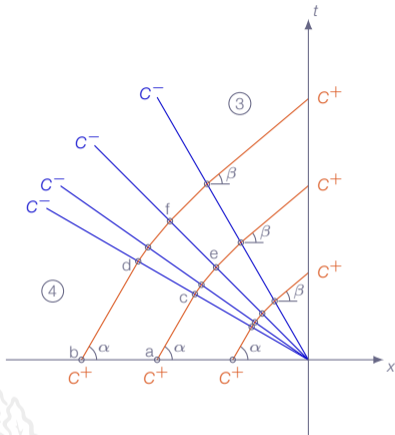
# Expansion Waves



# Expansion Waves



# Expansion Waves



constant flow properties in region 4:  $J_a^+ = J_b^+$

$J^+$  invariants constant along  $C^+$  characteristics:

$$J_a^+ = J_c^+ = J_e^+$$

$$J_b^+ = J_d^+ = J_f^+$$

since  $J_a^+ = J_b^+$  this also implies  $J_e^+ = J_f^+$

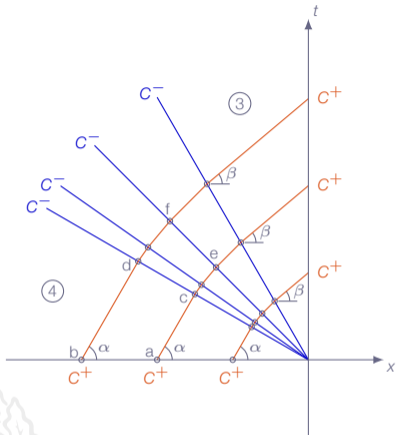
$J^-$  invariants constant along  $C^-$  characteristics:

$$J_c^- = J_d^-$$

$$J_e^- = J_f^-$$



# Expansion Waves



constant flow properties in region 4:  $J_a^+ = J_b^+$

$J^+$  invariants constant along  $C^+$  characteristics:

$$J_a^+ = J_c^+ = J_e^+$$

$$J_b^+ = J_d^+ = J_f^+$$

since  $J_a^+ = J_b^+$  this also implies  $J_e^+ = J_f^+$

$J^-$  invariants constant along  $C^-$  characteristics:

$$J_c^- = J_d^-$$

$$J_e^- = J_f^-$$

$$u_e = \frac{1}{2}(J_e^+ + J_e^-), u_f = \frac{1}{2}(J_f^+ + J_f^-), \Rightarrow u_e = u_f$$

$$a_e = \frac{\gamma - 1}{4}(J_e^+ - J_e^-), a_f = \frac{\gamma - 1}{4}(J_f^+ - J_f^-), \Rightarrow a_e = a_f$$

# Expansion Waves

Along each  $C^-$  line  $u$  and  $a$  are **constants** which means that

$$\frac{dx}{dt} = u - a = \text{const}$$

$C^-$  characteristics are **straight lines** in  $xt$ -space



# Expansion Waves

The start and end conditions are the same for all  $C^+$  lines

$J^+$  invariants have the same value for all  $C^+$  characteristics

$C^-$  characteristics are straight lines in  $xt$ -space

Simple expansion waves centered at  $(x, t) = (0, 0)$

# Expansion Waves

In a left-running expansion fan:

- ▶  $J^+$  is constant throughout expansion fan, which implies:

$$u + \frac{2a}{\gamma - 1} = u_4 + \frac{2a_4}{\gamma - 1} = u_3 + \frac{2a_3}{\gamma - 1}$$

- ▶  $J^-$  is constant along  $C^-$  lines, but varies from one line to the next, which means that

$$u - \frac{2a}{\gamma - 1}$$

is constant along each  $C^-$  line

# Expansion Waves

Since  $u_4 = 0$  we obtain:

$$u + \frac{2a}{\gamma - 1} = u_4 + \frac{2a_4}{\gamma - 1} = \frac{2a_4}{\gamma - 1} \Rightarrow$$

$$\frac{a}{a_4} = 1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}$$

with  $a = \sqrt{\gamma RT}$  we get

$$\frac{T}{T_4} = \left[ 1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4} \right]^2$$



# Expansion Wave Relations

Isentropic flow  $\Rightarrow$  we can use the isentropic relations

*complete description in terms of  $u/a_4$*

$$\frac{T}{T_4} = \left[ 1 - \frac{1}{2}(\gamma - 1) \frac{u}{a_4} \right]^2$$

$$\frac{\rho}{\rho_4} = \left[ 1 - \frac{1}{2}(\gamma - 1) \frac{u}{a_4} \right]^{\frac{2\gamma}{\gamma-1}}$$

$$\frac{p}{p_4} = \left[ 1 - \frac{1}{2}(\gamma - 1) \frac{u}{a_4} \right]^{\frac{2}{\gamma-1}}$$



# Expansion Wave Relations

Since  $C^-$  characteristics are straight lines, we have:

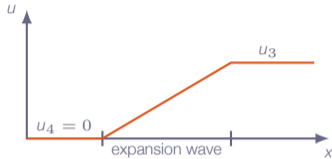
$$\frac{dx}{dt} = u - a \Rightarrow x = (u - a)t$$

$$\frac{a}{a_4} = 1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4} \Rightarrow a = a_4 - \frac{1}{2}(\gamma - 1)u \Rightarrow$$

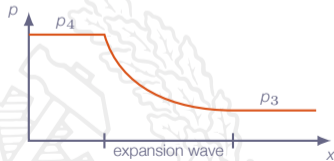
$$x = \left[ u - a_4 + \frac{1}{2}(\gamma - 1)u \right] t = \left[ \frac{1}{2}(\gamma - 1)u - a_4 \right] t \Rightarrow$$

$$u = \frac{2}{\gamma + 1} \left[ a_4 + \frac{x}{t} \right]$$

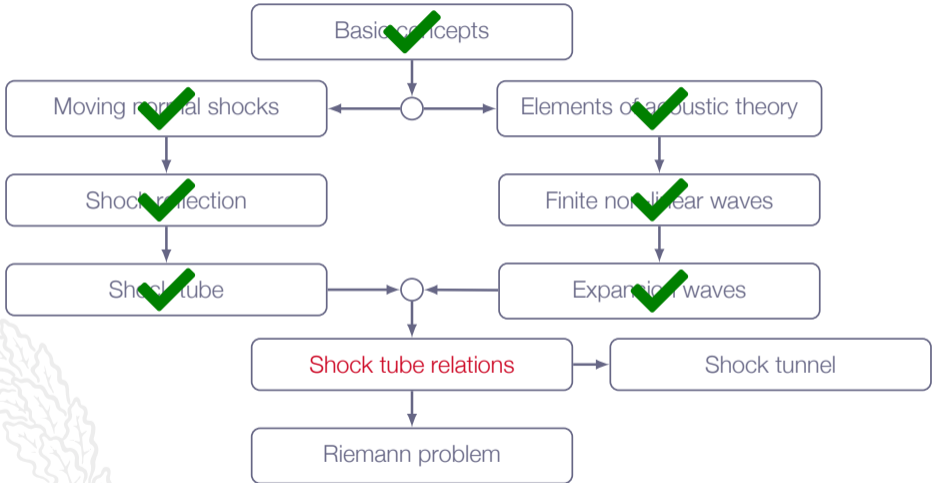
# Expansion Wave Relations



- ▶ Expansion wave head is advancing to the left with speed  $a_4$  into the stagnant gas
- ▶ Expansion wave tail is advancing with speed  $u_3 - a_3$ , which may be positive or negative, depending on the initial states



# Roadmap - Unsteady Wave Motion



# Chapter 7.8

## Shock Tube Relations



# Shock Tube Relations

$$u_p = u_2 = \frac{a_1}{\gamma} \left( \frac{p_2}{p_1} - 1 \right) \left[ \frac{\frac{2\gamma_1}{\gamma_1 + 1}}{\frac{p_2}{p_1} + \frac{\gamma_1 - 1}{\gamma_1 + 1}} \right]^{1/2}$$

$$\frac{p_3}{p_4} = \left[ 1 - \frac{\gamma_4 - 1}{2} \left( \frac{u_3}{a_4} \right) \right]^{2\gamma_4/(\gamma_4 - 1)}$$

solving for  $u_3$  gives

$$u_3 = \frac{2a_4}{\gamma_4 - 1} \left[ 1 - \left( \frac{p_3}{p_4} \right)^{(\gamma_4 - 1)/(2\gamma_4)} \right]$$

# Shock Tube Relations

But,  $p_3 = p_2$  and  $u_3 = u_2$  (no change in velocity and pressure over contact discontinuity)

$$\Rightarrow u_2 = \frac{2a_4}{\gamma_4 - 1} \left[ 1 - \left( \frac{p_2}{p_4} \right)^{(\gamma_4 - 1)/(2\gamma_4)} \right]$$

We have now two expressions for  $u_2$  which gives us

$$\frac{a_1}{\gamma} \left( \frac{p_2}{p_1} - 1 \right) \left[ \frac{\frac{2\gamma_1}{\gamma_1 + 1}}{\frac{p_2}{p_1} + \frac{\gamma_1 - 1}{\gamma_1 + 1}} \right]^{1/2} = \frac{2a_4}{\gamma_4 - 1} \left[ 1 - \left( \frac{p_2}{p_4} \right)^{(\gamma_4 - 1)/(2\gamma_4)} \right]$$

# Shock Tube Relations

Rearranging gives:

$$\frac{p_4}{p_1} = \frac{p_2}{p_1} \left\{ 1 - \frac{(\gamma_4 - 1)(a_1/a_4)(p_2/p_1 - 1)}{\sqrt{2\gamma_1 [2\gamma_1 + (\gamma_1 + 1)(p_2/p_1 - 1)]}} \right\}^{-2\gamma_4/(\gamma_4 - 1)}$$

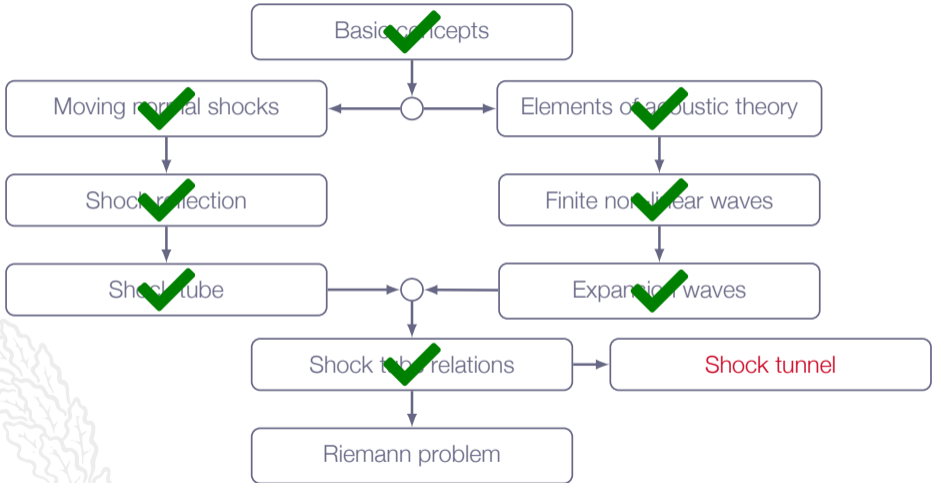
- ▶  $p_2/p_1$  as implicit function of  $p_4/p_1$
- ▶ for a given  $p_4/p_1$ ,  $p_2/p_1$  will increase with decreased  $a_1/a_4$

$$a = \sqrt{\gamma RT} = \sqrt{\gamma (R_u/M) T}$$

- ▶ the speed of sound in a light gas is higher than in a heavy gas
- ▶ driver gas: low molecular weight, high temperature
- ▶ driven gas: high molecular weight, low temperature

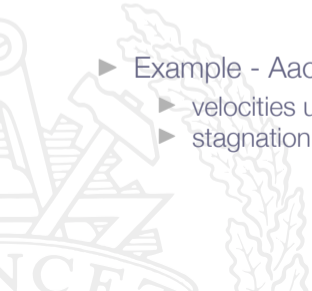


# Roadmap - Unsteady Wave Motion

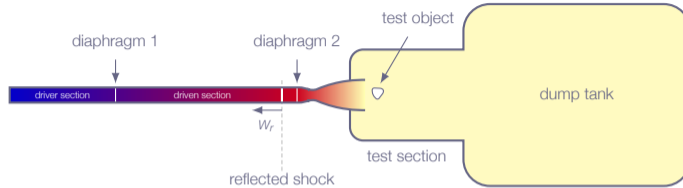


# Shock Tunnel

- ▶ Addition of a convergent-divergent nozzle to a shock tube configuration
- ▶ Capable of producing flow conditions which are close to those during the reentry of a space vehicles into the earth's atmosphere
  - ▶ high-enthalpy, hypersonic flows (short time)
  - ▶ real gas effects
- ▶ Example - Aachen TH2:
  - ▶ velocities up to 4 km/s
  - ▶ stagnation temperatures of several thousand degrees

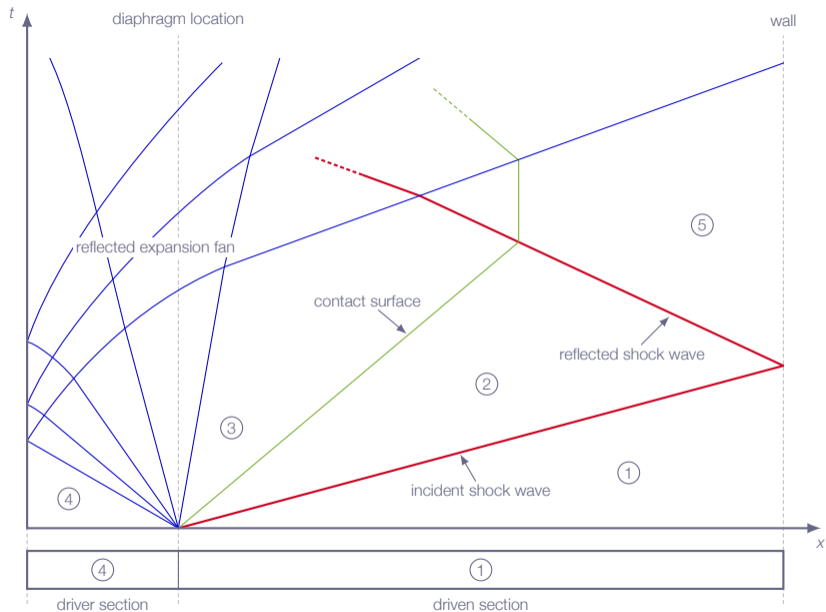


# Shock Tunnel



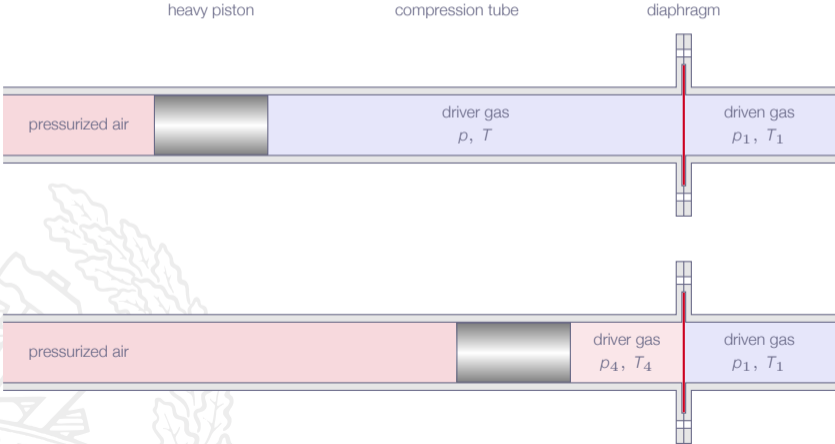
1. High pressure in region 4 (driver section)
  - ▶ diaphragm 1 burst
  - ▶ primary shock generated
2. Primary shock reaches end of shock tube
  - ▶ shock reflection
3. High pressure in region 5
  - ▶ diaphragm 2 burst
  - ▶ nozzle flow initiated
  - ▶ hypersonic flow in test section

# Shock Tunnel

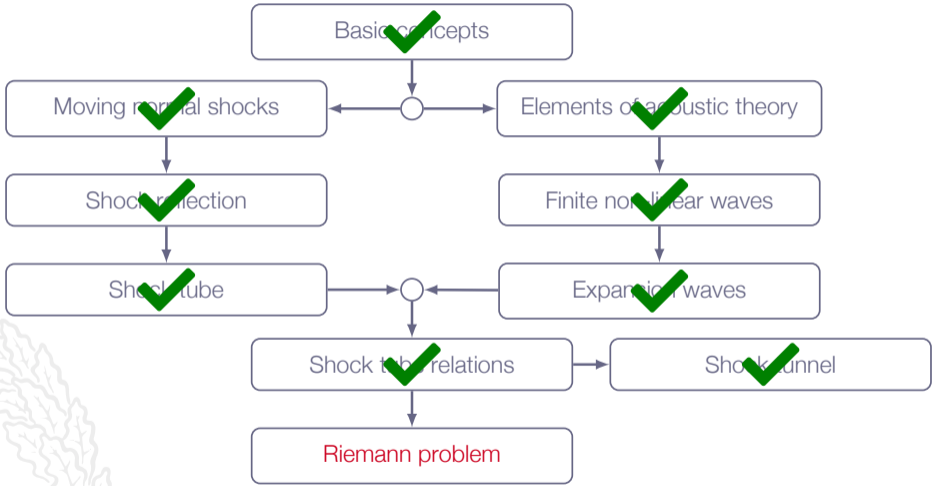


# Shock Tunnel

By adding a compression tube to the shock tube a very high  $p_4$  and  $T_4$  may be achieved for any gas in a fairly simple manner



# Roadmap - Unsteady Wave Motion



# Riemann Problem

The shock tube problem is a special case of the general **Riemann Problem**

*"... A Riemann problem, named after Bernhard Riemann, consists of an initial value problem composed by a conservation equation together with piece-wise constant data having a single discontinuity ..."*

Wikipedia



# Riemann Problem

May show that solutions to the shock tube problem have the general form:

$$p = p(x/t)$$

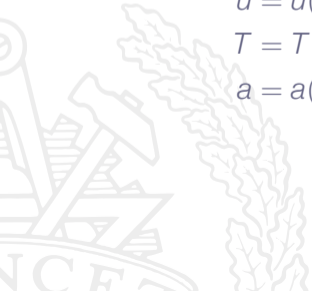
$$\rho = \rho(x/t)$$

$$u = u(x/t)$$

$$T = T(x/t)$$

$$a = a(x/t)$$

where  $x = 0$  denotes the position of the initial jump between states 1 and 4

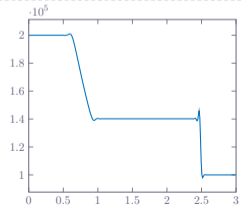
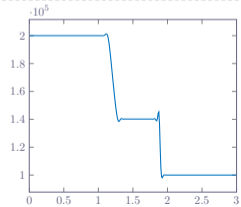
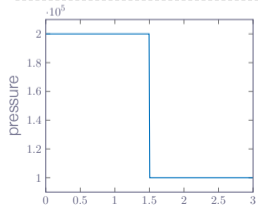
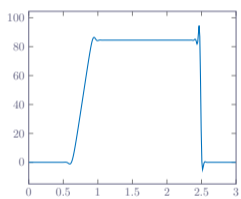
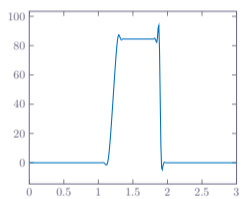
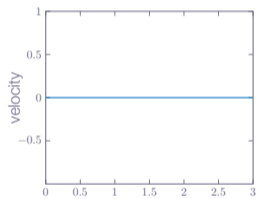
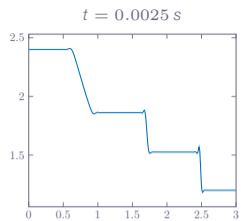
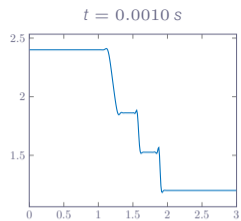
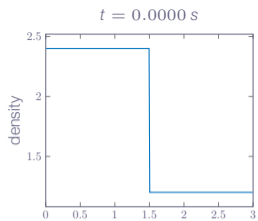


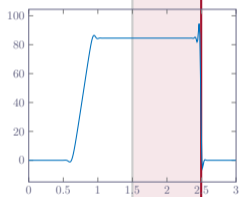
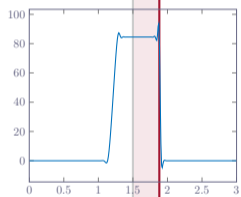
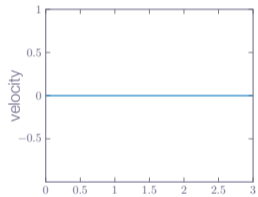
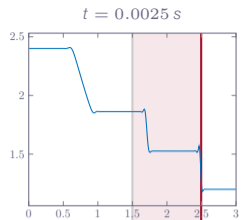
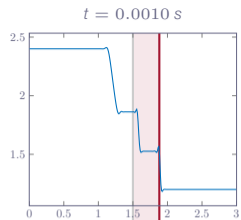
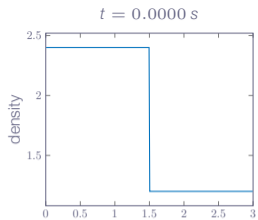


# Riemann Problem - Shock Tube

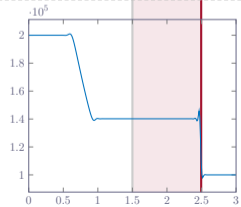
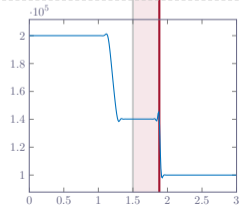
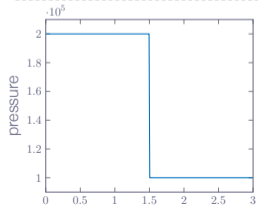
Shock tube simulation:

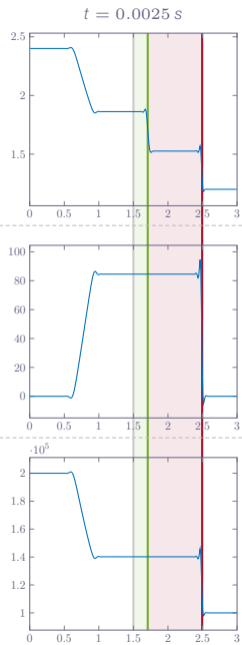
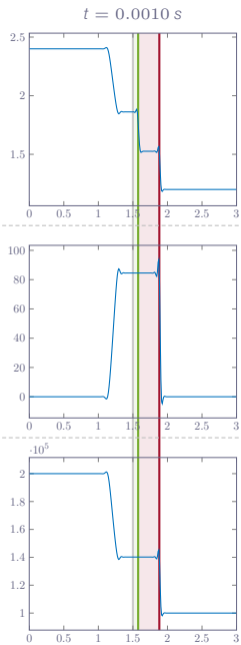
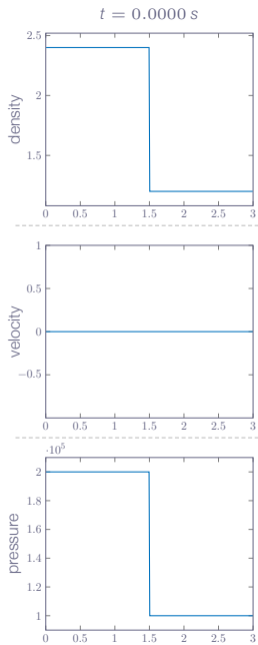
- ▶ left side conditions (state 4):
  - ▶  $\rho = 2.4 \text{ kg/m}^3$
  - ▶  $u = 0.0 \text{ m/s}$
  - ▶  $p = 2.0 \text{ bar}$
  
- ▶ right side conditions (state 1):
  - ▶  $\rho = 1.2 \text{ kg/m}^3$
  - ▶  $u = 0.0 \text{ m/s}$
  - ▶  $p = 1.0 \text{ bar}$
  
- ▶ Numerical method
  - ▶ Finite-Volume Method (FVM) solver
  - ▶ three-stage Runge-Kutta time stepping
  - ▶ third-order characteristic upwinding scheme
  - ▶ local artificial damping



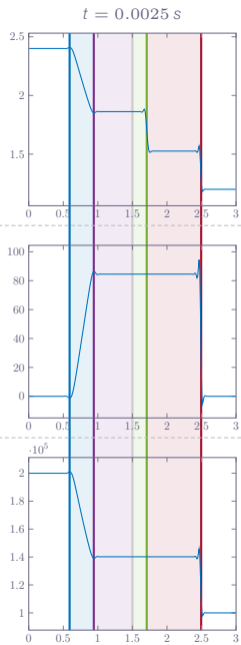
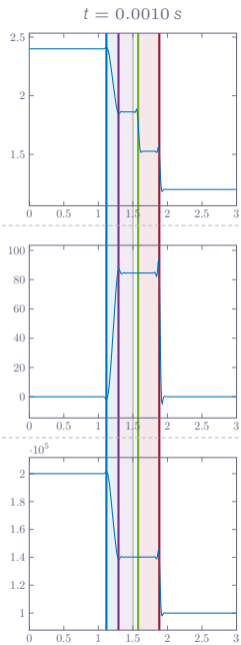
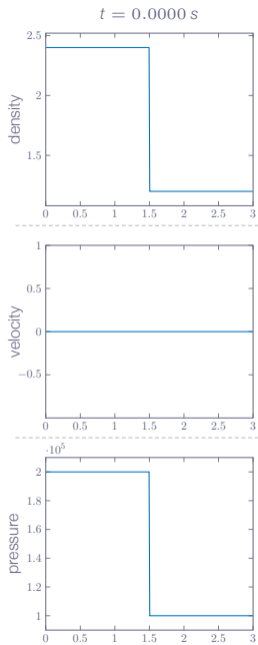


incident shock



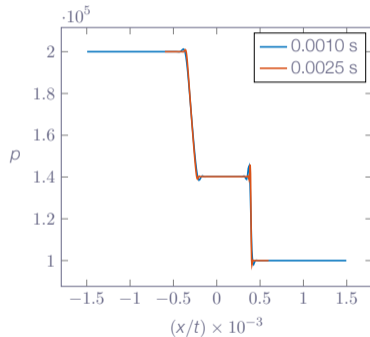
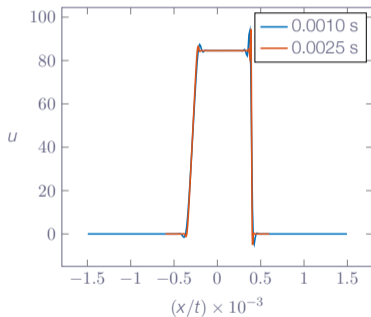
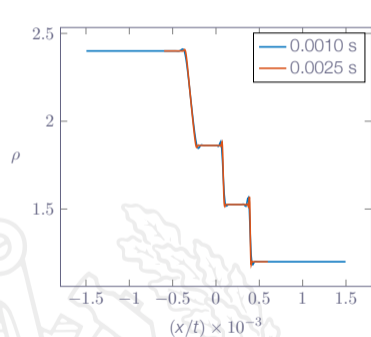


incident shock  
contact discontinuity



incident shock  
contact discontinuity  
expansion wave

# Riemann Problem - Shock Tube



The solution can be made self similar by plotting the flow field variables as function of  $x/t$

# Roadmap - Unsteady Wave Motion

