Compressible Flow - TME085

Lecture 6

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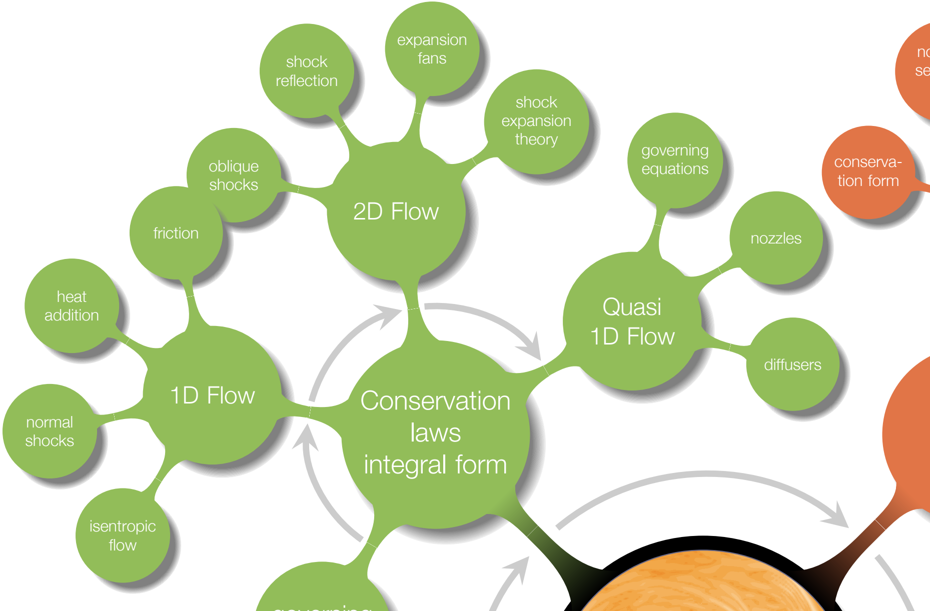
Chapter 4
Oblique Shocks and Expansion Waves

Overview


shock reflection
expansion fans
shock expansion theory
oblique shocks
2D Flow
friction
governing equations
nozzles
Quasi 1D Flow
diffusers
conserva-tion form
heat addition
1D Flow
normal shocks
isentropic flow
Conservation laws integral form

Learning Outcomes

4 Present at least two different formulations of the governing equations for compressible flows and explain what basic conservation principles they are based on

7 Explain why entropy is important for flow discontinuities

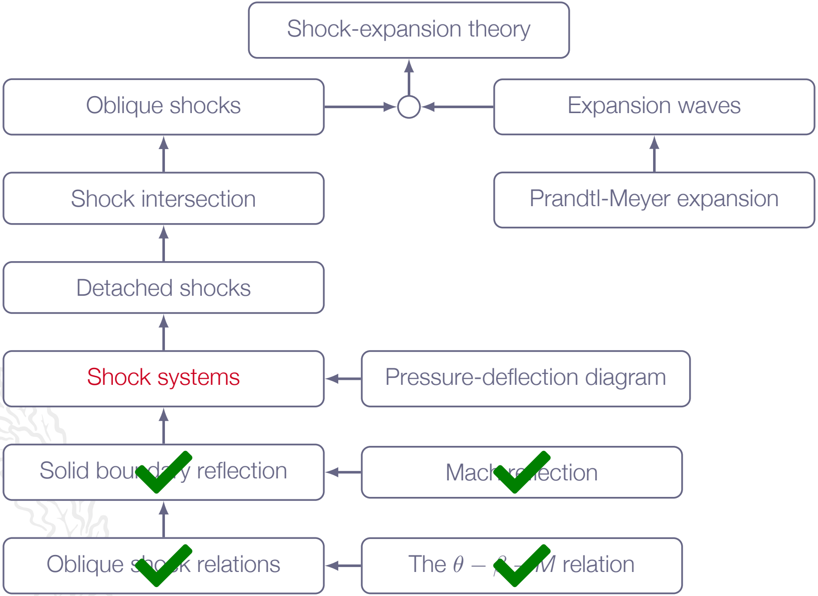
8 Derive (marked) and apply (all) of the presented mathematical formulae for classical gas dynamics

  b normal shocks*
  e oblique shocks in 2D*
  f shock reflection at solid walls*
  g contact discontinuities
  h Prandtl-Meyer expansion fans in 2D
  i detached blunt body shocks, nozzle flows

9 Solve engineering problems involving the above-mentioned phenomena (8a-8k)

why do we get normal shocks in some cases and oblique shocks in other?

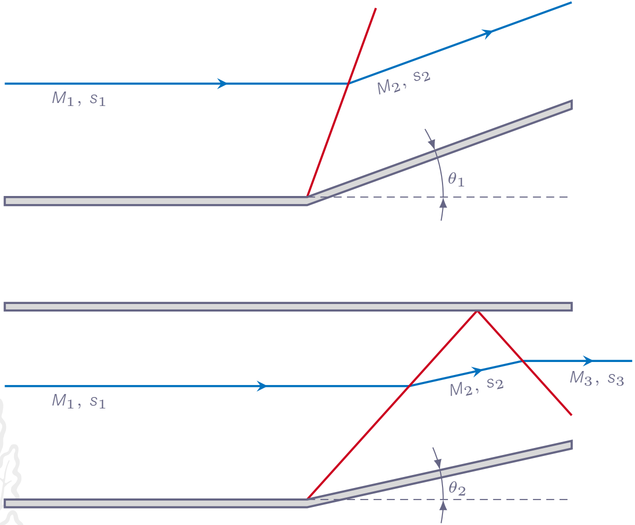
Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.7
Comments on Flow Through Multiple Shock Systems

Flow Through Multiple Shock Systems

Single-shock compression versus multiple-shock compression:



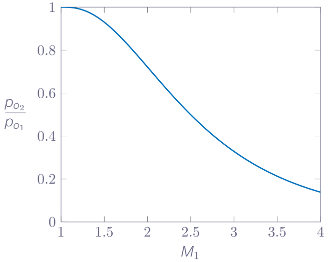
Flow Through Multiple Shock Systems

We may find θ_1 and θ_2 (for same M_1) which gives the same final Mach number

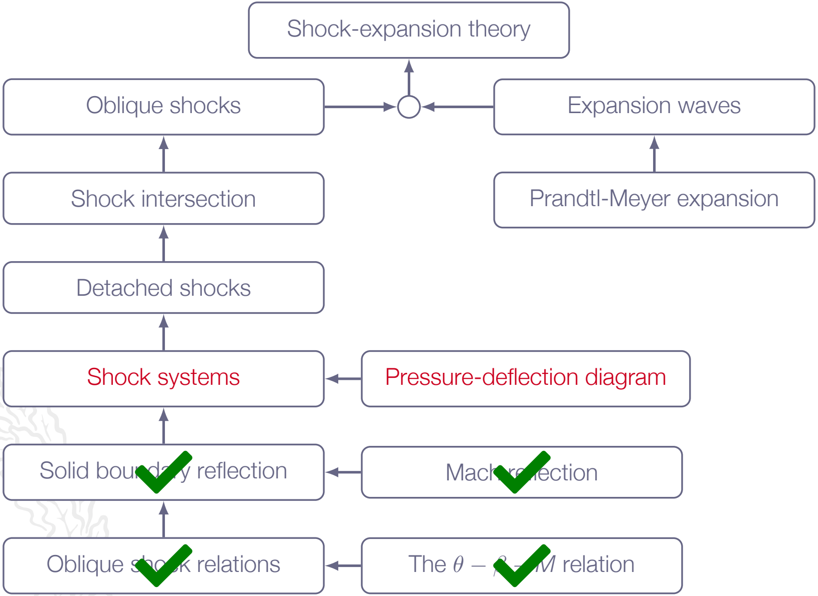
In such cases, the flow with multiple shocks has smaller losses

Explanation: entropy generation at a shock is a very non-linear function of shock strength

Note! the system of multiple shocks might very well result in a larger total flow deflection angle than the single-shock case

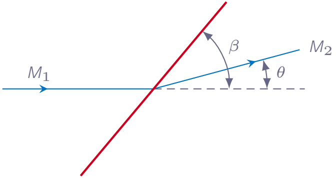


Roadmap - Oblique Shocks and Expansion Waves

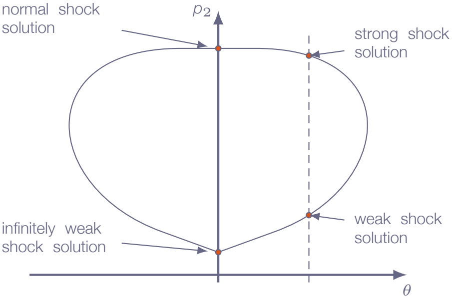


Chapter 4.8
Pressure Deflection Diagrams

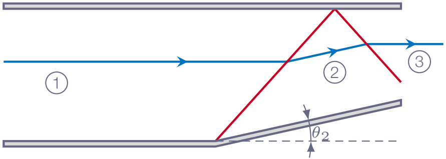
Pressure Deflection Diagrams

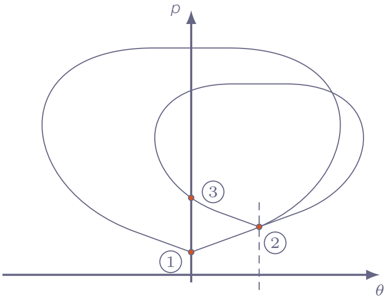


$\Rightarrow$ relation between p_2 and θ

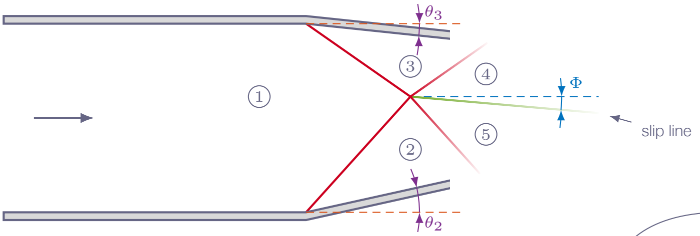


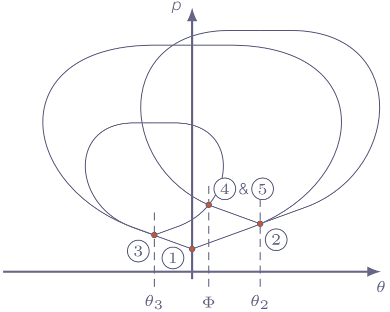
Pressure Deflection Diagrams - Shock Reflection





Pressure Deflection Diagrams - Shock Intersection

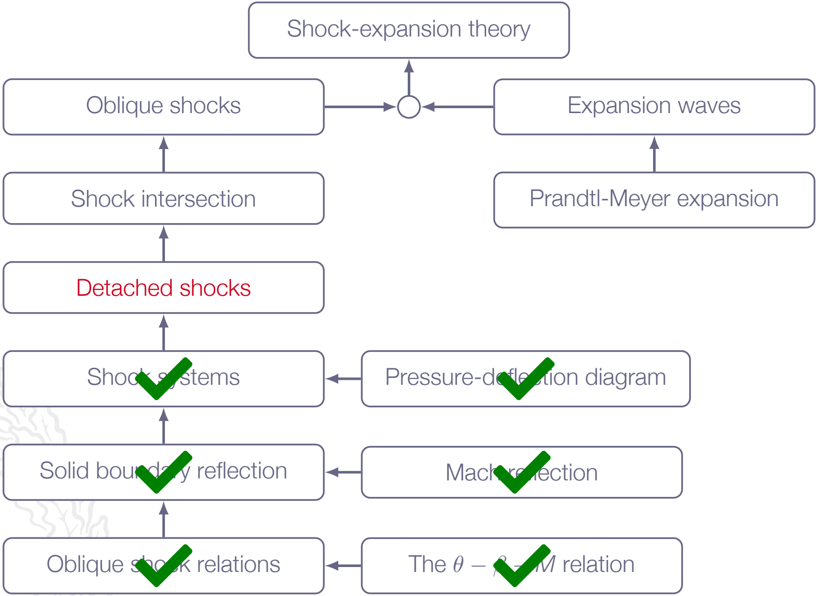




A slip line is a contact discontinuity

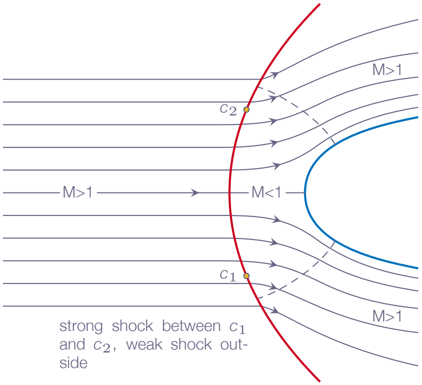
- discontinuity in ρ, T, s, v, and M
- continuous in p and flow angle

Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.12 Detached Shock Wave in Front of a Blunt Body

Detached Shocks

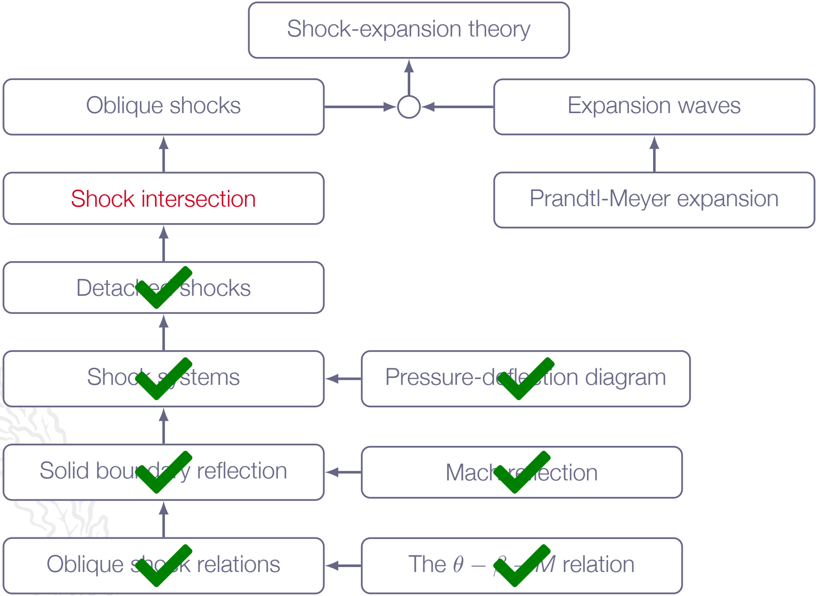


Detached Shocks

As we move along the detached shock form the centerline, the shock will change in nature as

- right in front of the body we will have a normal shock
- strong oblique shock
- weak oblique shock
- far away from the body it will approach a Mach wave, *i.e.* an infinitely weak oblique shock

Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.10
Intersection of Shocks of the Same Family

Mach Waves (*Repetition*)

Oblique shock, angle β, flow deflection θ:

$$M_{n_2}^2 = \frac{M_{n_1}^2 + [2/(\gamma - 1)]}{[2\gamma/(\gamma - 1)]M_{n_1}^2 - 1}$$

where

$$M_{n_1} = M_1 \sin(\beta)$$

and

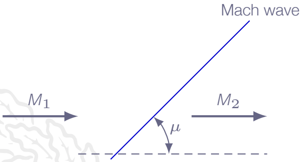
$$M_{n_2} = M_2 \sin(\beta - \theta)$$

Now, let $M_{n_1} \to 1$ and $M_{n_2} \to 1 \Rightarrow$ infinitely weak shock!

Such very weak shocks are called Mach waves

Mach Waves (*Repetition*)

$$M_{n_1} = 1 \Rightarrow M_1 \sin(\beta) = 1 \Rightarrow \beta = \arcsin(1/M_1)$$

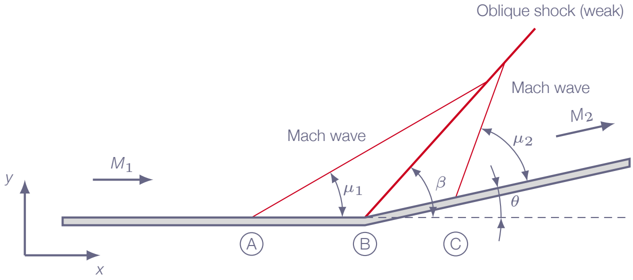


$$M_2 \approx M_1$$

$$\theta \approx 0$$

$$\mu = \arcsin(1/M_1)$$

Mach Waves

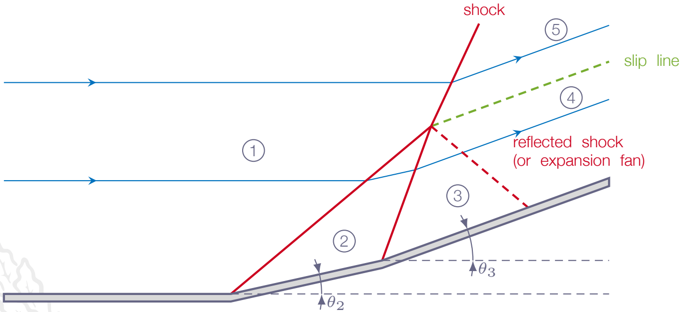


Oblique shock (weak)
Mach wave
M_2
μ_2
Mach wave
M_1
y
μ_1
β
θ
A
B
C
x

Mach Waves

- Mach wave at A: $\sin(\mu_1) = 1/M_1$
- Mach wave at C: $\sin(\mu_2) = 1/M_2$
- Oblique shock at B: $M_{n_1} = M_1 \sin(\beta) \Rightarrow \sin(\beta) = M_{n_1}/M_1$
  - Existence of shock requires $M_{n_1} > 1 \Rightarrow \beta > \mu_1$
  - Mach wave intercepts shock!
- Also, $M_{n_2} = M_2 \sin(\beta - \theta) \Rightarrow \sin(\beta - \theta) = M_{n_2}/M_2$
  - For finite shock strength $M_{n_2} < 1 \Rightarrow (\beta - \theta) < \mu_2$
  - Again, Mach wave intercepts shock

Shock Intersection - Same Family



Shock Intersection - Same Family

Case 1: Streamline going through regions 1, 2, 3, and 4
(through two oblique (weak) shocks)

Case 2: Streamline going through regions 1 and 5
(through one oblique (weak) shock)

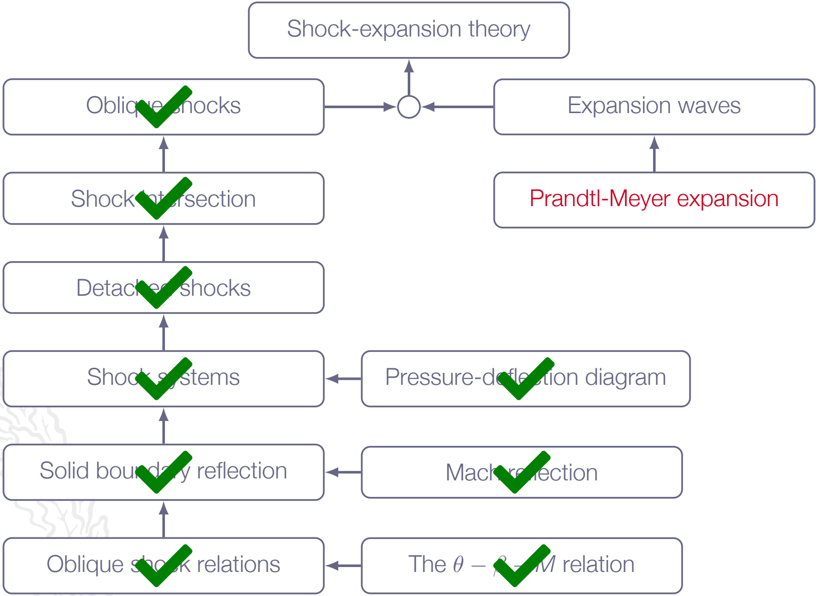
Problem: Find conditions 4 and 5 such that

a. $p_4 = p_5$
b. flow angle in 4 equals flow angle in 5

Solution may give either reflected shock or expansion fan, depending on actual conditions

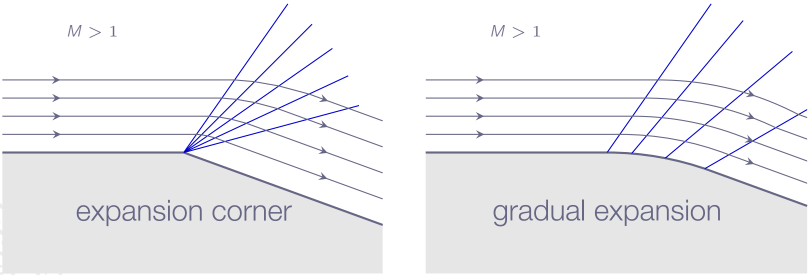
A slip line usually appears, across which there is a discontinuity in all variables except p and flow angle

Roadmap - Oblique Shocks and Expansion Waves



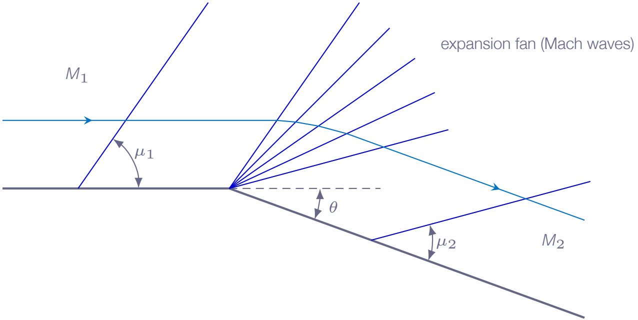
Chapter 4.14
Prandtl-Meyer Expansion Waves

Expansion Waves



Prandtl-Meyer Expansion Waves

An expansion fan is a centered simple wave (also called Prandl-Meyer expansion)



- $M_2 > M_1$ (the flow accelerates through the expansion fan)
- $p_2 < p_1$, $\rho_2 < \rho_1$, $T_2 < T_1$

Prandtl-Meyer Expansion Waves

- Continuous expansion region
- Infinite number of weak Mach waves
- Streamlines through the expansion wave are smooth curved lines
- $ds = 0$ for each Mach wave $\Rightarrow$ the expansion process is ISENTROPIC!

Prandtl-Meyer Expansion Waves

- upstream of expansion $M_1 > 1$, $\sin(\mu_1) = 1/M_1$
- flow accelerates as it curves through the expansion fan
- downstream of expansion $M_2 > M_1$, $\sin(\mu_2) = 1/M_2$
- flow is isentropic $\Rightarrow s$, p_o, T_o, ρ_o, a_o, ... are constant along streamlines
- flow deflection: θ

Prandtl-Meyer Expansion Waves

It can be shown that $d\theta = \sqrt{M^2 - 1}\dfrac{dv}{v}$, where $v = |\mathbf{v}|$ (valid for all gases)

Integration gives

$$\int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \sqrt{M^2 - 1}\frac{dv}{v}$$

the term $\dfrac{dv}{v}$ needs to be expressed in terms of Mach number

$v = Ma \Rightarrow \ln v = \ln M + \ln a \Rightarrow$

$$\frac{dv}{v} = \frac{dM}{M} + \frac{da}{a}$$

Prandtl-Meyer Expansion Waves

Calorically perfect gas and adiabatic flow gives

$$\frac{T_o}{T} = 1 + \frac{1}{2}(\gamma - 1)M^2$$

$$\left\{a = \sqrt{\gamma R T},\ a_o = \sqrt{\gamma R T_o}\right\} \Rightarrow \frac{T_o}{T} = \left(\frac{a_o}{a}\right)^2 \Rightarrow$$

$$\left(\frac{a_o}{a}\right)^2 = 1 + \frac{1}{2}(\gamma - 1)M^2 \Leftrightarrow a = a_o\left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-1/2}$$

Prandtl-Meyer Expansion Waves

Differentiation gives:

$$da = a_o \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-3/2} \left(-\frac{1}{2}\right)(\gamma - 1)MdM$$

or

$$da = a \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-1} \left(-\frac{1}{2}\right)(\gamma - 1)MdM$$

which gives

$$\frac{dv}{v} = \frac{dM}{M} + \frac{da}{a} = \frac{dM}{M} + \frac{-\frac{1}{2}(\gamma - 1)MdM}{1 + \frac{1}{2}(\gamma - 1)M^2} = \frac{1}{1 + \frac{1}{2}(\gamma - 1)M^2}\frac{dM}{M}$$

Prandtl-Meyer Expansion Waves

Thus,

$$\int_{\theta_1}^{\theta_2} d\theta = \theta_2 - \theta_1 = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{1}{2}(\gamma-1)M^2}\frac{dM}{M} = \nu(M_2) - \nu(M_1)$$

where

$$\nu(M) = \int \frac{\sqrt{M^2-1}}{1+\frac{1}{2}(\gamma-1)M^2}\frac{dM}{M}$$

is the so-called Prandtl-Meyer function

Prandtl-Meyer Expansion Waves

Performing the integration gives:

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1}$$

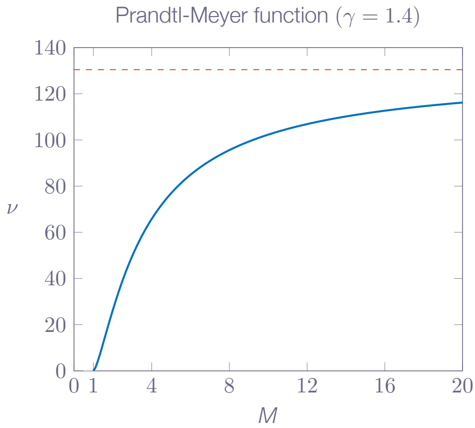
We can now calculate the deflection angle $\Delta\theta$ as:

$$\Delta\theta = \nu(M_2) - \nu(M_1)$$

Prandtl-Meyer Expansion Waves

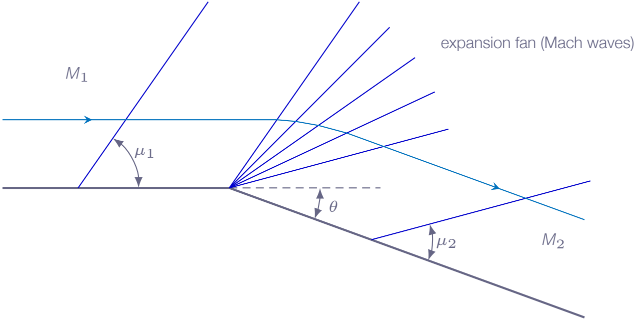
$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1}$$

$$\nu(M)|_{M\to\infty} = 130.45^\circ$$



Prandtl-Meyer Expansion Waves

Example:



- $\theta_1 = 0$, $M_1 > 1$ is given
- θ_2 is given
- problem: find M_2 such that $\theta_2 = \nu(M_2) - \nu(M_1)$
- $\nu(M)$ for $\gamma = 1.4$ can be found in Table A.5

Prandtl-Meyer Expansion Waves

Since the flow is isentropic, the usual isentropic relations apply:

(p_o and T_o are constant)

Calorically perfect gas:

$$\frac{p_o}{p} = \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_o}{T} = \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]$$

Prandtl-Meyer Expansion Waves

since $p_{o_1} = p_{o_2}$ and $T_{o_1} = T_{o_2}$

$$\frac{p_1}{p_2} = \frac{p_{o_2}}{p_{o_1}}\frac{p_1}{p_2} = \left(\frac{p_{o_2}}{p_2}\right) \Big/ \left(\frac{p_{o_1}}{p_1}\right) = \left[\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right]^{\frac{\gamma}{\gamma-1}}$$

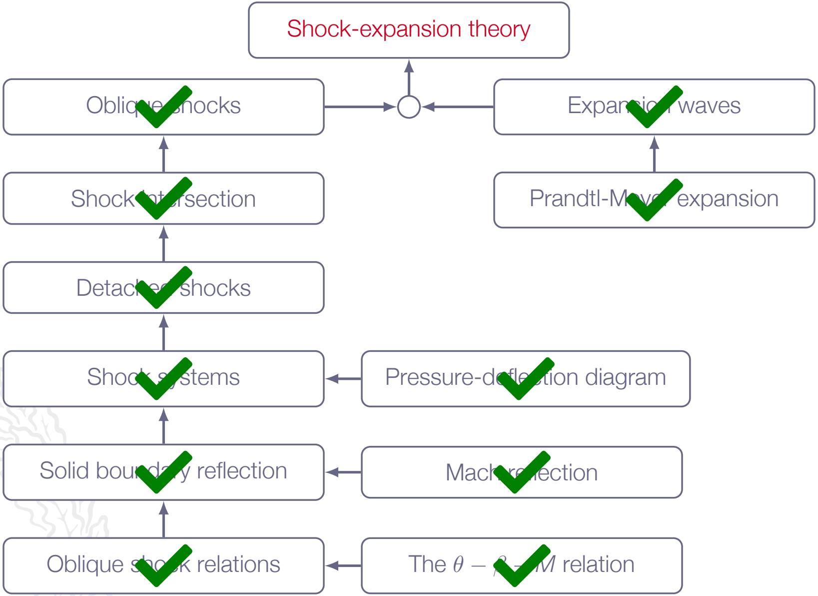
$$\frac{T_1}{T_2} = \frac{T_{o_2}}{T_{o_1}}\frac{T_1}{T_2} = \left(\frac{T_{o_2}}{T_2}\right) \Big/ \left(\frac{T_{o_1}}{T_1}\right) = \left[\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right]$$

Prandtl-Meyer Expansion Waves

Alternative solution:

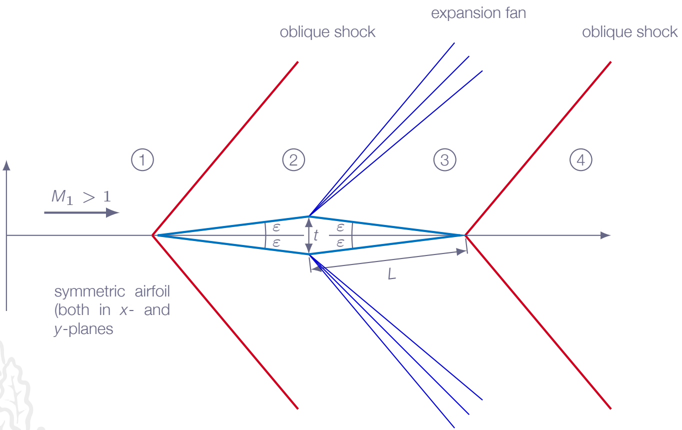
1. determine M_2 from $\theta_2 = \nu(M_2) - \nu(M_1)$
2. compute p_{o_1} and T_{o_1} from p_1, T_1, and M_1 (or use Table A.1)
3. set $p_{o_2} = p_{o_1}$ and $T_{o_2} = T_{o_1}$
4. compute p_2 and T_2 from p_{o_2}, T_{o_2}, and M_2 (or use Table A.1)

Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.15
Shock Expansion Theory

Diamond-Wedge Airfoil



Diamond-Wedge Airfoil

1-2 standard oblique shock calculation for flow deflection angle ε and upstream Mach number M_1

2-3 Prandtl-Meyer expansion for flow deflection angle 2ε and upstream Mach number M_2

3-4 standard oblique shock calculation for flow deflection angle ε and upstream Mach number M_3

Diamond-Wedge Airfoil

- symmetric airfoil
- zero incidence flow (freestream aligned with flow axis)

gives:

- symmetric flow field
- zero lift force on airfoil

Diamond-Wedge Airfoil

Drag force:

$$D = -\oiint_{\partial\Omega} p(\mathbf{n} \cdot \mathbf{e}_x)dS$$

$\partial\Omega$	airfoil surface
p	surface pressure
$\mathbf{n}$	outward facing unit normal vector
$\mathbf{e}_x$	unit vector in x-direction

Diamond-Wedge Airfoil

Since conditions 2 and 3 are constant in their respective regions, we obtain:

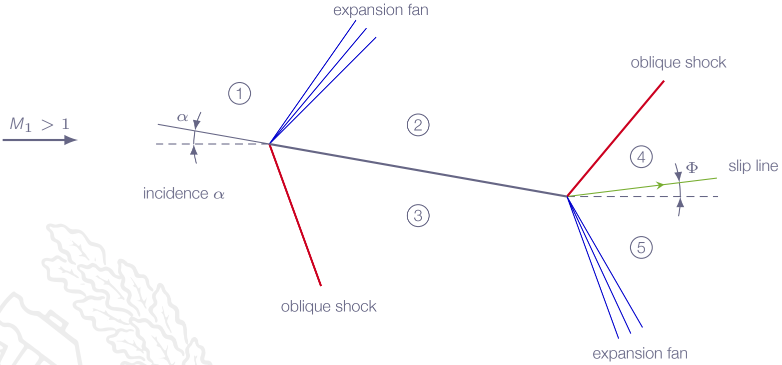
$$D = 2\left[p_2 L \sin(\varepsilon) - p_3 L \sin(\varepsilon)\right] = 2(p_2 - p_3)\frac{t}{2} = (p_2 - p_3)t$$

For supersonic free stream ($M_1 > 1$), with shocks and expansion fans according to figure we will always find that $p_2 > p_3$

which implies $D > 0$

Wave drag (drag due to flow loss at compression shocks)

Flat-Plate Airfoil



Flat-Plate Airfoil

It seems that the angle of the flow downstream of the flat plate would be different than the angle of the flow upstream of the plate. Can that really be correct?!

Flat-Plate Airfoil

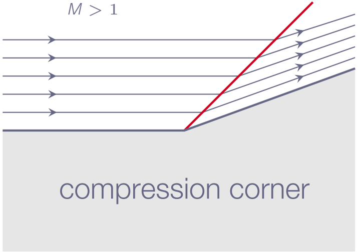
It seems that the angle of the flow downstream of the flat plate would be different than the angle of the flow upstream of the plate. Can that really be correct?!

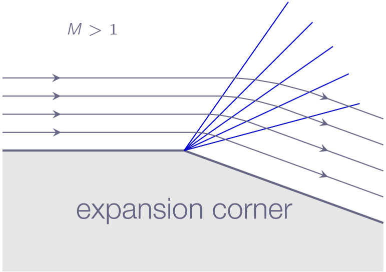
For the flow in the vicinity of the plate this is the correct picture. Further out from the plate, shock and expansion waves will interact and eventually sort the missmatch of flow angles out

Flat-Plate Airfoil

- Flow states 4 and 5 must satisfy:
  - $p_4 = p_5$
  - flow direction 4 equals flow direction 5 (Φ)
- Shock between 2 and 4 as well as expansion fan between 3 and 5 will adjust themselves to comply with the requirements
- For calculation of lift and drag only states 2 and 3 are needed
- States 2 and 3 can be obtained using standard oblique shock formulas and Prandtl-Meyer expansion

Oblique Shocks and Expansion Waves





M	decrease
V	decrease
p	increase
ρ	increase
T	increase

M	increase
V	increase
p	decrease
ρ	decrease
T	decrease

Roadmap - Oblique Shocks and Expansion Waves