Compressible Flow - TME085

Lecture 5

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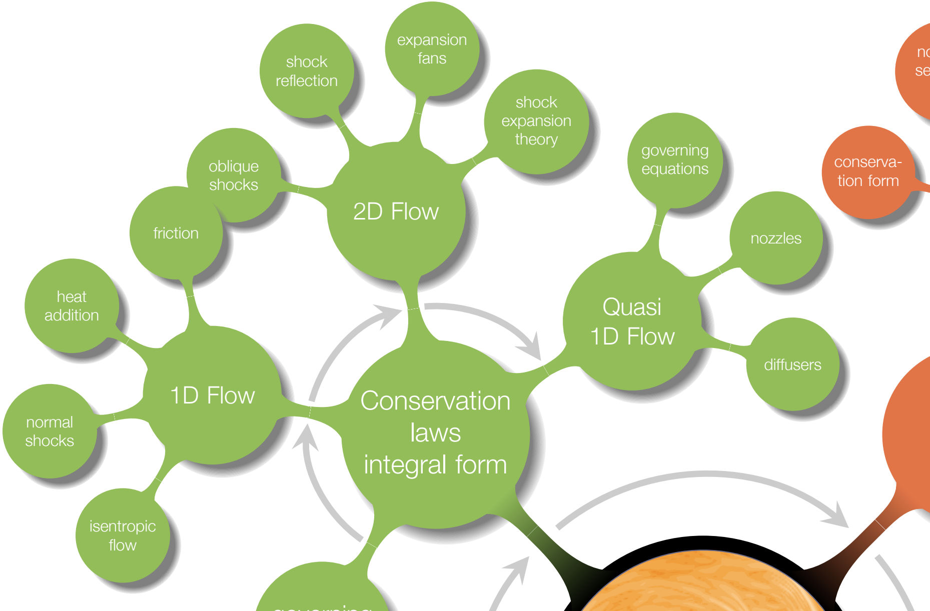
Chapter 4
Oblique Shocks and Expansion Waves

Overview


shock reflection
expansion fans
shock expansion theory
oblique shocks
2D Flow
governing equations
conserva-tion form
friction
nozzles
heat addition
Quasi 1D Flow
diffusers
1D Flow
Conservation laws integral form
normal shocks
isentropic flow

Learning Outcomes

4 Present at least two different formulations of the governing equations for compressible flows and explain what basic conservation principles they are based on

7 Explain why entropy is important for flow discontinuities

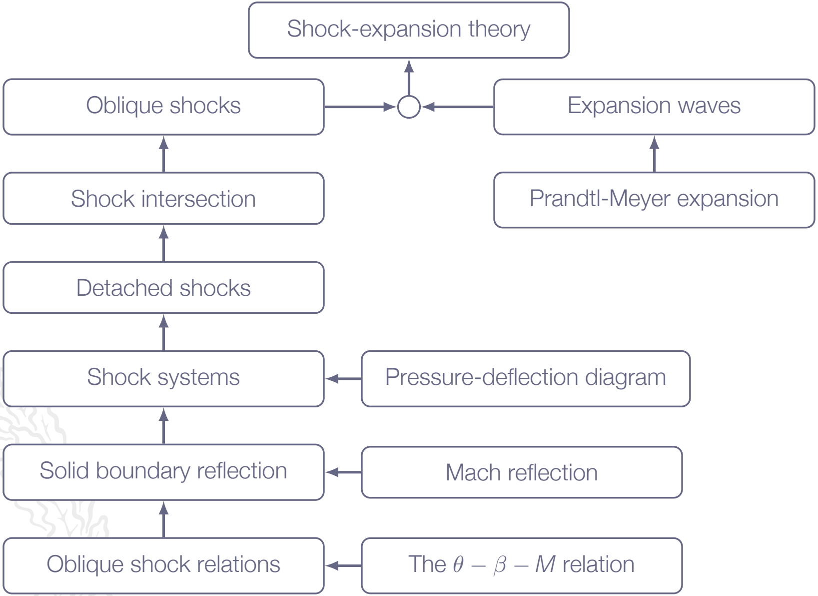
8 Derive (marked) and apply (all) of the presented mathematical formulae for classical gas dynamics

  b normal shocks*
  e oblique shocks in 2D*
  f shock reflection at solid walls*
  g contact discontinuities
  h Prandtl-Meyer expansion fans in 2D
  i detached blunt body shocks, nozzle flows

9 Solve engineering problems involving the above-mentioned phenomena (8a-8k)

why do we get normal shocks in some cases and oblique shocks in other?

Roadmap - Oblique Shocks and Expansion Waves



Motivation

Come on, two-dimensional flow, really?! Why not three-dimensional?

the normal shocks studied in chapter 3 are a special casees of the more general oblique shock waves that may be studied in two dimensions

in two dimensions, we can still analyze shock waves using a pen-and-paper approach

many practical problems or subsets of problems may be analyzed in two-dimensions

by going from one to two dimensions we will be able to introduce physical processes important for compressible flows

Oblique Shocks and Expansion Waves - Assumptions

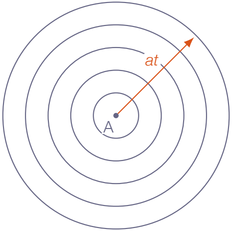
1. Supersonic
2. Steady-state
3. Two-dimensional
4. Inviscid flow (no wall friction)

In real flow, viscosity is non-zero $\Rightarrow$ boundary layers

For high-Reynolds-number flows, boundary layers are thin $\Rightarrow$ inviscid theory still relevant!

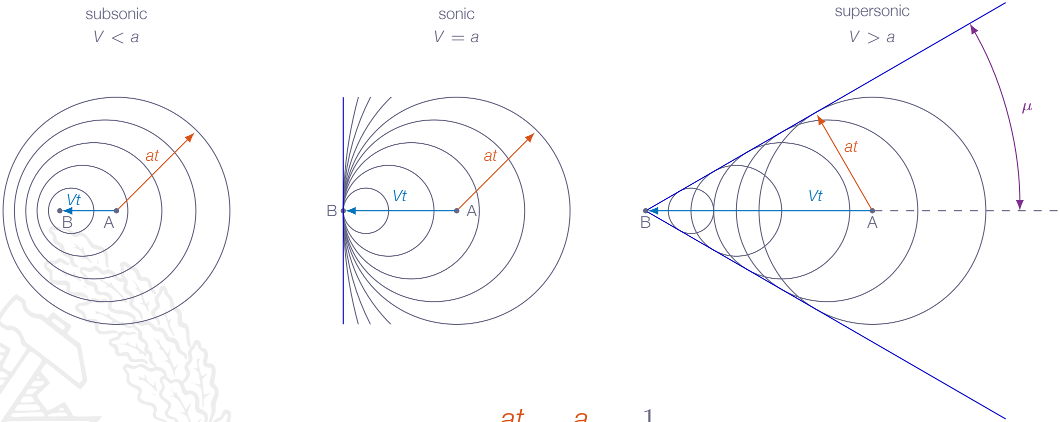
Mach Wave

Sound waves emitted from A (speed of sound a)



Mach Waves

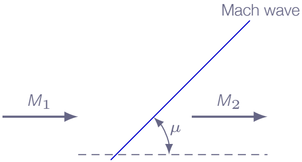
A Mach wave is an infinitely weak oblique shock



$$\sin\mu = \frac{at}{Vt} = \frac{a}{V} = \frac{1}{M}$$

Mach Wave

A Mach wave is an infinitely weak oblique shock

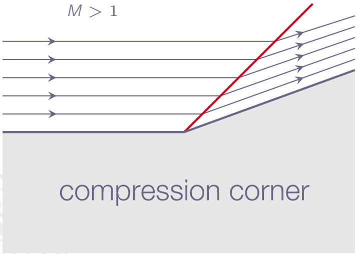


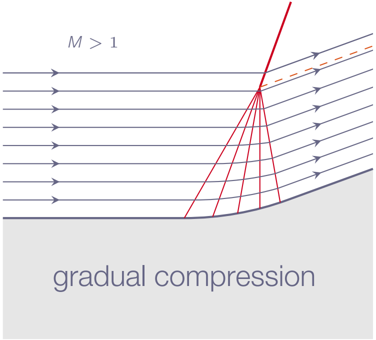
No substantial changes of flow properties over a single Mach wave

$M_1 > 1.0$ and $M_1 \approx M_2$

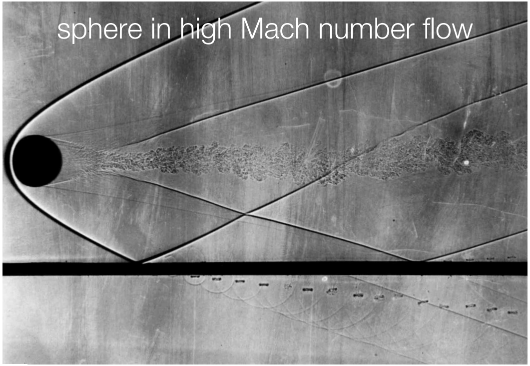
Isentropic

Oblique Shocks

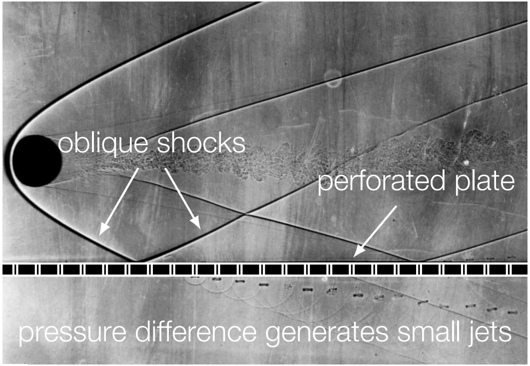




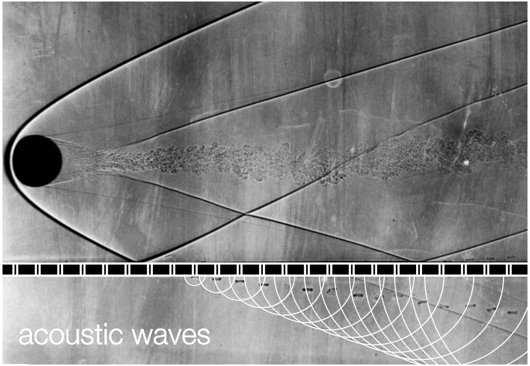
Oblique Shocks and Mach Waves



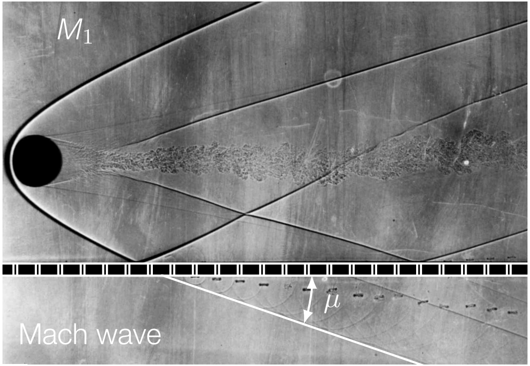
Oblique Shocks and Mach Waves



Oblique Shocks and Mach Waves

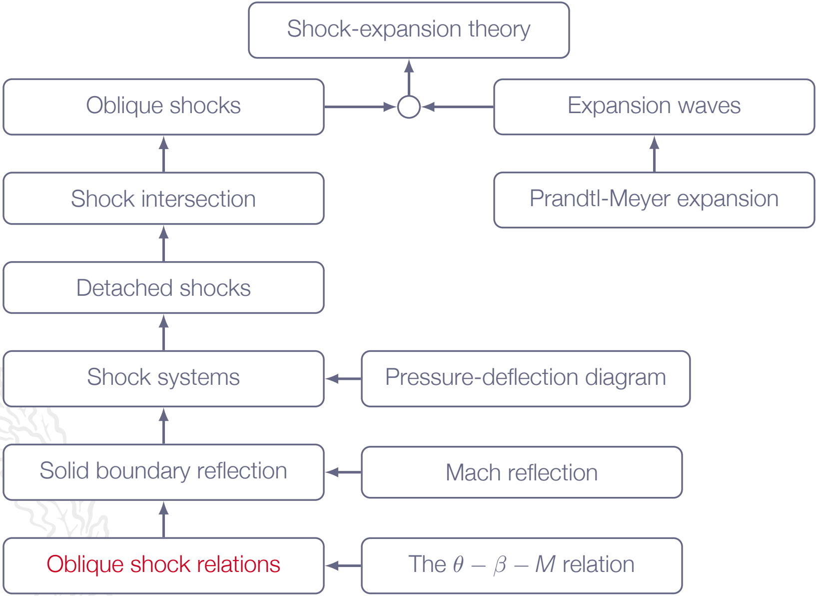


Oblique Shocks and Mach Waves



$$\mu = 19^\circ \Rightarrow M_1 = \frac{1}{\sin\mu} \approx 3.1$$

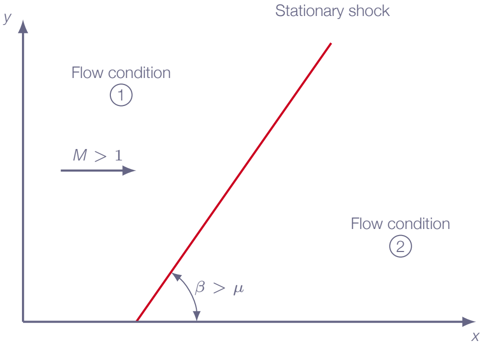
Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.3
Oblique Shock Relations

Oblique Shocks

Two-dimensional steady-state flow



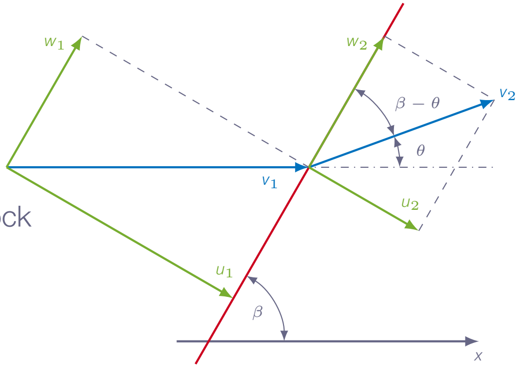
Significant changes of flow properties from 1 to 2

$M_1 > 1.0$, $\beta > \mu$, and $M_1 \neq M_2$

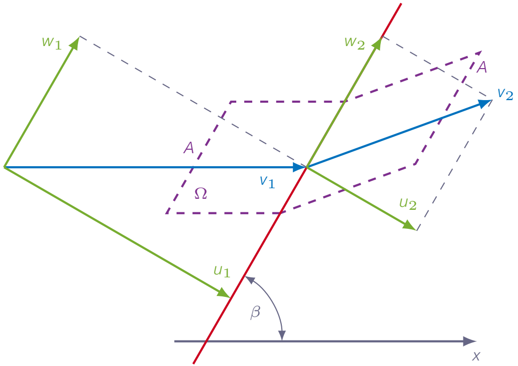
Not isentropic

Oblique Shocks

Stationary oblique shock



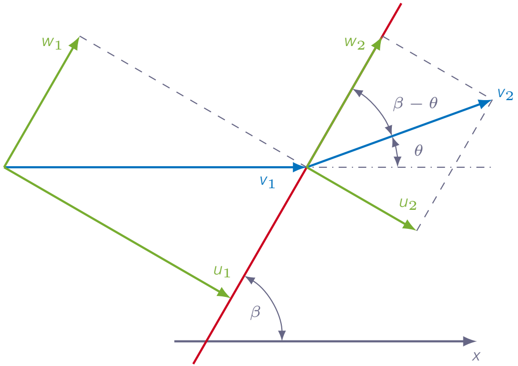
Oblique Shock Relations



Two-dimensional steady-state flow

Control volume aligned with flow stream lines

Oblique Shock Relations



Velocity notations:

$$M_{n_1} = \frac{u_1}{a_1} = M_1 \sin(\beta) \qquad M_1 = \frac{v_1}{a_1}$$

$$M_{n_2} = \frac{u_2}{a_2} = M_2 \sin(\beta - \theta) \qquad M_2 = \frac{v_2}{a_2}$$

Oblique Shock Relations

Conservation of mass:

$$\frac{d}{dt}\iiint_{\Omega} \rho d\mathscr{V} + \oiint_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS = 0$$

Mass conservation for control volume Ω:

$$0 - \rho_1 u_1 A + \rho_2 u_2 A = 0 \Rightarrow$$

$$\boxed{\rho_1 u_1 = \rho_2 u_2}$$

Oblique Shock Relations

Conservation of momentum:

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho\mathbf{v}d\mathscr{V}+\oiint\limits_{\partial\Omega}\left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}+p\mathbf{n}\right]dS=\iiint\limits_{\Omega}\rho\mathbf{f}d\mathscr{V}$$

Momentum in shock-normal direction:

$$0-(\rho_1u_1^2+p_1)A+(\rho_2u_2^2+p_2)A=0\Rightarrow$$

$$\boxed{\rho_1u_1^2+p_1=\rho_2u_2^2+p_2}$$

Oblique Shock Relations

Momentum in shock-tangential direction:

$$0 - \rho_1 u_1 w_1 A + \rho_2 u_2 w_2 A = 0 \Rightarrow$$

$$\boxed{w_1 = w_2}$$

Oblique Shock Relations

Conservation of energy:

$$\frac{d}{dt}\iiint_{\Omega} \rho e_o d\mathscr{V} + \oiint_{\partial\Omega} [\rho h_o \mathbf{v}\cdot\mathbf{n}]\, dS = \iiint_{\Omega} \rho \mathbf{f}\cdot\mathbf{v} d\mathscr{V}$$

Energy equation applied to the control volume Ω:

$$0 - \rho_1 u_1 [h_1 + \frac{1}{2}(u_1^2 + w_1^2)]A + \rho_2 u_2 [h_2 + \frac{1}{2}(u_2^2 + w_2^2)]A = 0 \Rightarrow$$

$$\boxed{h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2}$$

Oblique Shock Relations

We can use the same equations as for normal shocks if we replace M_1 with M_{n_1} and M_2 with M_{n_2}

$$M_{n_2}^2 = \frac{M_{n_1}^2 + [2/(\gamma - 1)]}{[2\gamma/(\gamma - 1)]M_{n_1}^2 - 1}$$

Ratios such as ρ_2/ρ_1, p_2/p_1, and T_2/T_1 can be calculated using the relations for normal shocks with M_1 replaced by M_{n_1}

Oblique Shock Relations

What about ratios involving stagnation flow properties, can we use the ones previously derived for normal shocks?

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The shock process is adiabatic and thus total temperature is not effected by the shock $\Rightarrow T_{o_2} = T_{o_1}$

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What about the total pressure?

Oblique Shock Relations

What about ratios involving stagnation flow properties, can we use the ones previously derived for normal shocks?

The shock process is adiabatic and thus total temperature is not effected by the shock $\Rightarrow T_{o_2} = T_{o_1}$

What about the total pressure?

$$s_2 - s_1 = C_p \ln\left(\frac{T_{o_2}}{T_{o_1}}\right) - R\ln\left(\frac{p_{o_2}}{p_{o_1}}\right) = \{T_{o_2} = T_{o_1}\} = -R\ln\left(\frac{p_{o_2}}{p_{o_1}}\right)$$

entropy is a thermodynamic flow property and $s_2 - s_1$ is dictated by the shock strength and thus the total pressure ratio is a function of the shock-normal Mach number

Oblique Shock Relations

Note! total pressure is always calculated using the flow Mach number, not the shock-normal Mach number

However, the ratio p_{o_2}/p_{o_1} may be calculated using the shock-normal Mach number

So, be careful when using relations derived for normal shocks for oblique shocks when it comes to total flow conditions...

Oblique Shock Relations

p_{o_2}/p_{o_1} is calculated as: $\dfrac{p_{o_2}}{p_{o_1}} = \dfrac{p_{o_2}}{p_2}\dfrac{p_2}{p_1}\dfrac{p_1}{p_{o_1}}$

where

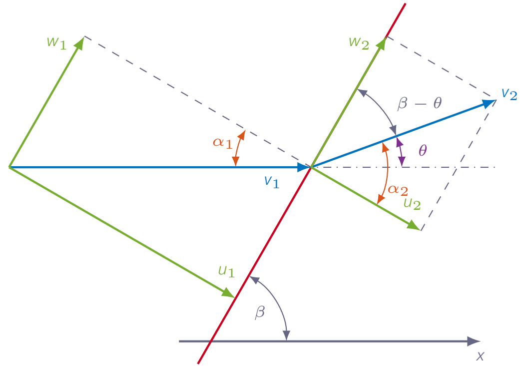
1. $\dfrac{p_{o_2}}{p_2} = f(M_2)$, $\dfrac{p_2}{p_1} = f(M_{n_1})$, and $\dfrac{p_1}{p_{o_1}} = f(M_1)$

or alternatively

2. $\dfrac{p_{o_2}}{p_2} = f(M_{n_2})$, $\dfrac{p_2}{p_1} = f(M_{n_1})$, and $\dfrac{p_1}{p_{o_1}} = f(M_{n_1})$

Note! in the second case the total pressures are **not** the true total pressures of the flow and therefore it is suggested to use the first approach

Deflection Angle (*for the interested*)



$$\theta = \alpha_2 - \alpha_1 = \tan^{-1}\left(\frac{w}{u_2}\right) - \tan^{-1}\left(\frac{w}{u_1}\right)$$

$$\frac{\partial \theta}{\partial w} = \frac{u_2}{w^2 + u_2^2} - \frac{u_1}{w^2 + u_1^2}$$

Deflection Angle (*for the interested*)

$$\frac{\partial \theta}{\partial w} = \frac{u_2}{w^2 + u_2^2} - \frac{u_1}{w^2 + u_1^2} = 0 \Rightarrow$$

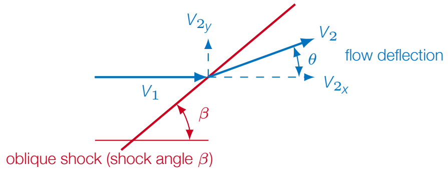
$$\frac{u_2(w^2 + u_1^2) - u_1(w^2 + u_2^2)}{(w^2 + u_2^2)(w^2 + u_1^2)} = 0 \Rightarrow \frac{(u_2 - u_1)(w^2 - u_1 u_2)}{(w^2 + u_2^2)(w^2 + u_1^2)} = 0$$

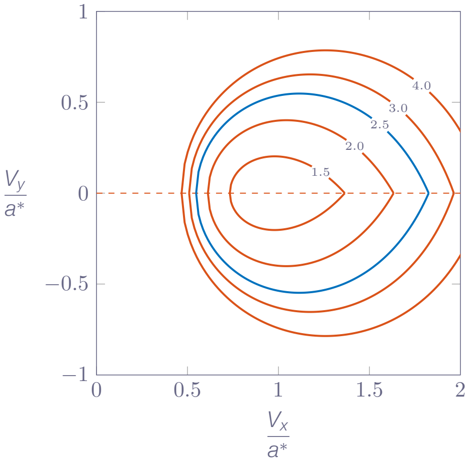
Two solutions:

- $u_2 = u_1$ (no deflection)
- $w^2 = u_1 u_2$ (max deflection)

Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number





No deflection cases:

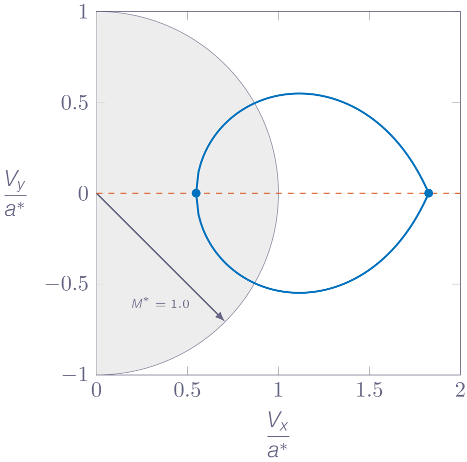
- normal shock
  (reduced shock-normal velocity)
- Mach wave
  (unchanged shock-normal velocity)

Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

$$M^* = \frac{\sqrt{V_x^2 + V_y^2}}{a^*}$$

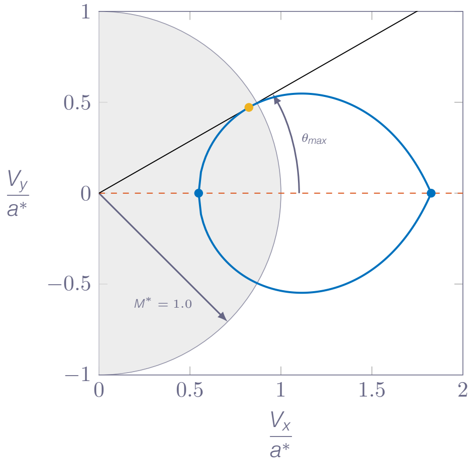
Solutions to the left of the sonic line are subsonic



Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

It is not possible to deflect the flow more than θ_{max}



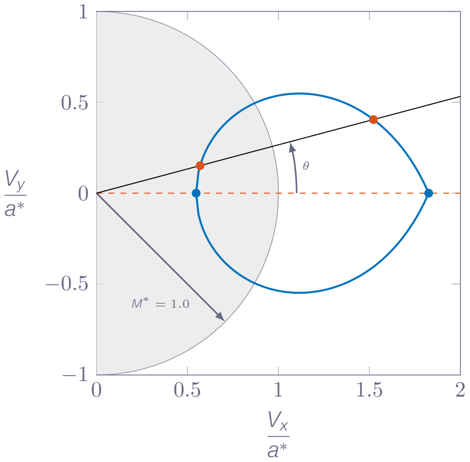
Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

For each deflection angle $\theta < \theta_{max}$, there are two solutions

- strong shock solution
- weak shock solution

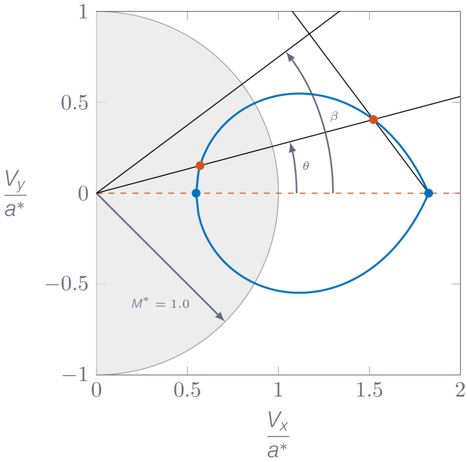
Weak shocks give lower losses and therefore the preferred solution



Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

The shock polar can be used to calculate the shock angle β for a given deflection angle θ



Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

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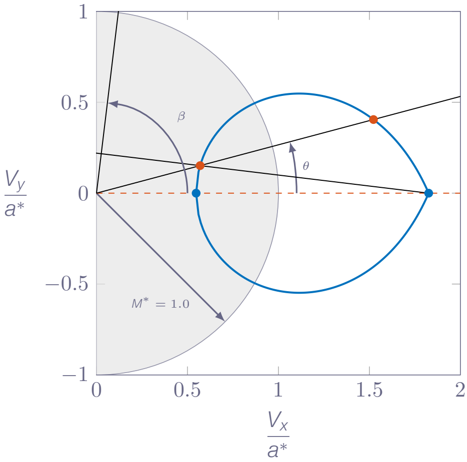
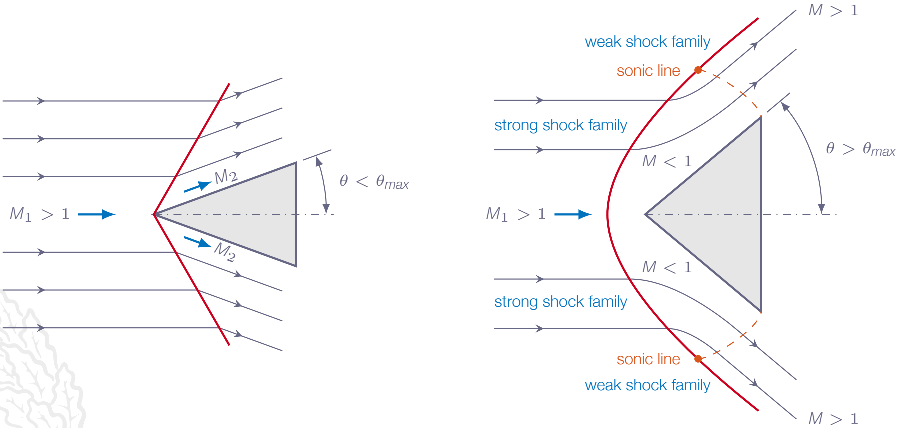
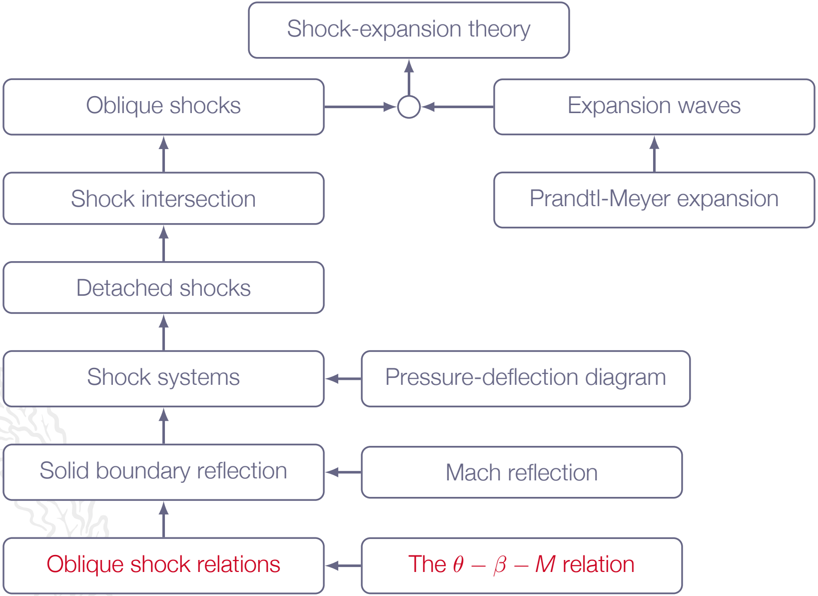


Flow Deflection



Roadmap - Oblique Shocks and Expansion Waves



The θ-β-M Relation

It can be shown that

$$\tan\theta = 2\cot\beta\left(\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}\right)$$

which is the θ-β-M relation

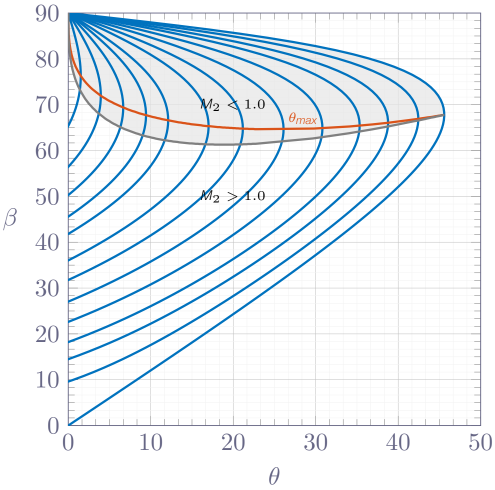
The θ-β-Mach Relation

A relation between:

- flow deflection angle θ
- shock angle β
- upstream flow Mach number M_1

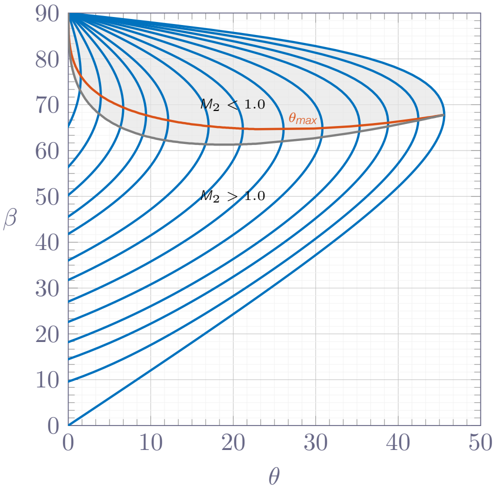
$$\tan(\theta) = 2\cot(\beta)\left(\frac{M_1^2\sin^2(\beta)-1}{M_1^2(\gamma+\cos(2\beta))+2}\right)$$

Note! in general there are two solutions for a given M_1 (or none)



The θ-β-Mach Relation

- There is a small region where we may find weak shock solutions for which $M_2 < 1$
- In most cases weak shock solutions have $M_2 > 1$
- Strong shock solutions always have $M_2 < 1$
- In practical situations, weak shock solutions are most common
- Strong shock solution may appear in special situations due to high back pressure, which forces $M_2 < 1$

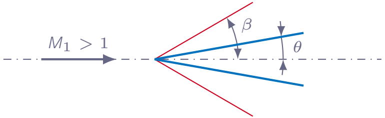


The θ-β-M Relation

Note! In Chapter 3 we learned that the Mach number always reduces to subsonic values behind a shock. This is true for normal shocks (shocks that are normal to the flow direction). It is also true for oblique shocks if looking in the shock-normal direction.

The θ-β-M Relation - Wedge Flow

Wedge flow oblique shock analysis:

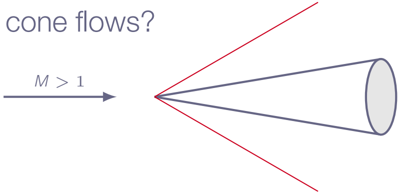


1. θ-β-M relation $\Rightarrow$ β for given M_1 and θ
2. β gives M_{n_1} according to: $M_{n_1} = M_1 \sin(\beta)$
3. normal shock formula with M_{n_1} instead of M_1 $\Rightarrow$ M_{n_2} (instead of M_2)
4. M_2 given by $M_2 = M_{n_2} / \sin(\beta - \theta)$
5. normal shock formula with M_{n_1} instead of M_1 $\Rightarrow$ ρ_2/ρ_1, p_2/p_1, etc
6. upstream conditions + ρ_2/ρ_1, p_2/p_1, etc $\Rightarrow$ downstream conditions

Chapter 4.4
Supersonic Flow over Wedges and Cones

Supersonic Flow over Wedges and Cones

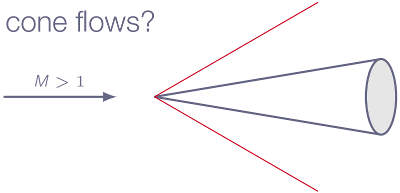
What about cone flows?



- Similar to wedge flow, we do get a constant-strength shock wave, attached at the cone tip (or else a detached curved shock)
- The attached shock is also cone-shaped

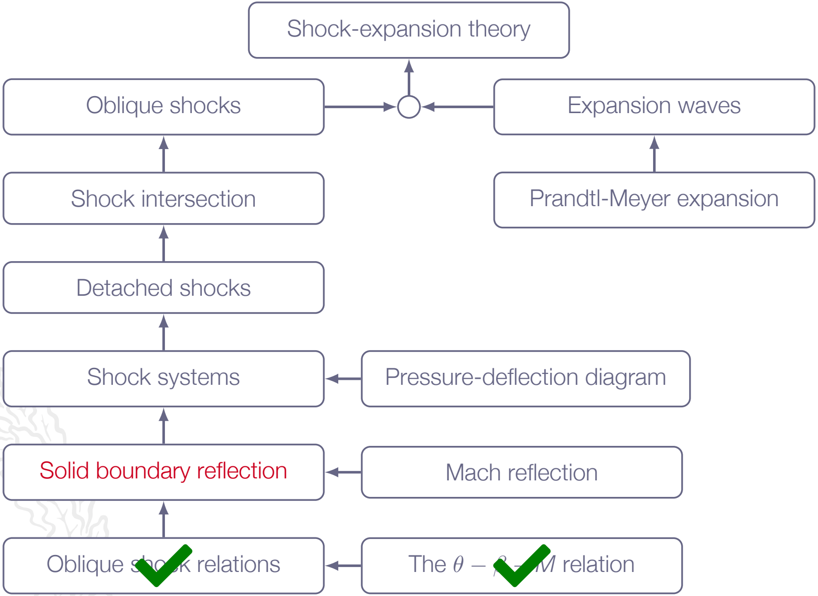
Supersonic Flow over Wedges and Cones

What about cone flows?



- The flow condition immediately downstream of the shock is uniform
- However, downstream of the shock the streamlines are curved and the flow varies in a more complex manner (3D relieving effect - as R increases there is more and more space around cone for the flow)
- β for cone shock is always smaller than that for wedge shock, if M_1 is the same

Roadmap - Oblique Shocks and Expansion Waves

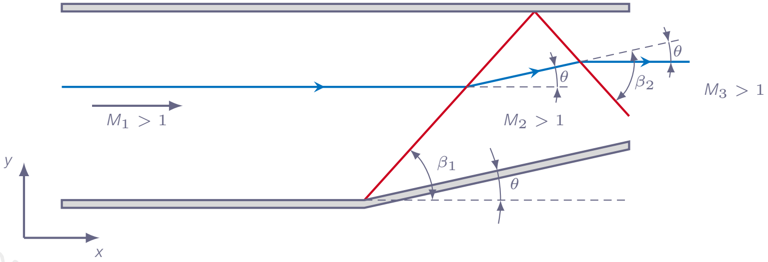


Chapter 4.6
Regular Reflection from a Solid Boundary

Shock Reflection

Regular reflection of oblique shock at solid wall

(see example 4.10)



Assumptions:

- steady-state inviscid flow
- weak shocks

Shock Reflection

first shock:

- upstream condition:

  $M_1 > 1$, flow in x-direction

- downstream condition:

  weak shock $\Rightarrow M_2 > 1$
  deflection angle θ
  shock angle β_1

second shock:

- upstream condition:

  same as downstream condition of first shock

- downstream condition:

  weak shock $\Rightarrow M_3 > 1$
  deflection angle θ
  shock angle β_2

Shock Reflection

Solution:

first shock:

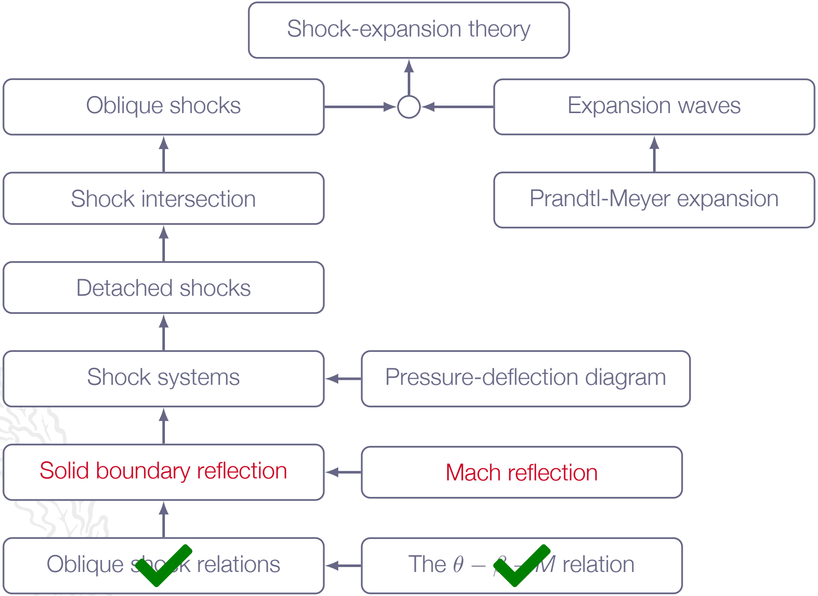
- β_1 calculated from θ-β-M relation for specified θ and M_1 (*weak solution*)
- flow condition 2 according to formulas for normal shocks ($M_{n_1} = M_1 \sin(\beta_1)$ and $M_{n_2} = M_2 \sin(\beta_1 - \theta)$)

second shock:

- β_2 calculated from θ-β-M relation for specified θ and M_2 (*weak solution*)
- flow condition 3 according to formulas for normal shocks ($M_{n_2} = M_2 \sin(\beta_2)$ and $M_{n_3} = M_3 \sin(\beta_2 - \theta)$)

$\Rightarrow$ complete description of flow and shock waves
(angle between upper wall and second shock: $\Phi = \beta_2 - \theta$)

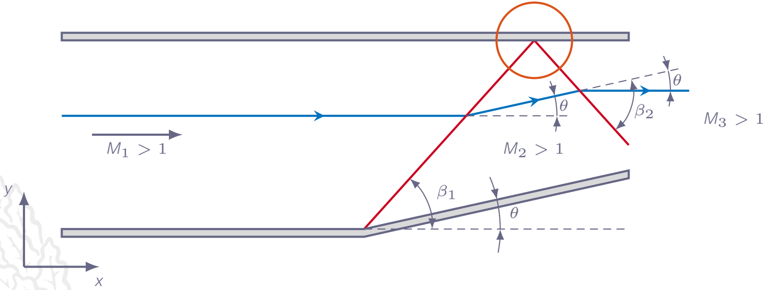
Roadmap - Oblique Shocks and Expansion Waves



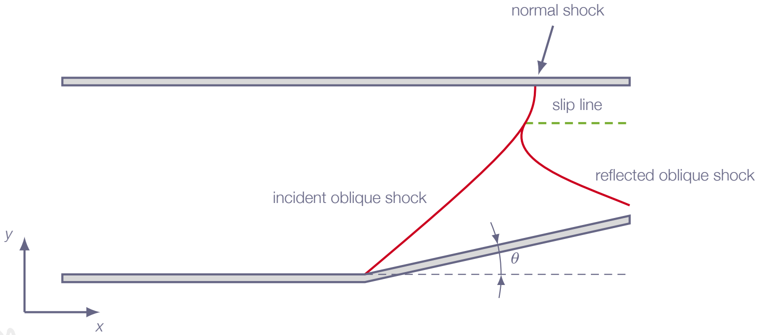
Chapter 4.11
Mach Reflection

Regular Shock Reflection

Regular reflection possible if both primary and reflected shocks are weak (see θ-β-M relation)



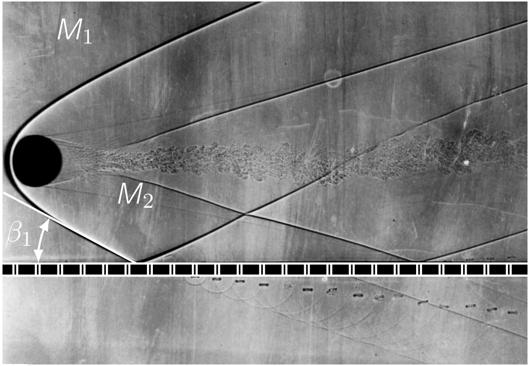
Mach Reflection



Mach reflection:

- appears when regular reflection is not possible
- more complex flow than for a regular reflection
- no analytic solution - numerical solution necessary

Oblique Shocks and Mach Waves

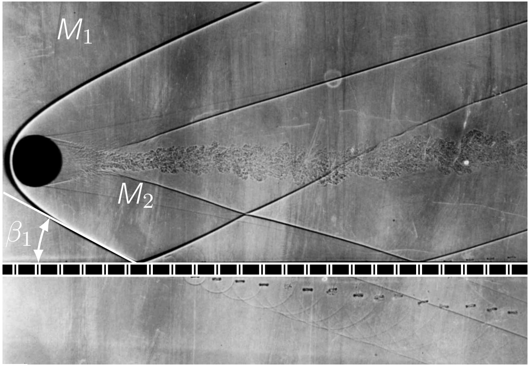


$M_1 > M_2$

$M_2 > 1.0$

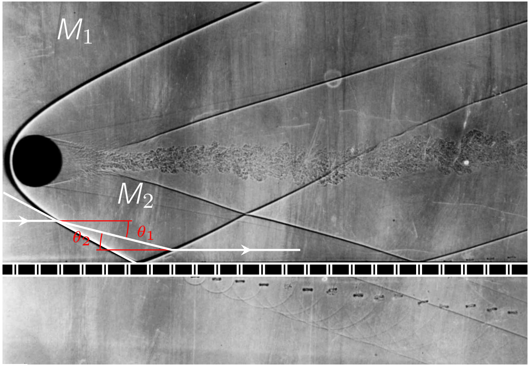
$$\theta_1 = f(M_1, \beta_1), \quad M_2 = f(M_1, \theta_1, \beta_1)$$

Oblique Shocks and Mach Waves



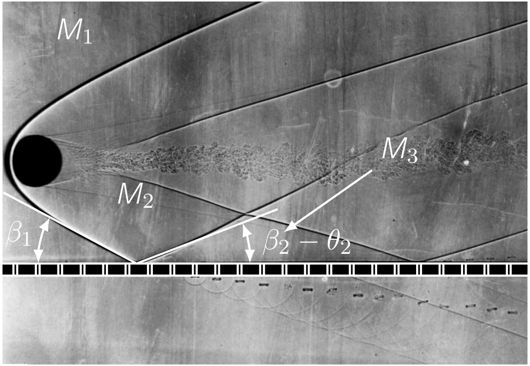
$$\left.\begin{array}{r}\beta_1 = 28^\circ \\ M_1 = 3.1\end{array}\right\} \Rightarrow \theta_1 \approx 11.2^\circ, \quad M_2 \approx 2.5$$

Oblique Shocks and Mach Waves



$$\theta_1 = \theta_2$$

Oblique Shocks and Mach Waves



$M_1 > M_2 > M_3$

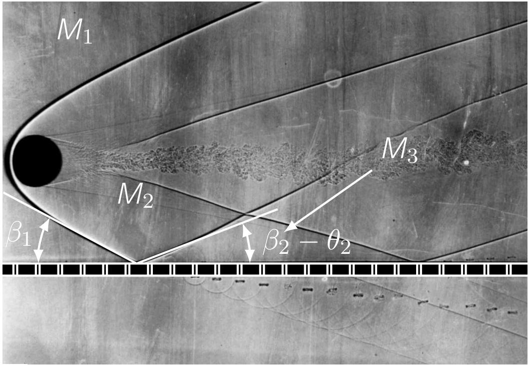
$M_3 > 1.0$

$\beta_2 > \beta_1$

$$\beta_2 = f(M_2, \theta_2), \quad M_3 = f(M_2, \theta_2, \beta_2)$$

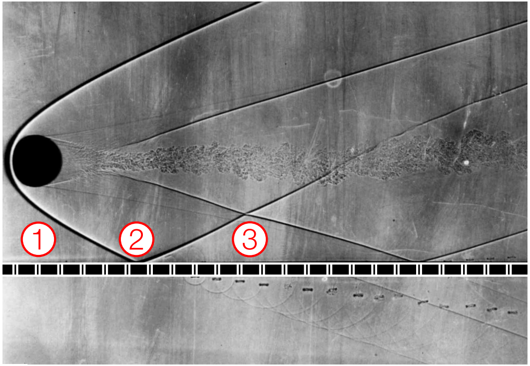
Note! Shock wave reflection at solid wall is **not** specular

Oblique Shocks and Mach Waves



$$\left.\begin{aligned} \theta_2 &= 11.2^\circ \\ M_2 &= 2.5 \end{aligned}\right\} \Rightarrow \beta_2 \approx 33^\circ, \quad M_3 \approx 2.0$$

Oblique Shocks and Mach Waves



$$\frac{p_3}{p_1} = \frac{p_2}{p_1}\frac{p_3}{p_2} \approx 4.52$$

$$\frac{T_3}{T_1} = \frac{T_2}{T_1}\frac{T_3}{T_2} \approx 1.57$$