Compressible Flow - TME085

Lecture 4

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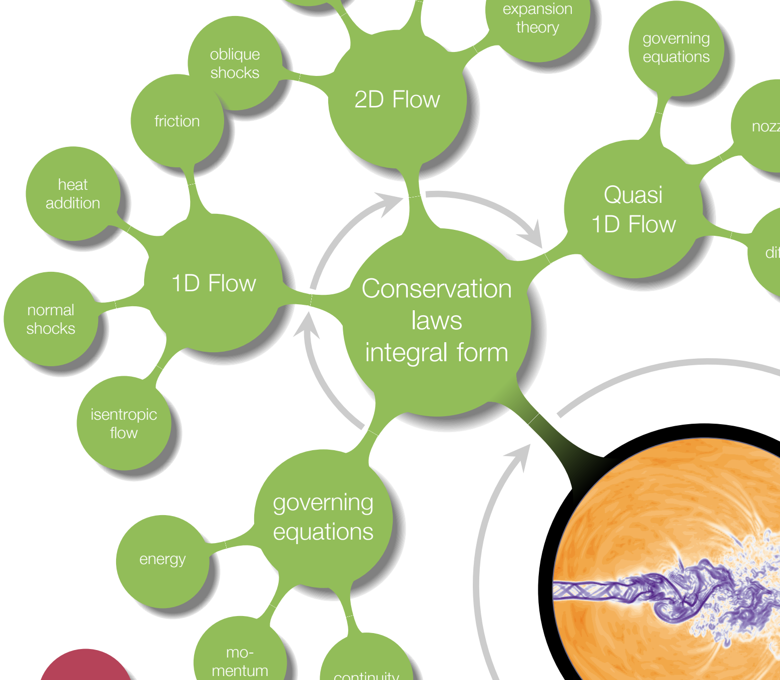
Chapter 3
One-Dimensional Flow

Overview



Conservation laws integral form
1D Flow
heat addition
normal shocks
isentropic flow
friction
2D Flow
oblique shocks
expansion theory
Quasi 1D Flow
governing equations
governing equations
energy
mo-
mentum
continuity

Learning Outcomes

4 Present at least two different formulations of the governing equations for compressible flows and explain what basic conservation principles they are based on

5 Explain how thermodynamic relations enter into the flow equations

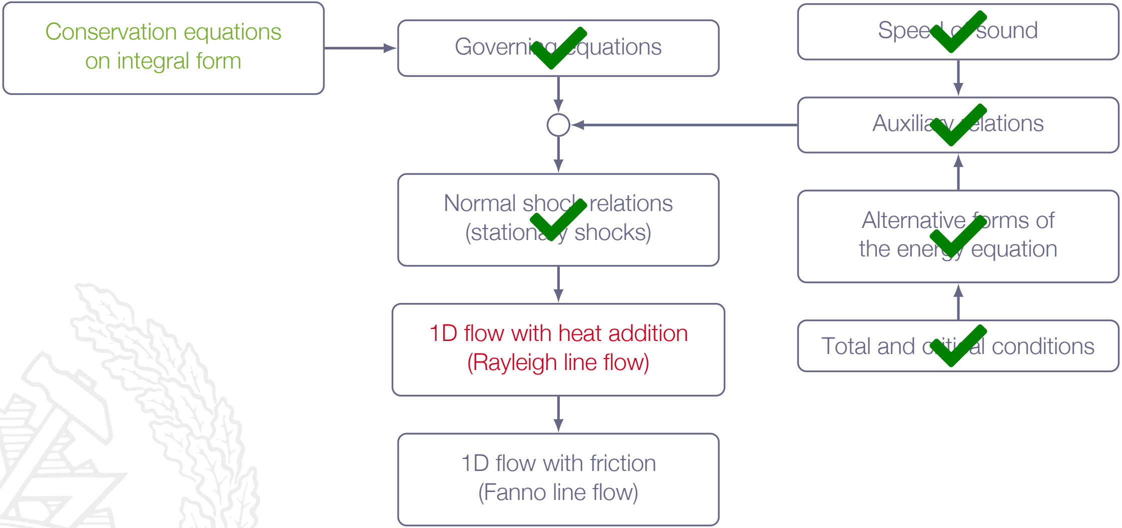
6 Define the special cases of calorically perfect gas, thermally perfect gas and real gas and explain the implication of each of these special cases

8 Derive (marked) and apply (all) of the presented mathematical formulae for classical gas dynamics

  a 1D isentropic flow*
  b normal shocks*
  c 1D flow with heat addition*
  d 1D flow with friction*

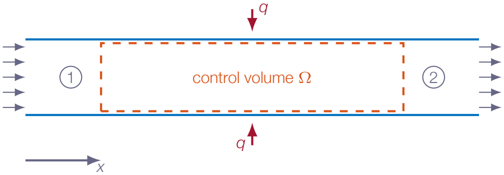
one-dimensional flows - isentropic and non-isentropic

Roadmap - One-dimensional Flow



Chapter 3.8
One-Dimensional Flow with Heat Addition

One-Dimensional Flow with Heat Addition



Pipe flow:

- no friction
- 1D steady-state $\Rightarrow$ all variables depend on x only
- q is the amount of heat per unit mass added between 1 and 2
- analyze by setting up a control volume between station 1 and 2

One-Dimensional Flow with Heat Addition

$$\boxed{\begin{aligned}\rho_1 u_1 &= \rho_2 u_2 \\ \rho_1 u_1^2 + p_1 &= \rho_2 u_2^2 + p_2 \\ h_1 + \frac{1}{2}u_1^2 + q &= h_2 + \frac{1}{2}u_2^2\end{aligned}}$$

Valid for all gases!

General gas $\Rightarrow$ Numerical solution necessary

Calorically perfect gas $\Rightarrow$ can be solved analytically

One-Dimensional Flow with Heat Addition

Calorically perfect gas ($h = C_pT$):

$$C_pT_1 + \frac{1}{2}u_1^2 + q = C_pT_2 + \frac{1}{2}u_2^2$$

$$q = \left(C_pT_2 + \frac{1}{2}u_2^2\right) - \left(C_pT_1 + \frac{1}{2}u_1^2\right)$$

$$C_pT_o = C_pT + \frac{1}{2}u^2 \Rightarrow$$

$$\boxed{q = C_p(T_{o_2} - T_{o_1})}$$

i.e. heat addition increases T_o downstream

One-Dimensional Flow with Heat Addition

Momentum equation:

$$p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2$$

$$\left\{ \rho u^2 = \rho a^2 M^2 = \rho \frac{\gamma p}{\rho} M^2 = \gamma p M^2 \right\}$$

$$p_2 - p_1 = \gamma p_1 M_1^2 - \gamma p_2 M_2^2 \Rightarrow$$

$$\boxed{\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}}$$

Normal Shock Relations

We used the momentum equation to derive the relation for p_2/p_1. In what way is this relation different than the one for normal shocks – the momentum equation is the same?

Normal Shock Relations

We used the momentum equation to derive the relation for p_2/p_1. In what way is this relation different than the one for normal shocks – the momentum equation is the same?

Answer: There is no difference. If we would insert $M_2 = f(M_1)$ from the normal shock relations, we would end up with the normal shock relation for p_2/p_1.

The relation for $M_2 = f(M_1)$ for normal shocks was derived assuming adiabatic flow

One-Dimensional Flow with Heat Addition

Ideal gas law:

$$T = \frac{p}{\rho R} \Rightarrow \frac{T_2}{T_1} = \frac{p_2}{\rho_2 R}\frac{\rho_1 R}{p_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}$$

Continuity equation:

$$\rho_1 u_1 = \rho_2 u_2 \Rightarrow \frac{\rho_1}{\rho_2} = \frac{u_2}{u_1}$$

$$\frac{u_2}{u_1} = \frac{M_2 a_2}{M_1 a_1} = \frac{M_2\sqrt{\gamma R T_2}}{M_1\sqrt{\gamma R T_1}} \Rightarrow \frac{\rho_1}{\rho_2} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}}$$

$$\boxed{\frac{T_2}{T_1} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2 \left(\frac{M_2}{M_1}\right)^2}$$

One-Dimensional Flow with Heat Addition

Calorically perfect gas, analytic solution:

$$\frac{T_2}{T_1} = \left[\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right]^2 \left(\frac{M_2}{M_1}\right)^2$$

$$\frac{\rho_2}{\rho_1} = \left[\frac{1+\gamma M_2^2}{1+\gamma M_1^2}\right] \left(\frac{M_1}{M_2}\right)^2$$

$$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}$$

One-Dimensional Flow with Heat Addition

Calorically perfect gas, analytic solution:

$$\frac{p_{o_2}}{p_{o_1}} = \left[\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right]\left(\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_{o_2}}{T_{o_1}} = \left[\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right]\left(\frac{M_2}{M_1}\right)^2\left(\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right)^{\frac{\gamma}{\gamma-1}}$$

One-Dimensional Flow with Heat Addition

Initially subsonic flow ($M < 1$)

- the Mach number, M, increases as more heat (per unit mass) is added to the gas
- for some limiting heat addition q^*, the flow will eventually become sonic $M = 1$

Initially supersonic flow ($M > 1$)

- the Mach number, M, decreases as more heat (per unit mass) is added to the gas
- for some limiting heat addition q^*, the flow will eventually become sonic $M = 1$

Note! The (*) condition in this context is not the same as the ”critical” condition discussed for isentropic flow

One-Dimensional Flow with Heat Addition

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}$$

Calculate the ratio between the pressure at a specific location in the flow p and the pressure at sonic conditions p^*

$p_1 = p$, $M_1 = M$, $p_2 = p^*$, and $M_2 = 1$

$$\frac{p^*}{p} = \frac{1 + \gamma M^2}{1 + \gamma}$$

One-Dimensional Flow with Heat Addition

$$\frac{T}{T^*} = \left[\frac{1+\gamma}{1+\gamma M^2}\right]^2 M^2$$

$$\frac{p_o}{p_o^*} = \left[\frac{1+\gamma}{1+\gamma M^2}\right]\left(\frac{2+(\gamma-1)M^2}{(\gamma+1)}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho}{\rho^*} = \left[\frac{1+\gamma M^2}{1+\gamma}\right]\left(\frac{1}{M^2}\right)$$

$$\frac{T_o}{T_o^*} = \frac{(\gamma+1)M^2}{(1+\gamma M^2)^2}(2+(\gamma-1)M^2)$$

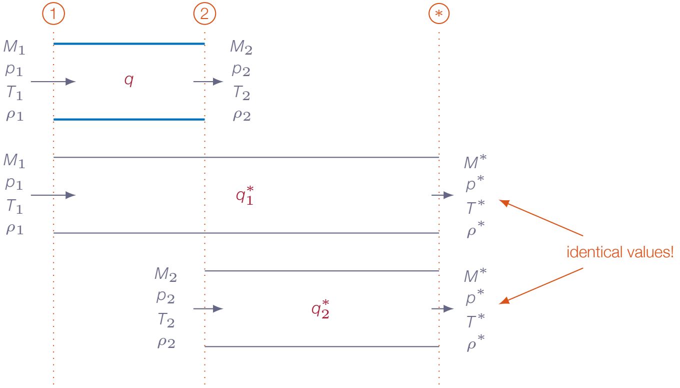
$$\frac{p}{p^*} = \frac{1+\gamma}{1+\gamma M^2}$$

One-Dimensional Flow with Heat Addition

Amount of heat per unit mass needed to choke the flow:

$$q^* = C_p(T_o^* - T_o) = C_p T_o \left(\frac{T_o^*}{T_o} - 1\right)$$

One-Dimensional Flow with Heat Addition



$$\boxed{q_2^* = q_1^* - q}$$

Note! for a given flow, the starred quantities are constant values

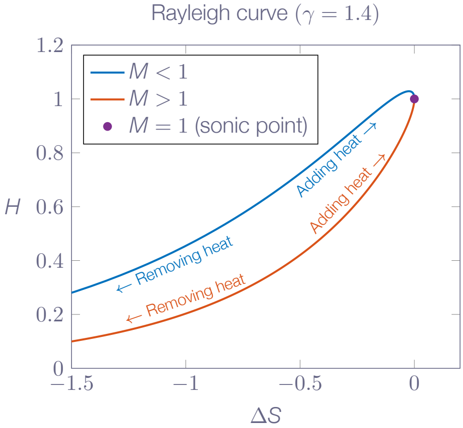
One-Dimensional Flow with Heat Addition

Lord Rayleigh 1842-1919
Nobel prize in physics 1904

Note! it is theoretically possible to heat an initially subsonic flow to reach sonic conditions and then continue to accelerate the flow by cooling

$$\Delta S = \frac{\Delta s}{C_p} = \ln\left[M^2\left(\frac{\gamma+1}{1+\gamma M^2}\right)^{\frac{\gamma+1}{\gamma}}\right]$$

$$H = \frac{h}{h^*} = \frac{C_pT}{C_pT^*} = \frac{T}{T^*} = \left[\frac{(\gamma+1)M}{1+\gamma M^2}\right]^2$$



One-Dimensional Flow with Heat Addition

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Answer: if the heat source or sink would have been included in the system studied, the system entropy would increase both when adding and removing heat.

One-Dimensional Flow with Heat Addition

$M < 1$: Adding heat will

increase M
decrease p
increase T_o
decrease p_o
increase s
increase u
decrease ρ

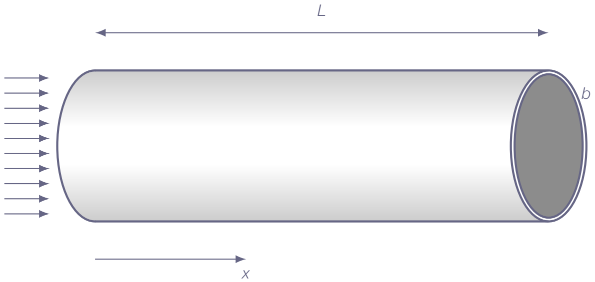
$M > 1$: Adding heat will

decrease M
increase p
increase T_o
decrease p_o
increase s
decrease u
increase ρ

Note! the flow is not isentropic, there will always be losses

One-Dimensional Flow with Heat Addition

Relation between added heat per unit mass (q) and heat per unit surface area and unit time ($\dot{q}_{wall}$)

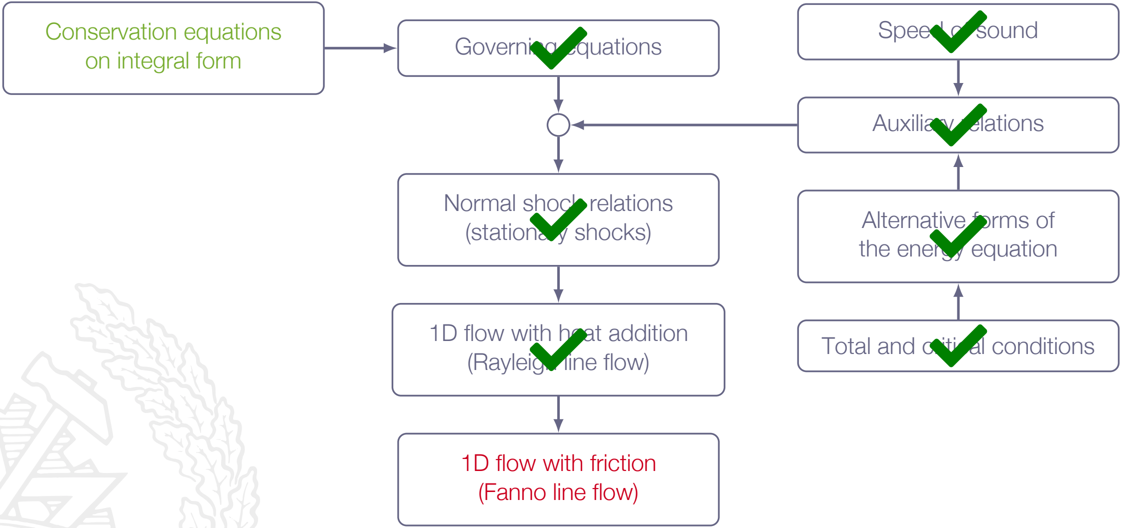


Pipe with arbitrary cross section (constant in x):

mass flow through pipe	$\dot{m}$
axial length of pipe	L
circumference of pipe	$b = 2\pi r$

$$q = \frac{Lb\dot{q}_{wall}}{\dot{m}}$$

Roadmap - One-dimensional Flow

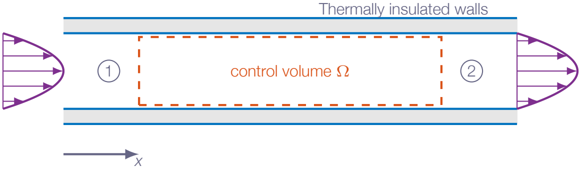


Chapter 3.9
One-Dimensional Flow with Friction

One-Dimensional Flow with Friction

inviscid flow with friction?!

One-Dimensional Flow with Friction



Pipe flow:

- adiabatic ($q = 0$)
- cross section area A is constant
- average all variables in each cross-section $\Rightarrow$ only x-dependence
- analyze by setting up a control volume between station 1 and 2

One-Dimensional Flow with Friction

Wall-friction contribution in momentum equation

$$\oiint_{\partial\Omega} \tau_w dS = b \int_0^L \tau_w dx$$

where L is the tube length and b is the circumference

One-Dimensional Flow with Friction

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 - \frac{4}{D}\int_0^L \tau_w dx = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

One-Dimensional Flow with Friction

τ_w varies with the distance x and thus complicating the integration

Solution: let L shrink to dx and we end up with relations on differential form

$$d(\rho u^2 + p) = -\frac{4}{D}\tau_w dx \Leftrightarrow \frac{d}{dx}(\rho u^2 + p) = -\frac{4}{D}\tau_w$$

One-Dimensional Flow with Friction

From the continuity equation we get

$$\rho_1 u_1 = \rho_2 u_2 = const \Rightarrow \frac{d}{dx}(\rho u) = 0$$

Writing out all terms in the momentum equation gives

$$\frac{d}{dx}(\rho u^2 + p) = \rho u \frac{du}{dx} + u \underbrace{\frac{d}{dx}(\rho u)}_{=0} + \frac{dp}{dx} = -\frac{4}{D}\tau_w \Rightarrow \rho u \frac{du}{dx} + \frac{dp}{dx} = -\frac{4}{D}\tau_w$$

Common approximation for τ_w:

$$\tau_w = f\frac{1}{2}\rho u^2 \Rightarrow \rho u \frac{du}{dx} + \frac{dp}{dx} = -\frac{2}{D}\rho u^2 f$$

One-Dimensional Flow with Friction

Energy conservation:

$$h_{o_1} = h_{o_2} \Rightarrow \frac{d}{dx} h_o = 0$$

One-Dimensional Flow with Friction

Summary:

$$\boxed{\begin{aligned} &\frac{d}{dx}(\rho u) = 0 \\ &\rho u \frac{du}{dx} + \frac{dp}{dx} = -\frac{2}{D}\rho u^2 f \\ &\frac{d}{dx} h_o = 0 \end{aligned}}$$

Valid for all gases!

General gas $\Rightarrow$ Numerical solution necessary

Calorically perfect gas $\Rightarrow$ Can be solved analytically (for constant f)

One-Dimensional Flow with Friction

Calorically perfect gas:

$$\int_{x_1}^{x_2} \frac{4f}{D} dx = \left[-\frac{1}{\gamma M^2} - \frac{\gamma+1}{2\gamma} \ln \left(\frac{M^2}{1 + \frac{\gamma-1}{2} M^2} \right) \right]_{M_1}^{M_2}$$

One-Dimensional Flow with Friction

Calorically perfect gas and adiabatic flow:

$$\frac{T_2}{T_1} = \frac{T_2}{T_{o_2}} \frac{T_{o_2}}{T_{o_1}} \frac{T_{o_1}}{T_1} = \{T_o = const\} = \frac{T_2}{T_o} \frac{T_o}{T_1} = \frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2}$$

Continuity:

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{a_1 M_1}{a_2 M_2} = \left\{a = \sqrt{\gamma R T}\right\} = \sqrt{\frac{T_1}{T_2}} \left(\frac{M_1}{M_2}\right)$$

Perfect gas:

$$\frac{p_2}{p_1} = \{p = \rho R T\} = \frac{\rho_2 T_2}{\rho_1 T_1}$$

Total pressure:

$$\frac{p_{o_2}}{p_{o_1}} = \frac{p_{o_2}}{p_2} \frac{p_2}{p_1} \frac{p_1}{p_{o_1}}$$

One-Dimensional Flow with Friction

Calorically perfect gas:

$$\frac{T_2}{T_1} = \frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}$$

$$\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\left[\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right]^{-1/2}$$

$$\frac{p_2}{p_1} = \frac{M_1}{M_2}\left[\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right]^{1/2}$$

$$\frac{p_{o_2}}{p_{o_1}} = \frac{M_1}{M_2}\left[\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

One-Dimensional Flow with Friction

Initially subsonic flow ($M_1 < 1$)

- M_2 will increase as L increases
- for a critical length L^*, the flow at point 2 will reach sonic conditions, *i.e.* $M_2 = 1$

Initially supersonic flow ($M_1 > 1$)

- M_2 will decrease as L increases
- for a critical length L^*, the flow at point 2 will reach sonic conditions, *i.e.* $M_2 = 1$

Note! The (*) condition in this context is not the same as the ”critical” condition discussed for isentropic flow

One-Dimensional Flow with Friction

$$\frac{T}{T^*} = \frac{(\gamma+1)}{2+(\gamma-1)M^2}$$

$$\frac{\rho}{\rho^*} = \frac{1}{M}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{1/2}$$

$$\frac{p}{p^*} = \frac{1}{M}\left[\frac{\gamma+1}{2+(\gamma-1)M^2}\right]^{1/2}$$

$$\frac{p_o}{p_o^*} = \frac{1}{M}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

see Table A.4

One-Dimensional Flow with Friction

and

$$\int_0^{L^*} \frac{4f}{D} dx = \left[-\frac{1}{\gamma M^2} - \frac{\gamma+1}{2\gamma} \ln \left(\frac{M^2}{1 + \frac{\gamma-1}{2} M^2} \right) \right]_M^1$$

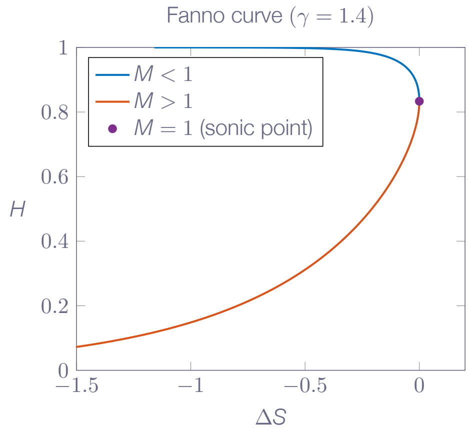
where L^* is the tube length needed to change current state to sonic conditions

Let $\bar{f}$ be the average friction coefficient over the length L^* $\Rightarrow$

$$\frac{4\bar{f}L^*}{D} = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma} \ln \left(\frac{(\gamma+1)M^2}{2+(\gamma-1)M^2} \right)$$

Turbulent pipe flow $\rightarrow \bar{f} \sim 0.005$ ($Re > 10^5$, roughness $\sim 0.001D$)

One-Dimensional Flow with Friction



$$H = \frac{h}{h_o} = \frac{C_p T}{C_p T_o} = \frac{T}{T_o} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{-1}$$

$$\Delta S = \frac{\Delta s}{C_p} = \ln\left[\left(\frac{1}{H} - 1\right)^{\frac{\gamma-1}{2\gamma}} \left(\frac{2}{\gamma - 1}\right)^{\frac{\gamma-1}{2\gamma}} \left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma+1}{2\gamma}} (H)^{\frac{\gamma+1}{2\gamma}}\right]$$

One-Dimensional Flow with Friction

$M < 1$: Friction will

increase M
decrease p
decrease T
decrease p_o
increase s
increase u
decrease ρ

$M > 1$: Friction will

decrease M
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Note! the flow is not isentropic, there will always be losses

Roadmap - One-dimensional Flow