Compressible Flow - TME085 Lecture 3

Niklas Andersson

Chalmers University of Technology Department of Mechanics and Maritime Sciences Division of Fluid Mechanics Gothenburg, Sweden

niklas.andersson@chalmers.se





Chapter 3 One-Dimensional Flow

Overview



Learning Outcomes

- 4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on
- 5 Explain how thermodynamic relations enter into the flow equations
- 6 Define the special cases of calorically perfect gas, thermally perfect gas and real gas and explain the implication of each of these special cases
- 8 Derive (marked) and apply (all) of the presented mathematical formulae for classical gas dynamics
 - a 1D isentropic flow*
 - b normal shocks*
 - c 1D flow with heat addition*
 - d 1D flow with friction*

one-dimensional flows - isentropic and non-isentropic

Roadmap - One-dimensional Flow



Motivation

Why one-dimensional flow?

many practical problems can be analyzed using a one-dimensional flow approach

a one-dimensional approach addresses the physical principles without adding the complexity of a full three-dimensional problem

the one-dimensional approach is a subset of the full three-dimensional counterpart

Roadmap - One-dimensional Flow



Chapter 3.2 One-Dimensional Flow Equations



Assumptions:

all flow variables only depend on *x* velocity aligned with *x*-axis



Control volume approach:

Define a rectangular control volume around shock, with upstream conditions denoted by 1 and downstream conditions by 2

Conservation of mass:

$$\underbrace{\frac{d}{dt}\iiint}_{=0}\rho d\mathscr{V} + \underbrace{\bigoplus}_{\frac{\partial\Omega}{\rho_2 u_2 A - \rho_1 u_1 A}}\rho \mathbf{v} \cdot \mathbf{n} dS = 0 \Rightarrow \rho_1 u_1 = \rho_2 u_2$$

Conservation of momentum:

$$\underbrace{\frac{d}{dt} \iiint \rho \mathbf{v} d\mathcal{V}}_{=0} + \underbrace{\bigoplus}_{\partial \Omega} \left[\rho(\mathbf{v} \cdot \mathbf{n}) \mathbf{v} + \rho \mathbf{n} \right] dS}_{(\rho_2 u_2^2 + \rho_2) A - (\rho_1 u_1^2 + \rho_1) A} = 0 \Rightarrow \rho_1 u_1^2 + \rho_1 = \rho_2 u_2^2 + \rho_2$$

Conservation of energy:

$$\underbrace{\frac{d}{dt}\iiint}_{=0}\rho \mathbf{e}_{o}d\mathcal{V} + \underbrace{\bigoplus}_{\frac{\partial\Omega}{\rho_{2}h_{o_{2}}u_{2}A - \rho_{1}h_{o_{1}}u_{1}A}} [\rho h_{o}\mathbf{v}\cdot\mathbf{n}] dS = 0 \Rightarrow \rho_{1}u_{1}h_{o_{1}} = \rho_{2}u_{2}h_{o_{2}}$$

Using the continuity equation this reduces to

$$h_{o_1} = h_{o_2}$$

or, if written out

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

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Summary:

$$\rho_1 u_1 = \rho_2 u_2$$
$$\rho_1 u_1^2 + \rho_1 = \rho_2 u_2^2 + \rho_2$$
$$h_1 + \frac{1}{2} u_1^2 = h_2 + \frac{1}{2} u_2^2$$

Note! These equations are valid regardless of whether or not there is a shock inside the control volume

Summary:

$$\rho_1 U_1 = \rho_2 U_2$$

$$\rho_1 U_1^2 + \rho_1 = \rho_2 U_2^2 + \rho_2$$

$$h_1 + \frac{1}{2} U_1^2 = h_2 + \frac{1}{2} U_2^2$$

Valid for all gases! General gas \Rightarrow Numerical solution necessary Calorically perfect gas \Rightarrow Can be solved analytically

Roadmap - One-dimensional Flow



Chapter 3.3 Speed of Sound and Mach Number



Sound wave / acoustic perturbation





Conservation of mass gives

$$\rho a = (\rho + d\rho)(a + da) = \rho a + \rho da + d\rho a + d\rho da$$

products of infinitesimal quantities are removed \Rightarrow



$$\rho da + d\rho a = 0$$

$$da = -a \frac{d\rho}{\rho}$$

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The momentum equation evaluated over the wave front gives

$$p + \rho a^2 = (p + dp) + (\rho + d\rho)(a + da)^2$$

Again, removing products of infinitesimal quantities gives



$$d\rho = -2a\rho da - a^2 d\rho$$

$$da = \frac{d\rho + a^2 d\rho}{-2a\rho}$$

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Continuity equation:

$$da = -a \frac{d\rho}{\rho}$$

Momentum equation:



$$da = \frac{d\rho + a^2 d\rho}{-2a\rho}$$

$$-arac{d
ho}{
ho}=rac{dp+a^2d
ho}{-2a
ho}\Rightarrow a^2=rac{dp}{d
ho}$$

Sound waves are small perturbations in ρ , **v**, ρ , T (with constant entropy *s*) propagating through gas with speed *a*



$$a^2 = \left(\frac{\partial \rho}{\partial \rho}\right)_s$$

Compressibility and speed of sound:

from before we have

$$\tau_{\rm s} = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial \rho} \right)_{\rm s}$$

insert in relation for speed of sound

$$a^{2} = \left(\frac{\partial \rho}{\partial \rho}\right)_{s} = \frac{1}{\rho \tau_{s}} \Rightarrow a = \sqrt{\frac{1}{\rho \tau_{s}}}$$

(valid for all gases)

Calorically perfect gas:

Isentropic process $\Rightarrow \rho = C \rho^{\gamma}$ (where *C* is a constant)

$$a^{2} = \left(\frac{\partial \rho}{\partial \rho}\right)_{s} = \gamma C \rho^{\gamma - 1} = \frac{\gamma \rho}{\rho}$$

which implies

$$a = \sqrt{\frac{\gamma \rho}{\rho}} \Rightarrow a = \sqrt{\gamma RT}$$

Sound wave / acoustic perturbation:

a weak wave

propagating through gas at speed of sound

small perturbations in velocity and thermodynamic properties

isentropic process

Mach Number

The mach number, M, is a local variable

$$M = \frac{v}{a}$$

where

 $v = |\mathbf{v}|$

and a is the local speed of sound

In the free stream, far away from solid objects, the flow is undisturbed and denoted by subscript ∞

$$M_{\infty} = \frac{V_{\infty}}{a_{\infty}}$$

For a fluid element moving along a streamline, the kinetic energy per unit mass and internal energy per unit mass are $V^2/2$ and *e*, respectively

$$\frac{V^2/2}{e} = \frac{V^2/2}{C_v T} = \frac{V^2/2}{RT/(\gamma - 1)} = \frac{(\gamma/2)V^2}{a^2/(\gamma - 1)} = \frac{\gamma(\gamma - 1)}{2}M^2$$

i.e. the Mach number is a measure of the ratio of the fluid motion (kinetic energy) and the random thermal motion of the molecules (internal energy)

Physical Consequences of Speed of Sound

Sound waves is the way gas molecules convey information about what is happening in the flow

In subsonic flow, sound waves are able to travel upstream, since v < a

In supersonic flow, sound waves are unable to travel upstream, since v > a



Physical Consequences of Speed of Sound



Roadmap - One-dimensional Flow



Chapter 3.4 Some Conveniently Defined Flow Parameters

Stagnation Flow Properties

Assumption: Steady inviscid flow

If the flow is slowed down isentropically (without flow losses) to zero velocity we get the so-called total conditions

(total pressure p_o , total temperature T_o , total density ρ_o)

Since the process is isentropic, we have (for calorically perfect gas)

$$\frac{p_o}{p} = \left(\frac{\rho_o}{\rho}\right)^{\gamma} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

Note! $v_o = 0$ and $M_o = 0$ by definition

Critical Conditions

If we accelerate/decelerate the flow adiabatically to the sonic point, where v = a, we obtain the so-called critical conditions, *e.g.* ρ^* , T^* , ρ^* , a^*

where, by definition, $v^* = a^*$

As for the total conditions, if the process is also reversible (entropy is preserved) we obtain the relations (for calorically perfect gas)

$$\frac{\rho^*}{\rho} = \left(\frac{\rho^*}{\rho}\right)^{\gamma} = \left(\frac{T^*}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

Total and Critical Conditions

For any given steady-state flow and location, we may think of an imaginary isentropic stagnation process or an imaginary adiabatic sonic flow process

We can compute total and critical conditions at any point

They represent conditions realizable under an isentropic/adiabatic deceleration or acceleration of the flow

Some variables like p_o and T_o will be conserved along streamlines under certain conditions

 \mathcal{T}_o is conserved along streamlines if the flow is adiabatic

conservation of p_o requires the flow to be isentropic (no viscous losses or shocks)

Note! The actual flow does not have to be adiabatic or isentropic from point to point, the total and critical conditions are results of an imaginary isentropic/adiabatic process at one point in the flow.

However, with isentropic flow T_o , p_o , ρ_o , etc are constants

In order for T_o to be constant it is only required that the flow is adiabatic.

Total and Critical Conditions

If A and B are two locations in a flow

1. Isentropic flow:

$$T_{o_A} = T_{o_B}$$
 and $p_{o_A} = p_{o_B}$

2. Adiabatic flow (not isentropic):

$$T_{o_{A}}=T_{o_{B}}$$
 and $p_{o_{A}}
eq p_{o_{B}}$

The flow is not isentropic nor adiabatic:

 $T_{o_A} \neq T_{o_B}$ and $p_{o_A} \neq p_{o_B}$

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Chapter 3.5 Alternative Forms of the Energy Equation

Alternative Forms of the Energy Equation

For steady-state adiabatic flow, we have already shown that conservation of energy gives that total enthalpy, h_o , is constant along streamlines

For a calorically perfect gas we have $h = C_{\rho}T$ which implies

$$C_{p}T + \frac{1}{2}v^{2} = C_{p}T_{o}$$

$$\frac{T_{o}}{T} = 1 + \frac{v^{2}}{2C_{p}T}$$
Inserting $C_{p} = \frac{\gamma R}{\gamma - 1}$ and $a^{2} = \gamma RT$ we get
$$\boxed{\frac{T_{o}}{T} = 1 + \frac{1}{2}(\gamma - 1)M^{2}}$$

Alternative Forms of the Energy Equation

For calorically perfect gas (1D/2D/3D flows):

$$\frac{T_o}{T} = 1 + \frac{1}{2}(\gamma - 1)M^2$$
$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{\frac{1}{\gamma - 1}}$$
$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma - 1}}$$

$$\left(\frac{a^*}{a_o}\right)^2 = \frac{T^*}{T_o} = \frac{2}{\gamma+1}$$
$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}$$
$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

Note! tabulated values for these relations can be found in Appendix A.1

The Characteristic Mach Number

$$M^* \equiv \frac{v}{a^*}$$

For a calorically perfect gas (1D/2D/3D flows)

$$M^2 = rac{2}{\left[(\gamma+1)/M^{*2}\right] - (\gamma-1)}$$

This relation between M and M^* gives:

 $M^* = 0 \Leftrightarrow M = 0$ $M^* = 1 \Leftrightarrow M = 1$ $M^* < 1 \Leftrightarrow M < 1$ $M^* > 1 \Leftrightarrow M > 1$

$$M^* \to \sqrt{rac{\gamma+1}{\gamma-1}}$$
 when $M \to \infty$

Roadmap - One-dimensional Flow



Chapter 3.6 Normal Shock Relations



One-Dimensional Flow Equations



Calorically perfect gas

$$h = C_{p}T, \quad p = \rho RT$$

with constant C_p

Assuming that state 1 is known and state 2 is unknown 5 unknown variables: ρ_2 , u_2 , p_2 , h_2 , T_2 5 equations \Rightarrow solution can be found

Divide the momentum equation by $\rho_1 u_1$

$$\frac{1}{\rho_1 u_1} \left(\rho_1 + \rho_1 u_1^2 \right) = \frac{1}{\rho_1 u_1} \left(\rho_2 + \rho_2 u_2^2 \right)$$
$$\{ \rho_1 u_1 = \rho_2 u_2 \} \Rightarrow$$
$$\frac{1}{\rho_1 u_1} \left(\rho_1 + \rho_1 u_1^2 \right) = \frac{1}{\rho_2 u_2} \left(\rho_2 + \rho_2 u_2^2 \right)$$
$$\frac{\rho_1}{\rho_1 u_1} - \frac{\rho_2}{\rho_2 u_2} = u_2 - u_1$$



$$\frac{\rho_1}{\rho_1 u_1} - \frac{\rho_2}{\rho_2 u_2} = u_2 - u_1$$

Recall that
$$\frac{a}{\rho} = \sqrt{\frac{\gamma \rho}{\rho}}$$
, which gives

$$\frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} = u_2 - u_1$$

Now, we will make use of the fact that the flow is adiabatic and thus a* is constant

Energy equation:





In any position in the flow we can get a relation between the local speed of sound *a*, the local velocity *u*, and the speed of sound at sonic conditions a^* by inserting in the equation on the previous slide. $u_1 = u$, $a_1 = a$, $u_2 = a_2 = a^* \Rightarrow$

$$\frac{a^2}{(\gamma - 1)} + \frac{1}{2}u^2 = \frac{a^{*2}}{(\gamma - 1)} + \frac{1}{2}a^{*2}$$
$$a^2 = \frac{\gamma + 1}{2}a^{*2} - \frac{\gamma - 1}{2}u^2$$

Evaluated in station 1 and 2, this gives

 $a_1^2 = \frac{\gamma+1}{2}a^{*2} - \frac{\gamma-1}{2}u_1^2$ $a_2^2 = \frac{\gamma+1}{2}a^{*2} - \frac{\gamma-1}{2}u_2^2$

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Now, inserting
$$\left\{a_1^2 = \frac{\gamma+1}{2}a^{*2} - \frac{\gamma-1}{2}u_1^2\right\}$$
 and $\left\{a_2^2 = \frac{\gamma+1}{2}a^{*2} - \frac{\gamma-1}{2}u_2^2\right\}$
in $\left\{\frac{a_1^2}{(\gamma-1)} + \frac{1}{2}u_1^2 = \frac{a_2^2}{(\gamma-1)} + \frac{1}{2}u_2^2\right\}$ and solve for a^* gives
 $a^{*2} = u_1u_2$

$$a^{*2} = U_1 U_2$$

A.K.A. the Prandtl relation. Divide by a^{*2} on both sides \Rightarrow

$$1 = \frac{U_1}{a^*} \frac{U_2}{a^*} = M_1^* M_2^*$$

Together with the relation between M and M^* , this gives

$$M_2^2 = \frac{1 + \frac{1}{2}(\gamma - 1)M_1^2}{\gamma M_1^2 - \frac{1}{2}(\gamma - 1)}$$

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Continuity equation and $a^{*2} = u_1 u_2$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1^2}{u_1 u_2} = \frac{u_1^2}{a^{*2}} = M_1^{*2}$$



$$\boxed{\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{2 + (\gamma-1)M_1^2}}$$

Now, once again back to the momentum equation

$$\rho_2 - \rho_1 = \rho_1 u_1^2 - \rho_2 u_2^2 = \{\rho_1 u_1 = \rho_2 u_2\} = \rho_1 u_1 (u_1 - u_2)$$

$$\frac{\rho_2}{\rho_1} - 1 = \frac{\rho_1 u_1^2}{\rho_1} \left(1 - \frac{u_2}{u_1} \right) = \left\{ a = \sqrt{\frac{\gamma \rho}{\rho}}, \ M^2 = \frac{u^2}{a^2} \right\} = \gamma M_1^2 \left(1 - \frac{u_2}{u_1} \right)$$

with the expression for u_2/u_1 derived previously, this gives

$$p_2 = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)$$

Are the normal shock relations valid for $M_1 < 1.0$?

Mathematically - yes!



Let's have a look at the 2^{nd} law of thermodynamics

$$s_2 - s_1 = C_{\rho} \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1}$$

We get the ratios (T_2/T_1) and (p_2/p_1) from the normal shock relations

$$s_{2} - s_{1} = C_{p} \ln \left[\left(1 + \frac{2\gamma}{\gamma+1} (M_{1}^{2} - 1) \right) \left(\frac{2 + (\gamma - 1)M_{1}^{2}}{(\gamma + 1)M_{1}^{2}} \right) \right] + R \ln \left(1 + \frac{2\gamma}{\gamma+1} (M_{1}^{2} - 1) \right)$$

 $M_1 = 1 \Rightarrow \Delta s = 0$ (Mach wave) $M_1 < 1 \Rightarrow \Delta s < 0$ (not physical) $M_1 > 1 \Rightarrow \Delta s > 0$



Normal shock $\Rightarrow M_1 > 1$

 $M_1^*M_2^* = 1$ $M_1 > 1 \Rightarrow M_1^* > 1$ $M_2^* = \frac{1}{M_1^*} \Rightarrow M_2^* < 1$ $M_2^* < 1 \Rightarrow M_2 < 1$

After a normal shock the Mach number must be lower than 1.0







Note! from before we know that M_1 must be greater than 1.0, which means that p_2/p_1 must be greater than 1.0

 $M_1 > 1.0$ gives $M_2 < 1.0$, $\rho_2 > \rho_1$, $\rho_2 > \rho_1$, and $T_2 > T_1$

What about T_o and p_o ?

Energy equation:
$$C_{\rho}T_1 + \frac{u_1^2}{2} = C_{\rho}T_2 + \frac{u_2^2}{2} \Rightarrow C_{\rho}T_{o_1} = C_{\rho}T_{o_2}$$

calorically perfect gas $\Rightarrow T_{o_1} = T_{o_2}$

or more general (as long as the shock is stationary): $h_{o_1} = h_{o_2}$

 2^{nd} law of thermodynamics and isentropic deceleration to zero velocity (Δs unchanged since isentropic) gives

$$s_2 - s_1 = C_p \ln \frac{T_{o_2}}{T_{o_1}} - R \ln \frac{p_{o_2}}{p_{o_1}} = \{T_{o_1} = T_{o_2}\} = -R \ln \frac{p_{o_2}}{p_{o_1}}$$
$$\frac{p_{o_2}}{p_{o_1}} = e^{-(s_2 - s_1)/R}$$

i.e. the total pressure decreases over a normal shock

Normal shock relations for calorically perfect gas (summary):

$$\begin{split} \mathcal{T}_{O_1} &= \mathcal{T}_{O_2} \\ a_{O_1} &= a_{O_2} \\ \mathcal{A}_1^* &= a_2^* = a^* \\ \mathcal{U}_1 \mathcal{U}_2 &= a^{*2} \text{ (the Prandtl relation)} \\ \mathcal{M}_2^* &= \frac{1}{M_1^*} \\ \mathcal{M}_2^* &= \frac{1}{M_1^*} \\ \mathcal{M}_2^{A_1} &= \frac{1}{2} \frac{1}{p_1} = \frac{1}{p_2} \frac{1}{p_1} \frac{1}{p_2} \\ \mathcal{M}_2^{A_1} &= \frac{1}{p_1} \frac{1}{p_2} \frac{1}{p_1} \frac{1}{p_2} \frac{1}{p_2} \\ \mathcal{M}_2^{A_1} &= \frac{1}{p_1} \frac{1}{p_2} \frac{1}{p_1} \frac{1}{p_2} \\ \mathcal{M}_2^{A_2} &= \frac{1}{p_1} \frac{1}{p_2} \frac{1}{p_1} \frac{1}{p_2} \frac{1}{p_1} \frac{1}{p_2} \\ \mathcal{M}_2^{A_2} &= \frac{1}{p_1} \frac{1}{p_2} \frac{1}{p_1} \frac{1}{p_1} \frac{1}{p_2} \frac{1}{p_1} \frac{1}{p_1} \frac{1}{p_2} \frac{1}{p_1} \frac{1}{p_2} \frac{1}{p_1} \frac{1}{p_1} \frac{1}{p_2} \frac{1}{p_1} \frac{1}{p_2} \frac{1}{p_1} \frac{1}{p_1} \frac{1}{p_1} \frac{1}{p_2} \frac{1}{p_1} \frac$$

As the flow passes a stationary normal shock, the following changes will take place discontinuously across the shock:









The normal shock relations for calorically perfect gases are valid for $M_1 \le 5$ (approximately) for air at standard conditions

Calorically perfect gas \Rightarrow Shock strength depends on M_1 only

Thermally perfect gas \Rightarrow Shock strength depends on M_1 and T_1

General real gas (non-perfect) \Rightarrow Shock strength depends on M_1 , p_1 , and T_1

And now to the question that probably bothers most of you but that no one dares to ask ...



And now to the question that probably bothers most of you but that no one dares to ask ...

When or where did we say that there was going to be a shock between 1 and 2?

And now to the question that probably bothers most of you but that no one dares to ask ...

When or where did we say that there was going to be a shock between 1 and 2?

Answer: We did not (explicitly)

The derivation is based on the fact that there should be a change in flow properties between 1 and 2

We are assuming steady state conditions

We have said that the flow is adiabatic (no added or removed heat)

There is no work done and no friction added

A normal shock is <u>the solution</u> provided by nature (and math) that fulfill these requirements!

Chapter 3.7 Hugoniot Equation

Hugoniot Equation

Starting point: governing equations for normal shocks

 $\rho_1 U_1 = \rho_2 U_2$

$$\rho_1 u_1^2 + \rho_1 = \rho_2 u_2^2 + \rho_2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

Eliminate u_1 and u_2 gives:

$$h_2 - h_1 = rac{
ho_2 -
ho_1}{2} \left(rac{1}{
ho_1} + rac{1}{
ho_2}
ight)$$

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Now, insert $h = e + p/\rho$ gives

$$e_2 - e_1 = \frac{\rho_2 + \rho_1}{2} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) = \frac{\rho_2 + \rho_1}{2} \left(\nu_1 - \nu_2 \right)$$

which is the Hugoniot relation

Stationary Normal Shock in One-Dimensional Flow

Normal shock:

$$e_2 - e_1 = -\frac{\rho_2 + \rho_1}{2} \left(\nu_2 - \nu_1\right)$$

- More effective than isentropic process
 - Gives entropy increase

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Isentropic process:

$$de = -pd\nu$$

 More efficient than normal shock process

see figure 3.11 p. 100
Stationary Normal Shock in One-Dimensional Flow

The Rankine-Hugoniot relation

$$\frac{\rho_2}{\rho_1} = \frac{1 + \left(\frac{\gamma+1}{\gamma-1}\right) \left(\frac{\rho_2}{\rho_1}\right)}{\left(\frac{\gamma+1}{\gamma-1}\right) + \left(\frac{\rho_2}{\rho_1}\right)}$$

The isentropic relation

$$\frac{\rho_2}{\rho_1} = \left(\frac{p_2}{\rho_1}\right)^{1/\gamma}$$

Pressure ratio ($\gamma = 1.4$)

















