Compressible Flow - TME085

Lecture 3

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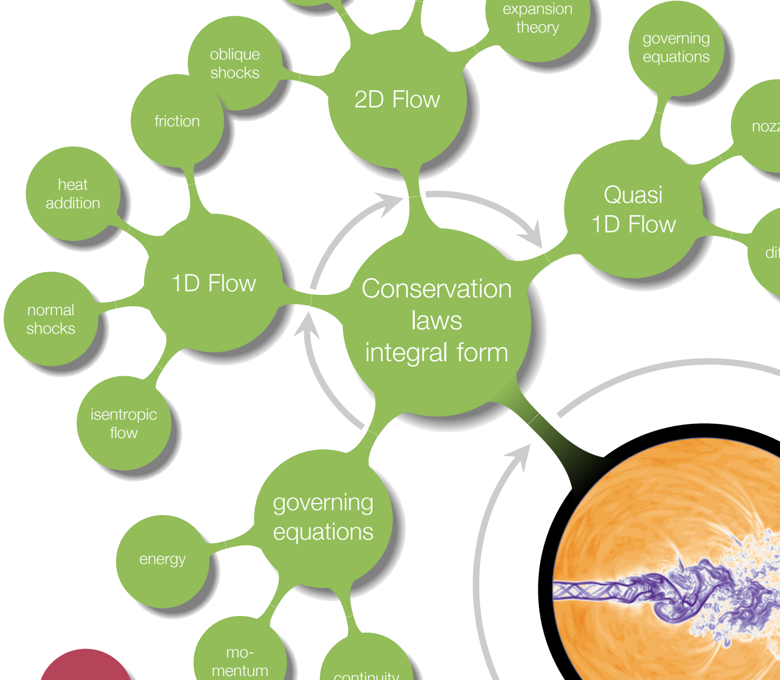
Chapter 3
One-Dimensional Flow

Overview



expansion theory
oblique shocks
2D Flow
governing equations
friction
heat addition
Quasi 1D Flow
1D Flow
Conservation laws integral form
normal shocks
isentropic flow
governing equations
energy
mo-mentum
continuity

Learning Outcomes

4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on

5 **Explain** how thermodynamic relations enter into the flow equations

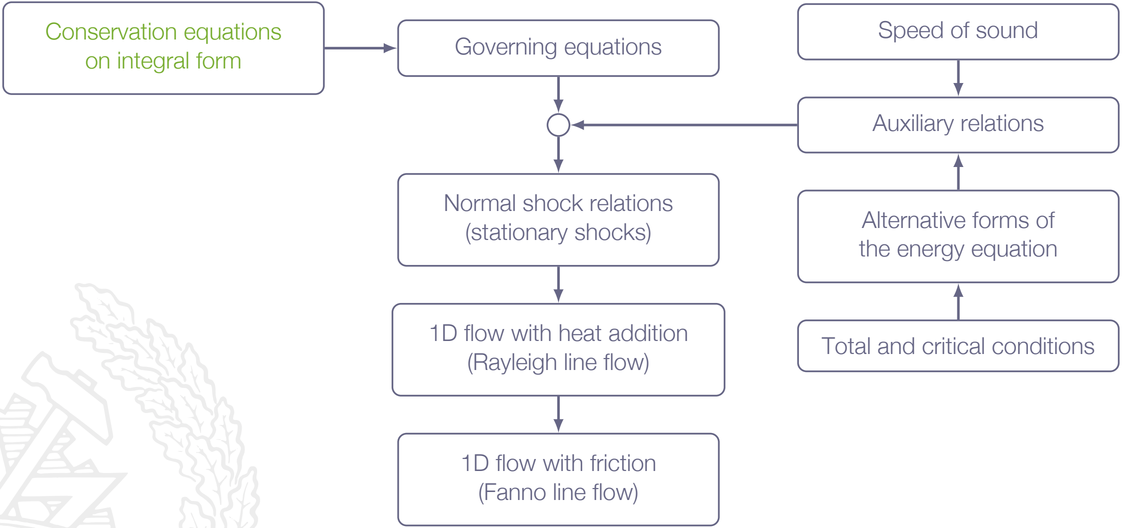
6 **Define** the special cases of calorically perfect gas, thermally perfect gas and real gas and **explain** the implication of each of these special cases

8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics

a 1D isentropic flow*
b normal shocks*
c 1D flow with heat addition*
d 1D flow with friction*

one-dimensional flows - isentropic and non-isentropic

Roadmap - One-dimensional Flow

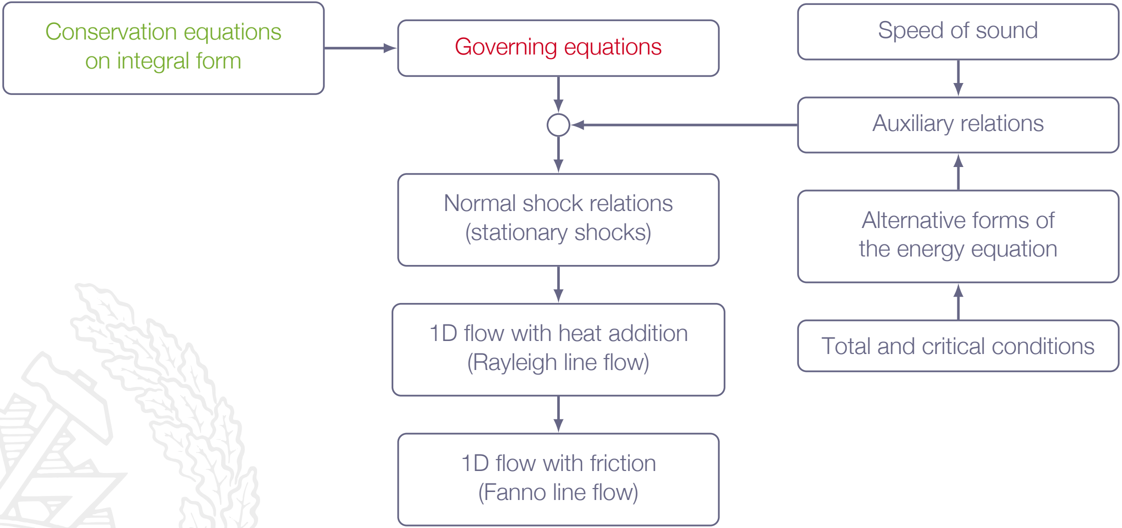


Motivation

Why one-dimensional flow?

- many practical problems can be analyzed using a one-dimensional flow approach
- a one-dimensional approach addresses the physical principles without adding the complexity of a full three-dimensional problem
- the one-dimensional approach is a subset of the full three-dimensional counterpart

Roadmap - One-dimensional Flow



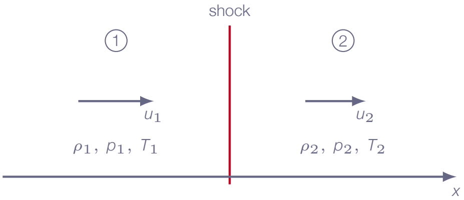
Chapter 3.2
One-Dimensional Flow Equations

One-Dimensional Flow Equations

Problems analyzed using the one-dimensional flow equations can be divided in to two categories:

1. problems with **wave solutions** (discontinuous)
   - acoustic wave
   - normal shock
2. problems with **continuous solutions**
   - flow with heat addition
   - flow with friction

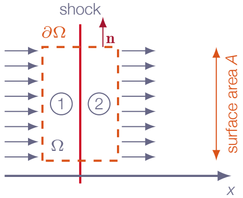
One-Dimensional Flow Equations



Assumptions:

- all flow variables only depend on x
- velocity aligned with x-axis

One-Dimensional Flow Equations



Control volume approach:

Define a rectangular control volume around shock, with upstream conditions denoted by 1 and downstream conditions by 2

One-Dimensional Flow Equations

Conservation of mass:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega} \rho d\mathscr{V}}_{=0} + \underbrace{\oint\!\!\!\oint\limits_{\partial\Omega} \rho \mathbf{v}\cdot\mathbf{n} dS}_{\rho_2 u_2 A - \rho_1 u_1 A} = 0 \Rightarrow \rho_1 u_1 = \rho_2 u_2$$

Conservation of momentum:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega} \rho \mathbf{v} d\mathscr{V}}_{=0} + \underbrace{\oint\!\!\!\oint\limits_{\partial\Omega} \left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v} + p\mathbf{n}\right] dS}_{(\rho_2 u_2^2 + p_2)A - (\rho_1 u_1^2 + p_1)A} = 0 \Rightarrow \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

One-Dimensional Flow Equations

Conservation of energy:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho e_o d\mathscr{V}}_{=0}+\underbrace{\oiint\limits_{\partial\Omega}[\rho h_o \mathbf{v}\cdot\mathbf{n}]\,dS}_{\rho_2 h_{o_2} u_2 A-\rho_1 h_{o_1} u_1 A}=0\Rightarrow \rho_1 u_1 h_{o_1}=\rho_2 u_2 h_{o_2}$$

Using the continuity equation this reduces to

$$h_{o_1}=h_{o_2}$$

or, if written out

$$h_1+\frac{1}{2}u_1^2=h_2+\frac{1}{2}u_2^2$$

One-Dimensional Flow Equations

Summary:

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

Note! These equations are valid regardless of whether or not there is a shock inside the control volume

One-Dimensional Flow Equations

Summary:

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

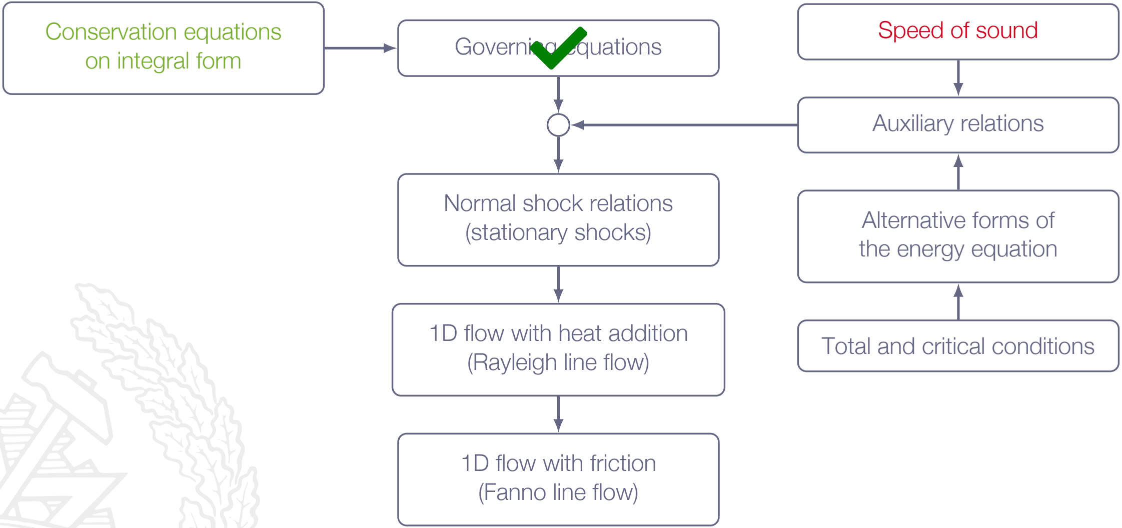
$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

Valid for all gases!

General gas $\Rightarrow$ Numerical solution necessary

Calorically perfect gas $\Rightarrow$ Can be solved analytically

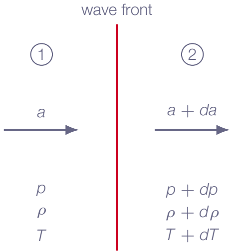
Roadmap - One-dimensional Flow



Chapter 3.3
Speed of Sound and Mach Number

Speed of Sound

Sound wave / acoustic perturbation



Speed of Sound

Conservation of mass gives

$$\rho a = (\rho + d\rho)(a + da) = \rho a + \rho da + d\rho a + d\rho da$$

products of infinitesimal quantities are removed $\Rightarrow$

$$\rho da + d\rho a = 0$$

solve for da $\Rightarrow$

$$da = -a\frac{d\rho}{\rho}$$

Speed of Sound

The momentum equation evaluated over the wave front gives

$$p + \rho a^2 = (p + dp) + (\rho + d\rho)(a + da)^2$$

Again, removing products of infinitesimal quantities gives

$$dp = -2a\rho da - a^2 d\rho$$

Solve for $da \Rightarrow$

$$da = \frac{dp + a^2 d\rho}{-2a\rho}$$

Speed of Sound

Continuity equation:

$$da = -a\frac{d\rho}{\rho}$$

Momentum equation:

$$da = \frac{dp + a^2 d\rho}{-2a\rho}$$

$$-a\frac{d\rho}{\rho} = \frac{dp + a^2 d\rho}{-2a\rho} \Rightarrow a^2 = \frac{dp}{d\rho}$$

Speed of Sound

Sound waves are **small perturbations** in ρ, $\mathbf{v}$, p, T (with constant entropy s) propagating through gas with speed a

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

(valid for all gases)

Speed of Sound

Compressibility and speed of sound:

from before we have

$$\tau_s = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_s$$

insert in relation for speed of sound

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{1}{\rho \tau_s} \Rightarrow a = \sqrt{\frac{1}{\rho \tau_s}}$$

(valid for all gases)

Speed of Sound

Calorically perfect gas:

Isentropic process $\Rightarrow p = C\rho^\gamma$ (where C is a constant)

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \gamma C \rho^{\gamma-1} = \frac{\gamma p}{\rho}$$

which implies

$$a = \sqrt{\frac{\gamma p}{\rho}} \Rightarrow a = \sqrt{\gamma R T}$$

Speed of Sound

Sound wave / acoustic perturbation:

- a **weak wave**
- propagating through gas at **speed of sound**
- **small perturbations** in velocity and thermodynamic properties
- **isentropic** process

Mach Number

The mach number, M, is a local variable

$$M = \frac{v}{a}$$

where

$$v = |\mathbf{v}|$$

and a is the local speed of sound

In the free stream, far away from solid objects, the flow is undisturbed and denoted by subscript ∞

$$M_\infty = \frac{v_\infty}{a_\infty}$$

Mach Number

For a fluid element moving along a streamline, the kinetic energy per unit mass and internal energy per unit mass are $V^2/2$ and e, respectively

$$\frac{V^2/2}{e} = \frac{V^2/2}{C_v T} = \frac{V^2/2}{RT/(\gamma-1)} = \frac{(\gamma/2)V^2}{a^2/(\gamma-1)} = \frac{\gamma(\gamma-1)}{2} M^2$$

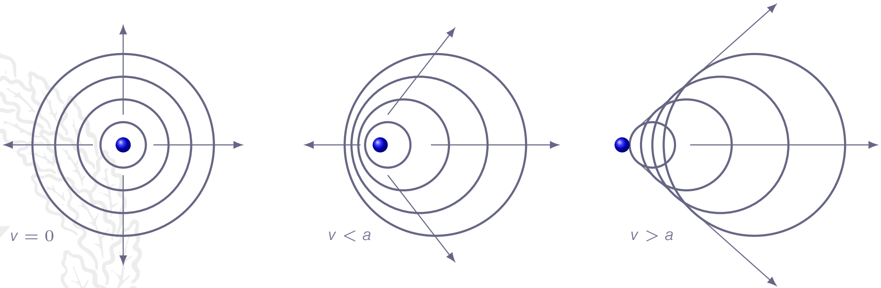
i.e. the Mach number is a measure of the ratio of the **fluid motion** (kinetic energy) and the **random thermal motion** of the molecules (internal energy)

Physical Consequences of Speed of Sound

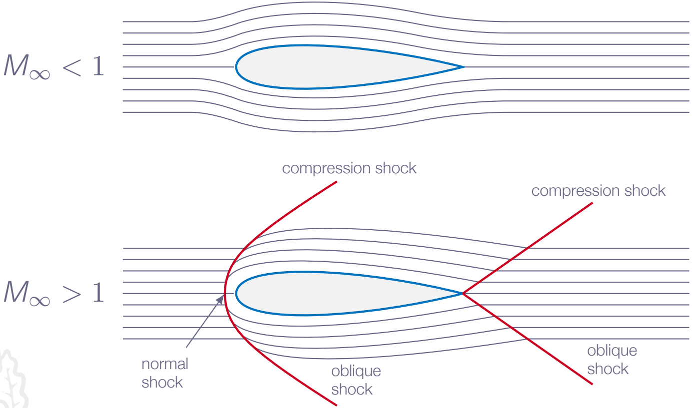
Sound waves is the way gas molecules convey information about what is happening in the flow

In subsonic flow, sound waves are able to travel upstream, since $v < a$

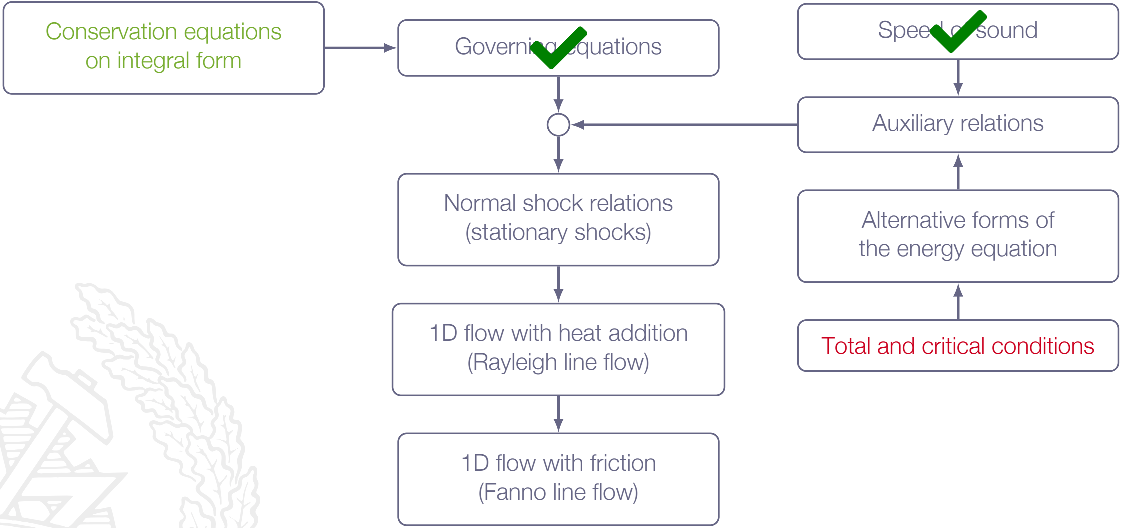
In supersonic flow, sound waves are unable to travel upstream, since $v > a$



Physical Consequences of Speed of Sound



Roadmap - One-dimensional Flow



Chapter 3.4
Some Conveniently Defined Flow Parameters

Stagnation Flow Properties

Assumption: Steady inviscid flow

If the flow is slowed down **isentropically** (without flow losses) to **zero velocity** we get the so-called **total conditions** (or stagnation flow properties)

(e.g. total pressure p_o, total temperature T_o, total density ρ_o, and total speed of sound a_o)

Since the process is isentropic, we have (for calorically perfect gas)

$$\frac{p_o}{p} = \left(\frac{\rho_o}{\rho}\right)^{\gamma} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

Note! T_o and a_o only requires an adiabatic deceleration process

Critical Conditions

If the flow is accelerated/decelerated **isentropically** to the **sonic point**, where $v = a$, we obtain the so-called **critical conditions**, *e.g.* p^*, T^*, ρ^*, a^*

where, by definition, $v^* = a^*$

As for the total conditions, if the process is also reversible (entropy is preserved) we obtain the relations (for calorically perfect gas)

$$\frac{p^*}{p_o} = \left(\frac{\rho^*}{\rho_o}\right)^{\gamma} = \left(\frac{T^*}{T_o}\right)^{\frac{\gamma}{\gamma-1}}$$

Note! T^* and a^* only requires an adiabatic acceleration/deceleration process

Total and Critical Conditions

For any given steady-state flow and location, we may think of an **imaginary** isentropic/adiabatic stagnation process or sonic flow process and thus

- We can obtain **total** and **critical** conditions at **any point** in a flow
- The total/critical conditions represent conditions realizable under an isentropic/adiabatic deceleration or acceleration of the flow
- In an adiabatic flow, T_o is conserved along streamlines
- Conservation of p_o along streamlines requires that the flow is isentropic (no viscous losses or shocks)

Total and Critical Conditions

Note! The actual flow does not have to be adiabatic or isentropic from point to point, the total and critical conditions are results of an imaginary isentropic/adiabatic process at one point in the flow.

However, with isentropic flow T_o, p_o, ρ_o, etc are constants

In order for T_o to be constant it is only required that the flow is adiabatic.

Total and Critical Conditions

If A and B are two locations in a flow

1. Isentropic flow:

$$T_{o_A} = T_{o_B} \text{ and } p_{o_A} = p_{o_B}$$

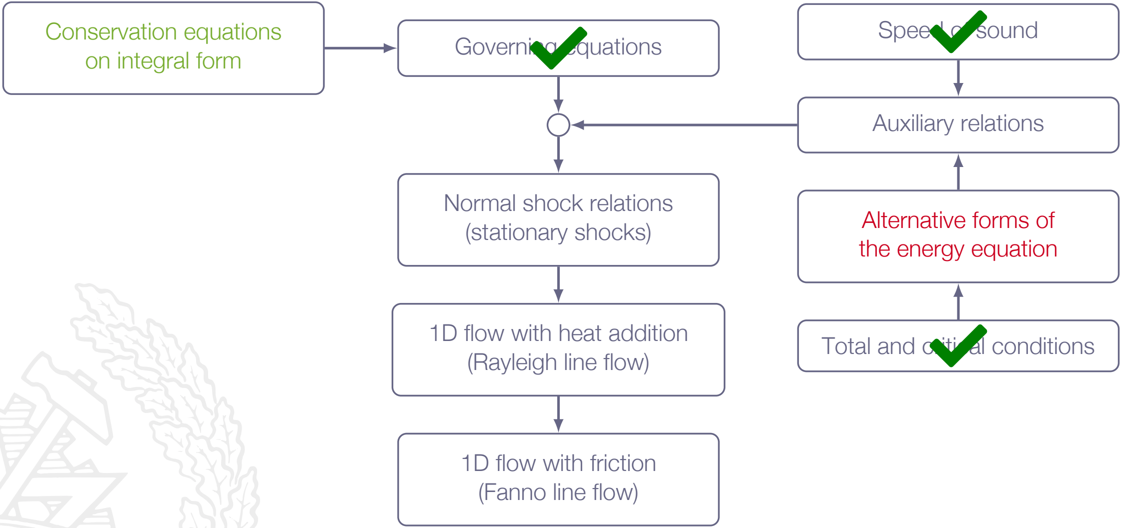
2. Adiabatic flow (not isentropic):

$$T_{o_A} = T_{o_B} \text{ and } p_{o_A} \neq p_{o_B}$$

3. The flow is not isentropic nor adiabatic:

$$T_{o_A} \neq T_{o_B} \text{ and } p_{o_A} \neq p_{o_B}$$

Roadmap - One-dimensional Flow



Chapter 3.5
Alternative Forms of the Energy Equation

Alternative Forms of the Energy Equation

For steady-state adiabatic flow, we have already shown that conservation of energy gives that total enthalpy, h_o, is constant along streamlines

For a calorically perfect gas we have $h = C_p T$ which implies

$$C_p T + \frac{1}{2} v^2 = C_p T_o$$

$$\frac{T_o}{T} = 1 + \frac{v^2}{2 C_p T}$$

Inserting $C_p = \dfrac{\gamma R}{\gamma - 1}$ and $a^2 = \gamma R T$ we get

$$\boxed{\frac{T_o}{T} = 1 + \frac{1}{2}(\gamma - 1) M^2}$$

Alternative Forms of the Energy Equation

For calorically perfect gas (1D/2D/3D flows):

$$\frac{T_o}{T} = 1 + \frac{1}{2}(\gamma - 1)M^2$$

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{\frac{1}{\gamma-1}}$$

$$\frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\left(\frac{a^*}{a_o}\right)^2 = \frac{T^*}{T_o} = \frac{2}{\gamma+1}$$

$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}$$

$$\frac{p^*}{p_o} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

Note! tabulated values for these relations can be found in Appendix A.1

The Characteristic Mach Number

$$M^* \equiv \frac{v}{a^*}$$

For a calorically perfect gas (1D/2D/3D flows)

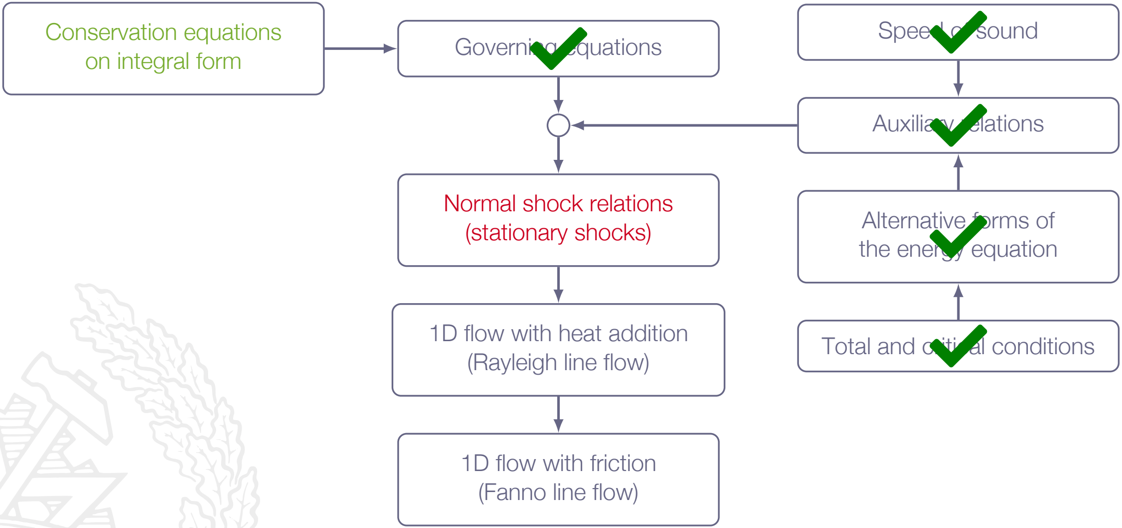
$$M^2 = \frac{2}{\left[(\gamma+1)/{M^*}^2\right] - (\gamma - 1)}$$

This relation between M and M^* gives:

$$M^* = 0 \Leftrightarrow M = 0$$
$$M^* = 1 \Leftrightarrow M = 1$$
$$M^* < 1 \Leftrightarrow M < 1$$
$$M^* > 1 \Leftrightarrow M > 1$$

$$M^* \rightarrow \sqrt{\frac{\gamma+1}{\gamma-1}} \text{ when } M \rightarrow \infty$$

Roadmap - One-dimensional Flow



Chapter 3.6
Normal Shock Relations

One-Dimensional Flow Equations

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

Normal Shock Relations

Calorically perfect gas

$$h = C_p T, \quad p = \rho R T$$

with constant C_p

Assuming that state 1 is known and state 2 is unknown

- 5 unknown variables: ρ_2, u_2, p_2, h_2, T_2
- 5 equations

$\Rightarrow$ solution can be found

Normal Shock Relations

Divide the momentum equation by $\rho_1 u_1$

$$\frac{1}{\rho_1 u_1}\left(p_1 + \rho_1 u_1^2\right) = \frac{1}{\rho_1 u_1}\left(p_2 + \rho_2 u_2^2\right)$$

$$\{\rho_1 u_1 = \rho_2 u_2\} \Rightarrow$$

$$\frac{1}{\rho_1 u_1}\left(p_1 + \rho_1 u_1^2\right) = \frac{1}{\rho_2 u_2}\left(p_2 + \rho_2 u_2^2\right)$$

$$\frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} = u_2 - u_1$$

Normal Shock Relations

$$\frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} = u_2 - u_1$$

Recall that $a = \sqrt{\frac{\gamma p}{\rho}}$, which gives

$$\frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} = u_2 - u_1$$

Now, we will make use of the fact that the flow is adiabatic and thus a^* is constant

Normal Shock Relations

Energy equation:

$$C_p T_1 + \frac{1}{2}u_1^2 = C_p T_2 + \frac{1}{2}u_2^2$$

$$\left\{ C_p = \frac{\gamma R}{\gamma - 1} \right\} \Rightarrow$$

$$\frac{\gamma R T_1}{(\gamma - 1)} + \frac{1}{2}u_1^2 = \frac{\gamma R T_2}{(\gamma - 1)} + \frac{1}{2}u_2^2$$

$$\left\{ a = \sqrt{\gamma R T} \right\} \Rightarrow$$

$$\frac{a_1^2}{(\gamma - 1)} + \frac{1}{2}u_1^2 = \frac{a_2^2}{(\gamma - 1)} + \frac{1}{2}u_2^2$$

Normal Shock Relations

In any position in the flow we can get a relation between the local speed of sound a, the local velocity u, and the speed of sound at sonic conditions a^* by inserting in the equation on the previous slide. $u_1 = u$, $a_1 = a$, $u_2 = a_2 = a^* \Rightarrow$

$$\frac{a^2}{(\gamma - 1)} + \frac{1}{2}u^2 = \frac{a^{*2}}{(\gamma - 1)} + \frac{1}{2}a^{*2}$$

$$a^2 = \frac{\gamma + 1}{2}a^{*2} - \frac{\gamma - 1}{2}u^2$$

Evaluated in station 1 and 2, this gives

$$a_1^2 = \frac{\gamma + 1}{2}a^{*2} - \frac{\gamma - 1}{2}u_1^2$$

$$a_2^2 = \frac{\gamma + 1}{2}a^{*2} - \frac{\gamma - 1}{2}u_2^2$$

Normal Shock Relations

Now, inserting $\left\{a_1^2 = \frac{\gamma+1}{2}a^{*2} - \frac{\gamma-1}{2}u_1^2\right\}$ and $\left\{a_2^2 = \frac{\gamma+1}{2}a^{*2} - \frac{\gamma-1}{2}u_2^2\right\}$

in $\left\{\frac{a_1^2}{(\gamma-1)} + \frac{1}{2}u_1^2 = \frac{a_2^2}{(\gamma-1)} + \frac{1}{2}u_2^2\right\}$ and solve for a^* gives

$$\boxed{a^{*2} = u_1 u_2}$$

Normal Shock Relations

$$a^{*2} = u_1 u_2$$

A.K.A. the Prandtl relation. Divide by a^{*2} on both sides $\Rightarrow$

$$1 = \frac{u_1}{a^*}\frac{u_2}{a^*} = M_1^* M_2^*$$

Together with the relation between M and M^*, this gives

$$\boxed{M_2^2 = \frac{1 + \frac{1}{2}(\gamma - 1)M_1^2}{\gamma M_1^2 - \frac{1}{2}(\gamma - 1)}}$$

Normal Shock Relations

Continuity equation and ${a^*}^2 = u_1 u_2$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1^2}{u_1 u_2} = \frac{u_1^2}{{a^*}^2} = {M_1^*}^2$$

which gives

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}}$$

Normal Shock Relations

Now, once again back to the momentum equation

$$p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2 = \{\rho_1 u_1 = \rho_2 u_2\} = \rho_1 u_1 (u_1 - u_2)$$

$$\frac{p_2}{p_1} - 1 = \frac{\rho_1 u_1^2}{p_1}\left(1 - \frac{u_2}{u_1}\right) = \left\{a = \sqrt{\frac{\gamma p}{\rho}},\ M^2 = \frac{u^2}{a^2}\right\} = \gamma M_1^2\left(1 - \frac{u_2}{u_1}\right)$$

with the expression for u_2/u_1 derived previously, this gives

$$\boxed{\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)}$$

Normal Shock Relations

Are the normal shock relations valid for $M_1 < 1.0$?

Mathematically - yes!

Physically - ?

Normal Shock Relations

Let's have a look at the 2^{nd} law of thermodynamics

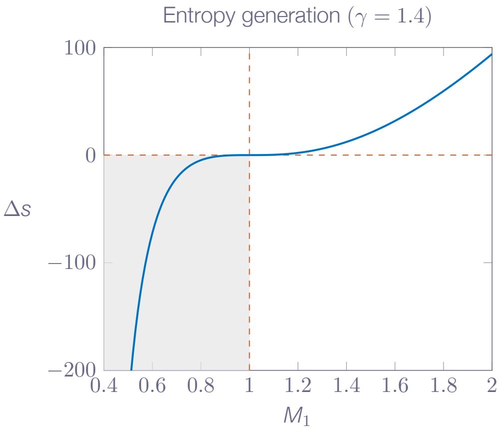
$$s_2 - s_1 = C_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

We get the ratios (T_2/T_1) and (p_2/p_1) from the normal shock relations

$$s_2 - s_1 = C_p \ln \left[\left(1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1) \right) \left(\frac{2 + (\gamma-1)M_1^2}{(\gamma+1)M_1^2} \right) \right] + \\ - R \ln \left(1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1) \right)$$

Normal Shock Relations

$$M_1 = 1 \Rightarrow \Delta s = 0 \text{ (Mach wave)}$$
$$M_1 < 1 \Rightarrow \Delta s < 0 \text{ (not physical)}$$
$$M_1 > 1 \Rightarrow \Delta s > 0$$



Normal Shock Relations

Normal shock $\Rightarrow M_1 > 1$

$$M_1^* M_2^* = 1$$

$$M_1 > 1 \Rightarrow M_1^* > 1$$

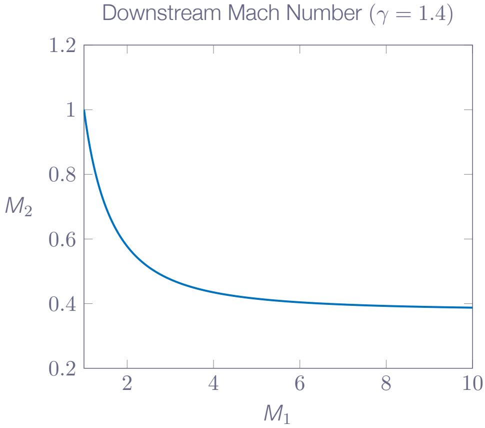
$$M_2^* = \frac{1}{M_1^*} \Rightarrow M_2^* < 1$$

$$M_2^* < 1 \Rightarrow M_2 < 1$$

After a normal shock the Mach number must be lower than 1.0

Normal Shock Relations

$$M_2^2 = \frac{1 + \frac{1}{2}(\gamma - 1)M_1^2}{\gamma M_1^2 - \frac{1}{2}(\gamma - 1)}$$



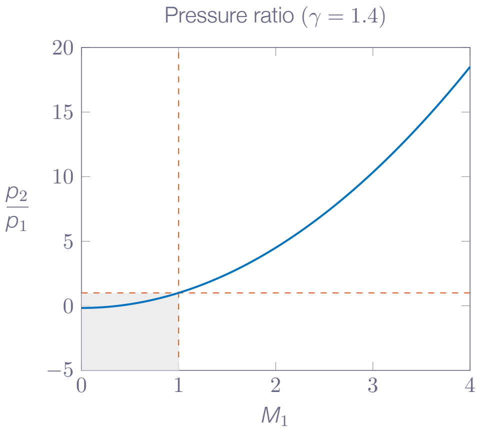
$$M_1 = 1.0 \Rightarrow M_2 = 1.0$$

$$M_1 > 1.0 \Rightarrow M_2 < 1.0$$

$$M_1 \to \infty \Rightarrow M_2 \to \sqrt{(\gamma - 1)/(2\gamma)} = \{\gamma = 1.4\} = 0.378$$

Normal Shock Relations

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right)$$



Note! from before we know that M_1 must be greater than 1.0, which means that p_2/p_1 must be greater than 1.0

Normal Shock Relations

$M_1 > 1.0$ gives $M_2 < 1.0$, $\rho_2 > \rho_1$, $p_2 > p_1$, and $T_2 > T_1$

What about T_o and p_o?

Energy equation: $C_p T_1 + \frac{u_1^2}{2} = C_p T_2 + \frac{u_2^2}{2} \Rightarrow C_p T_{o_1} = C_p T_{o_2}$

calorically perfect gas $\Rightarrow T_{o_1} = T_{o_2}$

or more general (as long as the shock is stationary): $h_{o_1} = h_{o_2}$

Normal Shock Relations

2^{nd} law of thermodynamics and isentropic deceleration to zero velocity (Δs unchanged since isentropic) gives

$$s_2 - s_1 = C_p \ln \frac{T_{o_2}}{T_{o_1}} - R \ln \frac{p_{o_2}}{p_{o_1}} = \{T_{o_1} = T_{o_2}\} = -R \ln \frac{p_{o_2}}{p_{o_1}}$$

$$\frac{p_{o_2}}{p_{o_1}} = e^{-(s_2 - s_1)/R}$$

i.e. the total pressure decreases over a normal shock

Normal Shock Relations

Normal shock relations for calorically perfect gas (summary):

$$T_{o_1} = T_{o_2}$$

$$a_{o_1} = a_{o_2}$$

$$a_1^* = a_2^* = a^*$$

$$u_1 u_2 = {a^*}^2 \quad \text{(the Prandtl relation)}$$

$$M_2^* = \frac{1}{M_1^*}$$

$$M_2^2 = \frac{1 + \frac{1}{2}(\gamma - 1)M_1^2}{\gamma M_1^2 - \frac{1}{2}(\gamma - 1)}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2}$$

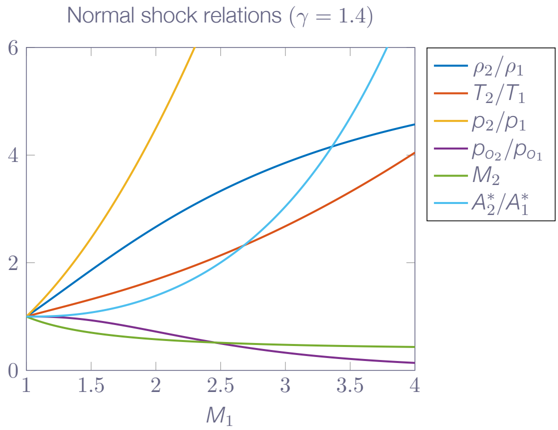
$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}$$

Normal Shock Relations

As the flow passes a stationary normal shock, the following changes will take place discontinuously across the shock:

ρ	increases
p	increases
u	decreases
M	decreases (from $M > 1$ to $M < 1$)
T	increases
p_o	decreases (due to shock loss)
s	increases (due to shock loss)
T_o	unaffected

Normal Shock Relations



Normal Shock Relations

The normal shock relations for calorically perfect gases are valid for $M_1 \leq 5$ (approximately) for air at standard conditions

Calorically perfect gas $\Rightarrow$ Shock strength depends on M_1 only

Thermally perfect gas $\Rightarrow$ Shock strength depends on M_1 and T_1

General real gas (non-perfect) $\Rightarrow$ Shock strength depends on M_1, p_1, and T_1

Normal Shock Relations

And now to the question that probably bothers most of you but that no one dares to ask ...

Normal Shock Relations

And now to the question that probably bothers most of you but that no one dares to ask ...

When or where did we say that there was going to be a shock between 1 and 2?

Normal Shock Relations

And now to the question that probably bothers most of you but that no one dares to ask ...

When or where did we say that there was going to be a shock between 1 and 2?

Answer: We did not (explicitly)

Normal Shock Relations

The derivation is based on the fact that there should be a change in flow properties between 1 and 2

We are assuming steady state conditions

We have said that the flow is adiabatic (no added or removed heat)

There is no work done and no friction added

A normal shock is <u>the solution</u> provided by nature (and math) that fulfill these requirements!

Chapter 3.7
Hugoniot Equation

Hugoniot Equation

Starting point: governing equations for normal shocks

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

Eliminate u_1 and u_2 gives:

$$h_2 - h_1 = \frac{p_2 - p_1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

Hugoniot Equation

Now, insert $h = e + p/\rho$ gives

$$e_2 - e_1 = \frac{p_2 + p_1}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = \frac{p_2 + p_1}{2}(\nu_1 - \nu_2)$$

which is the **Hugoniot relation**

Stationary Normal Shock in One-Dimensional Flow

Normal shock:

$$e_2 - e_1 = -\frac{p_2 + p_1}{2}(\nu_2 - \nu_1)$$

More effective than isentropic process

Gives entropy increase

Isentropic process:

$$de = -pd\nu$$

More efficient than normal shock process

see figure 3.11 p. 100

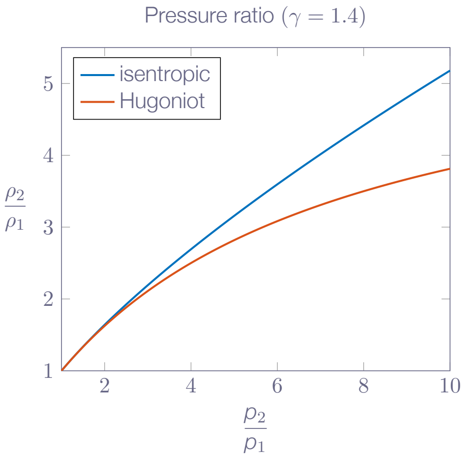
Stationary Normal Shock in One-Dimensional Flow

The Rankine-Hugoniot relation

$$\frac{\rho_2}{\rho_1} = \frac{1 + \left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right)}{\left(\frac{\gamma+1}{\gamma-1}\right) + \left(\frac{p_2}{p_1}\right)}$$

The isentropic relation

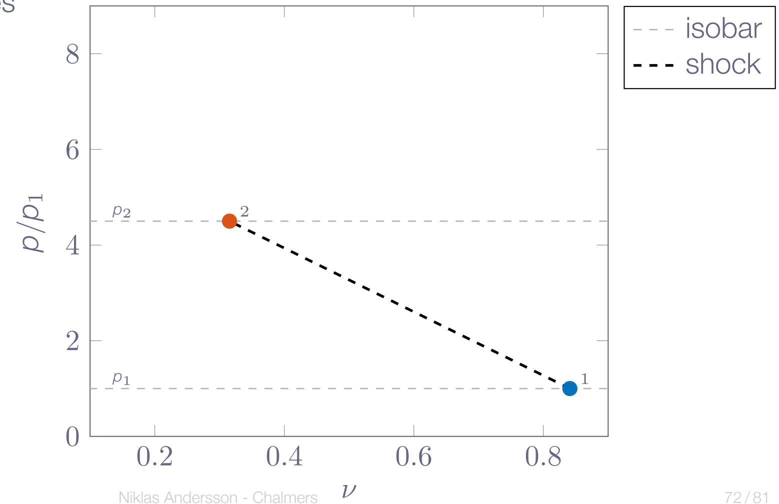
$$\frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{1/\gamma}$$



The Normal-shock Process

Note!

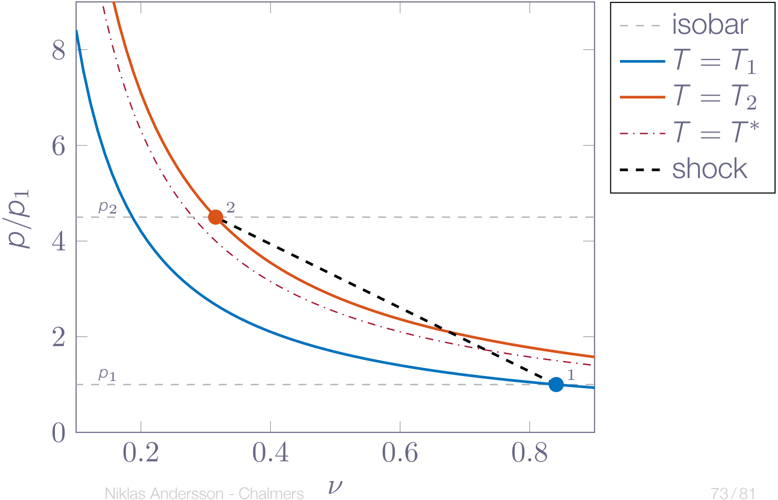
over the shock, the flow state changes discontinuously from 1 to 2 without passing any intermediate states



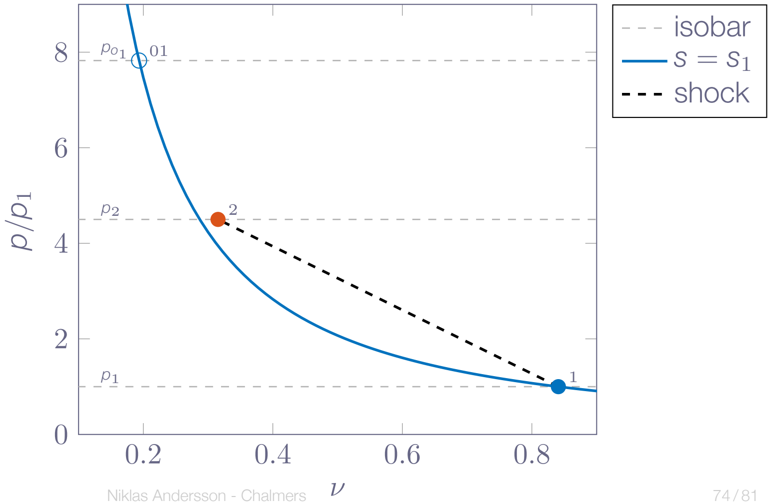
The Normal-shock Process

Note!

$M_1 > 1.0$ and $M_2 < 1.0 \Rightarrow T_1 < T^* < T_2$



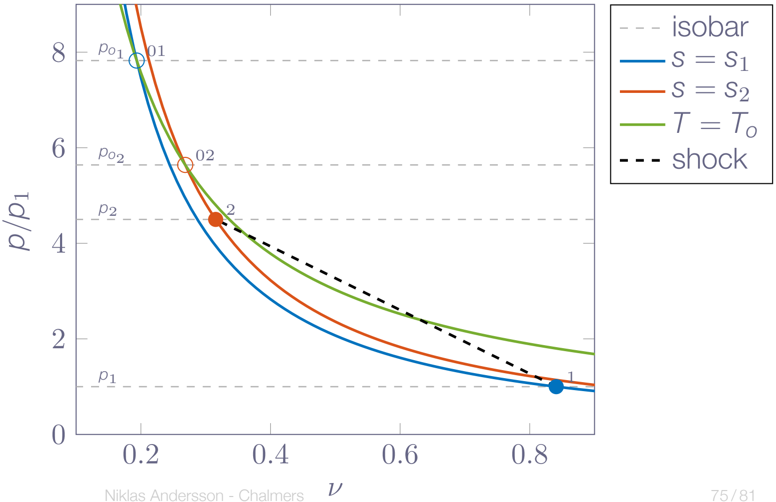
The Normal-shock Process



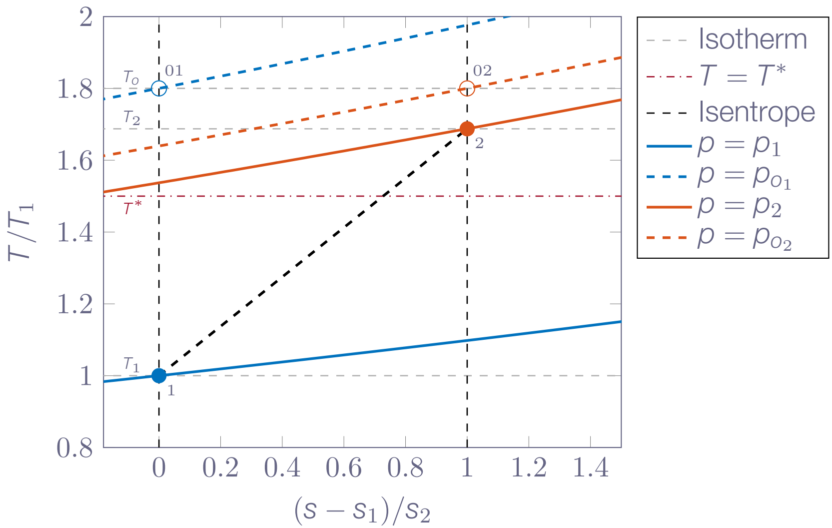
The Normal-shock Process

Note!

isotherms are less steep than isentropes $\Rightarrow p_{o_2} < p_{o_1}$



The Normal-shock Process



The Normal-shock Process

Continuity:

$$\rho_1 u_1 = \rho_2 u_2 = C > 0$$

Momentum:

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \Rightarrow p_1 + \frac{C^2}{\rho_1} = p_2 + \frac{C^2}{\rho_2} \Rightarrow p_1 + \nu_1 C^2 = p_2 + \nu_2 C^2$$

$$\frac{p_1 - p_2}{\nu_1 - \nu_2} = -C^2$$

a line in $p\nu$-space with negative slope

The Normal-shock Process

Energy equation:

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

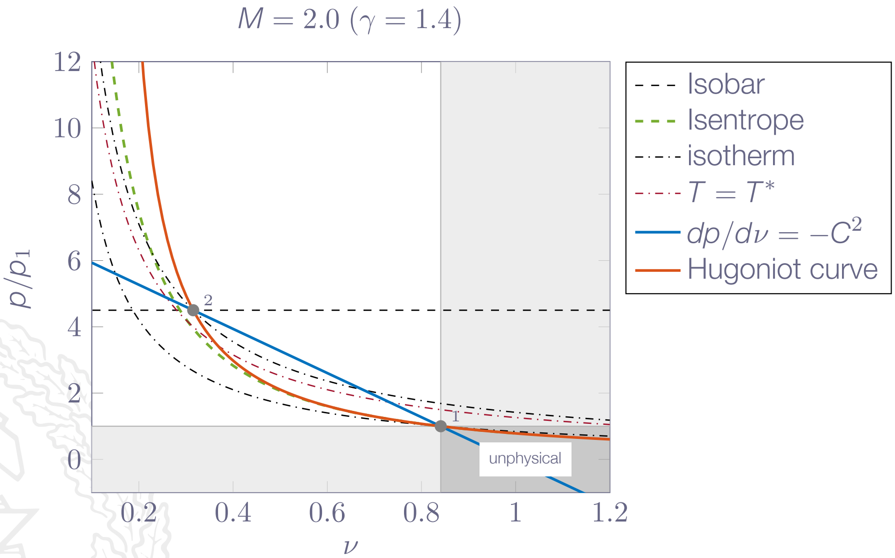
with $h = C_p T = \frac{\gamma R}{\gamma - 1} T$ and $u = \nu C$ we get

$$\frac{\gamma R}{\gamma - 1} T_1 + \frac{1}{2}\nu_1^2 C^2 = \frac{\gamma R}{\gamma - 1} T_2 + \frac{1}{2}\nu_2^2 C^2 \Rightarrow \ldots \Rightarrow \frac{p_2}{p_1}\left(\frac{\nu_2}{\nu_1} - \frac{\gamma + 1}{\gamma - 1}\right) / \left(1 - \frac{\nu_2}{\nu_1}\frac{\gamma + 1}{\gamma - 1}\right)$$

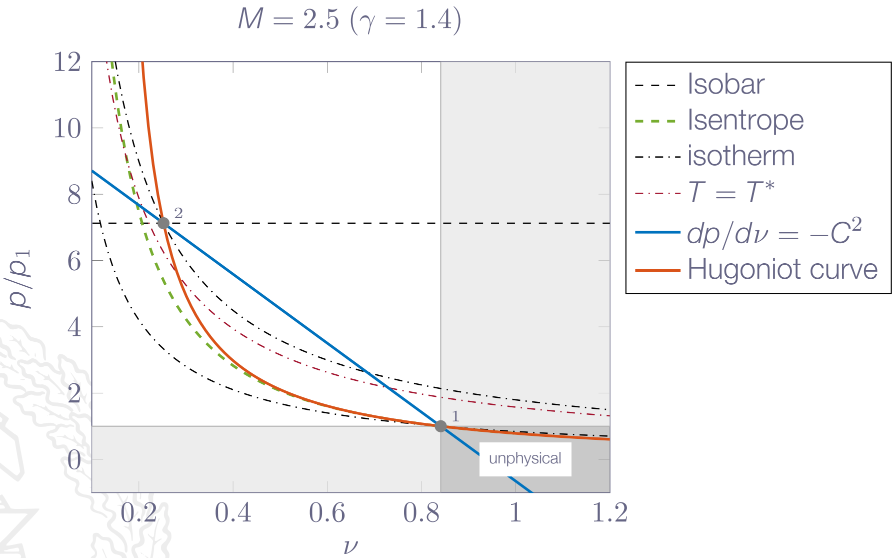
quadratic function in $p\nu$-space (Hugoniot curve)

only thermodynamic variables

The Normal-shock Process



The Normal-shock Process



The Normal-shock Process