Compressible Flow - TME085

Lecture Notes

Niklas Andersson

Chalmers University of Technology
Department of Mechanics and Maritime Sciences
Division of Fluid Mechanics
Gothenburg, Sweden

niklas.andersson@chalmers.se

CHALMERS
UNIVERSITY OF TECHNOLOGY

Compressible Flow

"Compressible flow (gas dynamics) is a branch of fluid mechanics that deals with flows having significant changes in fluid density"

Wikipedia

Gas Dynamics

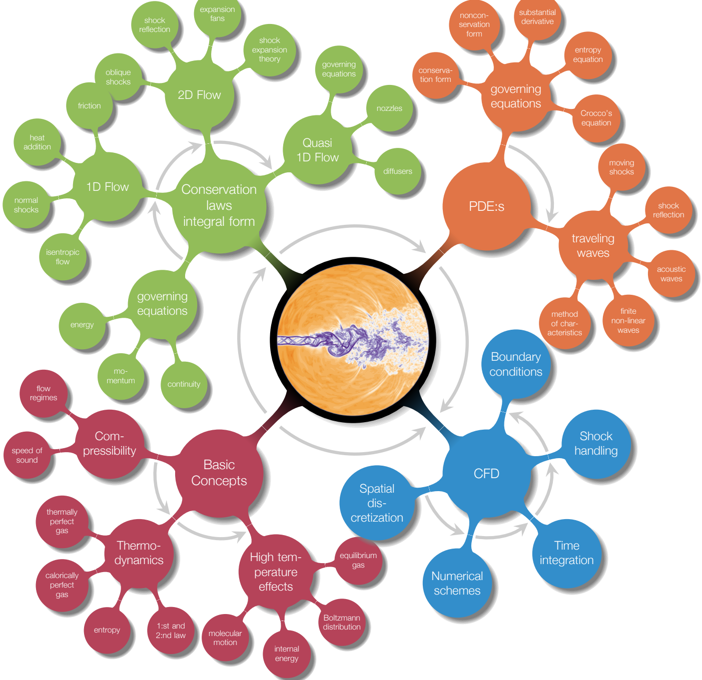
"... the study of motion of gases and its effects on physical systems ..."

"... based on the principles of fluid mechanics and thermodynamics ..."

"... gases flowing around or within physical objects at speeds comparable to the speed of sound ..."

Wikipedia

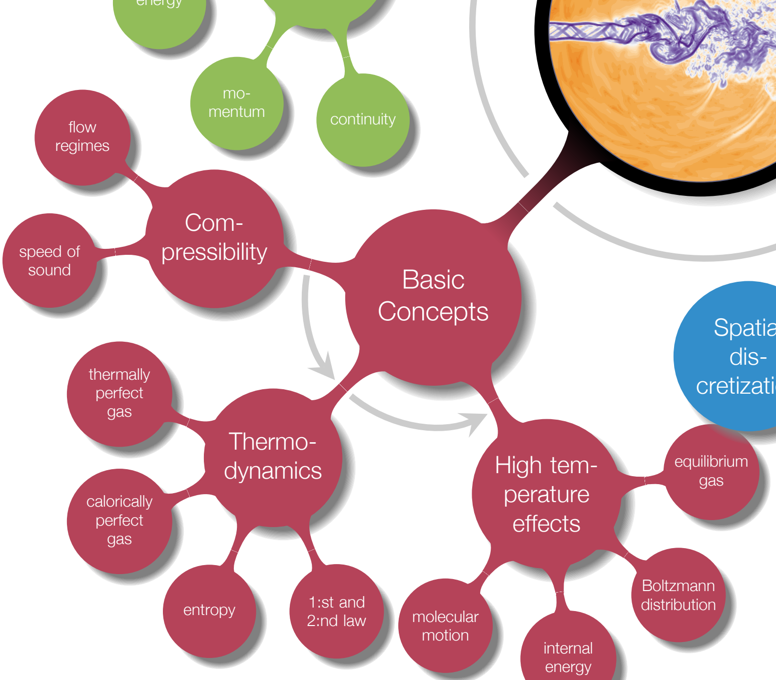
Overview



Conservation laws integral form
1D Flow
heat addition
normal shocks
isentropic flow
friction
2D Flow
oblique shocks
shock reflection
expansion fans
shock expansion theory
Quasi 1D Flow
governing equations
nozzles
diffusers
governing equations
energy
mo-mentum
continuity
PDE:s
governing equations
noncon-servation form
substantial derivative
entropy equation
conserva-tion form
Crocco's equation
traveling waves
moving shocks
shock reflection
acoustic waves
finite non-linear waves
method of char-acteristics
CFD
Boundary conditions
Shock handling
Time integration
Numerical schemes
Spatial dis-cretization
Basic Concepts
Com-pressibility
flow regimes
speed of sound
Thermo-dynamics
thermally perfect gas
calorically perfect gas
entropy
1:st and 2:nd law
High tem-perature effects
equilibrium gas
Boltzmann distribution
internal energy
molecular motion

Chapter 1
Introduction

Overview


energy
mo-
mentum
continuity
flow
regimes
Com-
pressibility
speed of
sound
Basic
Concepts
Spatia
dis-
cretizati
thermally
perfect
gas
Thermo-
dynamics
calorically
perfect
gas
entropy
1:st and
2:nd law
High tem-
perature
effects
equilibrium
gas
molecular
motion
internal
energy
Boltzmann
distribution

Learning Outcomes

1 **Define** the concept of compressibility for flows

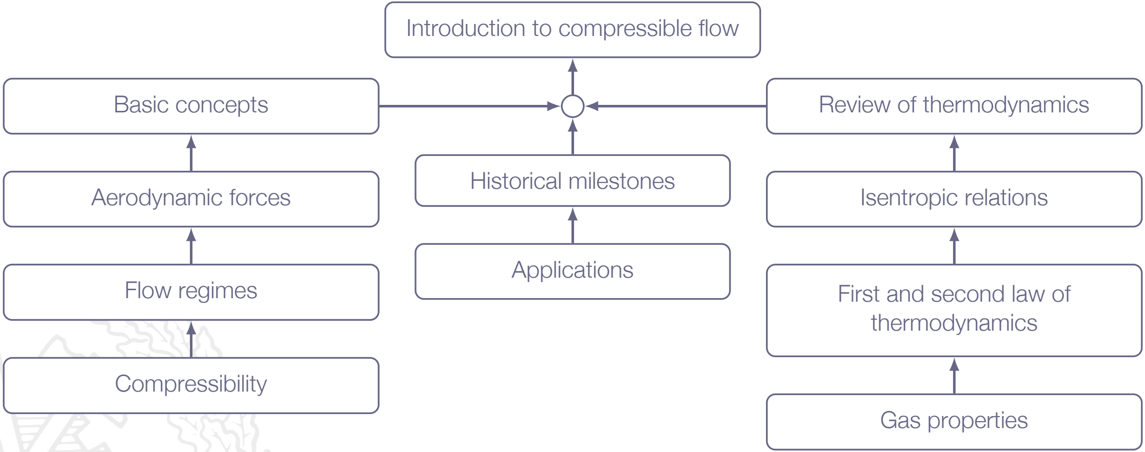
2 **Explain** how to find out if a given flow is subject to significant compressibility effects

3 **Describe** typical engineering flow situations in which compressibility effects are more or less predominant (e.g. Mach number regimes for steady-state flows)

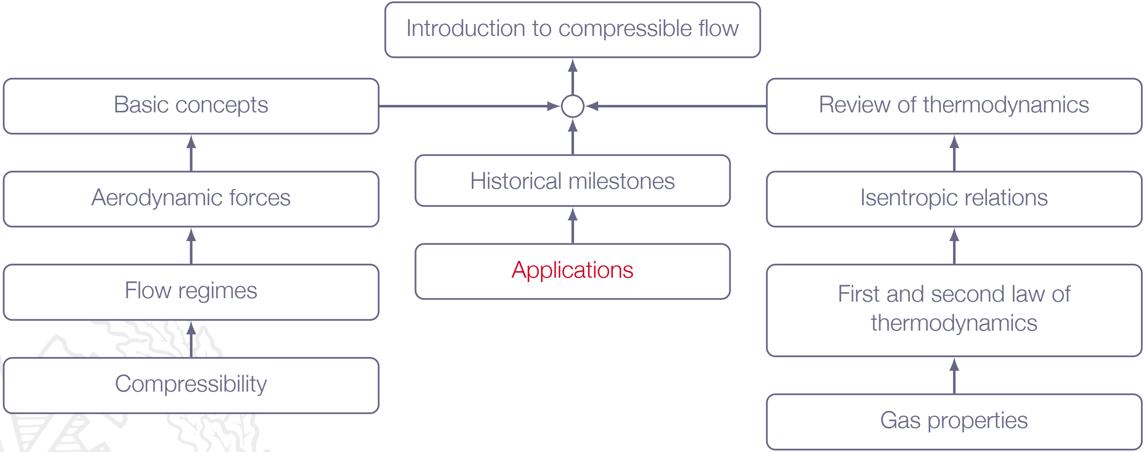
6 **Define** the special cases of calorically perfect gas, thermally perfect gas and real gas and **explain** the implication of each of these special cases

in this lecture we will find out what compressibility means and do a brief review of thermodynamics

Roadmap - Introduction to Compressible Flow



Roadmap - Introduction to Compressible Flow



Applications - Classical

- Treatment of calorically perfect gas
- Exact solutions of inviscid flow in 1D
- Shock-expansion theory for steady-state 2D flow
- Approximate closed form solutions to linearized equations in 2D and 3D
- Method of Characteristics (MOC) in 2D and axi-symmetric inviscid supersonic flows

Applications - Modern

Computational Fluid Dynamics (CFD)

Complex geometries (including moving boundaries)

Complex flow features (compression shocks, expansion waves, contact discontinuities)

Viscous effects

Turbulence modeling

High temperature effects (molecular vibration, dissociation, ionization)

Chemically reacting flow (equilibrium & non-equilibrium reactions)

Applications - Examples

Turbo-machinery flows:

Gas turbines, steam turbines, compressors
Aero engines (turbojets, turbofans, turboprops)

Aeroacoustics:

Flow induced noise (jets, wakes, moving surfaces)
Sound propagation in high speed flows

External flows:

Aircraft (airplanes, helicopters)
Space launchers (rockets, re-entry vehicles)

Internall flows:

Nozzle flows
Inlet flows, diffusers
Gas pipelines (natural gas, bio gas)

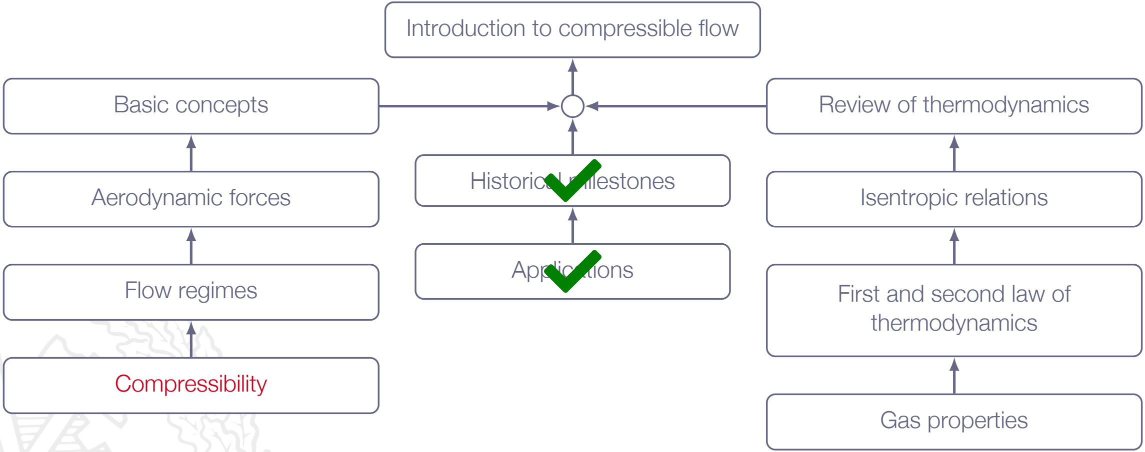
Free-shear flows:

High speed jets

Combustion:

Internal combustion engines (valve flow, in-cylinder flow, exhaust pipe flow, mufflers)
Combustion induced noise (turbulent combustion)
Combustion instabilities

Roadmap - Introduction to Compressible Flow



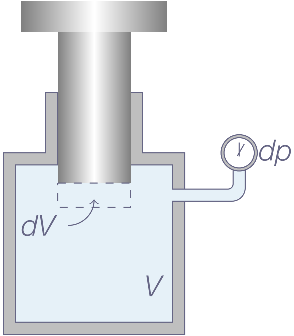
Chapter 1.2
Compressibility

Compressibility

$$\tau = -\frac{1}{\nu}\frac{\partial \nu}{\partial p}, \; (\nu = \frac{1}{\rho})$$

Not really precise!

Is T held constant during the compression or not?



Compressibility

Two fundamental cases:

Constant temperature

Heat is cooled off to keep T constant inside the cylinder

Adiabatic process

Thermal insulation prevents heat exchange

Compressibility

Isothermal process:

$$\tau_T = -\frac{1}{\nu}\left(\frac{\partial \nu}{\partial p}\right)_T$$

Adiabatic reversible (**isentropic**) process:

$$\tau_S = -\frac{1}{\nu}\left(\frac{\partial \nu}{\partial p}\right)_S$$

Air at normal conditions: $\tau_T \approx 1.0 \times 10^{-5} \quad [m^2/N]$

Water at normal conditions: $\tau_T \approx 5.0 \times 10^{-10} \quad [m^2/N]$

Compressibility

$$\tau = -\frac{1}{\nu}\frac{\partial \nu}{\partial p} \text{ where } \nu = \frac{1}{\rho} \text{ and thus}$$

$$\tau = -\rho\frac{\partial}{\partial p}\left(\frac{1}{\rho}\right) = -\rho\left(-\frac{1}{\rho^2}\right)\frac{\partial \rho}{\partial p} = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$$

$$\tau_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T$$

$$\tau_S = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_S$$

Compressibility

Definition of compressible flow:

If p changes with amount Δp over a characteristic length scale of the flow, such that the corresponding change in density, given by $\Delta\rho \sim \rho\tau\Delta$ p, is **too large to be neglected**, the flow is compressible (*typically* $\Delta\rho/\rho > 0.05$)

Note! Bernoulli´s equation is restricted to incompressible flow, *i.e.* it is **not valid** for compressible flow!

Compressibility - Mach Number

The freestream Mach number is defined as

$$M_\infty = \frac{U_\infty}{a_\infty}$$

where U_∞ is the freestream flow speed and a_∞ is the speed of sound at freestream conditions

Compressibility

Assume incompressible flow and estimate the maximum pressure difference using

$$\Delta p \approx \frac{1}{2}\rho_\infty U_\infty^2$$

For air at normal conditions we have

$$\tau_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T = \left\{p = \rho RT \Rightarrow \left(\frac{\partial \rho}{\partial p}\right)_T = \frac{1}{RT}\right\} = \frac{1}{\rho RT} = \frac{1}{p}$$

(ideal gas law for perfect gas $p = \rho RT$)

Compressibility

Using the relations on previous slide we get

$$\frac{\Delta\rho}{\rho} \approx \tau_T \Delta p \approx \frac{1}{p_\infty}\frac{1}{2}\rho_\infty U_\infty^2 = \frac{\frac{1}{2}\rho_\infty U_\infty^2}{\rho_\infty R T_\infty}$$

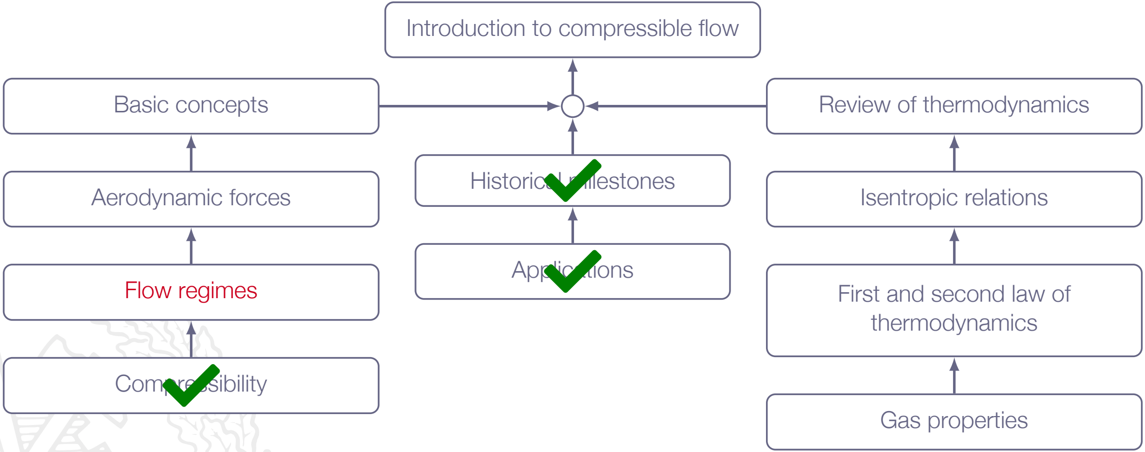
for a calorically perfect gas we have $a = \sqrt{\gamma R T}$

which gives us $\dfrac{\Delta\rho}{\rho} \approx \dfrac{\gamma U_\infty^2}{2a_\infty^2}$

now, using the definition of Mach number we get:

$$\boxed{\frac{\Delta\rho}{\rho} \approx \frac{\gamma}{2} M_\infty^2}$$

Roadmap - Introduction to Compressible Flow



Chapter 1.3
Flow Regimes

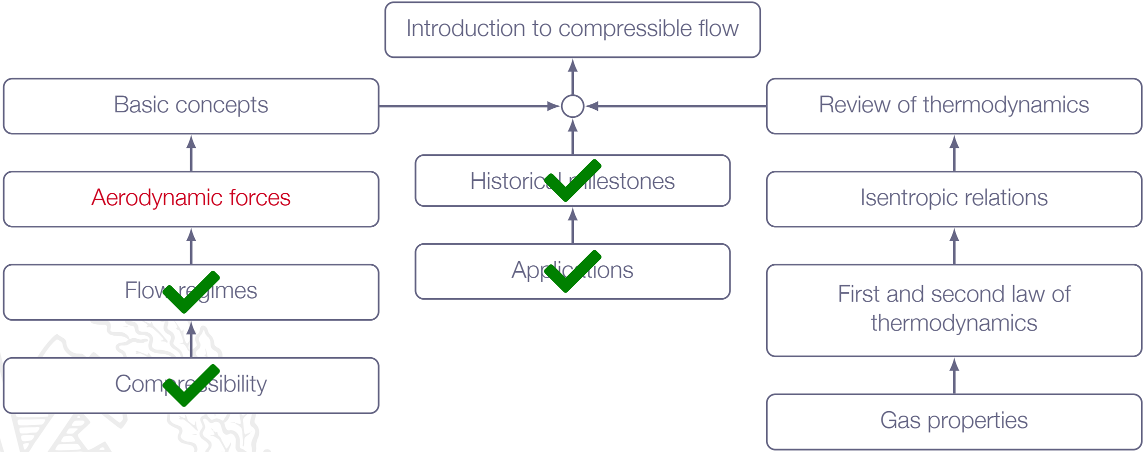
Flow Regimes

Incompressible	$M_\infty < 0.1$
Subsonic	$M_\infty < 1$ and $M < 1$ everywhere
Transonic	case 1: $M_\infty < 1$ and $M > 1$ locally case 2: $M_\infty > 1$ and $M < 1$ locally
Supersonic	$M_\infty > 1$ and $M > 1$ everywhere
Hypersonic	supersonic flow with high-temperature effects

Compressible

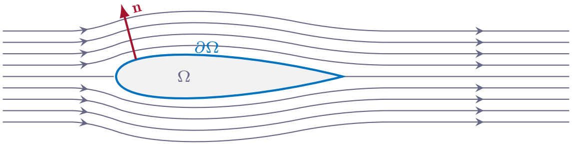
Local Mach number M is based on local flow speed, $U = |\mathbf{U}|$, and local speed of sound, a

Roadmap - Introduction to Compressible Flow



Chapter 1.5
Aerodynamic Forces

Aerodynamic Forces



Ω	region occupied by body
$\partial\Omega$	surface of body
$\mathbf{n}$	outward facing unit normal vector

Aerodynamic Forces

Overall forces on the body du to the flow

$$\mathbf{F} = \oiint (-p\mathbf{n} + \tau \cdot \mathbf{n}) dS$$

where p is static pressure and τ is a stress tensor

Aerodynamic Forces

Drag is the component of $\mathbf{F}$ which is **parallel** with the freestream direction:

$$D = D_p + D_f$$

where D_p is drag due to pressure and D_f is drag due to friction

Lift is the component of $\mathbf{F}$ which is **normal** to the free stream direction:

$$L = L_p + L_f$$

(L_f is usually negligible)

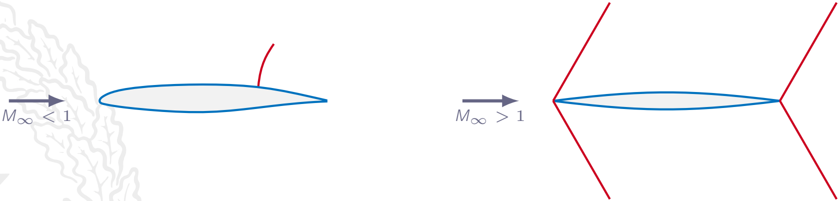
Aerodynamic Forces

Inviscid flow around slender body (*attached flow*)

subsonic flow: $D = 0$

transonic or supersonic flow: $D > 0$

Explanation: **Wave drag**



Aerodynamic Forces

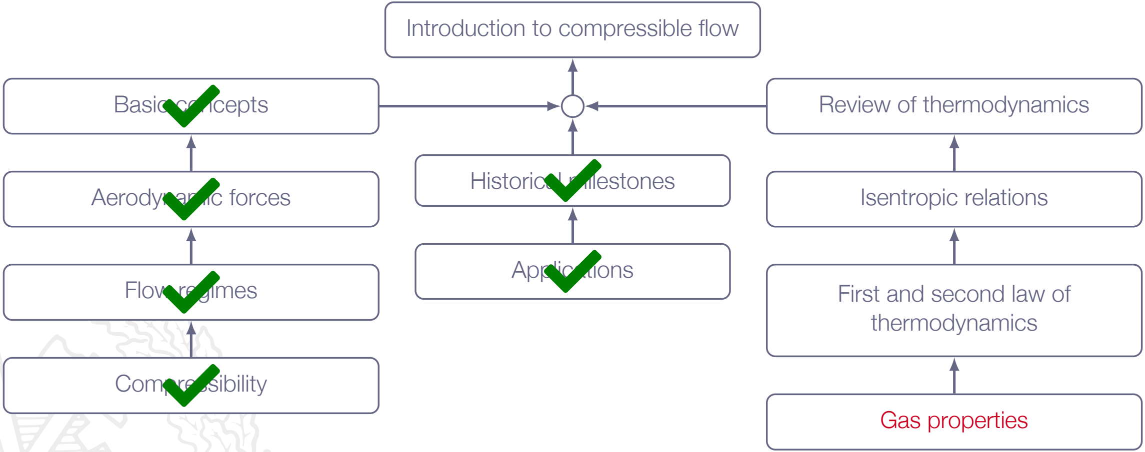
Wave drag is an **inviscid phenomena**, connected to the formation of compression shocks and entropy increase

Viscous effects are present in all Mach regimes

At transonic and supersonic conditions a particular phenomena named **shock/boundary-layer interaction** may appear

- shocks trigger flow separation
- usually leads to unsteady flow

Roadmap - Introduction to Compressible Flow



Chapter 1.4
Review of Thermodynamics

Thermodynamic Review

Compressible flow:

" strong interaction between flow and thermodynamics ... "

Perfect Gas

All intermolecular forces negligible

Only elastic collitions between molecules

$p\nu = RT$ or $\dfrac{p}{\rho} = RT$

where R is the gas constant $[R] = J/kgK$

Also, $R = R_{univ}/M$ where M is the molecular weight of gas molecules (in $kg/kmol$) and $R_{univ} = 8314\, J/kmol\, K$

Internal Energy and Enthalpy

Internal energy e ($[e] = J/kg$)

Enthalpy h ($[h] = J/kg$)

$$h = e + p\nu = e + \frac{p}{\rho} \text{ (valid for all gases)}$$

For any gas in thermodynamic equilibrium, e and h are functions of only two thermodynamic variables (*any two variables may be selected*) *e.g.*

$e = e(T, \rho)$ or $h = h(T, p)$

Internal Energy and Enthalpy

Special cases:

Thermally perfect gas:

$e = e(T)$ and $h = h(T)$

OK assumption for air at near atmospheric conditions and $100K < T < 2500K$

Calorically perfect gas:

$e = C_v T$ and $h = C_p T$ (C_v and C_p are constants)

OK assumption for air at near atmospheric pressure and $100K < T < 1000K$

Specific Heat

For thermally perfect (and calorically perfect) gas

$$C_p = \left(\frac{\partial h}{\partial T}\right)_p, \quad C_v = \left(\frac{\partial e}{\partial T}\right)_v$$

since $h = e + p/\rho = e + RT$ we obtain:

$$C_p = C_v + R$$

The ratio of specific heats, γ, is defined as:

$$\gamma \equiv \frac{C_p}{C_v}$$

Specific Heat

Important!

calorically perfect gas:

C_v, C_p, and γ are **constants**

thermally perfect gas:

C_v, C_p, and γ will **depend on temperature**

Specific Heat

$$C_p - C_v = R$$

$$C_p - C_v = R$$

Specific Heat

$$C_p - C_v = R$$

divide by C_v

$$C_p - C_v = R$$

divide by C_p

Specific Heat

$$C_p - C_v = R$$

divide by C_v

$$\gamma - 1 = \frac{R}{C_v}$$

$$C_p - C_v = R$$

divide by C_p

$$1 - \frac{1}{\gamma} = \frac{\gamma - 1}{\gamma} = \frac{R}{C_p}$$

Specific Heat

$$C_p - C_v = R$$

divide by C_v

$$\gamma - 1 = \frac{R}{C_v}$$

$$C_v = \frac{R}{\gamma - 1}$$

$$C_p - C_v = R$$

divide by C_p

$$1 - \frac{1}{\gamma} = \frac{\gamma - 1}{\gamma} = \frac{R}{C_p}$$

$$C_p = \frac{\gamma R}{\gamma - 1}$$

Specific Heat

$$C_p - C_v = R$$

divide by C_v

$$\gamma - 1 = \frac{R}{C_v}$$

$$C_v = \frac{R}{\gamma - 1}$$

$$C_p - C_v = R$$

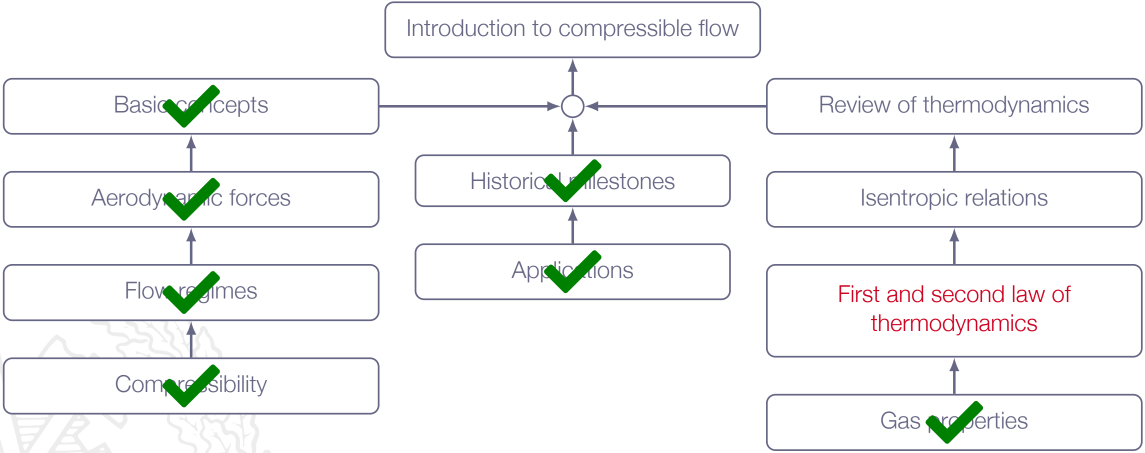
divide by C_p

$$1 - \frac{1}{\gamma} = \frac{\gamma - 1}{\gamma} = \frac{R}{C_p}$$

$$C_p = \frac{\gamma R}{\gamma - 1}$$

valid for both thermally perfect and calorically perfect gas!

Roadmap - Introduction to Compressible Flow



First Law of Thermodynamics

A fixed mass of gas, separated from its surroundings by an imaginary flexible boundary, is defined as a **system**. This system obeys the relation

$$de = \delta q - \delta w$$

where

de is a change in internal energy of system

δq is heat added to the system

δw is work done by the system (on its surroundings)

Note! de only depends on starting point and end point of the process while δq and δw depend on the actual process also

First Law of Thermodynamics

Examples:

Adiabatic process:

$\delta q = 0.$

Reversible process:

no dissipative phenomena (*no flow losses*)

Isentropic process:

a process which is both adiabatic and reversible

First Law of Thermodynamics

Reversible process:

$$\delta w = pd\nu = pd(1/\rho)$$

$$de = \delta q - pd(1/\rho)$$

Adiabatic & reversible process:

$$\delta q = 0.$$

$$de = -pd(1/\rho)$$

Entropy

Entropy s is a property of all gases, uniquely defined by any two thermodynamic variables, *e.g.*

$s = s(p, T)$ or $s = s(\rho, T)$ or $s = s(\rho, p)$ or $s = s(e, h)$ or ...

Second Law of Thermodynamics

Concept of entropy s:

$$ds = \frac{\delta q_{rev}}{T} = \frac{\delta q}{T} + ds_{ir} \text{ where } ds_{ir} > 0. \text{ and thus}$$

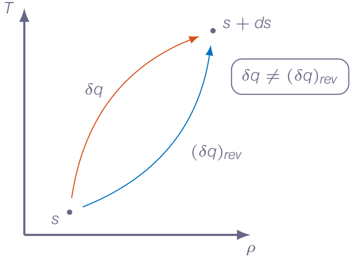
$$ds \geq \frac{\delta q}{T}$$

Second Law of Thermodynamics

Concept of entropy s:

$$ds = \frac{\delta q_{rev}}{T} = \frac{\delta q}{T} + ds_{ir} \text{ where } ds_{ir} > 0. \text{ and thus}$$

$$\boxed{ds \geq \frac{\delta q}{T}}$$



Second Law of Thermodynamics

In general:

$$ds \geq \frac{\delta q}{T}$$

For adiabatic processes:

$$ds \geq 0.$$

Calculation of Entropy

For reversible processes ($\delta w = pd(1/\rho)$ and $\delta q = Tds$):

$$de = Tds - pd\left(\frac{1}{\rho}\right) \Leftrightarrow Tds = de + pd\left(\frac{1}{\rho}\right)$$

from before we have $h = e + p/\rho \Rightarrow$

$$dh = de + pd\left(\frac{1}{\rho}\right) + \left(\frac{1}{\rho}\right)dp \Leftrightarrow de = dh - pd\left(\frac{1}{\rho}\right) - \left(\frac{1}{\rho}\right)dp$$

Calculation of Entropy

For thermally perfect gases, $p = \rho R T$ and $dh = C_p dT \Rightarrow ds = C_p \frac{dT}{T} - R \frac{dp}{p}$

Integration from starting point (1) to end point (2) gives:

$$s_2 - s_1 = \int_1^2 C_p \frac{dT}{T} - R \ln \left(\frac{p_2}{p_1} \right)$$

and for calorically perfect gases

$$\boxed{s_2 - s_1 = C_p \ln \left(\frac{T_2}{T_1} \right) - R \ln \left(\frac{p_2}{p_1} \right)}$$

Calculation of Entropy

If we instead use $de = C_v dT$ we get

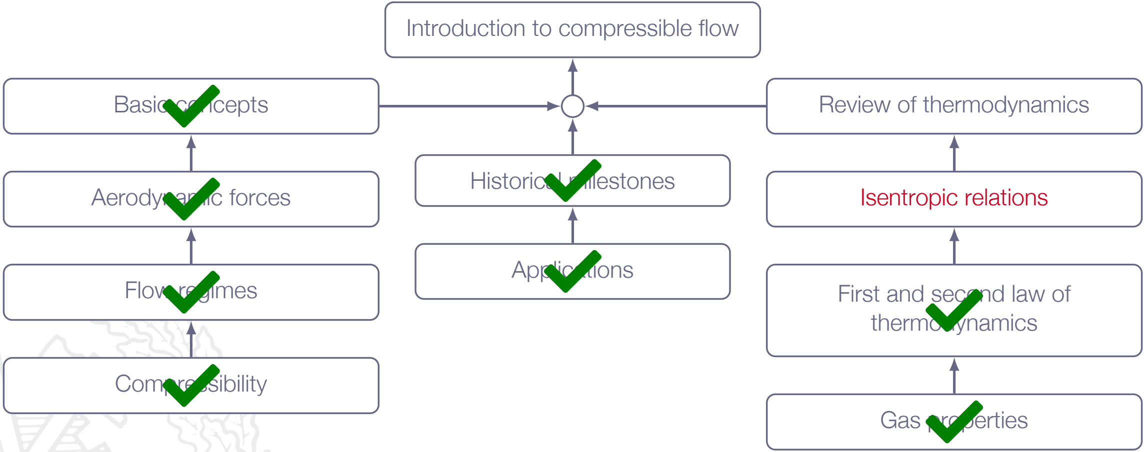
for thermally perfect gases

$$s_2 - s_1 = \int_1^2 C_v \frac{dT}{T} - R \ln \left(\frac{\rho_2}{\rho_1} \right)$$

and for calorically perfect gases

$$\boxed{s_2 - s_1 = C_v \ln \left(\frac{T_2}{T_1} \right) - R \ln \left(\frac{\rho_2}{\rho_1} \right)}$$

Roadmap - Introduction to Compressible Flow



Isentropic Relations

For calorically perfect gases, we have

$$s_2 - s_1 = C_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right)$$

For adiabatic reversible processes:

$$ds = 0. \Rightarrow s_1 = s_2 \Rightarrow C_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right) = 0 \Rightarrow$$

$$\ln\left(\frac{p_2}{p_1}\right) = \frac{C_p}{R} \ln\left(\frac{T_2}{T_1}\right)$$

Isentropic Relations

with $\frac{C_p}{R} = \frac{C_p}{C_p - C_v} = \frac{\gamma}{\gamma - 1} \Rightarrow$

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

Isentropic Relations

Alternatively, using $s_2 - s_1 = 0 = C_v \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{\rho_2}{\rho_1}\right) \Rightarrow$

$$\boxed{\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{\gamma-1}}}$$

Isentropic Relations - Summary

For an isentropic process and a calorically perfect gas we have

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

A.K.A. the **isentropic relations**

Thermodynamic Relations and Process Diagrams

Many times it's process diagrams makes it easier to understand physics

Examples of process diagrams: Ts-diagram and $p\nu$-diagram

We will use process diagrams in the following chapters to give insights into physical processes such as shocks, heat addition and friction

Thermodynamic Relations and Process Diagrams

From before:

$$ds = C_v \frac{dT}{T} + R\frac{d\nu}{\nu}$$

$$ds = C_p \frac{dT}{T} - R\frac{dp}{p}$$

Ts-diagram

$$ds = C_v \frac{dT}{T} + R \frac{d\nu}{\nu}$$

$$d\nu = \frac{\nu}{R} ds - C_v \frac{\nu}{RT} dT$$

$$d\nu = \frac{\nu}{R} ds - \frac{C_v}{p} dT$$

$ds = 0 \Rightarrow d\nu < 0$ for positive dT

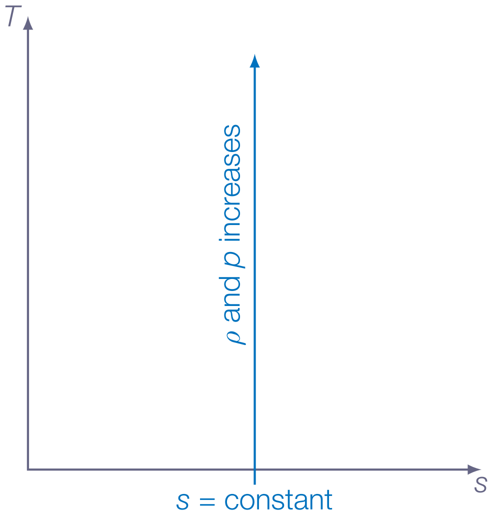
$$ds = C_p \frac{dT}{T} - R \frac{dp}{p}$$

$$dp = -\frac{p}{R} ds + C_p \frac{p}{RT} dT$$

$$dp = -\frac{p}{R} ds + C_p \rho dT$$

$ds = 0 \Rightarrow dp > 0$ for positive dT

Ts-diagram



Ts-diagram - Isochoric process

$$d\nu = \frac{\nu}{R}ds - \frac{C_v}{p}dT$$

From before: ν decreases with T and p increases with T and thus for a given dT, $d\nu$ will be greater at lower T than at higher T

ν=constant lines will be closely spaced at low T and more sparse at high T

ν=constant $\Rightarrow d\nu = 0$:

$$0 = \frac{\nu}{R}\left(ds - C_v\frac{dT}{T}\right) \Rightarrow \frac{dT}{ds} = \frac{T}{C_v}$$

slope is positive and increases with temperature

Ts-diagram - Isobaric process

$$ds = C_p \frac{dT}{T} - R\frac{dp}{p}$$

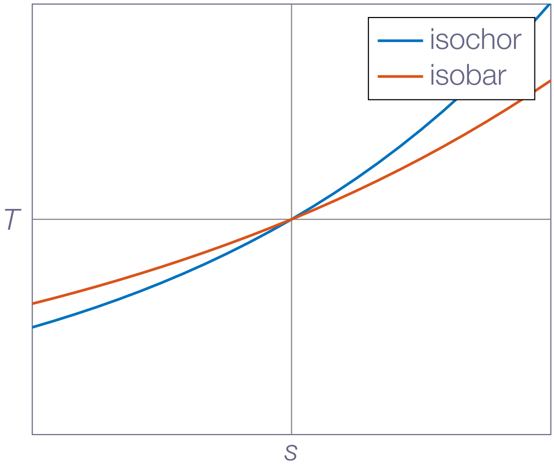
p=constant $\Rightarrow dp = 0$:

$$0 = \frac{p}{R}\left(C_p \frac{dT}{T} - ds\right) \Rightarrow \frac{dT}{ds} = \frac{T}{C_p}$$

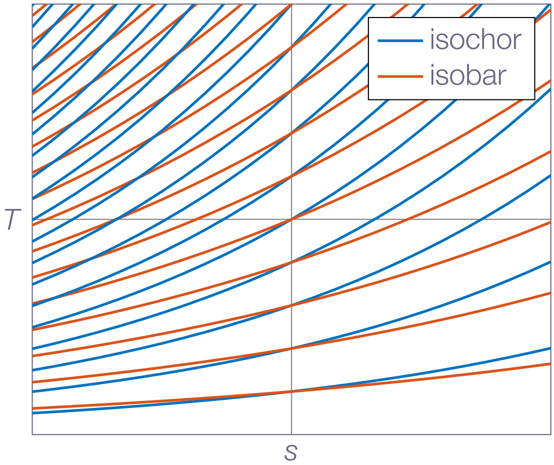
slope is positive and increases with temperature

$C_p > C_v \Rightarrow$ isobars are less steep than isochors

Ts-diagrams



Ts-diagrams



$p\nu$-diagrams

subtract

$$C_p \left[ds = C_v \frac{dT}{T} + R \frac{d\nu}{\nu} \right]$$

from

$$C_v \left[ds = C_p \frac{dT}{T} - R \frac{dp}{p} \right]$$

gives

$$ds \underbrace{(C_p - C_v)}_{R} = RC_p \frac{d\nu}{\nu} + RC_v \frac{dp}{p} \Rightarrow ds = C_p \frac{d\nu}{\nu} + C_v \frac{dp}{p}$$

$p\nu$-diagrams

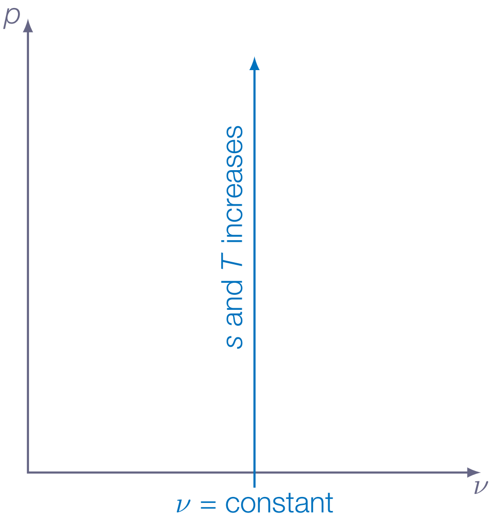
$$ds = C_p \frac{d\nu}{\nu} + C_v \frac{dp}{p}$$

$d\nu = 0$ (isochoric process) $\Rightarrow ds = C_v \dfrac{dp}{p}$

entropy increases with increasing pressure

from before: temperature increases with increasing pressure

$p\nu$-diagrams



$p\nu$-diagrams - isentropic process

$$ds = C_p \frac{d\nu}{\nu} + C_v \frac{dp}{p}$$

s=constant ($ds = 0$):

$$C_p \frac{d\nu}{\nu} + C_v \frac{dp}{p} = 0 \Rightarrow \frac{dp}{d\nu} = -\gamma \frac{p}{\nu}$$

negative slope

slope becomes steeper with increased pressure and decreased ν

$p\nu$-diagrams - isothermal process

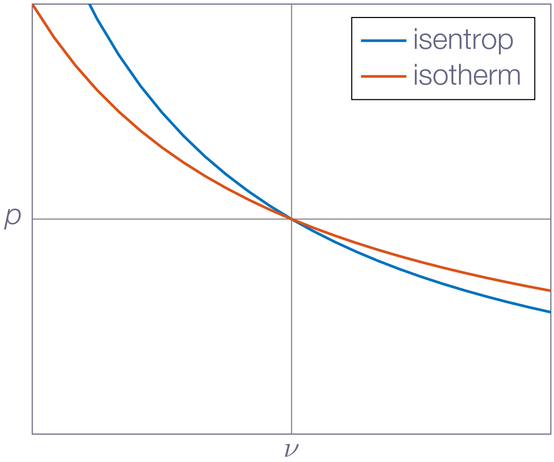
$$ds = C_v \frac{dT}{T} + R\frac{d\nu}{\nu} = C_p \frac{dT}{T} - R\frac{dp}{p}$$

T=constant ($dT = 0$):

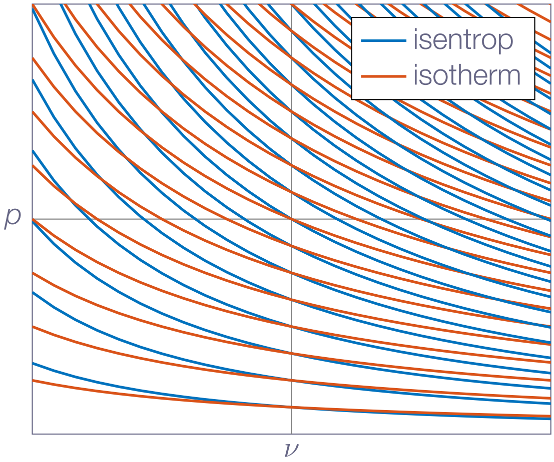
$$\frac{d\nu}{\nu} = -\frac{dp}{p} \Rightarrow \frac{dp}{d\nu} = -\frac{p}{\nu}$$

$\gamma > 0 \Rightarrow$ isentropes are steeper than isotherms

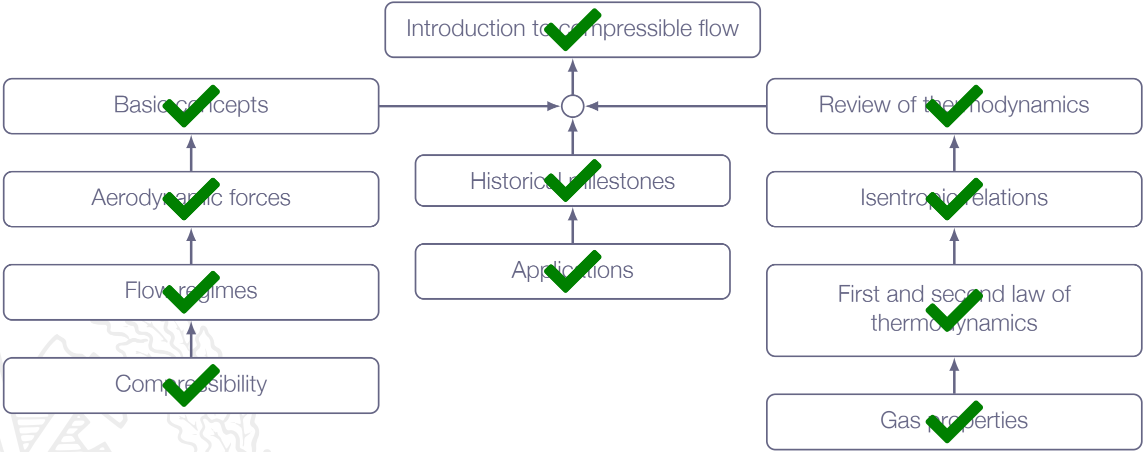
$p\nu$-diagrams - isothermal process



$p\nu$-diagrams - isothermal process

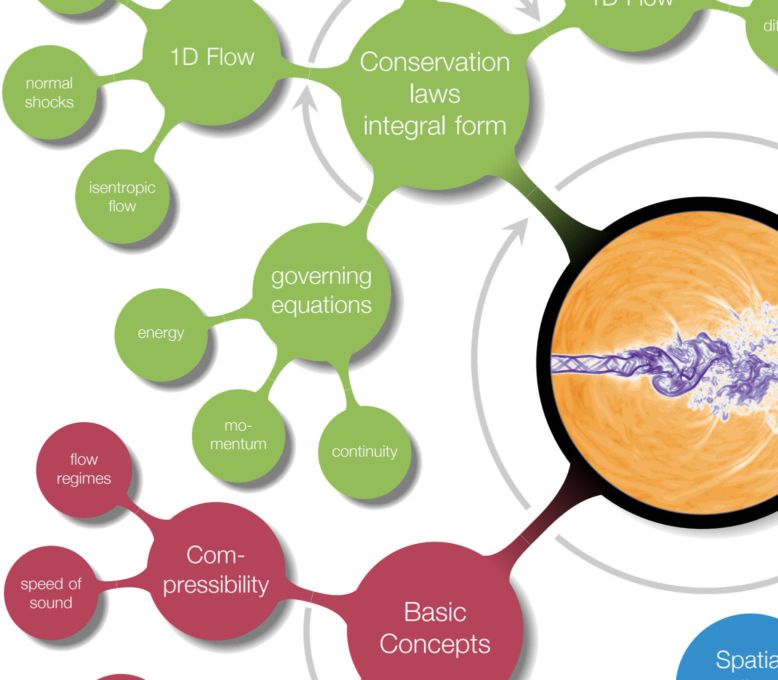


Roadmap - Introduction to Compressible Flow



Chapter 2
Integral Forms of the Conservation Equations for Inviscid Flows

Overview


1D Flow
normal shocks
isentropic flow
Conservation laws integral form
governing equations
energy
mo-mentum
continuity
flow regimes
Com-pressibility
speed of sound
Basic Concepts

Learning Outcomes

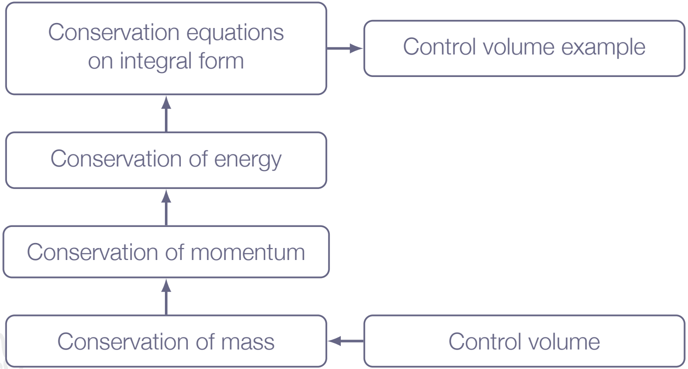
4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on

5 **Explain** how thermodynamic relations enter into the flow equations

7 **Explain** why entropy is important for flow discontinuities

equations, equations and more equations

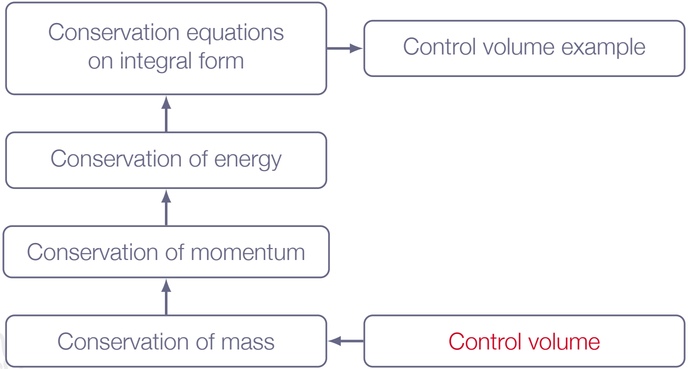
Roadmap - Integral Relations



Motivation

We need to formulate the basic form of the governing equations for compressible flow before we get to the applications

Roadmap - Integral Relations



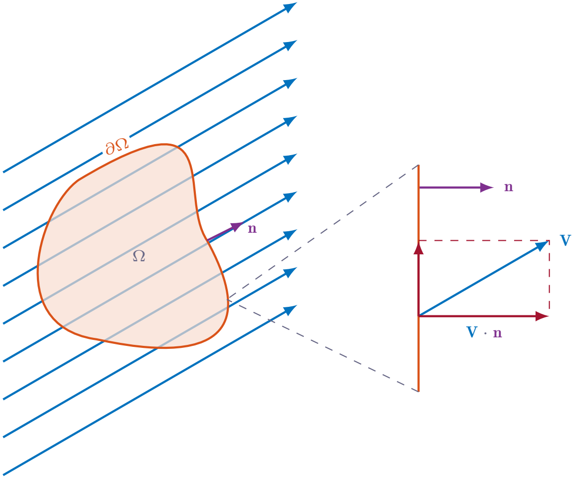
Integral Forms of the Conservation Equations

Conservation principles:

1. conservation of mass
2. conservation of momentum (*Newton's second law*)
3. conservation of energy (*first law of thermodynamics*)

Integral Forms of the Conservation Equations

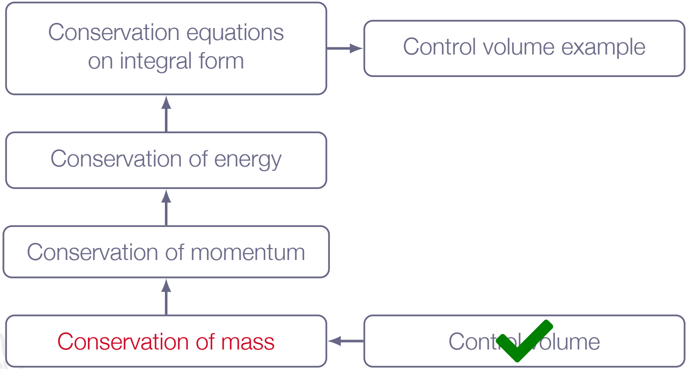
The control volume approach



Notation:

Ω fixed control volume
$\partial\Omega$ boundary of Ω
$\mathbf{n}$ outward facing unit normal vector
$\mathbf{v}$ fluid velocity ($v = |\mathbf{v}|$)

Roadmap - Integral Relations



Chapter 2.3
Continuity Equation

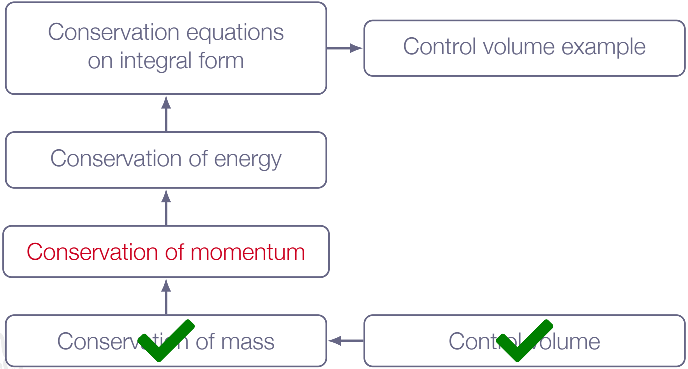
Continuity Equation

Conservation of mass:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega} \rho d\mathscr{V}}_{\substack{\text{rate of change of}\\ \text{total mass in } \Omega}} + \underbrace{\oint\!\!\!\oint\limits_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS}_{\substack{\text{net mass flow out}\\ \text{from } \Omega}} = 0$$

Note! notation in the text book $\mathbf{n} \cdot dS = d\mathbf{S}$

Roadmap - Integral Relations



Chapter 2.4
Momentum Equation

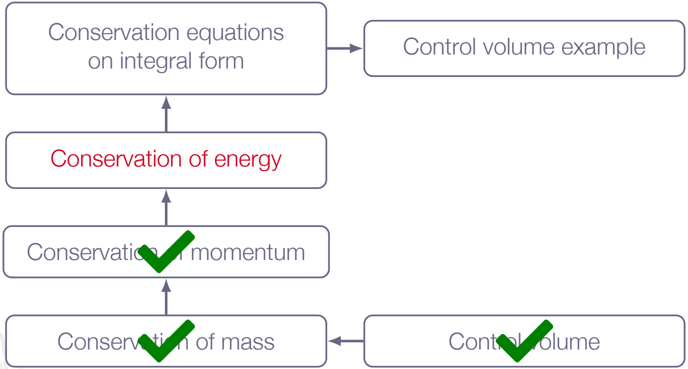
Momentum Equation

Conservation of momentum:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega} \rho \mathbf{v} d\mathscr{V}}_{\substack{\text{rate of change of total} \\ \text{momentum in } \Omega}} + \underbrace{\oiint\limits_{\partial\Omega} \left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v} + p\mathbf{n}\right] dS}_{\substack{\text{net momentum flow out from} \\ \Omega \text{ plus surface force on } \partial\Omega \\ \text{due to pressure}}} = \underbrace{\iiint\limits_{\Omega} \rho \mathbf{f} d\mathscr{V}}_{\substack{\text{rate of momentum} \\ \text{generation due to} \\ \text{forces inside } \Omega}}$$

Note! friction forces due to viscosity are not included here. To account for these forces, the term $-(\tau\cdot\mathbf{n})$ must be added to the surface integral term.
The body force, f, is force per unit mass.

Roadmap - Integral Relations



Chapter 2.5
Energy Equation

Energy Equation

Conservation of energy:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho e_o d\mathscr{V}}_{\text{rate of change of total internal energy in } \Omega} + \underbrace{\oiint\limits_{\partial\Omega}\left[\rho e_o(\mathbf{v}\cdot\mathbf{n}) + p\mathbf{v}\cdot\mathbf{n}\right]dS}_{\text{net flow of total internal energy out from } \Omega \text{ plus work due to surface pressure on } \partial\Omega} = \underbrace{\iiint\limits_{\Omega}\rho\mathbf{f}\cdot\mathbf{v}d\mathscr{V}}_{\text{work due to forces inside } \Omega}$$

where

$$\rho e_o = \rho\left(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) = \rho\left(e + \frac{1}{2}v^2\right)$$

is the total internal energy

Energy Equation

The surface integral term may be rewritten as follows:

$$\iint_{\partial\Omega} \left[\rho\left(e+\frac{1}{2}v^2\right)(\mathbf{v}\cdot\mathbf{n})+p\mathbf{v}\cdot\mathbf{n}\right]dS$$

$$\Leftrightarrow$$

$$\iint_{\partial\Omega} \left[\rho\left(e+\frac{p}{\rho}+\frac{1}{2}v^2\right)(\mathbf{v}\cdot\mathbf{n})\right]dS$$

$$\Leftrightarrow$$

$$\iint_{\partial\Omega} \left[\rho\left(h+\frac{1}{2}v^2\right)(\mathbf{v}\cdot\mathbf{n})\right]dS$$

Energy Equation

Introducing total enthalpy

$$h_o = h + \frac{1}{2}v^2$$

we get

$$\frac{d}{dt}\iiint\limits_{\Omega} \rho e_o d\mathscr{V} + \oiint\limits_{\partial\Omega} \left[\rho h_o \mathbf{v}\cdot\mathbf{n}\right] dS = \iiint\limits_{\Omega} \rho\mathbf{f}\cdot\mathbf{v} d\mathscr{V}$$

Energy Equation

Note 1: to include friction work on $\partial\Omega$, the energy equation is extended as

$$\frac{d}{dt}\iiint_{\Omega} \rho e_o d\mathscr{V} + \oiint_{\partial\Omega} \left[\rho h_o \mathbf{v}\cdot\mathbf{n} - (\tau\cdot\mathbf{n})\cdot\mathbf{v}\right] dS = \iiint_{\Omega} \rho\mathbf{f}\cdot\mathbf{v} d\mathscr{V}$$

Note 2: to include heat transfer on $\partial\Omega$, the energy equation is further extended

$$\frac{d}{dt}\iiint_{\Omega} \rho e_o d\mathscr{V} + \oiint_{\partial\Omega} \left[\rho h_o \mathbf{v}\cdot\mathbf{n} - (\tau\cdot\mathbf{n})\cdot\mathbf{v} + \mathbf{q}\cdot\mathbf{n}\right] dS = \iiint_{\Omega} \rho\mathbf{f}\cdot\mathbf{v} d\mathscr{V}$$

where $\mathbf{q}$ is the heat flux vector

Energy Equation

Note 3: the force $\mathbf{f}$ inside Ω may be a distributed body force field

Examples:

- Gravity
- Coriolis and centrifugal acceleration terms in a rotating frame of reference

Energy Equation

Note 4: there may be objects inside Ω which we choose to represent as sources of momentum and energy.

For example, there may be a solid object inside Ω which acts on the fluid with a force $\mathbf{F}$ and performs work $\dot{W}$ on the fluid

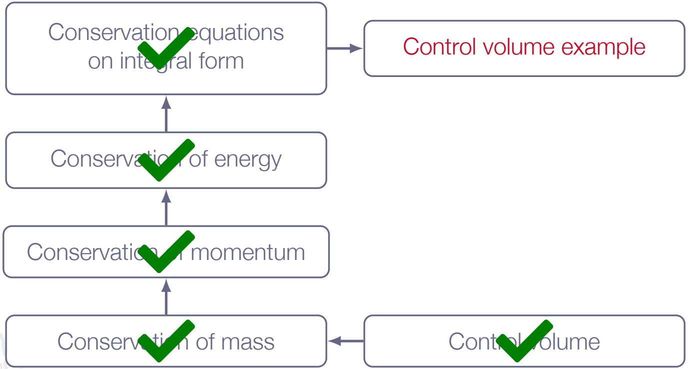
Momentum equation:

$$\frac{d}{dt}\iiint_{\Omega} \rho \mathbf{v} d\mathscr{V} + \oiint_{\partial\Omega} \left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v} + p\mathbf{n}\right] dS = \iiint_{\Omega} \rho \mathbf{f} d\mathscr{V} + \mathbf{F}$$

Energy equation:

$$\frac{d}{dt}\iiint_{\Omega} \rho e_o d\mathscr{V} + \oiint_{\partial\Omega} \left[\rho h_o \mathbf{v}\cdot\mathbf{n}\right] dS = \iiint_{\Omega} \rho \mathbf{f}\cdot\mathbf{v} d\mathscr{V} + \dot{W}$$

Roadmap - Integral Relations



Integral Equations - Applications

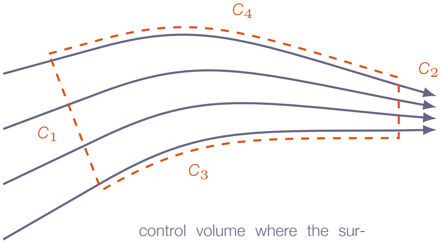
How can we use control volume formulations of conservation laws?

Let $\Omega \to 0$: In the limit of vanishing volume the control volume formulations give the **P**artial **D**ifferential **E**quations (**PDE**:s) for mass, momentum and energy conservation (see Chapter 6)

Apply in a ”smart” way $\Rightarrow$ Analysis tool for many practical problems involving compressible flow (see Chapter 2, Section 2.8)

Integral Equations - Applications

Example: Steady-state adiabatic inviscid flow



control volume where the surfaces C_1 and C_2 are normal to the flow and C_3 and C_4 are parallel to the stream lines

Integral Equations - Applications

Conservation of mass:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega} \rho d\mathscr{V}}_{=0} + \underbrace{\oint\!\!\!\oint\limits_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS}_{-\rho_1 v_1 A_1 + \rho_2 v_2 A_2} = 0$$

Conservation of energy:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega} \rho e_o d\mathscr{V}}_{=0} + \underbrace{\oint\!\!\!\oint\limits_{\partial\Omega} \left[\rho h_o \mathbf{v} \cdot \mathbf{n}\right] dS}_{-\rho_1 h_{o_1} v_1 A_1 + \rho_2 h_{o_2} v_2 A_2} = 0$$

Integral Equations - Applications

Conservation of mass:

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2$$

Conservation of energy:

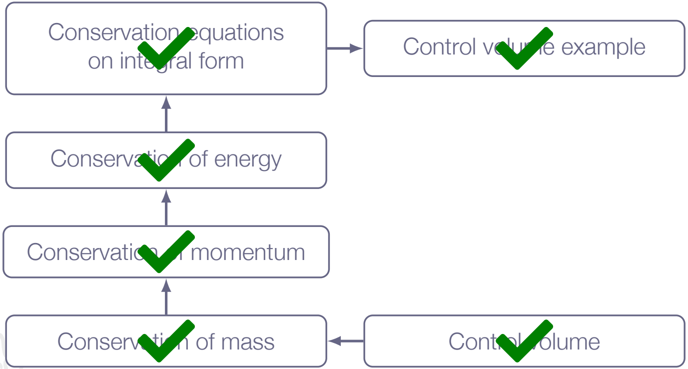
$$\rho_1 h_{o_1} v_1 A_1 = \rho_2 h_{o_2} v_2 A_2$$

$$\Leftrightarrow$$

$$h_{o_1} = h_{o_2}$$

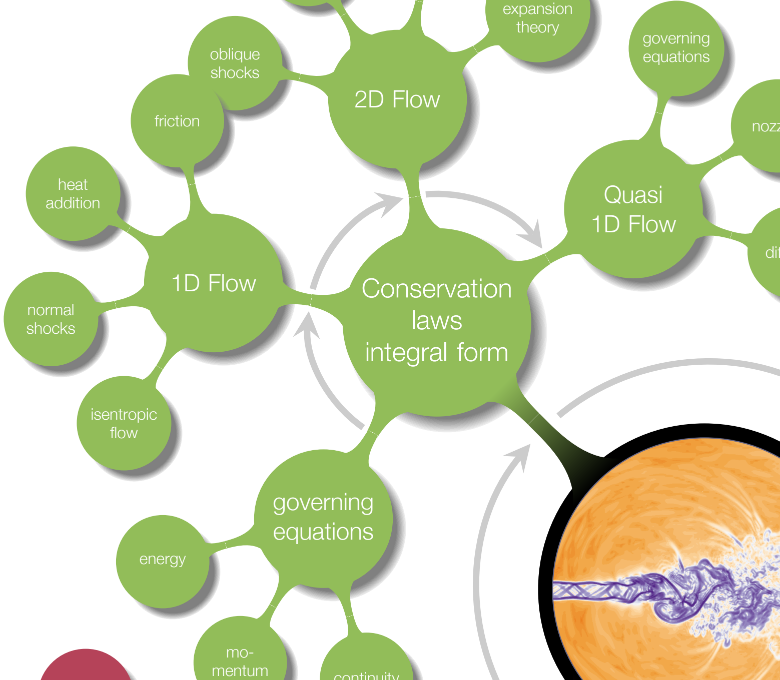
Total enthalpy h_o is conserved along streamlines in steady-state adiabatic inviscid flow

Roadmap - Integral Relations



Chapter 3
One-Dimensional Flow

Overview



expansion theory
governing equations
oblique shocks
2D Flow
friction
nozz
heat addition
Quasi 1D Flow
1D Flow
Conservation laws integral form
normal shocks
isentropic flow
governing equations
energy
mo-mentum
continuity

Learning Outcomes

4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on

5 **Explain** how thermodynamic relations enter into the flow equations

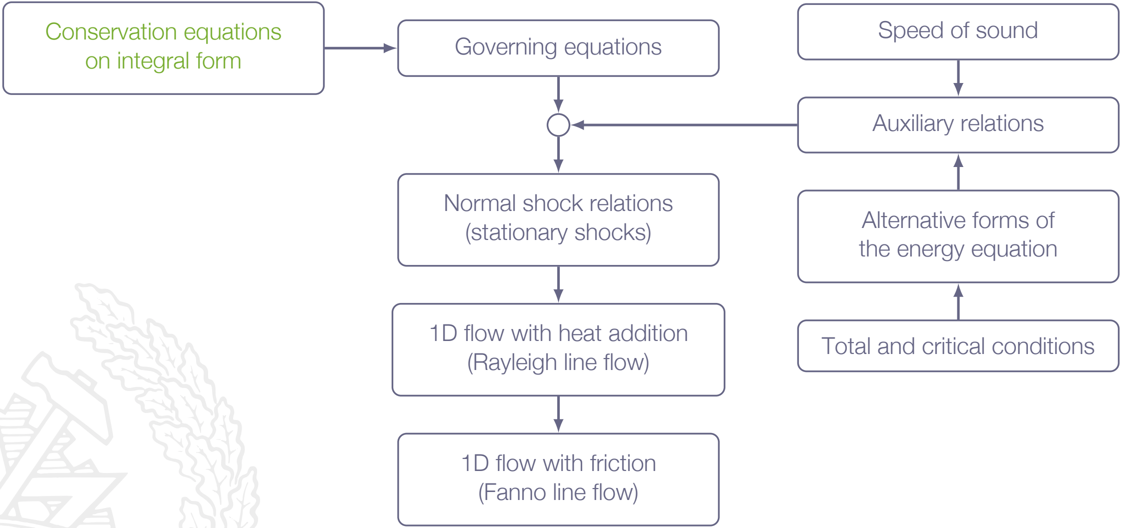
6 **Define** the special cases of calorically perfect gas, thermally perfect gas and real gas and **explain** the implication of each of these special cases

8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics

- a 1D isentropic flow*
- b normal shocks*
- c 1D flow with heat addition*
- d 1D flow with friction*

one-dimensional flows - isentropic and non-isentropic

Roadmap - One-dimensional Flow

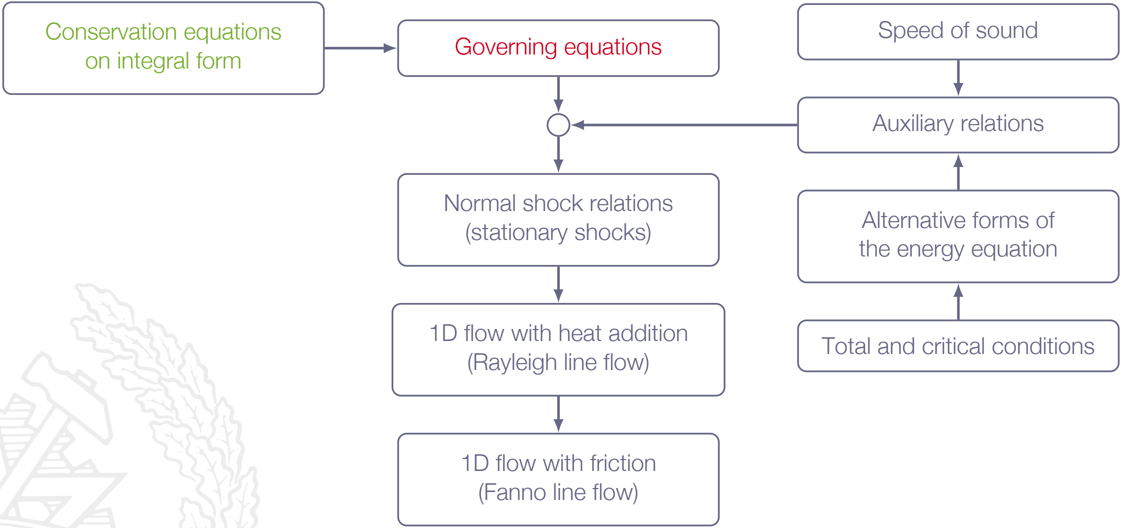


Motivation

Why one-dimensional flow?

- many practical problems can be analyzed using a one-dimensional flow approach
- a one-dimensional approach addresses the physical principles without adding the complexity of a full three-dimensional problem
- the one-dimensional approach is a subset of the full three-dimensional counterpart

Roadmap - One-dimensional Flow



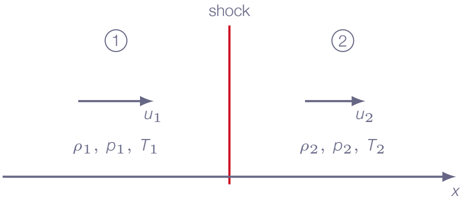
Chapter 3.2
One-Dimensional Flow Equations

One-Dimensional Flow Equations

Problems analyzed using the one-dimensional flow equations can be divided in to two categories:

1. problems with **wave solutions** (discontinuous)
   - acoustic wave
   - normal shock
2. problems with **continuous solutions**
   - flow with heat addition
   - flow with friction

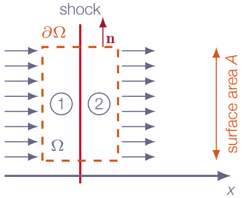
One-Dimensional Flow Equations



Assumptions:

- all flow variables only depend on x
- velocity aligned with x-axis

One-Dimensional Flow Equations



Control volume approach:

Define a rectangular control volume around shock, with upstream conditions denoted by 1 and downstream conditions by 2

One-Dimensional Flow Equations

Conservation of mass:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho d\mathscr{V}}_{=0}+\underbrace{\oiint\limits_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS}_{\rho_2u_2A-\rho_1u_1A}=0\Rightarrow\rho_1u_1=\rho_2u_2$$

Conservation of momentum:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho\mathbf{v}d\mathscr{V}}_{=0}+\underbrace{\oiint\limits_{\partial\Omega}\left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}+p\mathbf{n}\right]dS}_{(\rho_2u_2^2+p_2)A-(\rho_1u_1^2+p_1)A}=0\Rightarrow\rho_1u_1^2+p_1=\rho_2u_2^2+p_2$$

One-Dimensional Flow Equations

Conservation of energy:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho e_o d\mathscr{V}}_{=0}+\underbrace{\oiint\limits_{\partial\Omega}\left[\rho h_o \mathbf{v}\cdot\mathbf{n}\right]dS}_{\rho_2 h_{o_2} u_2 A - \rho_1 h_{o_1} u_1 A}=0\Rightarrow \rho_1 u_1 h_{o_1}=\rho_2 u_2 h_{o_2}$$

Using the continuity equation this reduces to

$$h_{o_1}=h_{o_2}$$

or, if written out

$$h_1+\frac{1}{2}u_1^2=h_2+\frac{1}{2}u_2^2$$

One-Dimensional Flow Equations

Summary:

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

Note! These equations are valid regardless of whether or not there is a shock inside the control volume

One-Dimensional Flow Equations

Summary:

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

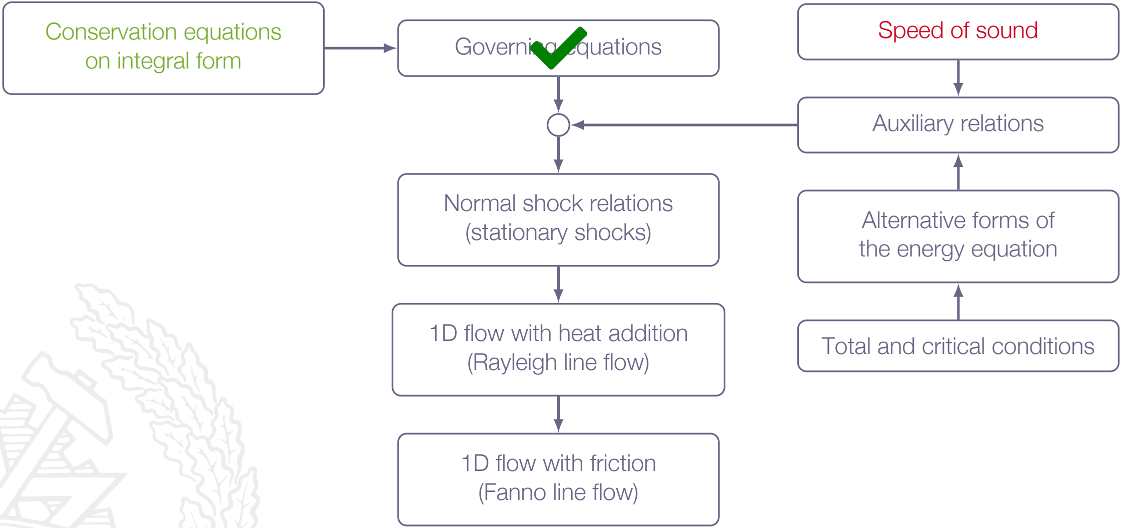
$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

Valid for all gases!

General gas $\Rightarrow$ Numerical solution necessary

Calorically perfect gas $\Rightarrow$ Can be solved analytically

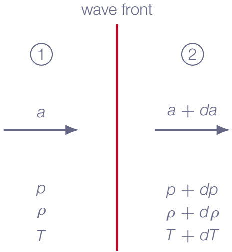
Roadmap - One-dimensional Flow



Chapter 3.3
Speed of Sound and Mach Number

Speed of Sound

Sound wave / acoustic perturbation



Speed of Sound

Conservation of mass gives

$$\rho a = (\rho + d\rho)(a + da) = \rho a + \rho da + d\rho a + d\rho da$$

products of infinitesimal quantities are removed $\Rightarrow$

$$\rho da + d\rho a = 0$$

solve for da $\Rightarrow$

$$da = -a\frac{d\rho}{\rho}$$

Speed of Sound

The momentum equation evaluated over the wave front gives

$$p + \rho a^2 = (p + dp) + (\rho + d\rho)(a + da)^2$$

Again, removing products of infinitesimal quantities gives

$$dp = -2a\rho da - a^2 d\rho$$

Solve for $da \Rightarrow$

$$da = \frac{dp + a^2 d\rho}{-2a\rho}$$

Speed of Sound

Continuity equation:

$$da = -a\frac{d\rho}{\rho}$$

Momentum equation:

$$da = \frac{dp + a^2 d\rho}{-2a\rho}$$

$$-a\frac{d\rho}{\rho} = \frac{dp + a^2 d\rho}{-2a\rho} \Rightarrow a^2 = \frac{dp}{d\rho}$$

Speed of Sound

Sound waves are **small perturbations** in ρ, $\mathbf{v}$, p, T (with constant entropy s) propagating through gas with speed a

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

(valid for all gases)

Speed of Sound

Compressibility and speed of sound:

from before we have

$$\tau_s = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_s$$

insert in relation for speed of sound

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{1}{\rho \tau_s} \Rightarrow a = \sqrt{\frac{1}{\rho \tau_s}}$$

(valid for all gases)

Speed of Sound

Calorically perfect gas:

Isentropic process $\Rightarrow p = C\rho^{\gamma}$ (where C is a constant)

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \gamma C \rho^{\gamma-1} = \frac{\gamma p}{\rho}$$

which implies

$$a = \sqrt{\frac{\gamma p}{\rho}} \Rightarrow a = \sqrt{\gamma R T}$$

Speed of Sound

Sound wave / acoustic perturbation:

a **weak wave**

propagating through gas at **speed of sound**

small perturbations in velocity and thermodynamic properties

isentropic process

Mach Number

The mach number, M, is a local variable

$$M = \frac{v}{a}$$

where

$$v = |\mathbf{v}|$$

and a is the local speed of sound

In the free stream, far away from solid objects, the flow is undisturbed and denoted by subscript ∞

$$M_\infty = \frac{v_\infty}{a_\infty}$$

Mach Number

For a fluid element moving along a streamline, the kinetic energy per unit mass and internal energy per unit mass are $V^2/2$ and e, respectively

$$\frac{V^2/2}{e} = \frac{V^2/2}{C_v T} = \frac{V^2/2}{RT/(\gamma-1)} = \frac{(\gamma/2)V^2}{a^2/(\gamma-1)} = \frac{\gamma(\gamma-1)}{2}M^2$$

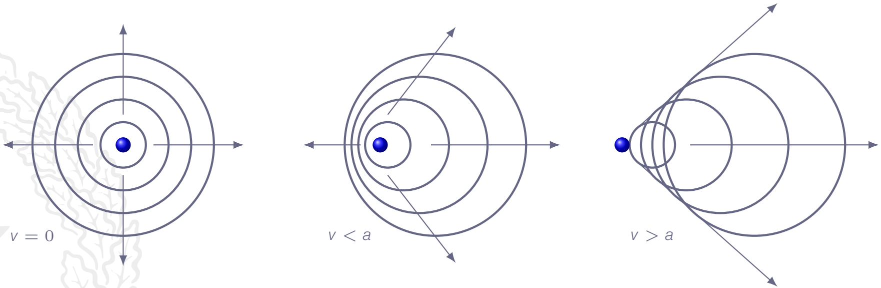
i.e. the Mach number is a measure of the ratio of the **fluid motion** (kinetic energy) and the **random thermal motion** of the molecules (internal energy)

Physical Consequences of Speed of Sound

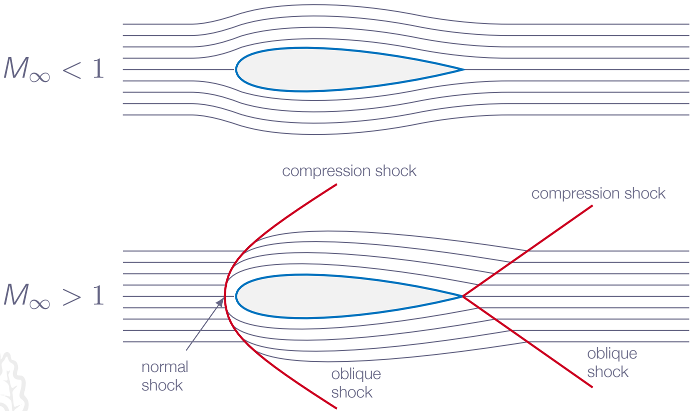
Sound waves is the way gas molecules convey information about what is happening in the flow

In subsonic flow, sound waves are able to travel upstream, since $v < a$

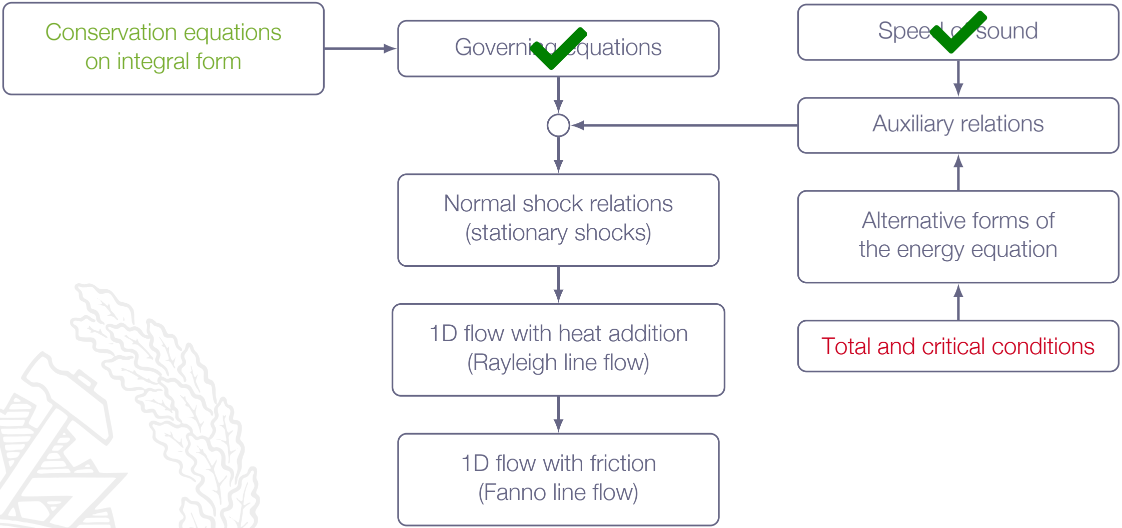
In supersonic flow, sound waves are unable to travel upstream, since $v > a$



Physical Consequences of Speed of Sound



Roadmap - One-dimensional Flow



Chapter 3.4
Some Conveniently Defined Flow Parameters

Stagnation Flow Properties

Assumption: Steady inviscid flow

If the flow is slowed down **isentropically** (without flow losses) to **zero velocity** we get the so-called **total conditions** (or stagnation flow properties)

(e.g. total pressure p_o, total temperature T_o, total density ρ_o, and total speed of sound a_o)

Since the process is isentropic, we have (for calorically perfect gas)

$$\frac{p_o}{p} = \left(\frac{\rho_o}{\rho}\right)^{\gamma} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

Note! T_o and a_o only requires an adiabatic deceleration process

Critical Conditions

If the flow is accelerated/decelerated **isentropically** to the **sonic point**, where $v = a$, we obtain the so-called **critical conditions**, *e.g.* p^*, T^*, ρ^*, a^*

where, by definition, $v^* = a^*$

As for the total conditions, if the process is also reversible (entropy is preserved) we obtain the relations (for calorically perfect gas)

$$\frac{p^*}{p_o} = \left(\frac{\rho^*}{\rho_o}\right)^{\gamma} = \left(\frac{T^*}{T_o}\right)^{\frac{\gamma}{\gamma-1}}$$

Note! T^* and a^* only requires an adiabatic acceleration/deceleration process

Total and Critical Conditions

For any given steady-state flow and location, we may think of an **imaginary** isentropic/adiabatic stagnation process or sonic flow process and thus

We can obtain **total** and **critical** conditions at **any point** in a flow

The total/critical conditions represent conditions realizable under an isentropic/adiabatic deceleration or acceleration of the flow

In an adiabatic flow, T_o is conserved along streamlines

Conservation of p_o along streamlines requires that the flow is isentropic (no viscous losses or shocks)

Total and Critical Conditions

Note! The actual flow does not have to be adiabatic or isentropic from point to point, the total and critical conditions are results of an imaginary isentropic/adiabatic process at one point in the flow.

However, with isentropic flow T_o, p_o, ρ_o, etc are constants

In order for T_o to be constant it is only required that the flow is adiabatic.

Total and Critical Conditions

If A and B are two locations in a flow

1. Isentropic flow:

$$T_{o_A} = T_{o_B} \text{ and } p_{o_A} = p_{o_B}$$

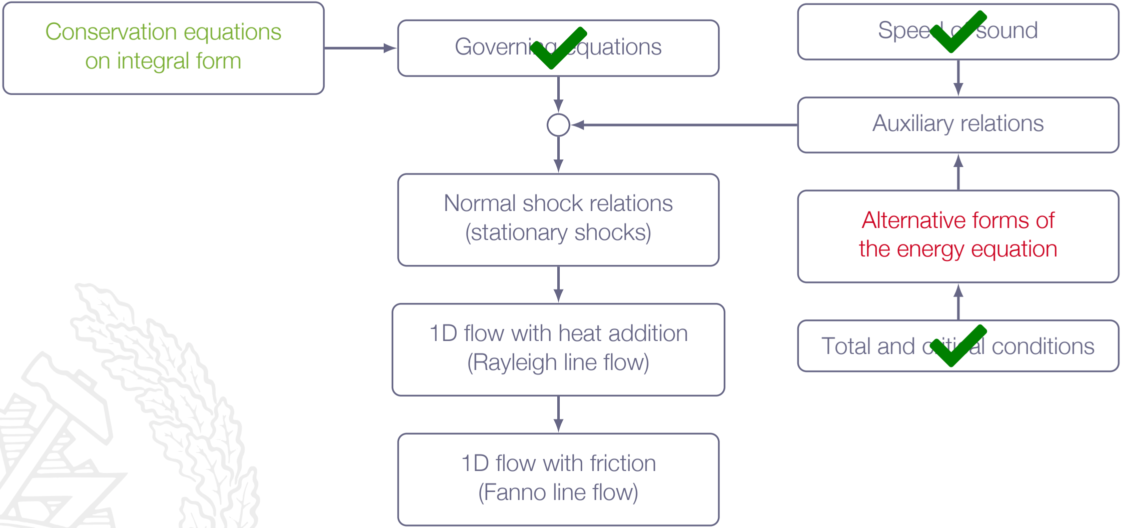
2. Adiabatic flow (not isentropic):

$$T_{o_A} = T_{o_B} \text{ and } p_{o_A} \neq p_{o_B}$$

3. The flow is not isentropic nor adiabatic:

$$T_{o_A} \neq T_{o_B} \text{ and } p_{o_A} \neq p_{o_B}$$

Roadmap - One-dimensional Flow



Chapter 3.5
Alternative Forms of the Energy Equation

Alternative Forms of the Energy Equation

For steady-state adiabatic flow, we have already shown that conservation of energy gives that total enthalpy, h_o, is constant along streamlines

For a calorically perfect gas we have $h = C_p T$ which implies

$$C_p T + \frac{1}{2}v^2 = C_p T_o$$

$$\frac{T_o}{T} = 1 + \frac{v^2}{2C_p T}$$

Inserting $C_p = \dfrac{\gamma R}{\gamma - 1}$ and $a^2 = \gamma R T$ we get

$$\boxed{\frac{T_o}{T} = 1 + \frac{1}{2}(\gamma - 1)M^2}$$

Alternative Forms of the Energy Equation

For calorically perfect gas (1D/2D/3D flows):

$$\frac{T_o}{T} = 1 + \frac{1}{2}(\gamma - 1)M^2$$

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{\frac{1}{\gamma-1}}$$

$$\frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\left(\frac{a^*}{a_o}\right)^2 = \frac{T^*}{T_o} = \frac{2}{\gamma + 1}$$

$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma-1}}$$

$$\frac{p^*}{p_o} = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma-1}}$$

Note! tabulated values for these relations can be found in Appendix A.1

The Characteristic Mach Number

$$M^* \equiv \frac{v}{a^*}$$

For a calorically perfect gas (1D/2D/3D flows)

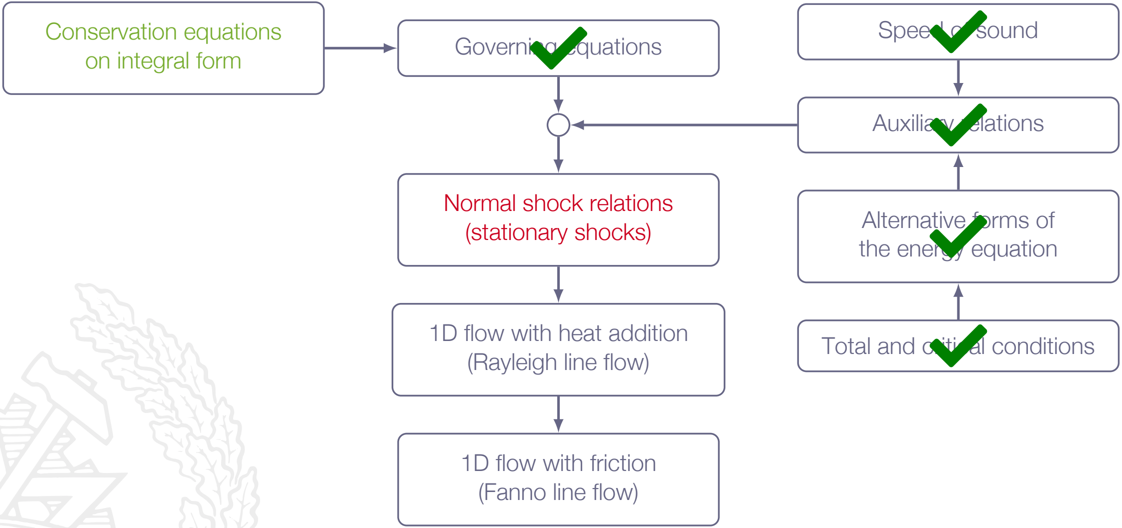
$$M^2 = \frac{2}{\left[(\gamma+1)/M^{*2}\right] - (\gamma-1)}$$

This relation between M and M^* gives:

$$M^* = 0 \Leftrightarrow M = 0$$
$$M^* = 1 \Leftrightarrow M = 1$$
$$M^* < 1 \Leftrightarrow M < 1$$
$$M^* > 1 \Leftrightarrow M > 1$$

$M^* \to \sqrt{\frac{\gamma+1}{\gamma-1}}$ when $M \to \infty$

Roadmap - One-dimensional Flow



Chapter 3.6
Normal Shock Relations

One-Dimensional Flow Equations

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

Normal Shock Relations

Calorically perfect gas

$$h = C_p T, \quad p = \rho R T$$

with constant C_p

Assuming that state 1 is known and state 2 is unknown

- 5 unknown variables: ρ_2, u_2, p_2, h_2, T_2
- 5 equations

$\Rightarrow$ solution can be found

Normal Shock Relations

Divide the momentum equation by $\rho_1 u_1$

$$\frac{1}{\rho_1 u_1}\left(p_1 + \rho_1 u_1^2\right) = \frac{1}{\rho_1 u_1}\left(p_2 + \rho_2 u_2^2\right)$$

$$\{\rho_1 u_1 = \rho_2 u_2\} \Rightarrow$$

$$\frac{1}{\rho_1 u_1}\left(p_1 + \rho_1 u_1^2\right) = \frac{1}{\rho_2 u_2}\left(p_2 + \rho_2 u_2^2\right)$$

$$\frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} = u_2 - u_1$$

Normal Shock Relations

$$\frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} = u_2 - u_1$$

Recall that $a = \sqrt{\frac{\gamma p}{\rho}}$, which gives

$$\frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} = u_2 - u_1$$

Now, we will make use of the fact that the flow is adiabatic and thus a^* is constant

Normal Shock Relations

Energy equation:

$$C_p T_1 + \frac{1}{2}u_1^2 = C_p T_2 + \frac{1}{2}u_2^2$$

$$\left\{ C_p = \frac{\gamma R}{\gamma - 1} \right\} \Rightarrow$$

$$\frac{\gamma R T_1}{(\gamma - 1)} + \frac{1}{2}u_1^2 = \frac{\gamma R T_2}{(\gamma - 1)} + \frac{1}{2}u_2^2$$

$$\left\{ a = \sqrt{\gamma R T} \right\} \Rightarrow$$

$$\frac{a_1^2}{(\gamma - 1)} + \frac{1}{2}u_1^2 = \frac{a_2^2}{(\gamma - 1)} + \frac{1}{2}u_2^2$$

Normal Shock Relations

In any position in the flow we can get a relation between the local speed of sound a, the local velocity u, and the speed of sound at sonic conditions a^* by inserting in the equation on the previous slide. $u_1 = u$, $a_1 = a$, $u_2 = a_2 = a^* \Rightarrow$

$$\frac{a^2}{(\gamma-1)} + \frac{1}{2}u^2 = \frac{a^{*2}}{(\gamma-1)} + \frac{1}{2}a^{*2}$$

$$a^2 = \frac{\gamma+1}{2}a^{*2} - \frac{\gamma-1}{2}u^2$$

Evaluated in station 1 and 2, this gives

$$a_1^2 = \frac{\gamma+1}{2}a^{*2} - \frac{\gamma-1}{2}u_1^2$$

$$a_2^2 = \frac{\gamma+1}{2}a^{*2} - \frac{\gamma-1}{2}u_2^2$$

Normal Shock Relations

Now, inserting $\left\{a_1^2 = \frac{\gamma+1}{2}a^{*2} - \frac{\gamma-1}{2}u_1^2\right\}$ and $\left\{a_2^2 = \frac{\gamma+1}{2}a^{*2} - \frac{\gamma-1}{2}u_2^2\right\}$

in $\left\{\frac{a_1^2}{(\gamma-1)} + \frac{1}{2}u_1^2 = \frac{a_2^2}{(\gamma-1)} + \frac{1}{2}u_2^2\right\}$ and solve for a^* gives

$$\boxed{a^{*2} = u_1 u_2}$$

Normal Shock Relations

$$a^{*2} = u_1 u_2$$

A.K.A. the Prandtl relation. Divide by a^{*2} on both sides $\Rightarrow$

$$1 = \frac{u_1}{a^*}\frac{u_2}{a^*} = M_1^* M_2^*$$

Together with the relation between M and M^*, this gives

$$\boxed{M_2^2 = \frac{1 + \frac{1}{2}(\gamma - 1)M_1^2}{\gamma M_1^2 - \frac{1}{2}(\gamma - 1)}}$$

Normal Shock Relations

Continuity equation and $a^{*2} = u_1 u_2$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1^2}{u_1 u_2} = \frac{u_1^2}{a^{*2}} = M_1^{*2}$$

which gives

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}$$

Normal Shock Relations

Now, once again back to the momentum equation

$$p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2 = \{\rho_1 u_1 = \rho_2 u_2\} = \rho_1 u_1 (u_1 - u_2)$$

$$\frac{p_2}{p_1} - 1 = \frac{\rho_1 u_1^2}{p_1}\left(1 - \frac{u_2}{u_1}\right) = \left\{a = \sqrt{\frac{\gamma p}{\rho}},\ M^2 = \frac{u^2}{a^2}\right\} = \gamma M_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

with the expression for u_2/u_1 derived previously, this gives

$$\boxed{\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)}$$

Normal Shock Relations

Are the normal shock relations valid for $M_1 < 1.0$?

Mathematically - yes!

Physically - ?

Normal Shock Relations

Let's have a look at the 2^{nd} law of thermodynamics

$$s_2 - s_1 = C_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

We get the ratios (T_2/T_1) and (p_2/p_1) from the normal shock relations

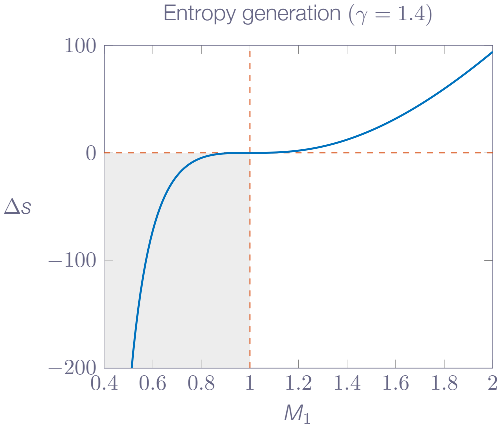
$$s_2 - s_1 = C_p \ln \left[\left(1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1) \right) \left(\frac{2 + (\gamma - 1)M_1^2}{(\gamma+1)M_1^2} \right) \right] + - R \ln \left(1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1) \right)$$

Normal Shock Relations

$$M_1 = 1 \Rightarrow \Delta s = 0 \text{ (Mach wave)}$$

$$M_1 < 1 \Rightarrow \Delta s < 0 \text{ (not physical)}$$

$$M_1 > 1 \Rightarrow \Delta s > 0$$



Normal Shock Relations

Normal shock $\Rightarrow M_1 > 1$

$$M_1^* M_2^* = 1$$

$$M_1 > 1 \Rightarrow M_1^* > 1$$

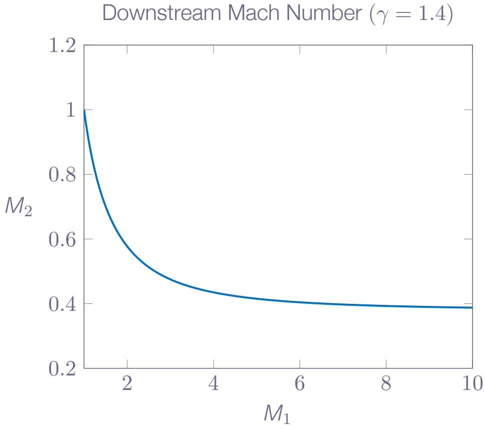
$$M_2^* = \frac{1}{M_1^*} \Rightarrow M_2^* < 1$$

$$M_2^* < 1 \Rightarrow M_2 < 1$$

After a normal shock the Mach number must be lower than 1.0

Normal Shock Relations

$$M_2^2 = \frac{1 + \frac{1}{2}(\gamma - 1)M_1^2}{\gamma M_1^2 - \frac{1}{2}(\gamma - 1)}$$



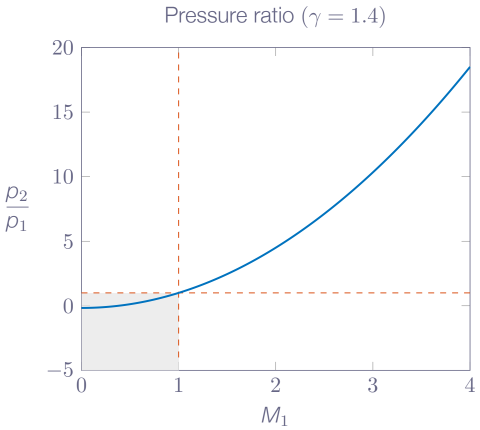
$$M_1 = 1.0 \Rightarrow M_2 = 1.0$$

$$M_1 > 1.0 \Rightarrow M_2 < 1.0$$

$$M_1 \to \infty \Rightarrow M_2 \to \sqrt{(\gamma - 1)/(2\gamma)} = \{\gamma = 1.4\} = 0.378$$

Normal Shock Relations

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right)$$



Note! from before we know that M_1 must be greater than 1.0, which means that p_2/p_1 must be greater than 1.0

Normal Shock Relations

$M_1 > 1.0$ gives $M_2 < 1.0$, $\rho_2 > \rho_1$, $p_2 > p_1$, and $T_2 > T_1$

What about T_o and p_o?

Energy equation: $C_pT_1 + \frac{u_1^2}{2} = C_pT_2 + \frac{u_2^2}{2} \Rightarrow C_pT_{o_1} = C_pT_{o_2}$

calorically perfect gas $\Rightarrow T_{o_1} = T_{o_2}$

or more general (as long as the shock is stationary): $h_{o_1} = h_{o_2}$

Normal Shock Relations

2^{nd} law of thermodynamics and isentropic deceleration to zero velocity (Δs unchanged since isentropic) gives

$$s_2 - s_1 = C_p \ln \frac{T_{o_2}}{T_{o_1}} - R \ln \frac{p_{o_2}}{p_{o_1}} = \{T_{o_1} = T_{o_2}\} = -R \ln \frac{p_{o_2}}{p_{o_1}}$$

$$\frac{p_{o_2}}{p_{o_1}} = e^{-(s_2 - s_1)/R}$$

i.e. the total pressure decreases over a normal shock

Normal Shock Relations

Normal shock relations for calorically perfect gas (summary):

$$T_{o_1} = T_{o_2}$$

$$a_{o_1} = a_{o_2}$$

$$a_1^* = a_2^* = a^*$$

$$u_1 u_2 = {a^*}^2$$ (the Prandtl relation)

$$M_2^* = \frac{1}{M_1^*}$$

$$M_2^2 = \frac{1 + \frac{1}{2}(\gamma - 1)M_1^2}{\gamma M_1^2 - \frac{1}{2}(\gamma - 1)}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2}$$

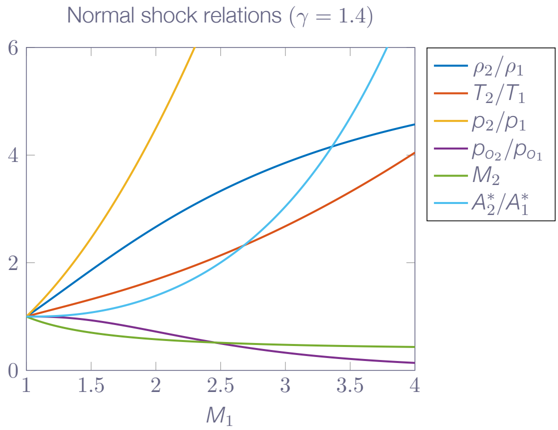
$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}$$

Normal Shock Relations

As the flow passes a stationary normal shock, the following changes will take place discontinuously across the shock:

ρ	increases
p	increases
u	decreases
M	decreases (from $M > 1$ to $M < 1$)
T	increases
p_o	decreases (due to shock loss)
s	increases (due to shock loss)
T_o	unaffected

Normal Shock Relations



Normal Shock Relations

The normal shock relations for calorically perfect gases are valid for $M_1 \leq 5$ (approximately) for air at standard conditions

Calorically perfect gas $\Rightarrow$ Shock strength depends on M_1 only

Thermally perfect gas $\Rightarrow$ Shock strength depends on M_1 and T_1

General real gas (non-perfect) $\Rightarrow$ Shock strength depends on M_1, p_1, and T_1

Normal Shock Relations

And now to the question that probably bothers most of you but that no one dares to ask ...

Normal Shock Relations

And now to the question that probably bothers most of you but that no one dares to ask ...

When or where did we say that there was going to be a shock between 1 and 2?

Normal Shock Relations

And now to the question that probably bothers most of you but that no one dares to ask ...

When or where did we say that there was going to be a shock between 1 and 2?

Answer: We did not (explicitly)

Normal Shock Relations

The derivation is based on the fact that there should be a change in flow properties between 1 and 2

We are assuming steady state conditions

We have said that the flow is adiabatic (no added or removed heat)

There is no work done and no friction added

A normal shock is <u>the solution</u> provided by nature (and math) that fulfill these requirements!

Chapter 3.7
Hugoniot Equation

Hugoniot Equation

Starting point: governing equations for normal shocks

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

Eliminate u_1 and u_2 gives:

$$h_2 - h_1 = \frac{p_2 - p_1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

Hugoniot Equation

Now, insert $h = e + p/\rho$ gives

$$e_2 - e_1 = \frac{p_2 + p_1}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = \frac{p_2 + p_1}{2}(\nu_1 - \nu_2)$$

which is the **Hugoniot relation**

Stationary Normal Shock in One-Dimensional Flow

Normal shock:

$$e_2 - e_1 = -\frac{p_2 + p_1}{2}(\nu_2 - \nu_1)$$

More effective than isentropic process

Gives entropy increase

Isentropic process:

$$de = -pd\nu$$

More efficient than normal shock process

see figure 3.11 p. 100

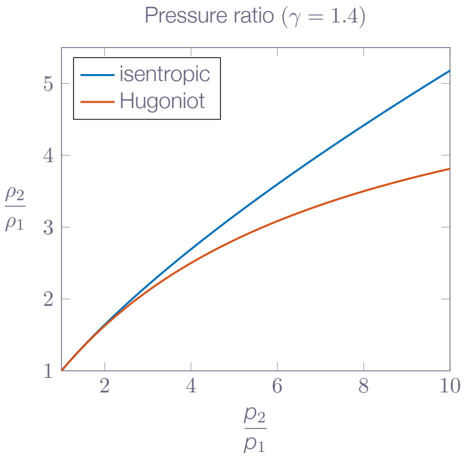
Stationary Normal Shock in One-Dimensional Flow

The Rankine-Hugoniot relation

$$\frac{\rho_2}{\rho_1} = \frac{1 + \left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right)}{\left(\frac{\gamma+1}{\gamma-1}\right) + \left(\frac{p_2}{p_1}\right)}$$

The isentropic relation

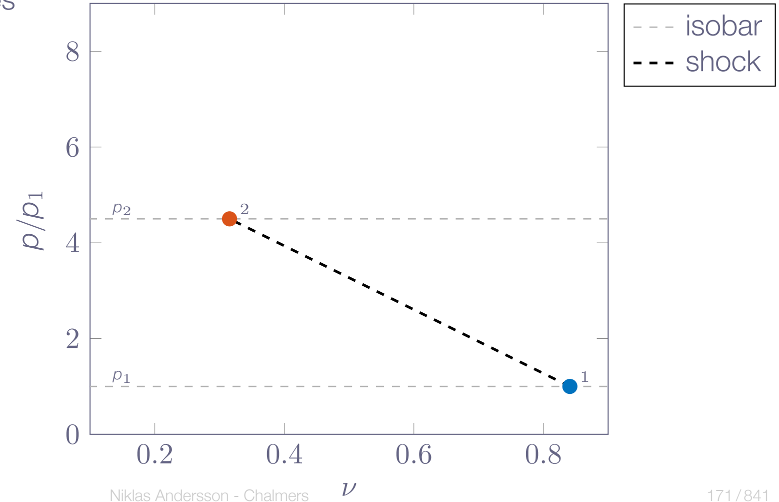
$$\frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{1/\gamma}$$



The Normal-shock Process

Note!

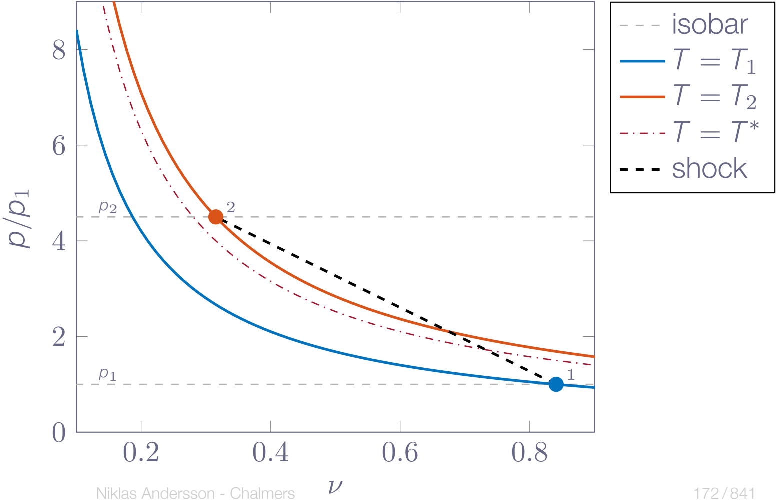
over the shock, the flow state changes discontinuously from 1 to 2 without passing any intermediate states



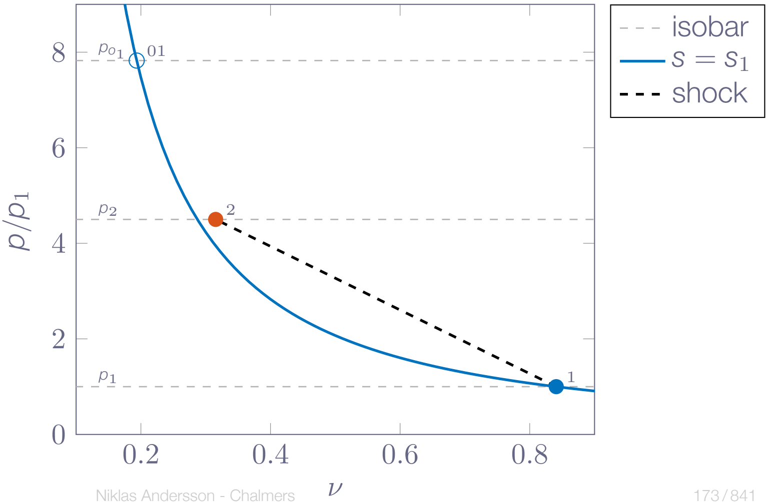
The Normal-shock Process

Note!

$M_1 > 1.0$ and $M_2 < 1.0 \Rightarrow T_1 < T^* < T_2$



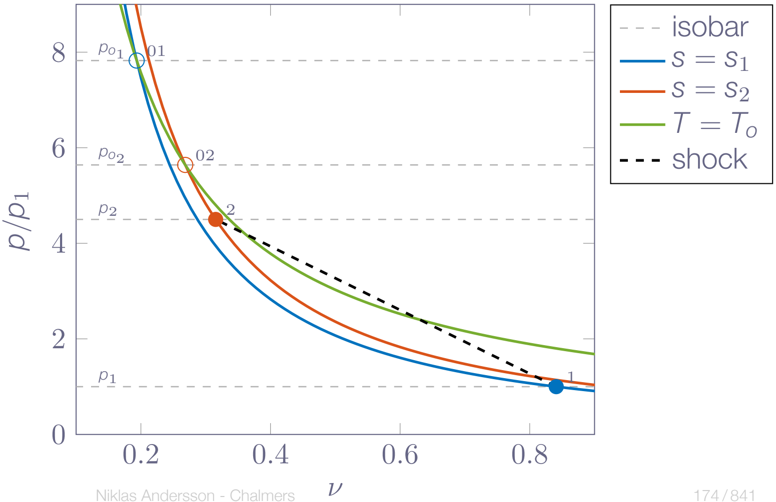
The Normal-shock Process



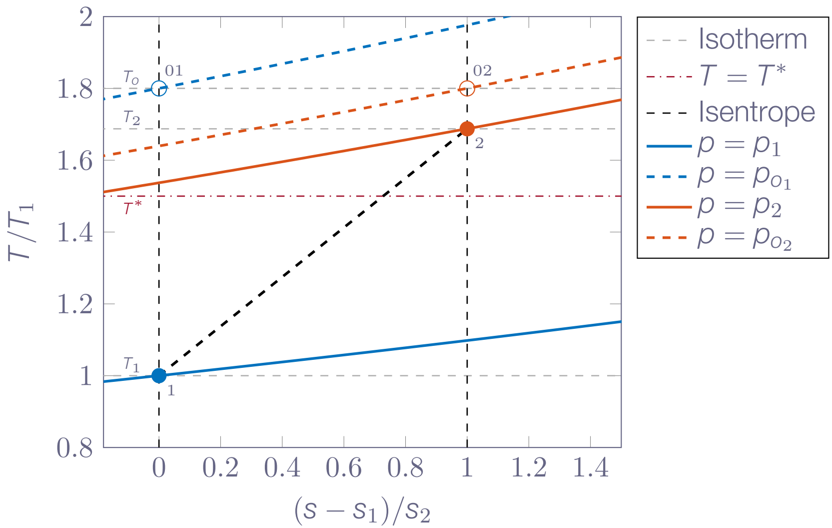
The Normal-shock Process

Note!

isotherms are less steep than isentropes $\Rightarrow p_{o_2} < p_{o_1}$



The Normal-shock Process



The Normal-shock Process

Continuity:

$$\rho_1 u_1 = \rho_2 u_2 = C > 0$$

Momentum:

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \Rightarrow p_1 + \frac{C^2}{\rho_1} = p_2 + \frac{C^2}{\rho_2} \Rightarrow p_1 + \nu_1 C^2 = p_2 + \nu_2 C^2$$

$$\frac{p_1 - p_2}{\nu_1 - \nu_2} = -C^2$$

a line in $p\nu$-space with negative slope

The Normal-shock Process

Energy equation:

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

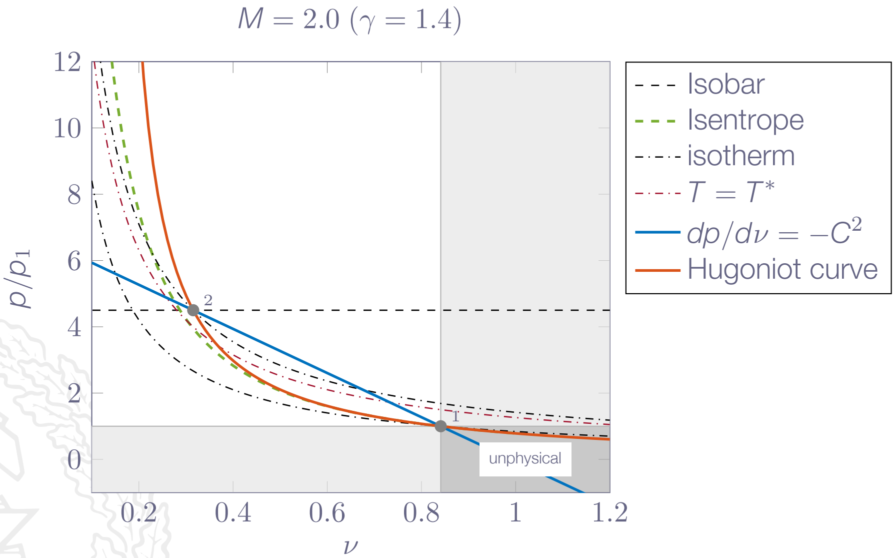
with $h = C_p T = \dfrac{\gamma R}{\gamma - 1} T$ and $u = \nu C$ we get

$$\frac{\gamma R}{\gamma - 1} T_1 + \frac{1}{2}\nu_1^2 C^2 = \frac{\gamma R}{\gamma - 1} T_2 + \frac{1}{2}\nu_2^2 C^2 \Rightarrow \ldots \Rightarrow \frac{p_2}{p_1} \left(\frac{\nu_2}{\nu_1} - \frac{\gamma + 1}{\gamma - 1} \right) / \left(1 - \frac{\nu_2}{\nu_1} \frac{\gamma + 1}{\gamma - 1} \right)$$

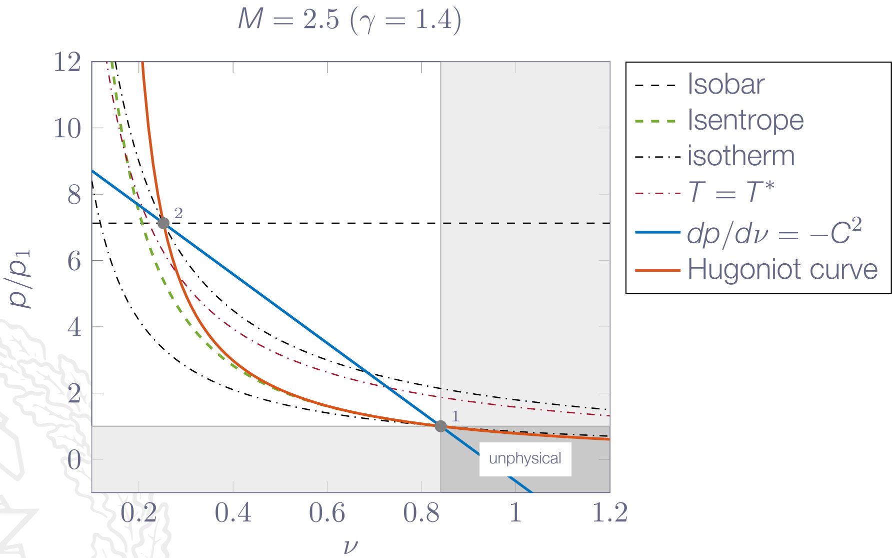
quadratic function in $p\nu$-space (Hugoniot curve)

only thermodynamic variables

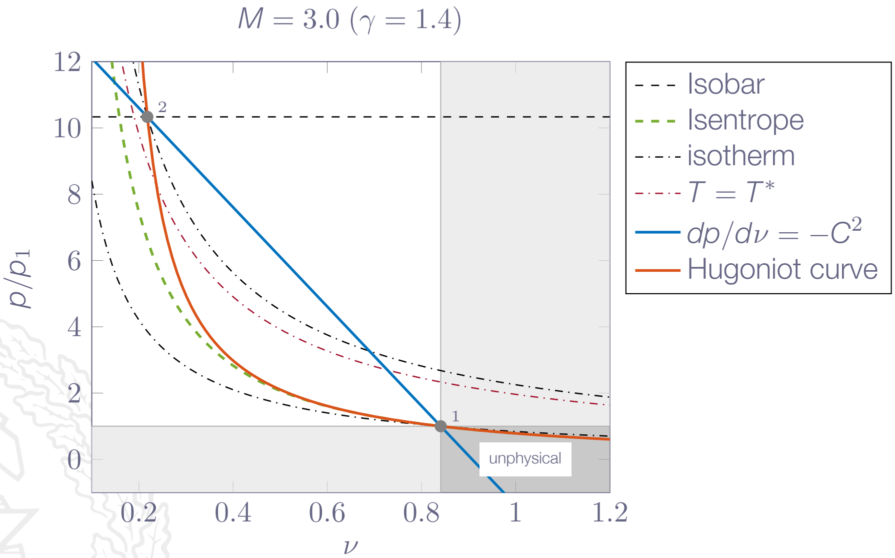
The Normal-shock Process



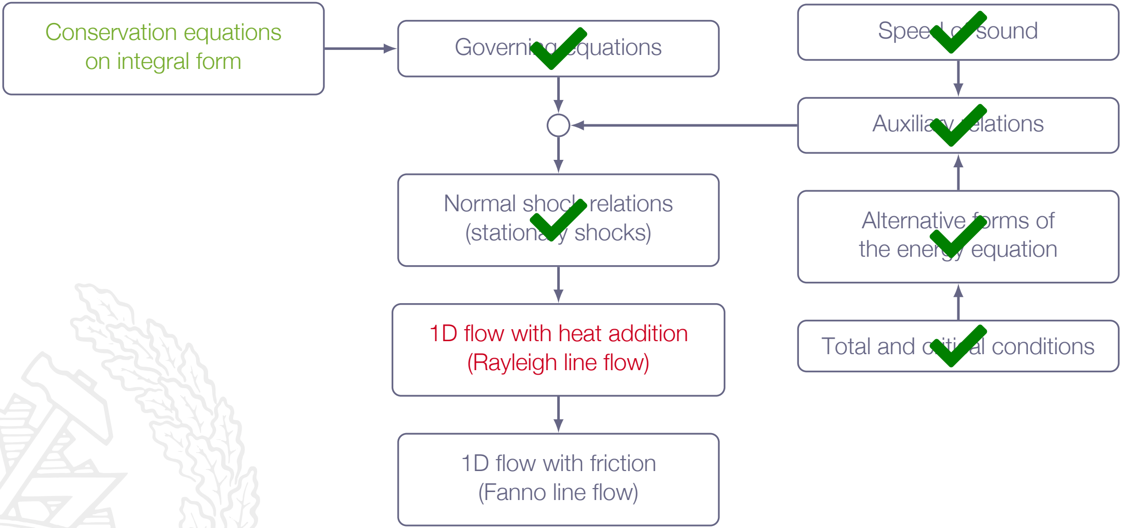
The Normal-shock Process



The Normal-shock Process

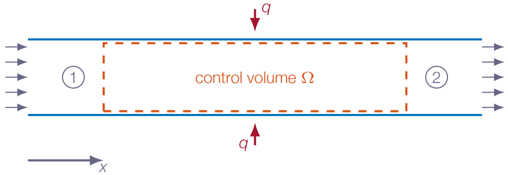


Roadmap - One-dimensional Flow



Chapter 3.8
One-Dimensional Flow with Heat Addition

One-Dimensional Flow with Heat Addition



1D pipe flow with heat addition:

1. no friction
2. 1D steady-state ⇒ all variables depend on x only
3. q is the amount of heat per unit mass added between 1 and 2
4. analyze by setting up a control volume between station 1 and 2

One-Dimensional Flow with Heat Addition

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 + q = h_2 + \frac{1}{2}u_2^2$$

Valid for all gases!

General gas $\Rightarrow$ Numerical solution necessary

Calorically perfect gas $\Rightarrow$ can be solved analytically

One-Dimensional Flow with Heat Addition

Calorically perfect gas ($h = C_pT$):

$$C_pT_1 + \frac{1}{2}u_1^2 + q = C_pT_2 + \frac{1}{2}u_2^2$$

$$q = \left(C_pT_2 + \frac{1}{2}u_2^2\right) - \left(C_pT_1 + \frac{1}{2}u_1^2\right)$$

$$C_pT_o = C_pT + \frac{1}{2}u^2 \Rightarrow$$

$$\boxed{q = C_p(T_{o_2} - T_{o_1})}$$

i.e. heat addition increases T_o downstream

One-Dimensional Flow with Heat Addition

Momentum equation:

$$p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2$$

$$\left\{ \rho u^2 = \rho a^2 M^2 = \rho \frac{\gamma p}{\rho} M^2 = \gamma p M^2 \right\}$$

$$p_2 - p_1 = \gamma p_1 M_1^2 - \gamma p_2 M_2^2 \Rightarrow$$

$$\boxed{\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}}$$

Normal Shock Relations

We used the momentum equation to derive the relation for p_2/p_1. In what way is this relation different than the one for normal shocks – the momentum equation is the same?

Normal Shock Relations

We used the momentum equation to derive the relation for p_2/p_1. In what way is this relation different than the one for normal shocks – the momentum equation is the same?

Answer: There is no difference. If we would insert $M_2 = f(M_1)$ from the normal shock relations, we would end up with the normal shock relation for p_2/p_1.

The relation for $M_2 = f(M_1)$ for normal shocks was derived assuming adiabatic flow

One-Dimensional Flow with Heat Addition

Ideal gas law:

$$T = \frac{p}{\rho R} \Rightarrow \frac{T_2}{T_1} = \frac{p_2}{\rho_2 R}\frac{\rho_1 R}{p_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}$$

Continuity equation:

$$\rho_1 u_1 = \rho_2 u_2 \Rightarrow \frac{\rho_1}{\rho_2} = \frac{u_2}{u_1}$$

$$\frac{u_2}{u_1} = \frac{M_2 a_2}{M_1 a_1} = \frac{M_2\sqrt{\gamma R T_2}}{M_1\sqrt{\gamma R T_1}} \Rightarrow \frac{\rho_1}{\rho_2} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}}$$

$$\boxed{\frac{T_2}{T_1} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2 \left(\frac{M_2}{M_1}\right)^2}$$

One-Dimensional Flow with Heat Addition

Calorically perfect gas, analytic solution:

$$\frac{T_2}{T_1} = \left[\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right]^2 \left(\frac{M_2}{M_1}\right)^2$$

$$\frac{\rho_2}{\rho_1} = \left[\frac{1+\gamma M_2^2}{1+\gamma M_1^2}\right] \left(\frac{M_1}{M_2}\right)^2$$

$$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}$$

One-Dimensional Flow with Heat Addition

Calorically perfect gas, analytic solution:

$$\frac{p_{o_2}}{p_{o_1}} = \left[\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right]\left(\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_{o_2}}{T_{o_1}} = \left[\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right]\left(\frac{M_2}{M_1}\right)^2\left(\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right)^{\frac{\gamma}{\gamma-1}}$$

One-Dimensional Flow with Heat Addition

Initially subsonic flow ($M < 1$)

the Mach number, M, increases as more heat (per unit mass) is added to the gas

for some limiting heat addition q^*, the flow will eventually become sonic $M = 1$

Initially supersonic flow ($M > 1$)

the Mach number, M, decreases as more heat (per unit mass) is added to the gas

for some limiting heat addition q^*, the flow will eventually become sonic $M = 1$

Note! The (*) condition in this context **is not** the same as the ”critical” condition discussed for isentropic flow

One-Dimensional Flow with Heat Addition

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}$$

Calculate the ratio between the pressure at a specific location in the flow p and the pressure at sonic conditions p^*

$p_1 = p$, $M_1 = M$, $p_2 = p^*$, and $M_2 = 1$

$$\frac{p^*}{p} = \frac{1 + \gamma M^2}{1 + \gamma}$$

One-Dimensional Flow with Heat Addition

$$\frac{T}{T^*} = \left[\frac{1+\gamma}{1+\gamma M^2}\right]^2 M^2$$

$$\frac{\rho}{\rho^*} = \left[\frac{1+\gamma M^2}{1+\gamma}\right]\left(\frac{1}{M^2}\right)$$

$$\frac{p}{p^*} = \frac{1+\gamma}{1+\gamma M^2}$$

$$\frac{p_o}{p_o^*} = \left[\frac{1+\gamma}{1+\gamma M^2}\right]\left(\frac{2+(\gamma-1)M^2}{(\gamma+1)}\right)^{\frac{\gamma}{\gamma-1}}$$

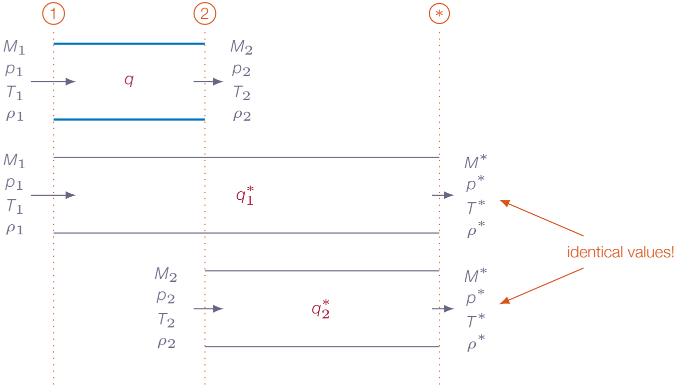
$$\frac{T_o}{T_o^*} = \frac{(\gamma+1)M^2}{(1+\gamma M^2)^2}(2+(\gamma-1)M^2)$$

One-Dimensional Flow with Heat Addition

Amount of heat per unit mass needed to choke the flow:

$$q^* = C_p(T_o^* - T_o) = C_p T_o \left(\frac{T_o^*}{T_o} - 1 \right)$$

One-Dimensional Flow with Heat Addition



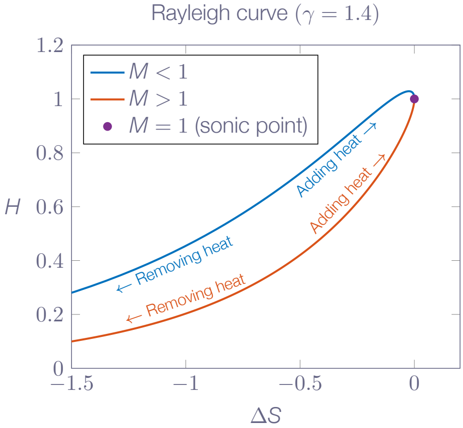
$$q_2^* = q_1^* - q$$

Note! for a given flow, the starred quantities are constant values

One-Dimensional Flow with Heat Addition

Lord Rayleigh 1842-1919
Nobel prize in physics 1904

Note! it is theoretically possible to heat an initially subsonic flow to reach sonic conditions and then continue to accelerate the flow by cooling



$$\Delta S = \frac{\Delta s}{C_p} = \ln\left[M^2\left(\frac{\gamma+1}{1+\gamma M^2}\right)^{\frac{\gamma+1}{\gamma}}\right]$$

$$H = \frac{h}{h^*} = \frac{C_p T}{C_p T^*} = \frac{T}{T^*} = \left[\frac{(\gamma+1)M}{1+\gamma M^2}\right]^2$$

One-Dimensional Flow with Heat Addition

And now, the million-dollar question ...

One-Dimensional Flow with Heat Addition

And now, the million-dollar question ...

Removing heat seems to reduce the entropy. Isn't that a violation of the second law of thermodynamics?!

One-Dimensional Flow with Heat Addition

And now, the million-dollar question ...

Removing heat seems to reduce the entropy. Isn't that a violation of the second law of thermodynamics?!

Answer: if the heat source or sink would have been included in the system studied, the system entropy would increase both when adding and removing heat.

One-Dimensional Flow with Heat Addition

$M < 1$: Adding heat will

increase M
decrease p
increase T_o
decrease p_o
increase s
increase u
decrease ρ

$M > 1$: Adding heat will

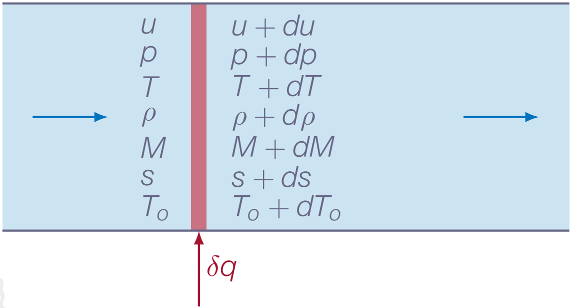
decrease M
increase p
increase T_o
decrease p_o
increase s
decrease u
increase ρ

Note! the flow is not isentropic, there will always be losses

The Rayleigh-flow Process

Unlike the normal shock, Rayleigh flow has **continuous** solutions

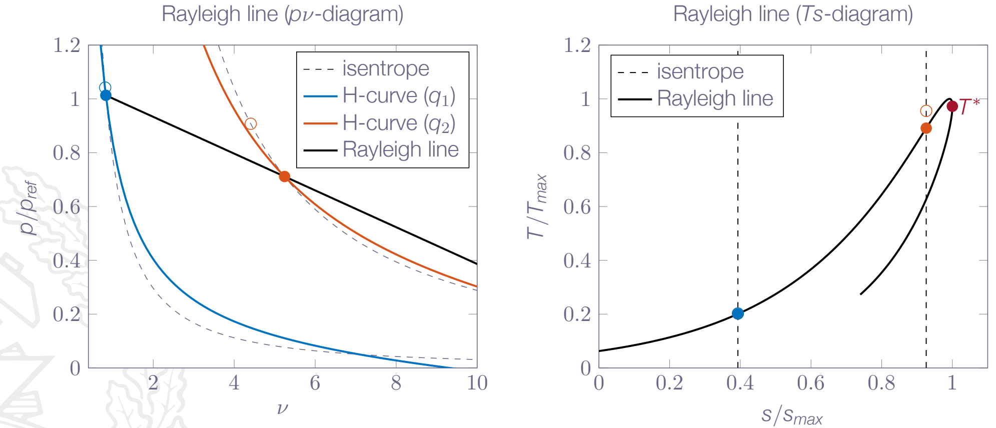
A small addition of heat δq will change flow properties slightly



The Rayleigh-flow Process - Subsonic Heat Addition

Note!

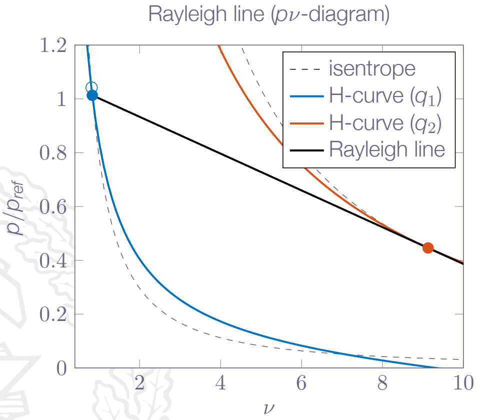
Heat addition moves the H-curve in the direction of increasing pressure and increasing specific volume

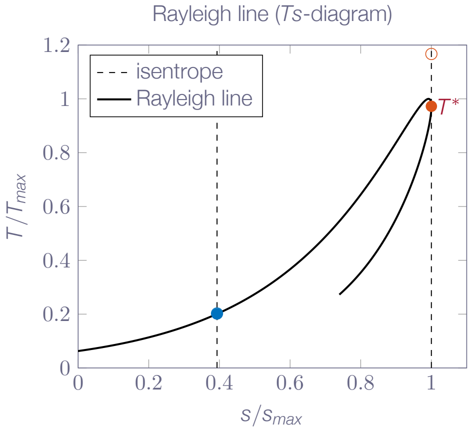


The Rayleigh-flow Process - Choked Subsonic Flow

Note!

When $q = q^*$, the H-curve is tangent to the Rayleigh line (thermal choking)
Further heat addition will move the H-curve away from the Rayleigh line



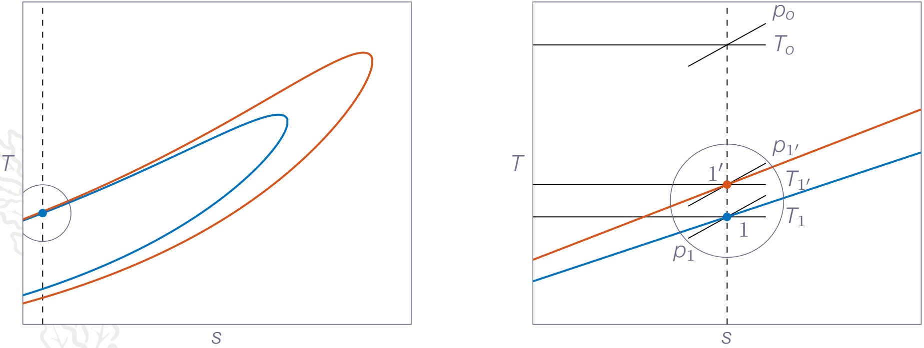


The Rayleigh-flow Process - Choked Subsonic Flow

Note!

If is added such that $q > q^*$, the inlet static flow properties will change (new mass-flow) such that the new q^* is equal to the added heat q

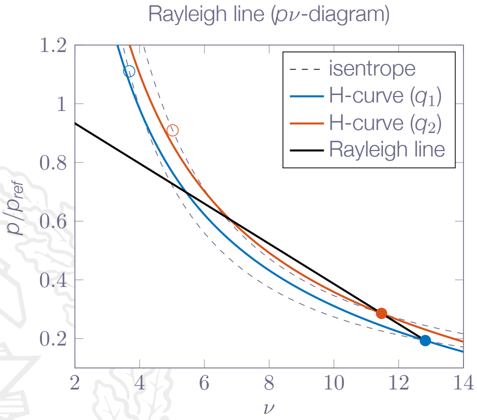
Total flow properties at the inlet remains the same (only work or heat addition can change the total flow properties)

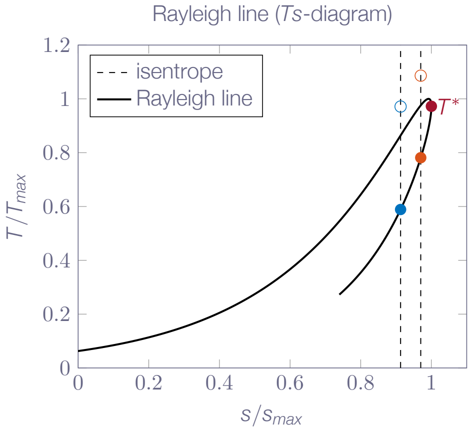


The Rayleigh-flow Process - Supersonic Heat Addition

Note!

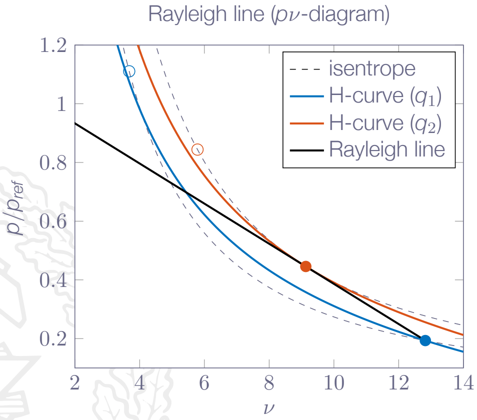
A supersonic flow is in general closer to thermal choking than a subsonic flow due to the high energy level (and thereby high T_o)

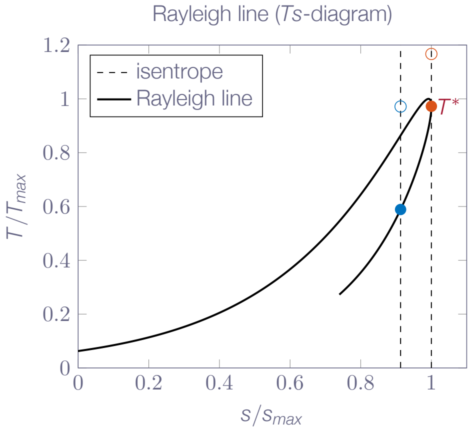




The Rayleigh-flow Process - Choked Supersonic Flow

Note!
When heat is added to a thermally choked supersonic flow, a shock will be generated at the exit of the pipe





The Rayleigh-flow Process - Choked Supersonic Flow

The shock generated at the exit will be infinitely weak ($M = 1$)

As the shock does not affect T_o, T^*, p^* etc, it does not affect the thermal choking condition (remember: T^* and p^* are **not the critical conditions**)

The heat process and the normal shock process operates along the **same line** in $p\nu$-space

The shock will travel upstream through the pipe

If the supersonic flow is generated in a convergent-diveregent nozzle, the shock will propagate upstream in the nozzle until the resulting pipe inlet condition allows for the heat to be added with thermal choking at the pipe exit

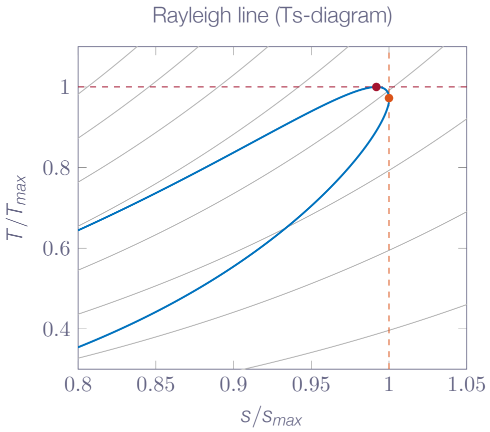
The Rayleigh-flow Process - Maxumim Temperature

It can be showed that $\dfrac{dT}{ds} = \dfrac{1-\gamma M^2}{1-M^2}\dfrac{T}{C_p}$

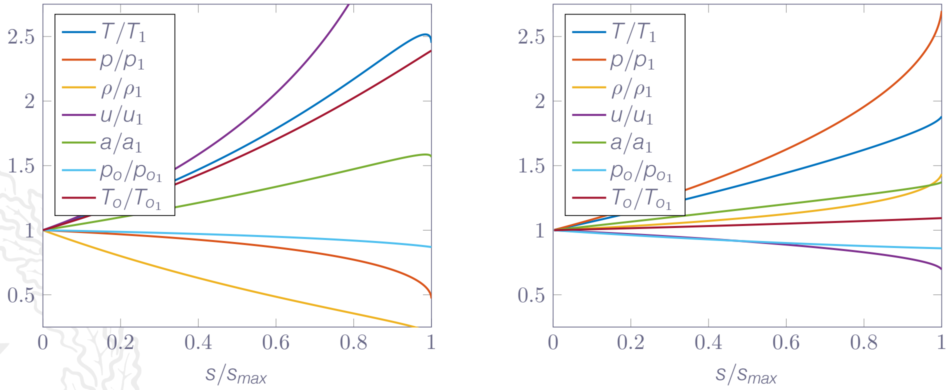
$$\frac{dT}{ds} = 0 \Rightarrow M = \sqrt{\frac{1}{\gamma}}$$

we will have the maximum temperature for a subsonic Mach number

$$M = 1.0 \Rightarrow \frac{dT}{ds} = \infty$$

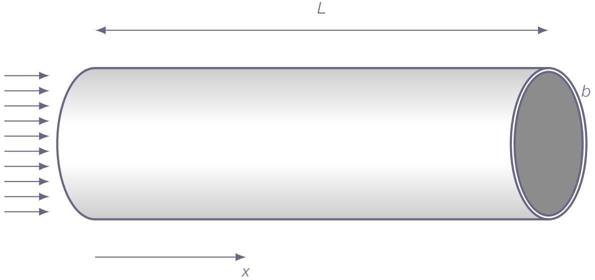


Rayleigh Flow Trends



One-Dimensional Flow with Heat Addition

Relation between added heat per unit mass (q) and heat per unit surface area and unit time ($\dot{q}_{wall}$)

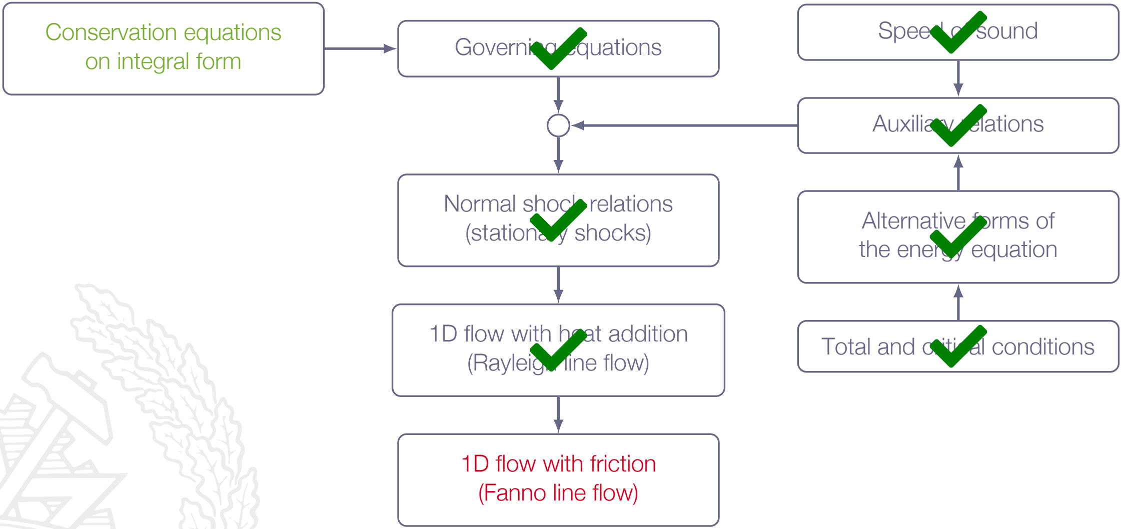


Pipe with arbitrary cross section (constant in x):

mass flow through pipe	$\dot{m}$
axial length of pipe	L
circumference of pipe	$b = 2\pi r$

$$q = \frac{Lb\dot{q}_{wall}}{\dot{m}}$$

Roadmap - One-dimensional Flow

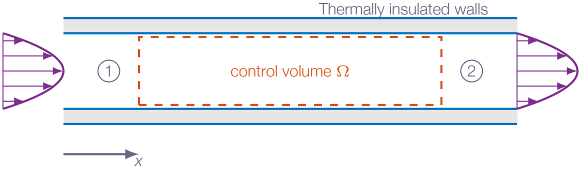


Chapter 3.9
One-Dimensional Flow with Friction

One-Dimensional Flow with Friction

inviscid flow with friction?!

One-Dimensional Flow with Friction



1D pipe flow with friction:

1. adiabatic ($q = 0$)
2. cross section area A is constant
3. average all variables in each cross-section $\Rightarrow$ only x-dependence
4. analyze by setting up a control volume between station 1 and 2

One-Dimensional Flow with Friction

Wall-friction contribution in momentum equation

$$\iint_{\partial\Omega} \tau_W dS = b \int_0^L \tau_W dx$$

where L is the tube length and b is the circumference

One-Dimensional Flow with Friction

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 - \frac{4}{D}\int_0^L \tau_w dx = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

One-Dimensional Flow with Friction

τ_w varies with the distance x and thus complicating the integration

Solution: let L shrink to dx and we end up with relations on differential form

$$d(\rho u^2 + p) = -\frac{4}{D}\tau_w dx \Leftrightarrow \frac{d}{dx}(\rho u^2 + p) = -\frac{4}{D}\tau_w$$

One-Dimensional Flow with Friction

From the continuity equation we get

$$\rho_1 u_1 = \rho_2 u_2 = const \Rightarrow \frac{d}{dx}(\rho u) = 0$$

Writing out all terms in the momentum equation gives

$$\frac{d}{dx}(\rho u^2 + p) = \rho u \frac{du}{dx} + u \underbrace{\frac{d}{dx}(\rho u)}_{=0} + \frac{dp}{dx} = -\frac{4}{D}\tau_w \Rightarrow \rho u \frac{du}{dx} + \frac{dp}{dx} = -\frac{4}{D}\tau_w$$

Common approximation for τ_w:

$$\tau_w = f\frac{1}{2}\rho u^2 \Rightarrow \rho u \frac{du}{dx} + \frac{dp}{dx} = -\frac{2}{D}\rho u^2 f$$

One-Dimensional Flow with Friction

Energy conservation:

$$h_{o_1} = h_{o_2} \Rightarrow \frac{d}{dx} h_o = 0$$

One-Dimensional Flow with Friction

Summary:

$$\frac{d}{dx}(\rho u) = 0$$

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = -\frac{2}{D}\rho u^2 f$$

$$\frac{d}{dx}h_o = 0$$

Valid for all gases!

General gas $\Rightarrow$ Numerical solution necessary

Calorically perfect gas $\Rightarrow$ Can be solved analytically (for constant f)

One-Dimensional Flow with Friction

Calorically perfect gas:

$$\int_{x_1}^{x_2} \frac{4f}{D}dx = \left[-\frac{1}{\gamma M^2} - \frac{\gamma+1}{2\gamma}\ln\left(\frac{M^2}{1+\frac{\gamma-1}{2}M^2}\right)\right]_{M_1}^{M_2}$$

One-Dimensional Flow with Friction

Calorically perfect gas and adiabatic flow:

$$\frac{T_2}{T_1} = \frac{T_2}{T_{o_2}}\frac{T_{o_2}}{T_{o_1}}\frac{T_{o_1}}{T_1} = \{T_o = const\} = \frac{T_2}{T_o}\frac{T_o}{T_1} = \frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}$$

Continuity:

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{a_1 M_1}{a_2 M_2} = \left\{a = \sqrt{\gamma R T}\right\} = \sqrt{\frac{T_1}{T_2}}\left(\frac{M_1}{M_2}\right)$$

Perfect gas:

$$\frac{p_2}{p_1} = \{p = \rho R T\} = \frac{\rho_2 T_2}{\rho_1 T_1}$$

Total pressure:

$$\frac{p_{o_2}}{p_{o_1}} = \frac{p_{o_2}}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_{o_1}}$$

One-Dimensional Flow with Friction

Calorically perfect gas:

$$\frac{T_2}{T_1} = \frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2}$$

$$\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\left[\frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2}\right]^{-1/2}$$

$$\frac{p_2}{p_1} = \frac{M_1}{M_2}\left[\frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2}\right]^{1/2}$$

$$\frac{p_{o_2}}{p_{o_1}} = \frac{M_1}{M_2}\left[\frac{2 + (\gamma - 1)M_2^2}{2 + (\gamma - 1)M_1^2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

One-Dimensional Flow with Friction

Initially subsonic flow ($M_1 < 1$)

M_2 will increase as L increases

for a critical length L^*, the flow at point 2 will reach sonic conditions, *i.e.* $M_2 = 1$

Initially supersonic flow ($M_1 > 1$)

M_2 will decrease as L increases

for a critical length L^*, the flow at point 2 will reach sonic conditions, *i.e.* $M_2 = 1$

Note! The (*) condition in this context **is not** the same as the "critical" condition discussed for isentropic flow

One-Dimensional Flow with Friction

$$\frac{T}{T^*} = \frac{(\gamma+1)}{2+(\gamma-1)M^2}$$

$$\frac{\rho}{\rho^*} = \frac{1}{M}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{1/2}$$

$$\frac{p}{p^*} = \frac{1}{M}\left[\frac{\gamma+1}{2+(\gamma-1)M^2}\right]^{1/2}$$

$$\frac{p_o}{p_o^*} = \frac{1}{M}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

see Table A.4

One-Dimensional Flow with Friction

and

$$\int_0^{L^*} \frac{4f}{D} dx = \left[-\frac{1}{\gamma M^2} - \frac{\gamma+1}{2\gamma} \ln \left(\frac{M^2}{1 + \frac{\gamma-1}{2} M^2} \right) \right]_M^1$$

where L^* is the tube length needed to change current state to sonic conditions

Let $\bar{f}$ be the average friction coefficient over the length L^* $\Rightarrow$

$$\frac{4\bar{f}L^*}{D} = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma} \ln \left(\frac{(\gamma+1)M^2}{2+(\gamma-1)M^2} \right)$$

Turbulent pipe flow $\rightarrow \bar{f} \sim 0.005$ ($Re > 10^5$, roughness $\sim 0.001D$)

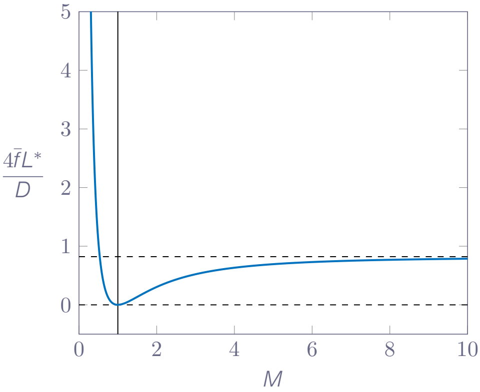
One-Dimensional Flow with Friction - Choking Length

Note!

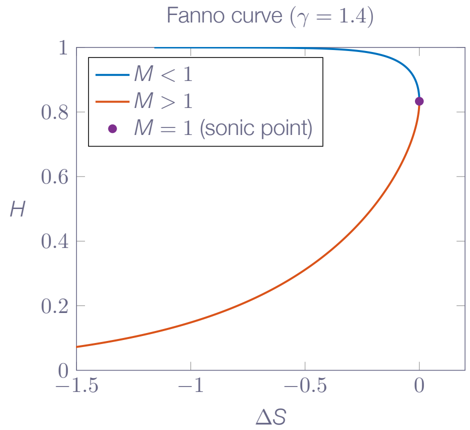
Supersonic flow is much more prone to choke than subsonic flow

There is an upper limit for supersonic choking length L^*

$$\left.\frac{4\bar{f}L^*}{D}(M_1)\right|_{M_1\to\infty} = \frac{1}{\gamma} + \left(\frac{\gamma+1}{2\gamma}\right)\ln\left(\frac{\gamma+1}{\gamma-1}\right)$$



One-Dimensional Flow with Friction



$$H = \frac{h}{h_o} = \frac{C_p T}{C_p T_o} = \frac{T}{T_o} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{-1}$$

$$\Delta S = \frac{\Delta s}{C_p} = \ln\left[\left(\frac{1}{H} - 1\right)^{\frac{\gamma-1}{2\gamma}} \left(\frac{2}{\gamma - 1}\right)^{\frac{\gamma-1}{2\gamma}} \left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma+1}{2\gamma}} (H)^{\frac{\gamma+1}{2\gamma}}\right]$$

One-Dimensional Flow with Friction

$M < 1$: Friction will

- increase M
- decrease p
- decrease T
- **decrease** p_o
- **increase** s
- increase u
- decrease ρ

$M > 1$: Friction will

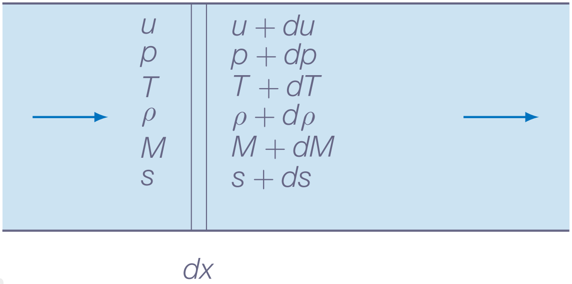
- decrease M
- increase p
- increase T
- **decrease** p_o
- **increase** s
- decrease u
- increase ρ

Note! the flow is not isentropic, there will always be losses

The Fanno-flow Process

Just like the Rayleigh flow, Fanno flow has **continuous** solutions

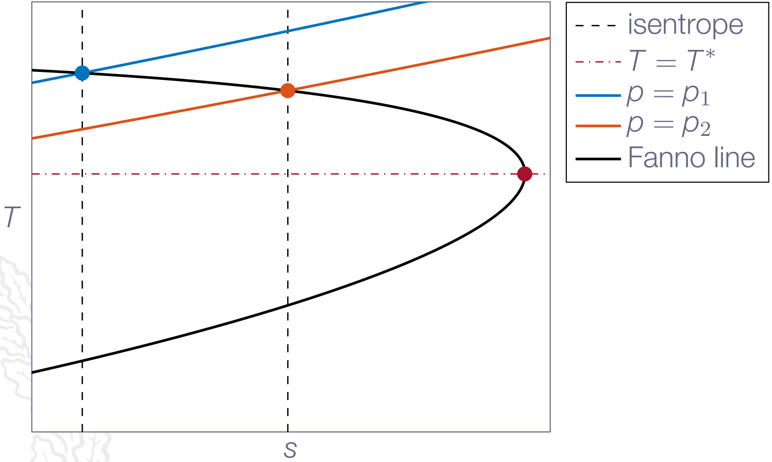
A small pipe section with length *dx* will change flow properties slightly



The Fanno-flow Process - Subsonic Flow

Note!

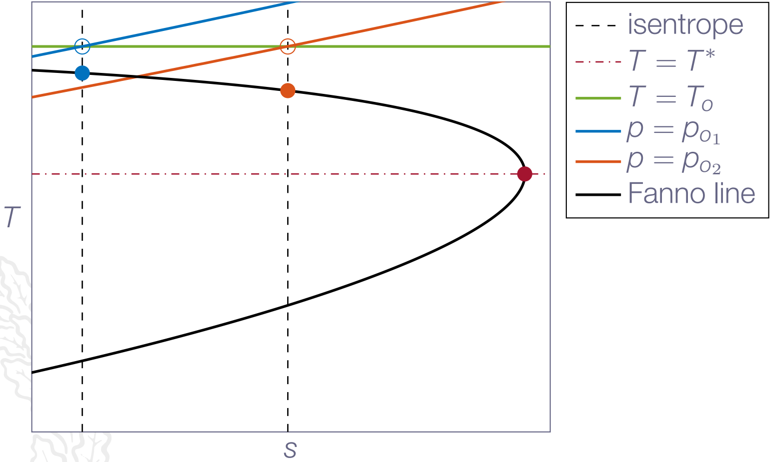
Pressure and temperature decreases when friction is added to a subsonic flow



The Fanno-flow Process - Subsonic Flow

Note!

The Fanno flow process is adiabatic $\Rightarrow$ T_o is constant $\Rightarrow$ p_o increases

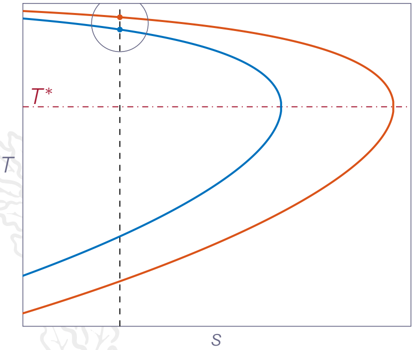


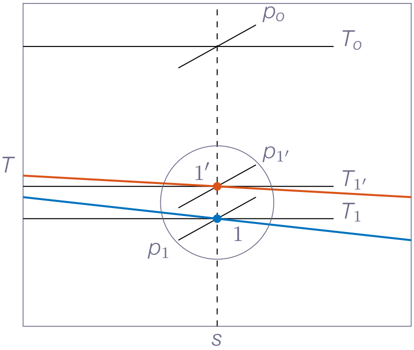
The Fanno-flow Process - Choked Subsonic Flow

Note!

If the pipe length is increased such that $L > L^*$, the inlet static flow properties will change (new massflow) such that the new L^* is equal to the pipe length

Total flow properties at the inlet remains the same (only work or heat addition can change the total flow properties)

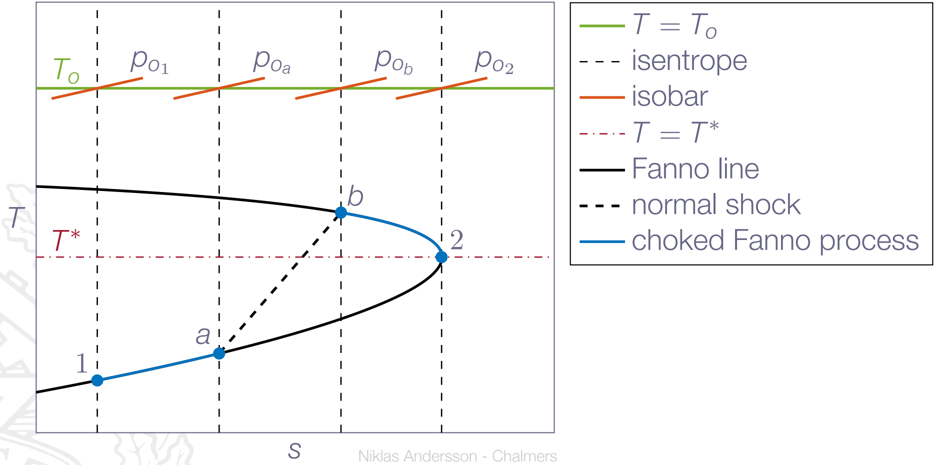




The Fanno-flow Process - Choked Supersonic

Note!

Choked supersonic flow will lead to the formation of a shock inside the pipe (shock location depends on flow conditions)



The Fanno-flow Process - Choked Supersonic

Why does the normal shock change the choking condition for Fanno flow but not for Rayleigh flow?

As for Rayleigh flow, T_o, T^*, p^*, etc are not affected by the shock

The **momentum equation is not the same** as for normal shocks $\Rightarrow$ the Fanno-flow process does not operate along the same line as the normal-shock process in $p\nu$-space

The Fanno-flow Process - Choked Supersonic

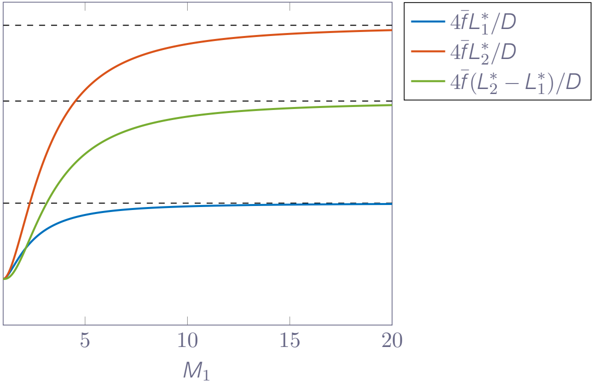
Note!

An internal shock will always increase the choking length L^*

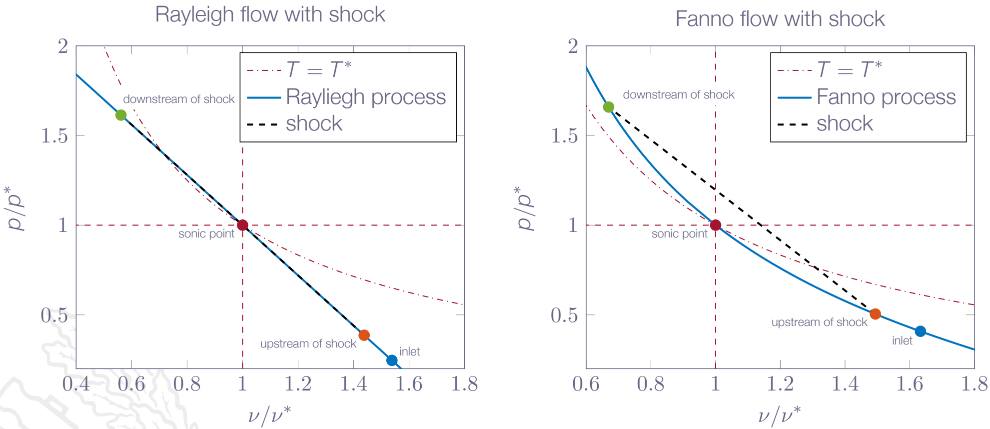
$$L_1^* = f(M_1)$$

$$\left.\begin{array}{l} L_2^* = f(M_2) \\ M_2 = f(M_1) \end{array}\right\} \Rightarrow L_2^* = f(M_1)$$

$$L_2^* - L_1^* = f(M_1)$$



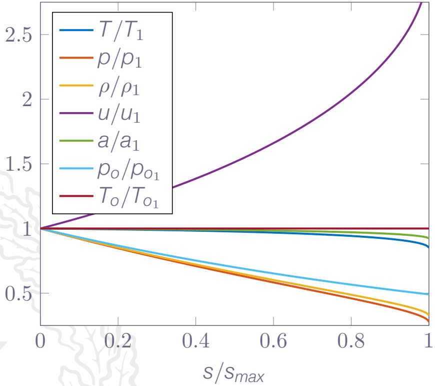
Friction Choking vs Thermal Choking

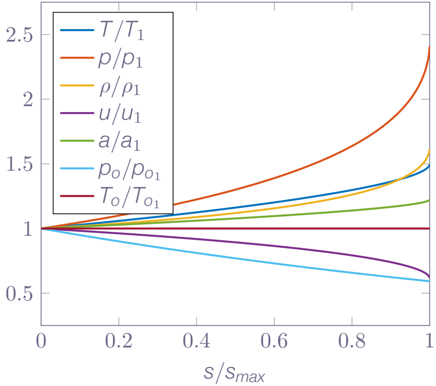


Rayleigh flow: a shock does not affect T_o^* or $T_o \Rightarrow q^*$ will not change over the shock

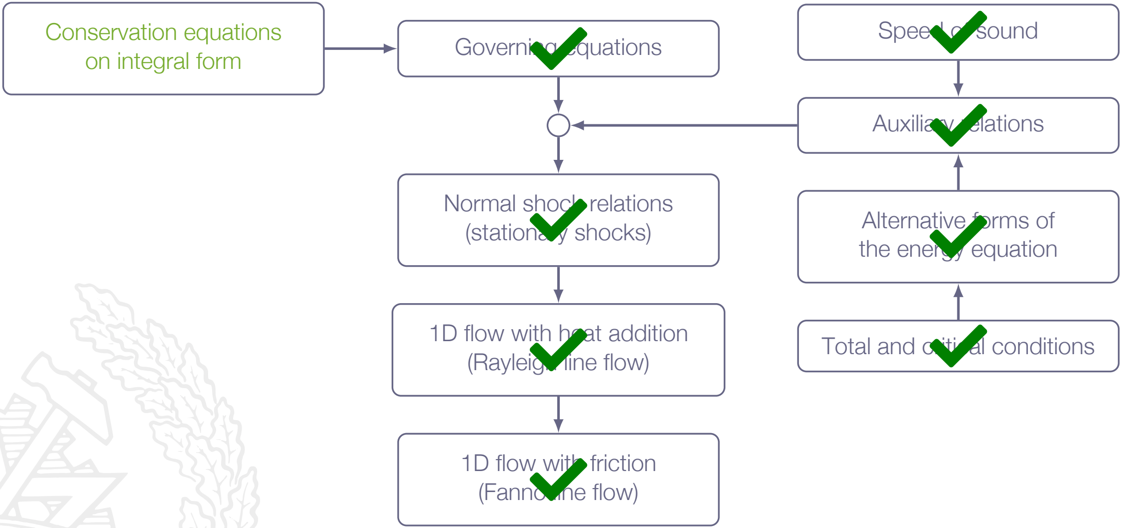
Fanno flow: L^* changes discontinuously over the shock $\Rightarrow$
L^* will always increase over a shock $\Rightarrow$ possible to extend pipe for supersonic flow

Fanno-flow Trends



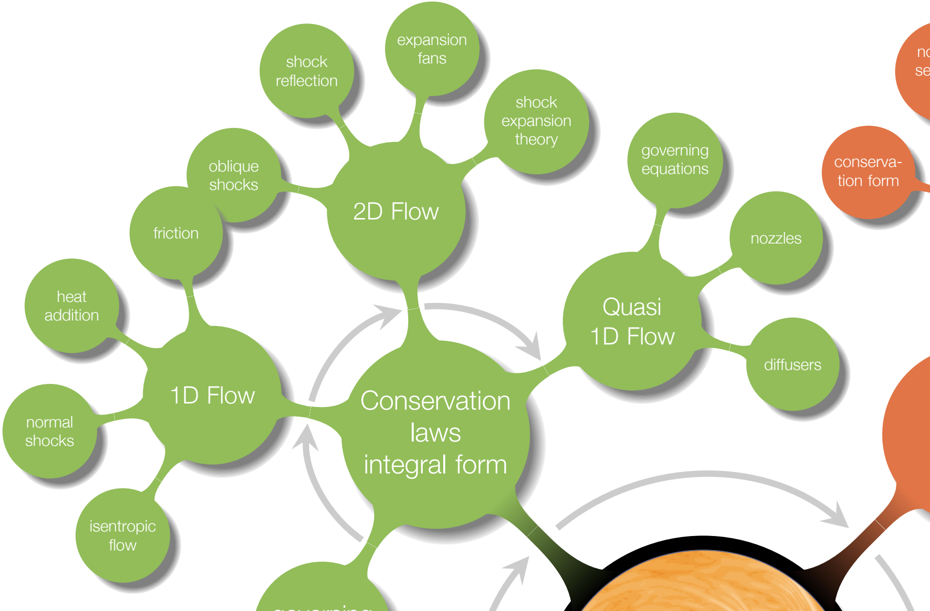


Roadmap - One-dimensional Flow



Chapter 4
Oblique Shocks and Expansion Waves

Overview



shock reflection
expansion fans
shock expansion theory
oblique shocks
2D Flow
friction
governing equations
nozzles
conserva-tion form
Quasi 1D Flow
diffusers
heat addition
1D Flow
normal shocks
Conservation laws integral form
isentropic flow

Learning Outcomes

4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on

7 **Explain** why entropy is important for flow discontinuities

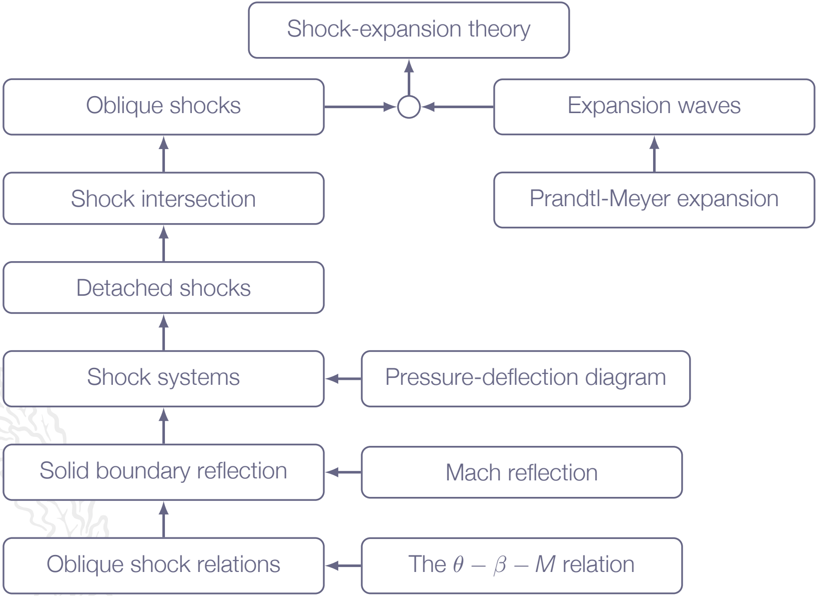
8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics

- b normal shocks*
- e oblique shocks in 2D*
- f shock reflection at solid walls*
- g contact discontinuities
- h Prandtl-Meyer expansion fans in 2D
- i detached blunt body shocks, nozzle flows

9 **Solve** engineering problems involving the above-mentioned phenomena (8a-8k)

why do we get normal shocks in some cases and oblique shocks in other?

Roadmap - Oblique Shocks and Expansion Waves



Motivation

Come on, two-dimensional flow, really?! Why not three-dimensional?

the normal shocks studied in chapter 3 are a special casees of the more general oblique shock waves that may be studied in two dimensions

in two dimensions, we can still analyze shock waves using a pen-and-paper approach

many practical problems or subsets of problems may be analyzed in two-dimensions

by going from one to two dimensions we will be able to introduce physical processes important for compressible flows

Oblique Shocks and Expansion Waves - Assumptions

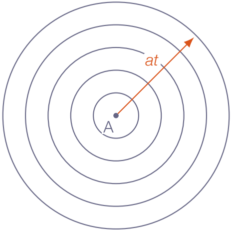
1. Supersonic
2. Steady-state
3. Two-dimensional
4. Inviscid flow (no wall friction)

In real flow, viscosity is non-zero $\Rightarrow$ boundary layers

For high-Reynolds-number flows, boundary layers are thin $\Rightarrow$ inviscid theory still relevant!

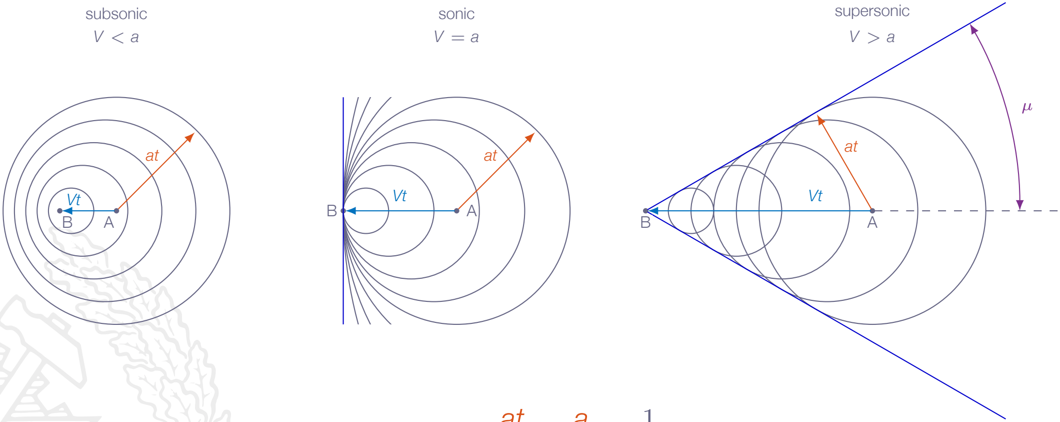
Mach Wave

Sound waves emitted from A (speed of sound a)



Mach Waves

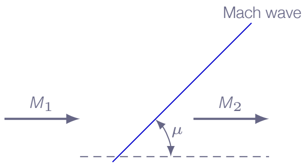
A Mach wave is an infinitely weak oblique shock



$$\sin \mu = \frac{at}{Vt} = \frac{a}{V} = \frac{1}{M}$$

Mach Wave

A Mach wave is an infinitely weak oblique shock

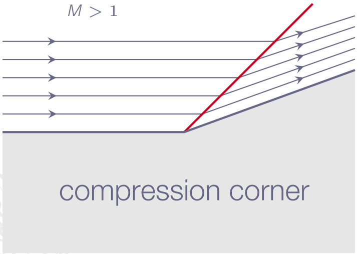


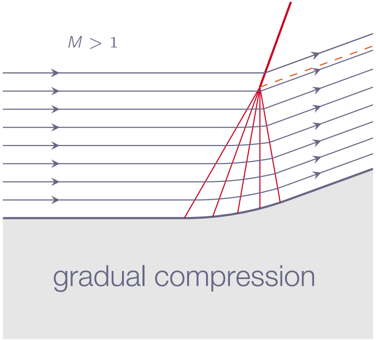
No substantial changes of flow properties over a single Mach wave

$M_1 > 1.0$ and $M_1 \approx M_2$

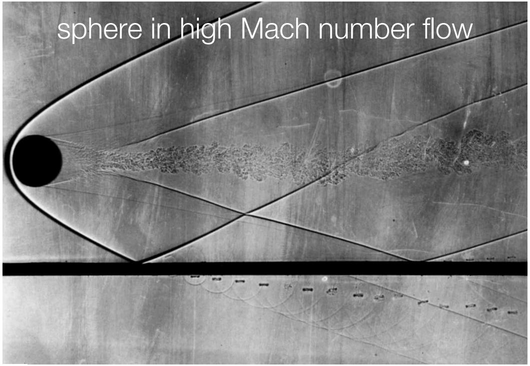
Isentropic

Oblique Shocks

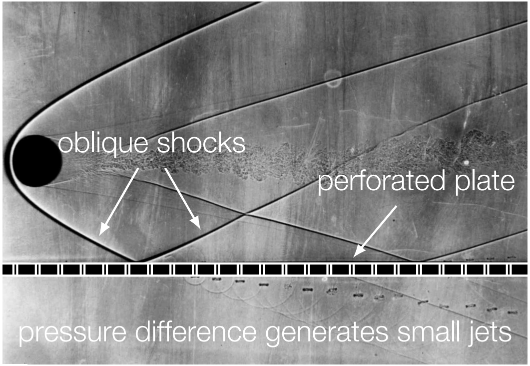




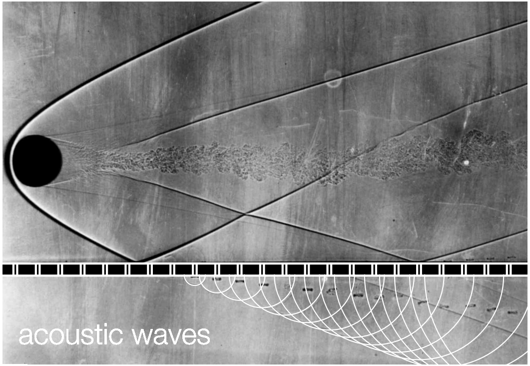
Oblique Shocks and Mach Waves



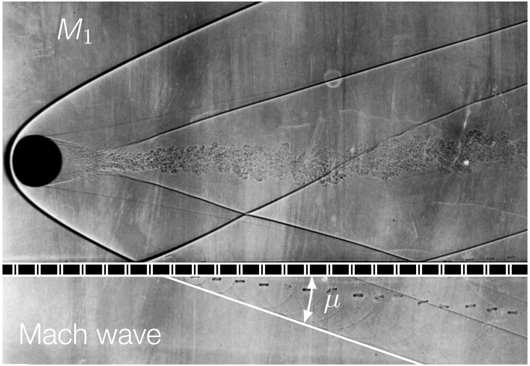
Oblique Shocks and Mach Waves



Oblique Shocks and Mach Waves

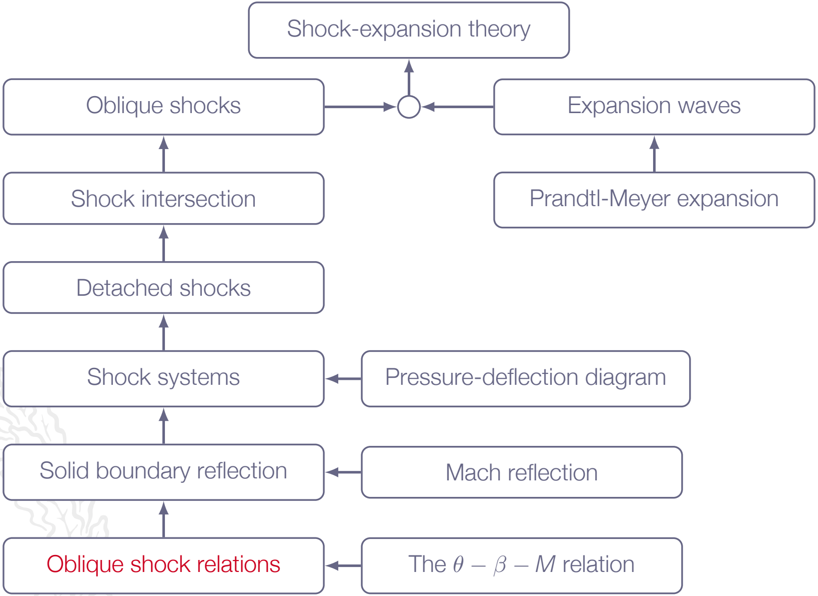


Oblique Shocks and Mach Waves



$$\mu = 19^\circ \Rightarrow M_1 = \frac{1}{\sin\mu} \approx 3.1$$

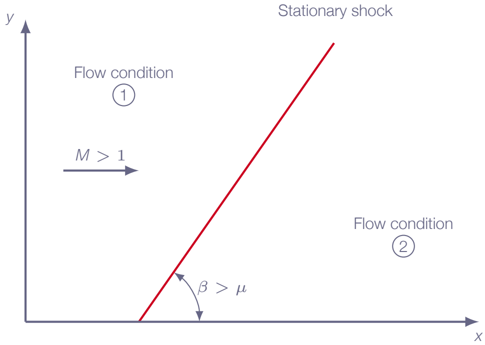
Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.3
Oblique Shock Relations

Oblique Shocks

Two-dimensional steady-state flow



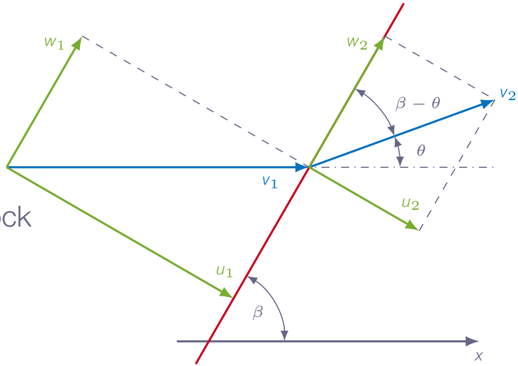
Significant changes of flow properties from 1 to 2

$M_1 > 1.0$, $\beta > \mu$, and $M_1 \neq M_2$

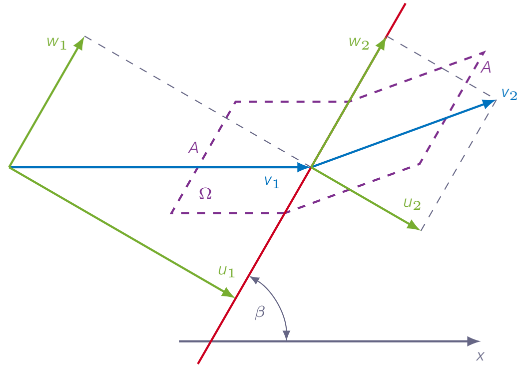
Not isentropic

Oblique Shocks

Stationary oblique shock



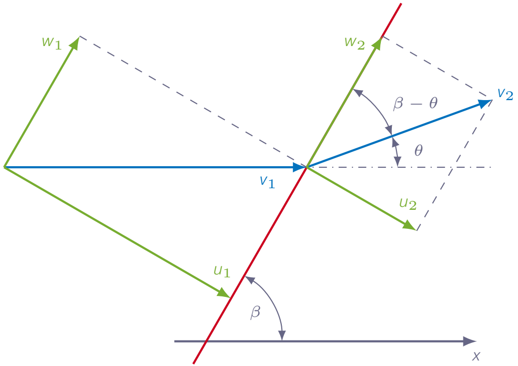
Oblique Shock Relations



Two-dimensional steady-state flow

Control volume aligned with flow stream lines

Oblique Shock Relations



Velocity notations:

$$M_{n_1} = \frac{u_1}{a_1} = M_1 \sin(\beta) \qquad M_1 = \frac{v_1}{a_1}$$

$$M_{n_2} = \frac{u_2}{a_2} = M_2 \sin(\beta - \theta) \qquad M_2 = \frac{v_2}{a_2}$$

Oblique Shock Relations

Conservation of mass:

$$\frac{d}{dt}\iiint\limits_{\Omega} \rho d\mathscr{V} + \oiint\limits_{\partial\Omega} \rho\mathbf{v}\cdot\mathbf{n}dS = 0$$

Mass conservation for control volume Ω:

$$0 - \rho_1 u_1 A + \rho_2 u_2 A = 0 \Rightarrow$$

$$\boxed{\rho_1 u_1 = \rho_2 u_2}$$

Oblique Shock Relations

Conservation of momentum:

$$\frac{d}{dt}\iiint\limits_{\Omega} \rho \mathbf{v} d\mathscr{V} + \oiint\limits_{\partial\Omega} [\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v} + p\mathbf{n}]\, dS = \iiint\limits_{\Omega} \rho \mathbf{f} d\mathscr{V}$$

Momentum in shock-normal direction:

$$0 - (\rho_1 u_1^2 + p_1)A + (\rho_2 u_2^2 + p_2)A = 0 \Rightarrow$$

$$\boxed{\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2}$$

Oblique Shock Relations

Momentum in shock-tangential direction:

$$0 - \rho_1 u_1 w_1 A + \rho_2 u_2 w_2 A = 0 \Rightarrow$$

$$\boxed{w_1 = w_2}$$

Oblique Shock Relations

Conservation of energy:

$$\frac{d}{dt}\iiint_{\Omega}\rho e_o d\mathscr{V} + \oiint_{\partial\Omega}[\rho h_o \mathbf{v}\cdot\mathbf{n}]\,dS = \iiint_{\Omega}\rho\mathbf{f}\cdot\mathbf{v}d\mathscr{V}$$

Energy equation applied to the control volume Ω:

$$0 - \rho_1 u_1[h_1 + \frac{1}{2}(u_1^2 + w_1^2)]A + \rho_2 u_2[h_2 + \frac{1}{2}(u_2^2 + w_2^2)]A = 0 \Rightarrow$$

$$\boxed{h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2}$$

Oblique Shock Relations

We can use the same equations as for normal shocks if we replace M_1 with M_{n_1} and M_2 with M_{n_2}

$$M_{n_2}^2 = \frac{M_{n_1}^2 + [2/(\gamma - 1)]}{[2\gamma/(\gamma - 1)]M_{n_1}^2 - 1}$$

Ratios such as ρ_2/ρ_1, p_2/p_1, and T_2/T_1 can be calculated using the relations for normal shocks with M_1 replaced by M_{n_1}

Oblique Shock Relations

What about ratios involving stagnation flow properties, can we use the ones previously derived for normal shocks?

Oblique Shock Relations

What about ratios involving stagnation flow properties, can we use the ones previously derived for normal shocks?

The shock process is adiabatic and thus total temperature is not effected by the shock $\Rightarrow T_{o_2} = T_{o_1}$

Oblique Shock Relations

What about ratios involving stagnation flow properties, can we use the ones previously derived for normal shocks?

The shock process is adiabatic and thus total temperature is not effected by the shock $\Rightarrow T_{o_2} = T_{o_1}$

What about the total pressure?

Oblique Shock Relations

What about ratios involving stagnation flow properties, can we use the ones previously derived for normal shocks?

The shock process is adiabatic and thus total temperature is not effected by the shock $\Rightarrow T_{o_2} = T_{o_1}$

What about the total pressure?

$$s_2 - s_1 = C_p \ln\left(\frac{T_{o_2}}{T_{o_1}}\right) - R\ln\left(\frac{p_{o_2}}{p_{o_1}}\right) = \{T_{o_2} = T_{o_1}\} = -R\ln\left(\frac{p_{o_2}}{p_{o_1}}\right)$$

entropy is a thermodynamic flow property and $s_2 - s_1$ is dictated by the shock strength and thus the total pressure ratio is a function of the shock-normal Mach number

Oblique Shock Relations

Note! total pressure is always calculated using the flow Mach number, not the shock-normal Mach number

However, the ratio p_{o_2}/p_{o_1} may be calculated using the shock-normal Mach number

So, be careful when using relations derived for normal shocks for oblique shocks when it comes to total flow conditions...

Oblique Shock Relations

p_{o_2}/p_{o_1} is calculated as: $\dfrac{p_{o_2}}{p_{o_1}} = \dfrac{p_{o_2}}{p_2}\dfrac{p_2}{p_1}\dfrac{p_1}{p_{o_1}}$

where

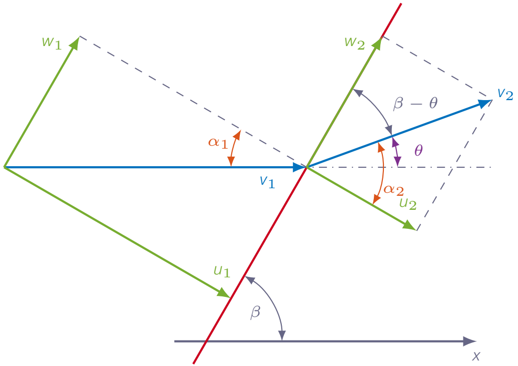
1. $\dfrac{p_{o_2}}{p_2} = f(M_2)$, $\dfrac{p_2}{p_1} = f(M_{n_1})$, and $\dfrac{p_1}{p_{o_1}} = f(M_1)$

or alternatively

2. $\dfrac{p_{o_2}}{p_2} = f(M_{n_2})$, $\dfrac{p_2}{p_1} = f(M_{n_1})$, and $\dfrac{p_1}{p_{o_1}} = f(M_{n_1})$

Note! in the second case the total pressures are **not** the true total pressures of the flow and therefore it is suggested to use the first approach

Deflection Angle (*for the interested*)



$$\theta = \alpha_2 - \alpha_1 = \tan^{-1}\left(\frac{w}{u_2}\right) - \tan^{-1}\left(\frac{w}{u_1}\right)$$

$$\frac{\partial\theta}{\partial w} = \frac{u_2}{w^2 + u_2^2} - \frac{u_1}{w^2 + u_1^2}$$

Deflection Angle (*for the interested*)

$$\frac{\partial \theta}{\partial w} = \frac{u_2}{w^2 + u_2^2} - \frac{u_1}{w^2 + u_1^2} = 0 \Rightarrow$$

$$\frac{u_2(w^2 + u_1^2) - u_1(w^2 + u_2^2)}{(w^2 + u_2^2)(w^2 + u_1^2)} = 0 \Rightarrow \frac{(u_2 - u_1)(w^2 - u_1 u_2)}{(w^2 + u_2^2)(w^2 + u_1^2)} = 0$$

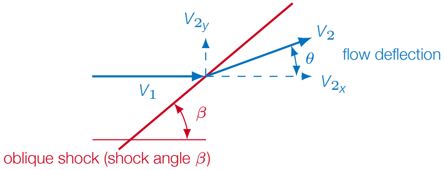
Two solutions:

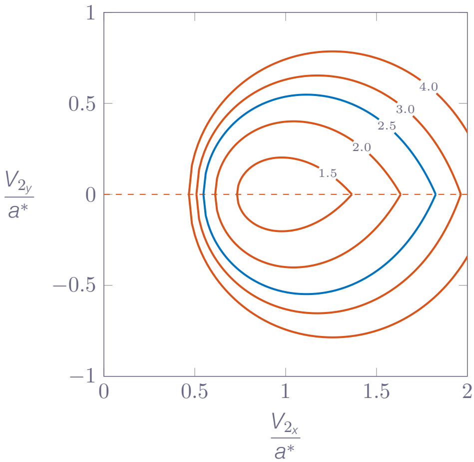
$u_2 = u_1$ (no deflection)

$w^2 = u_1 u_2$ (max deflection)

Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number





Note!

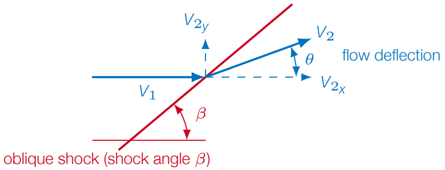
In the shock polar, V_{2_x} and V_{2_y} are normalized by a^*

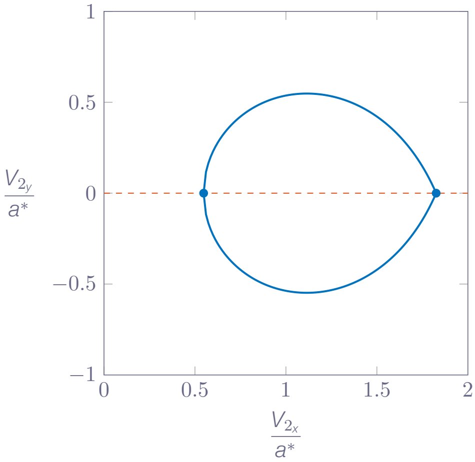
a^* is a constant in a adiabatic flow

a^* is not affected by shocks

Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number





No deflection cases:

normal shock
(reduced shock-normal velocity)

Mach wave
(unchanged shock-normal velocity)

Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

$$M^* = \frac{\sqrt{V_{2_x}^2 + V_{2_y}^2}}{a^*}$$

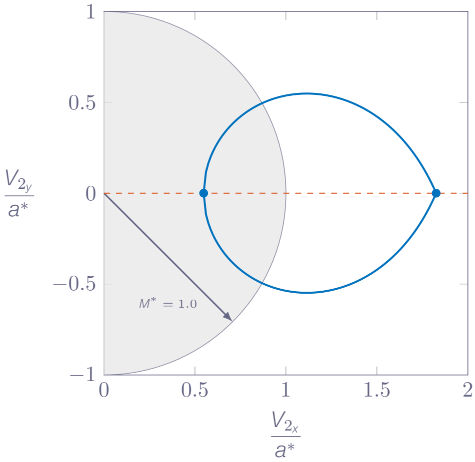
Solutions to the left of the sonic line are subsonic

Recall

$M^* = 1 \Leftrightarrow M = 1$

$M^* < 1 \Leftrightarrow M < 1$

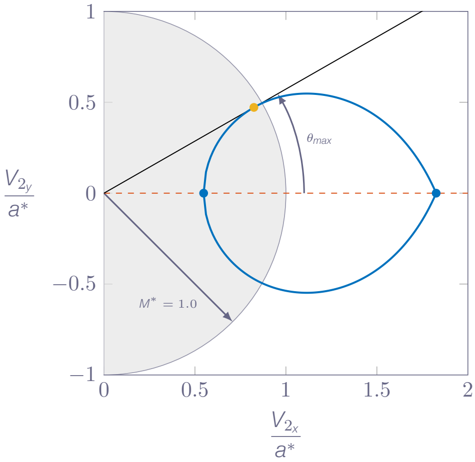
$M^* > 1 \Leftrightarrow M > 1$



Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

It is not possible to deflect the flow more than θ_{max}



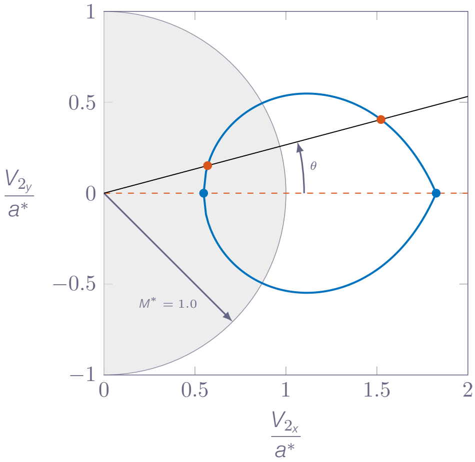
Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

For each deflection angle $\theta < \theta_{max}$, there are two solutions

1. **strong shock** solution
2. **weak shock** solution

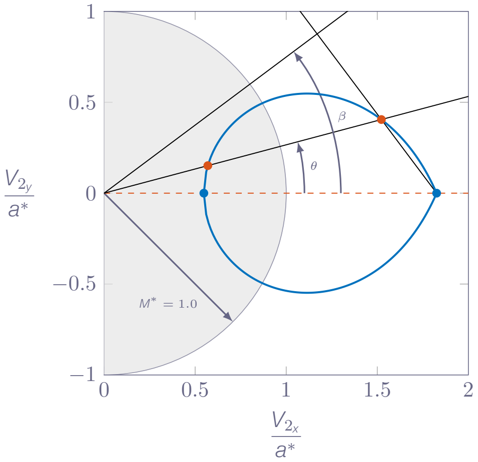
Weak shocks give lower losses and therefore the preferred solution



Shock Polar

Graphical representation of all possible deflection angles for a specific Mach number

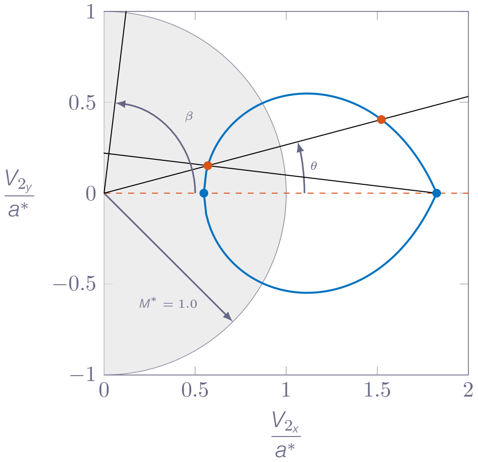
The shock polar can be used to calculate the shock angle β for a given deflection angle θ



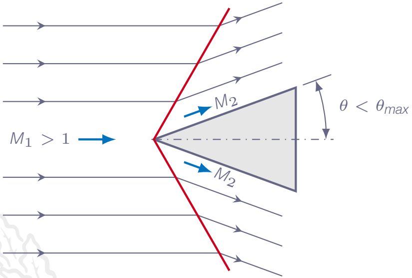
Shock Polar

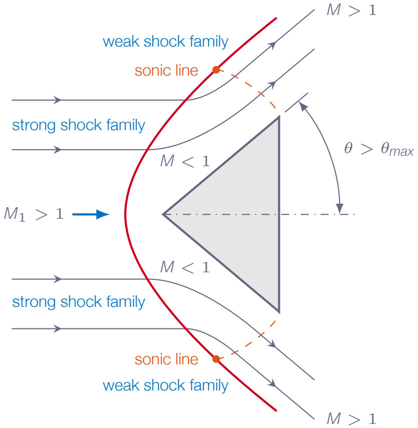
Graphical representation of all possible deflection angles for a specific Mach number

The shock polar can be used to calculate the shock angle β for a given deflection angle θ

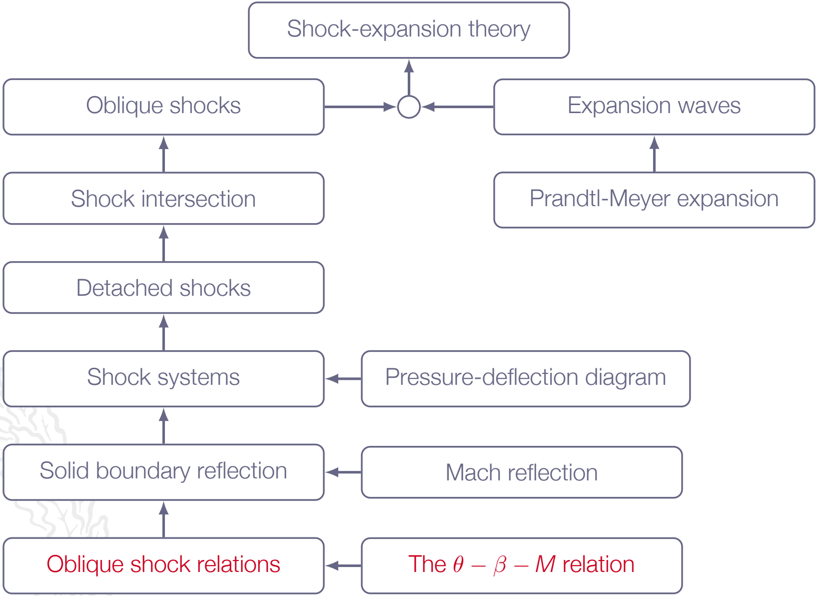


Flow Deflection





Roadmap - Oblique Shocks and Expansion Waves



The θ-β-M Relation

It can be shown that

$$\tan\theta = 2\cot\beta\left(\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}\right)$$

which is the θ-β-M relation

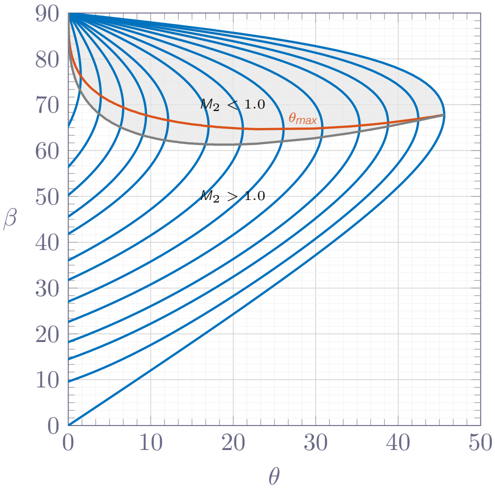
The θ-β-Mach Relation

A relation between:

1. flow deflection angle θ
2. shock angle β
3. upstream flow Mach number M_1

$$\tan(\theta) = 2\cot(\beta)\left(\frac{M_1^2\sin^2(\beta) - 1}{M_1^2(\gamma + \cos(2\beta)) + 2}\right)$$

Note! in general there are two solutions for a given M_1 (or none)



The θ-β-Mach Relation

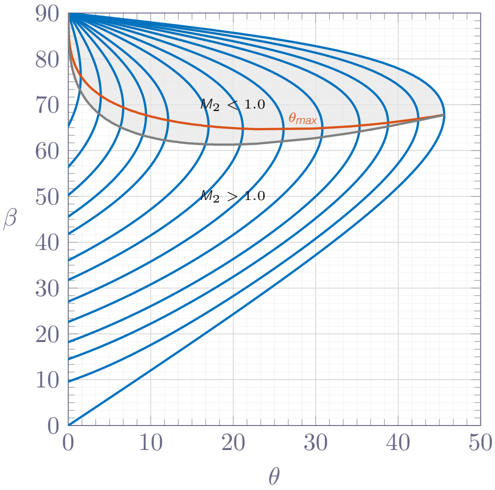
There is a small region where we may find weak shock solutions for which $M_2 < 1$

In most cases weak shock solutions have $M_2 > 1$

Strong shock solutions always have $M_2 < 1$

In practical situations, **weak shock solutions are most common**

Strong shock solution may appear in special situations due to high back pressure, which forces $M_2 < 1$

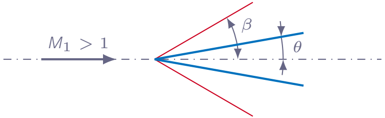


The θ-β-M Relation

Note! In Chapter 3 we learned that the Mach number always reduces to subsonic values behind a shock. This is true for normal shocks (shocks that are normal to the flow direction). It is also true for oblique shocks if looking in the shock-normal direction.

The θ-β-M Relation - Wedge Flow

Wedge flow oblique shock analysis:

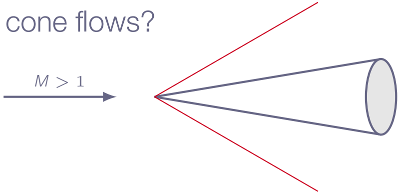


1. θ-β-M relation $\Rightarrow$ β for given M_1 and θ
2. β gives M_{n_1} according to: $M_{n_1} = M_1 \sin(\beta)$
3. normal shock formula with M_{n_1} instead of M_1 $\Rightarrow$ M_{n_2} (instead of M_2)
4. M_2 given by $M_2 = M_{n_2} / \sin(\beta - \theta)$
5. normal shock formula with M_{n_1} instead of M_1 $\Rightarrow$ ρ_2/ρ_1, p_2/p_1, etc
6. upstream conditions + ρ_2/ρ_1, p_2/p_1, etc $\Rightarrow$ downstream conditions

Chapter 4.4
Supersonic Flow over Wedges and Cones

Supersonic Flow over Wedges and Cones

What about cone flows?

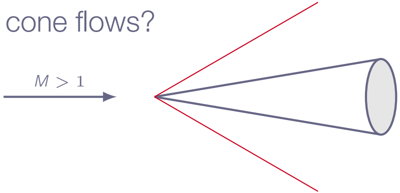


Similar to wedge flow, we do get a constant-strength shock wave, attached at the cone tip (or else a detached curved shock)

The attached shock is also cone-shaped

Supersonic Flow over Wedges and Cones

What about cone flows?

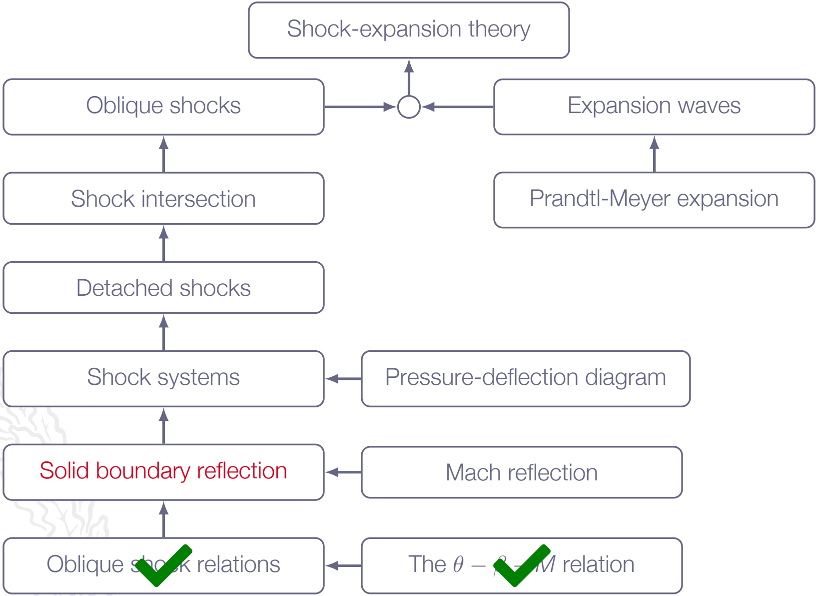


The flow condition immediately downstream of the shock is uniform

However, downstream of the shock the streamlines are curved and the flow varies in a more complex manner (3D relieving effect - as R increases there is more and more space around cone for the flow)

β for cone shock is always smaller than that for wedge shock, if M_1 is the same

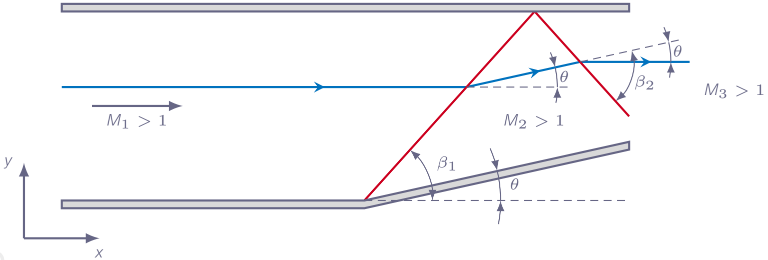
Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.6 Regular Reflection from a Solid Boundary

Shock Reflection

Regular reflection of oblique shock at solid wall
(see example 4.10)



Assumptions:

- steady-state inviscid flow
- weak shocks

Shock Reflection

first shock	second shock
upstream condition	**upstream condition**
$M_1 > 1$ flow in x-direction	downstream of first shock
downstream condition	**downstream condition**
weak shock $\Rightarrow M_2 > 1$ deflection angle θ shock angle β_1	weak shock $\Rightarrow M_3 > 1$ deflection angle θ shock angle β_2

Shock Reflection

Solution:

first shock:

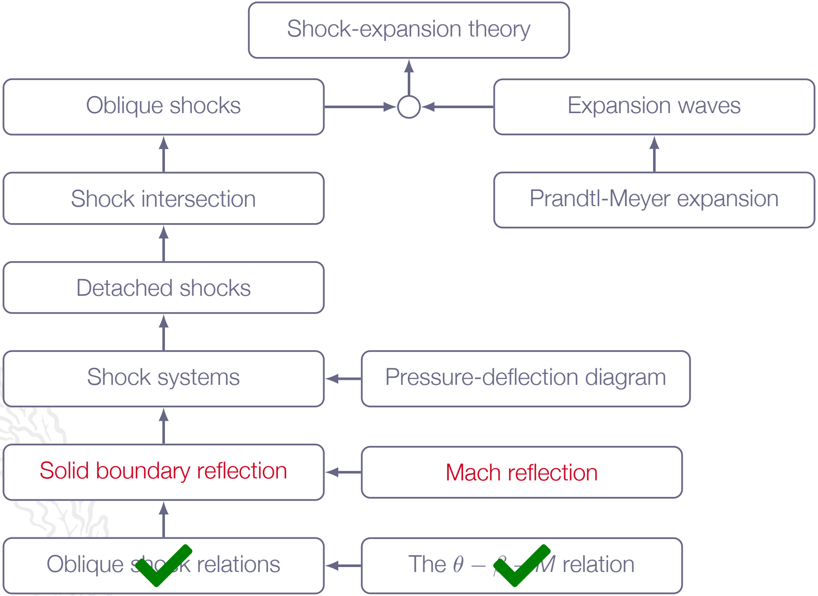
1. β_1 calculated from θ-β-M relation for specified θ and M_1 (*weak solution*)
2. flow condition 2 according to formulas for normal shocks ($M_{n_1} = M_1 \sin(\beta_1)$ and $M_{n_2} = M_2 \sin(\beta_1 - \theta)$)

second shock:

1. β_2 calculated from θ-β-M relation for specified θ and M_2 (*weak solution*)
2. flow condition 3 according to formulas for normal shocks ($M_{n_2} = M_2 \sin(\beta_2)$ and $M_{n_3} = M_3 \sin(\beta_2 - \theta)$)

$\Rightarrow$ complete description of flow and shock waves
(angle between upper wall and second shock: $\Phi = \beta_2 - \theta$)

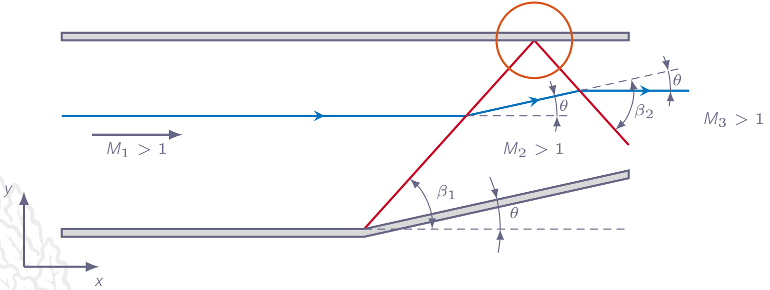
Roadmap - Oblique Shocks and Expansion Waves



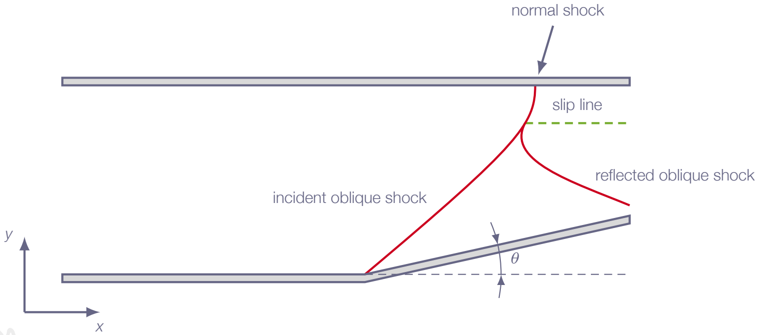
Chapter 4.11
Mach Reflection

Regular Shock Reflection

Regular reflection possible if both primary and reflected shocks are weak (see θ-β-M relation)



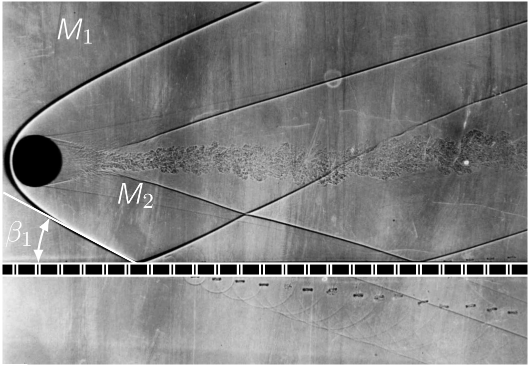
Mach Reflection



Mach reflection:

- appears when regular reflection is not possible
- more complex flow than for a regular reflection
- no analytic solution - numerical solution necessary

Oblique Shocks and Mach Waves

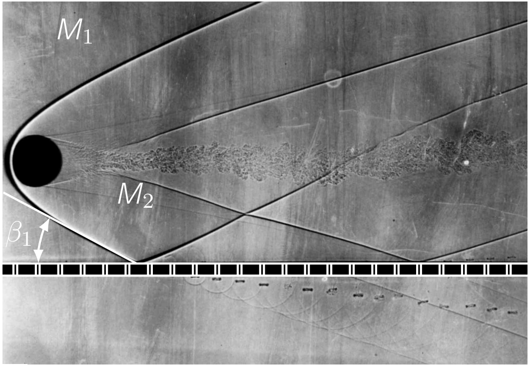


$M_1 > M_2$

$M_2 > 1.0$

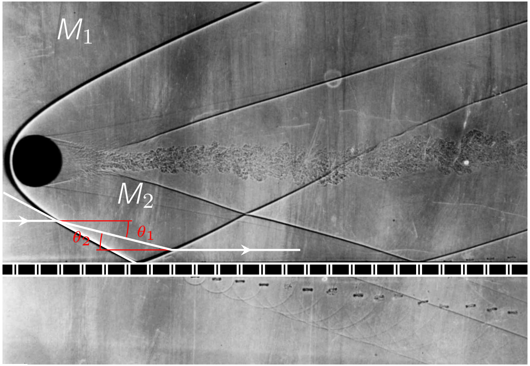
$$\theta_1 = f(M_1, \beta_1), \quad M_2 = f(M_1, \theta_1, \beta_1)$$

Oblique Shocks and Mach Waves



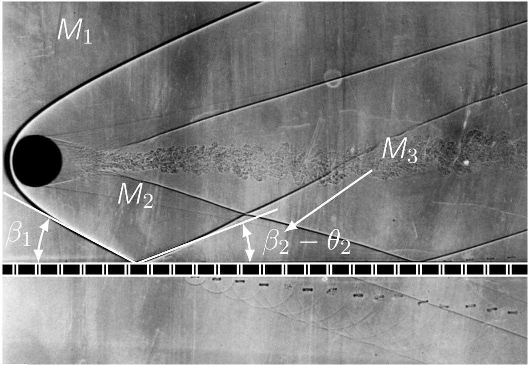
$$\left.\begin{aligned} \beta_1 &= 28^\circ \\ M_1 &= 3.1 \end{aligned}\right\} \Rightarrow \theta_1 \approx 11.2^\circ, \quad M_2 \approx 2.5$$

Oblique Shocks and Mach Waves



$$\theta_1 = \theta_2$$

Oblique Shocks and Mach Waves



$$M_1 > M_2 > M_3$$

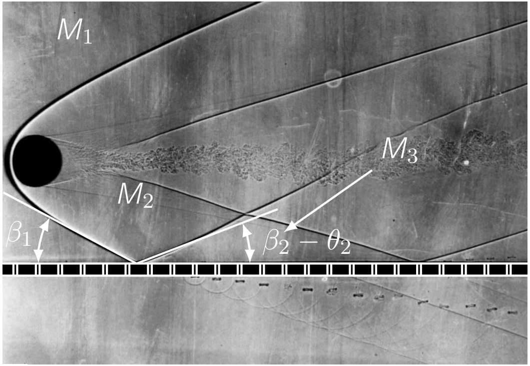
$$M_3 > 1.0$$

$$\beta_2 > \beta_1$$

$$\beta_2 = f(M_2, \theta_2), \quad M_3 = f(M_2, \theta_2, \beta_2)$$

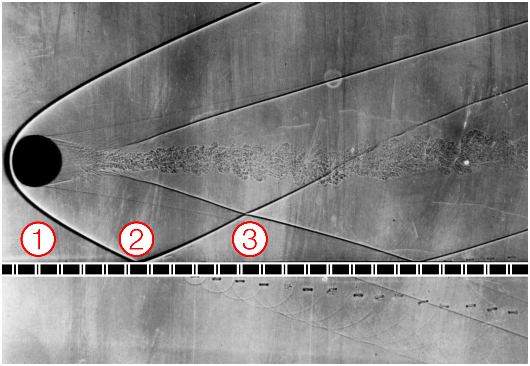
Note! Shock wave reflection at solid wall is **not** specular

Oblique Shocks and Mach Waves



$$\left.\begin{aligned} \theta_2 &= 11.2^\circ \\ M_2 &= 2.5 \end{aligned}\right\} \Rightarrow \beta_2 \approx 33^\circ, \quad M_3 \approx 2.0$$

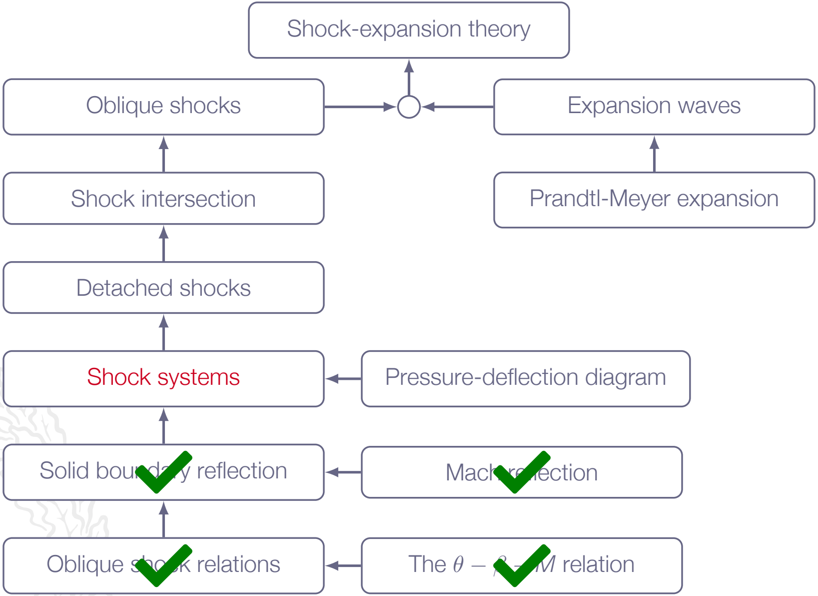
Oblique Shocks and Mach Waves



$$\frac{p_3}{p_1} = \frac{p_2}{p_1}\frac{p_3}{p_2} \approx 4.52$$

$$\frac{T_3}{T_1} = \frac{T_2}{T_1}\frac{T_3}{T_2} \approx 1.57$$

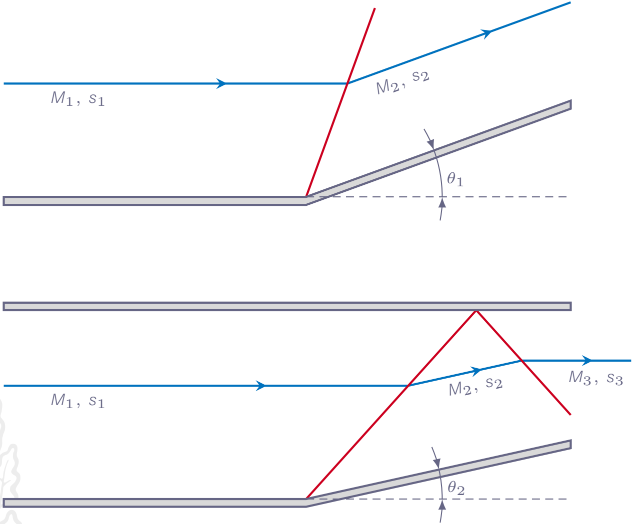
Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.7
Comments on Flow Through Multiple Shock Systems

Flow Through Multiple Shock Systems

Single-shock compression versus multiple-shock compression:



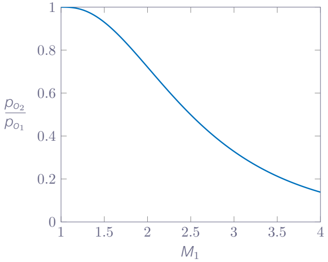
Flow Through Multiple Shock Systems

We may find θ_1 and θ_2 (for same M_1) which gives the same final Mach number

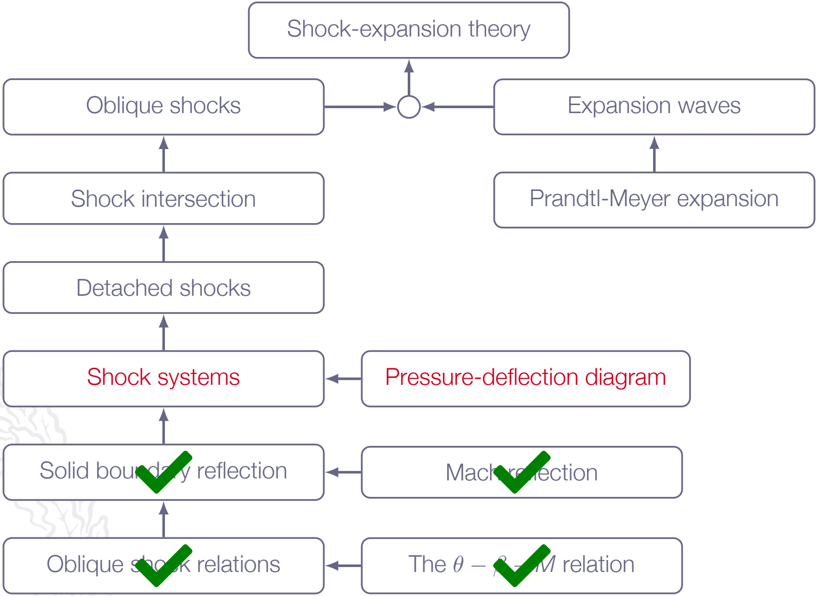
In such cases, the flow with multiple shocks has smaller losses

Explanation: entropy generation at a shock is a very non-linear function of shock strength

Note! the system of multiple shocks might very well result in a larger total flow deflection angle than the single-shock case

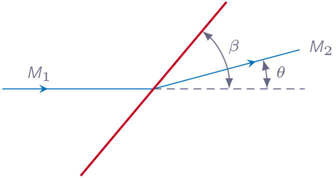


Roadmap - Oblique Shocks and Expansion Waves

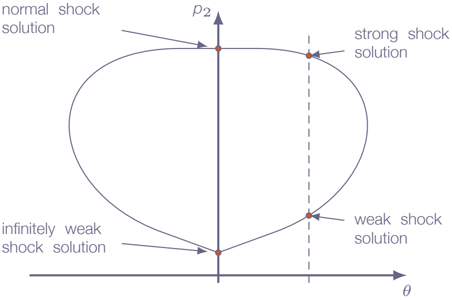


Chapter 4.8
Pressure Deflection Diagrams

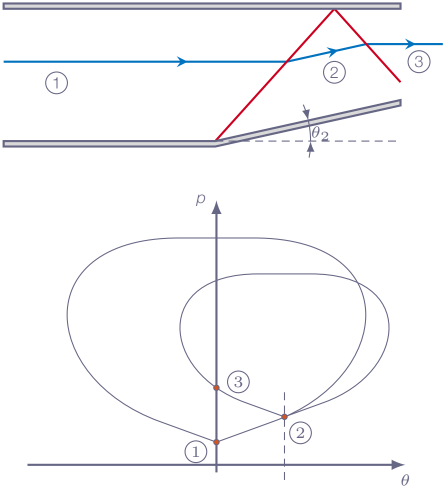
Pressure Deflection Diagrams



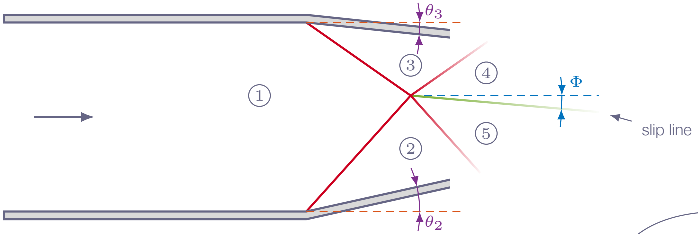
$\Rightarrow$ relation between p_2 and θ

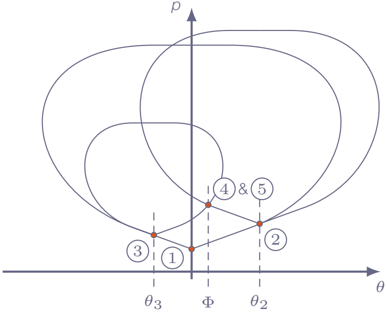


Pressure Deflection Diagrams - Shock Reflection



Pressure Deflection Diagrams - Shock Intersection

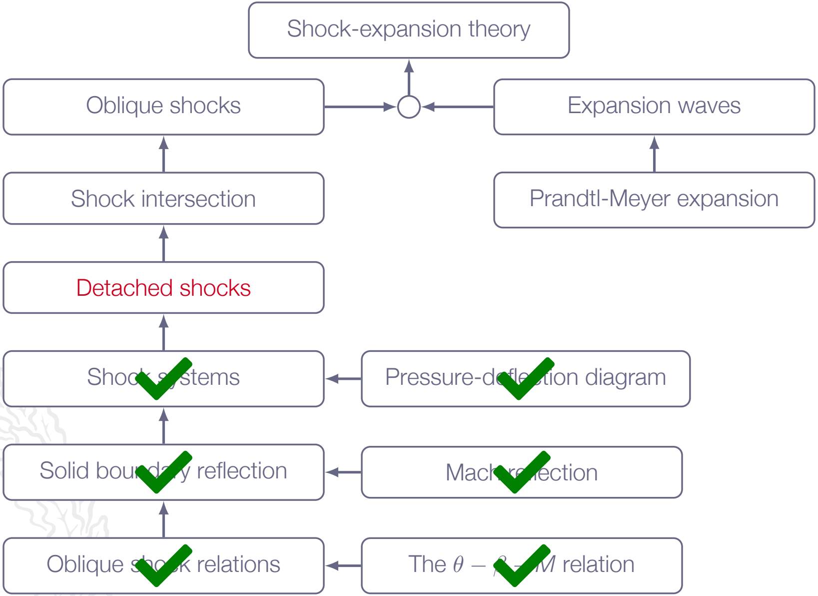




A slip line is a contact discontinuity:

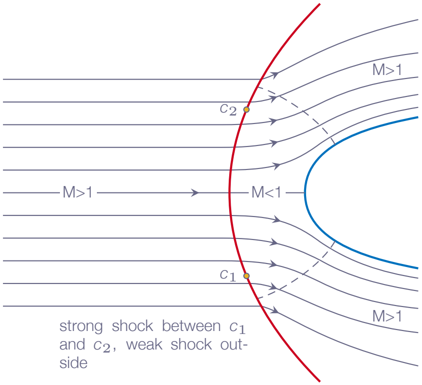
discontinuity in ρ, T, s, v, and M
continuous in p and flow angle

Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.12
Detached Shock Wave in Front of a Blunt Body

Detached Shocks

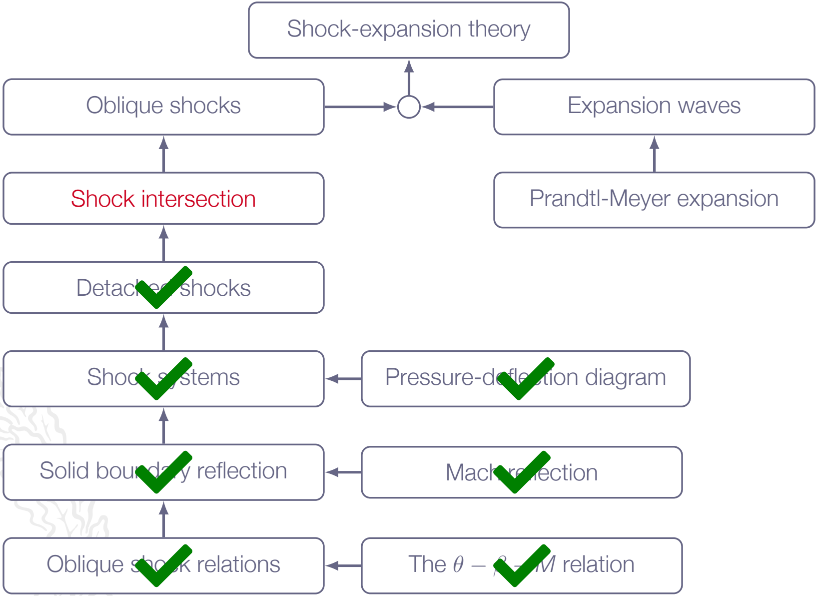


Detached Shocks

As we move along the detached shock form the centerline, the shock will change in nature as

1. right in front of the body we will have a normal shock
2. strong oblique shock
3. weak oblique shock
4. far away from the body it will approach a **Mach wave**, *i.e.* an infinitely weak oblique shock

Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.10
Intersection of Shocks of the Same Family

Mach Waves (*Repetition*)

Oblique shock, angle β, flow deflection θ:

$$M_{n_2}^2 = \frac{M_{n_1}^2 + [2/(\gamma - 1)]}{[2\gamma/(\gamma - 1)]M_{n_1}^2 - 1}$$

where

$$M_{n_1} = M_1 \sin(\beta)$$

and

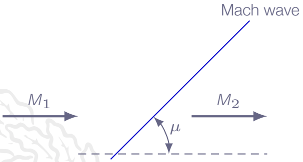
$$M_{n_2} = M_2 \sin(\beta - \theta)$$

Now, let $M_{n_1} \rightarrow 1$ and $M_{n_2} \rightarrow 1 \Rightarrow$ infinitely weak shock!

Such very weak shocks are called **Mach waves**

Mach Waves (*Repetition*)

$$M_{n_1} = 1 \Rightarrow M_1 \sin(\beta) = 1 \Rightarrow \beta = \arcsin(1/M_1)$$

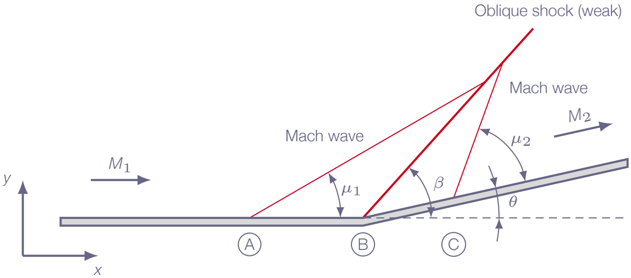


$$M_2 \approx M_1$$

$$\theta \approx 0$$

$$\mu = \arcsin(1/M_1)$$

Mach Waves



Oblique shock (weak)
Mach wave
M_2
μ_2
Mach wave
M_1
μ_1
β
θ
y
x
A
B
C

Mach Waves

1. Mach wave at A: $\sin(\mu_1) = 1/M_1$
2. Mach wave at C: $\sin(\mu_2) = 1/M_2$
3. Oblique shock at B: $M_{n_1} = M_1 \sin(\beta) \Rightarrow \sin(\beta) = M_{n_1}/M_1$

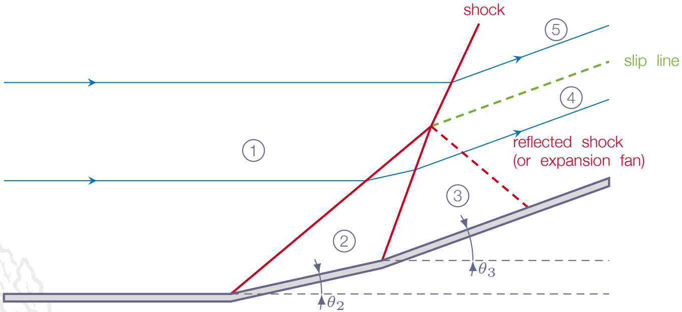
   Existence of shock requires $M_{n_1} > 1 \Rightarrow \beta > \mu_1$

   Mach wave intercepts shock!
4. Also, $M_{n_2} = M_2 \sin(\beta - \theta) \Rightarrow \sin(\beta - \theta) = M_{n_2}/M_2$

   For finite shock strength $M_{n_2} < 1 \Rightarrow (\beta - \theta) < \mu_2$

   Again, Mach wave intercepts shock

Shock Intersection - Same Family



Shock Intersection - Same Family

Case 1: Streamline going through regions 1, 2, 3, and 4 (through two oblique (weak) shocks)

Case 2: Streamline going through regions 1 and 5 (through one oblique (weak) shock)

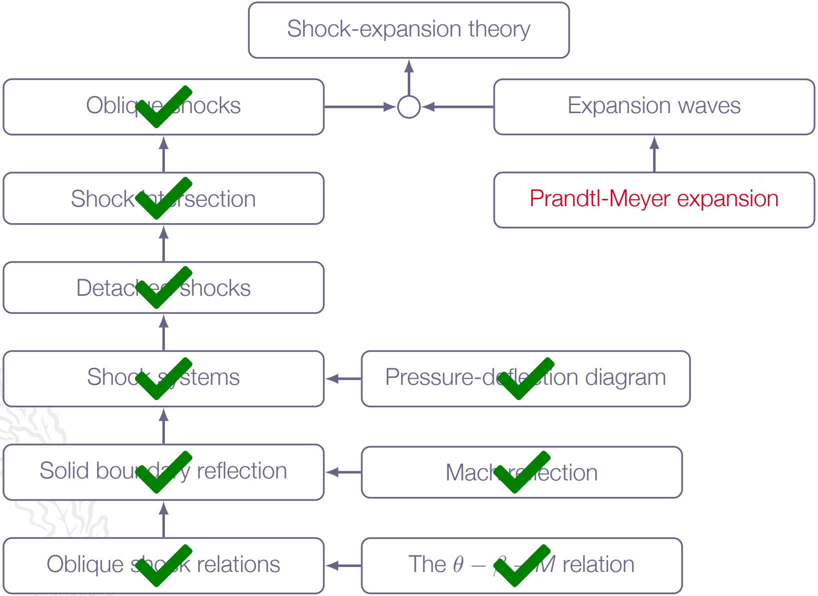
Problem: Find conditions 4 and 5 such that

a. $p_4 = p_5$
b. flow angle in 4 equals flow angle in 5

Solution may give either **reflected shock** or **expansion fan**, depending on actual conditions

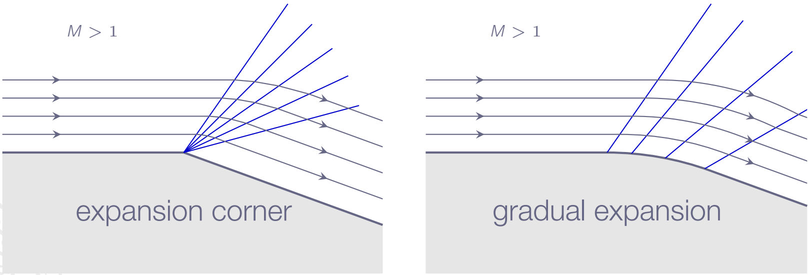
A **slip line** usually appears, across which there is a discontinuity in all variables except p and flow angle

Roadmap - Oblique Shocks and Expansion Waves



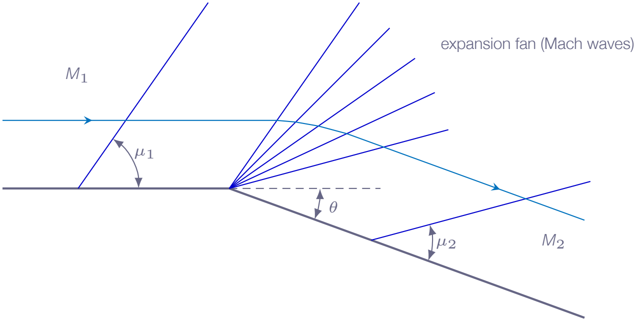
Chapter 4.14
Prandtl-Meyer Expansion Waves

Expansion Waves



Prandtl-Meyer Expansion Waves

An expansion fan is a centered simple wave (also called Prandl-Meyer expansion)



$M_2 > M_1$ (the flow accelerates through the expansion fan)

$p_2 < p_1$, $\rho_2 < \rho_1$, $T_2 < T_1$

Prandtl-Meyer Expansion Waves

Continuous expansion region

Infinite number of weak Mach waves

Streamlines through the expansion wave are smooth curved lines

$ds = 0$ for each Mach wave $\Rightarrow$ the expansion process is **isentropic!**

Prandtl-Meyer Expansion Waves

upstream of expansion $M_1 > 1$, $\sin(\mu_1) = 1/M_1$

flow accelerates as it curves through the expansion fan

downstream of expansion $M_2 > M_1$, $\sin(\mu_2) = 1/M_2$

flow is isentropic $\Rightarrow s, p_o, T_o, \rho_o, a_o$, ... are constant along streamlines

flow deflection: θ

Prandtl-Meyer Expansion Waves

It can be shown that $d\theta = \sqrt{M^2 - 1}\dfrac{dv}{v}$, where $v = |\mathbf{v}|$ (valid for all gases)

Integration gives

$$\int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \sqrt{M^2 - 1}\frac{dv}{v}$$

the term $\dfrac{dv}{v}$ needs to be expressed in terms of Mach number

$v = Ma \Rightarrow \ln v = \ln M + \ln a \Rightarrow$

$$\frac{dv}{v} = \frac{dM}{M} + \frac{da}{a}$$

Prandtl-Meyer Expansion Waves

Calorically perfect gas and adiabatic flow gives

$$\frac{T_o}{T} = 1 + \frac{1}{2}(\gamma - 1)M^2$$

$$\left\{a = \sqrt{\gamma R T},\ a_o = \sqrt{\gamma R T_o}\right\} \Rightarrow \frac{T_o}{T} = \left(\frac{a_o}{a}\right)^2 \Rightarrow$$

$$\left(\frac{a_o}{a}\right)^2 = 1 + \frac{1}{2}(\gamma - 1)M^2 \Leftrightarrow a = a_o\left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-1/2}$$

Prandtl-Meyer Expansion Waves

Differentiation gives:

$$da = a_o \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-3/2} \left(-\frac{1}{2}\right)(\gamma - 1)MdM$$

or

$$da = a \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-1} \left(-\frac{1}{2}\right)(\gamma - 1)MdM$$

which gives

$$\frac{dv}{v} = \frac{dM}{M} + \frac{da}{a} = \frac{dM}{M} + \frac{-\frac{1}{2}(\gamma - 1)MdM}{1 + \frac{1}{2}(\gamma - 1)M^2} = \frac{1}{1 + \frac{1}{2}(\gamma - 1)M^2}\frac{dM}{M}$$

Prandtl-Meyer Expansion Waves

Thus,

$$\int_{\theta_1}^{\theta_2} d\theta = \theta_2 - \theta_1 = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{1}{2}(\gamma-1)M^2}\frac{dM}{M} = \nu(M_2) - \nu(M_1)$$

where

$$\nu(M) = \int \frac{\sqrt{M^2-1}}{1+\frac{1}{2}(\gamma-1)M^2}\frac{dM}{M}$$

is the so-called **Prandtl-Meyer function**

Prandtl-Meyer Expansion Waves

Performing the integration gives:

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1}$$

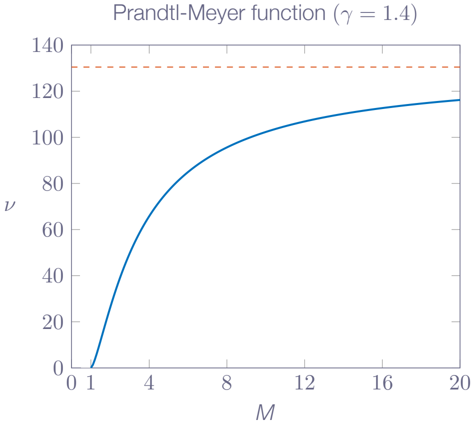
We can now calculate the deflection angle $\Delta\theta$ as:

$$\Delta\theta = \nu(M_2) - \nu(M_1)$$

Prandtl-Meyer Expansion Waves

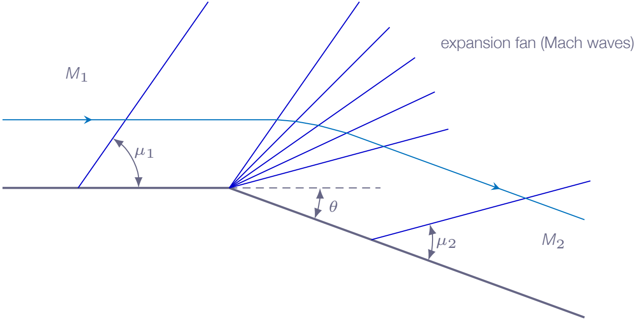
$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1}$$

$$\nu(M)|_{M\to\infty} = 130.45^\circ$$



Prandtl-Meyer Expansion Waves

Example:



1. $\theta_1 = 0$, $M_1 > 1$ is given
2. θ_2 is given
3. problem: find M_2 such that $\theta_2 = \nu(M_2) - \nu(M_1)$
4. $\nu(M)$ for $\gamma = 1.4$ can be found in Table A.5

Prandtl-Meyer Expansion Waves

Since the flow is isentropic, the usual isentropic relations apply:

(p_o and T_o are constant)

Calorically perfect gas:

$$\frac{p_o}{p} = \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_o}{T} = \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]$$

Prandtl-Meyer Expansion Waves

since $p_{o_1} = p_{o_2}$ and $T_{o_1} = T_{o_2}$

$$\frac{p_1}{p_2} = \frac{p_{o_2}}{p_{o_1}}\frac{p_1}{p_2} = \left(\frac{p_{o_2}}{p_2}\right) \Big/ \left(\frac{p_{o_1}}{p_1}\right) = \left[\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right]^{\frac{\gamma}{\gamma-1}}$$

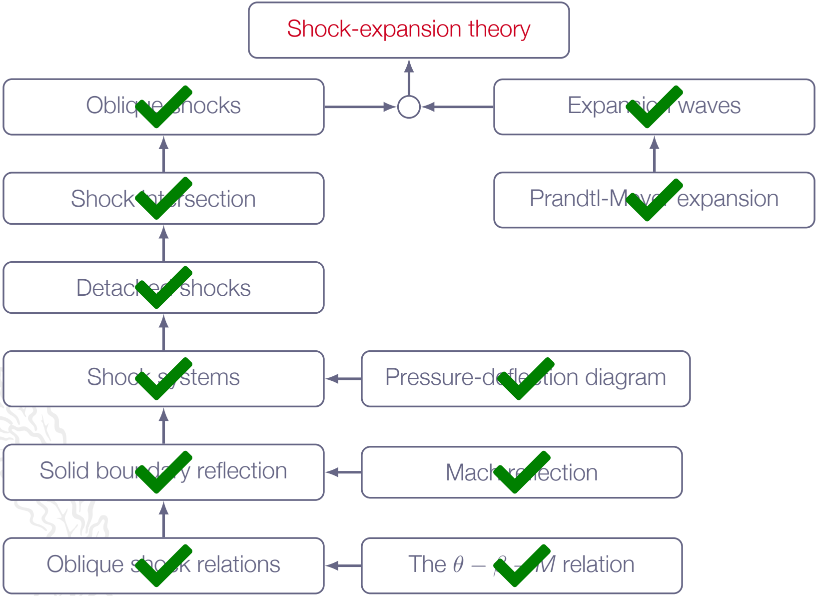
$$\frac{T_1}{T_2} = \frac{T_{o_2}}{T_{o_1}}\frac{T_1}{T_2} = \left(\frac{T_{o_2}}{T_2}\right) \Big/ \left(\frac{T_{o_1}}{T_1}\right) = \left[\frac{1+\frac{1}{2}(\gamma-1)M_2^2}{1+\frac{1}{2}(\gamma-1)M_1^2}\right]$$

Prandtl-Meyer Expansion Waves

Alternative solution:

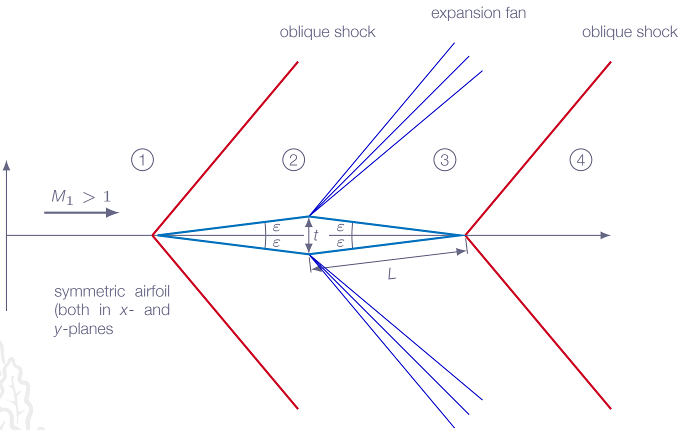
1. determine M_2 from $\theta_2 = \nu(M_2) - \nu(M_1)$
2. compute p_{o_1} and T_{o_1} from p_1, T_1, and M_1 (or use Table A.1)
3. set $p_{o_2} = p_{o_1}$ and $T_{o_2} = T_{o_1}$
4. compute p_2 and T_2 from p_{o_2}, T_{o_2}, and M_2 (or use Table A.1)

Roadmap - Oblique Shocks and Expansion Waves



Chapter 4.15
Shock Expansion Theory

Diamond-Wedge Airfoil



Diamond-Wedge Airfoil

1-2 standard oblique shock calculation for flow deflection angle ε and upstream Mach number M_1

2-3 Prandtl-Meyer expansion for flow deflection angle 2ε and upstream Mach number M_2

3-4 standard oblique shock calculation for flow deflection angle ε and upstream Mach number M_3

Diamond-Wedge Airfoil

symmetric airfoil

zero incidence flow (freestream aligned with flow axis)

gives:

symmetric flow field

zero lift force on airfoil

Diamond-Wedge Airfoil

Drag force:

$$D = -\oiint_{\partial\Omega} p(\mathbf{n} \cdot \mathbf{e}_x) dS$$

$\partial\Omega$	airfoil surface
p	surface pressure
$\mathbf{n}$	outward facing unit normal vector
$\mathbf{e}_x$	unit vector in x-direction

Diamond-Wedge Airfoil

Since conditions 2 and 3 are constant in their respective regions, we obtain:

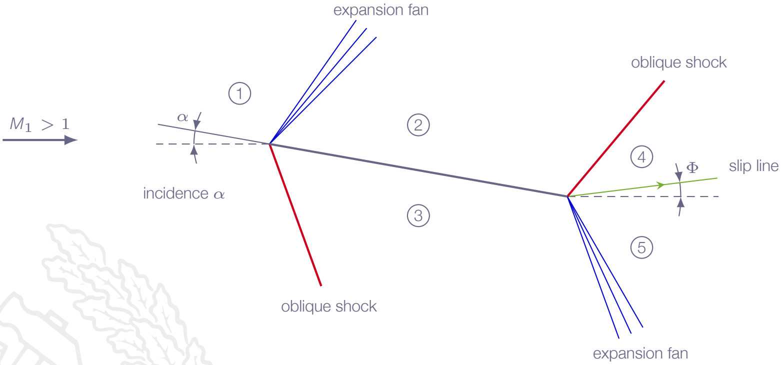
$$D = 2\left[p_2 L \sin(\varepsilon) - p_3 L \sin(\varepsilon)\right] = 2(p_2 - p_3)\frac{t}{2} = (p_2 - p_3)t$$

For supersonic free stream ($M_1 > 1$), with shocks and expansion fans according to figure we will always find that $p_2 > p_3$

which implies $D > 0$

Wave drag (drag due to flow loss at compression shocks)

Flat-Plate Airfoil



Flat-Plate Airfoil

It seems that the angle of the flow downstream of the flat plate would be different than the angle of the flow upstream of the plate. Can that really be correct?!

Flat-Plate Airfoil

It seems that the angle of the flow downstream of the flat plate would be different than the angle of the flow upstream of the plate. Can that really be correct?!

For the flow in the vicinity of the plate this is the correct picture. Further out from the plate, shock and expansion waves will interact and eventually sort the missmatch of flow angles out

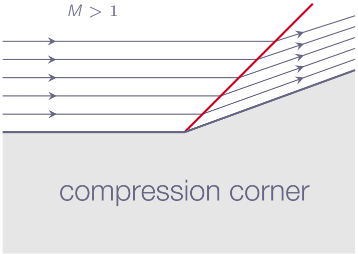
Flat-Plate Airfoil

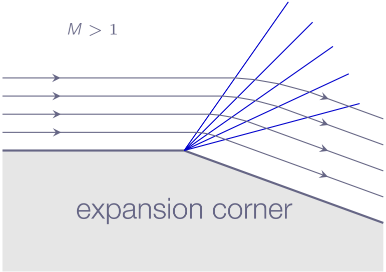
1. Flow states 4 and 5 must satisfy:

   $p_4 = p_5$

   flow direction 4 equals flow direction 5 (Φ)
2. Shock between 2 and 4 as well as expansion fan between 3 and 5 will adjust themselves to comply with the requirements
3. For calculation of lift and drag only states 2 and 3 are needed
4. States 2 and 3 can be obtained using standard oblique shock formulas and Prandtl-Meyer expansion

Oblique Shocks and Expansion Waves

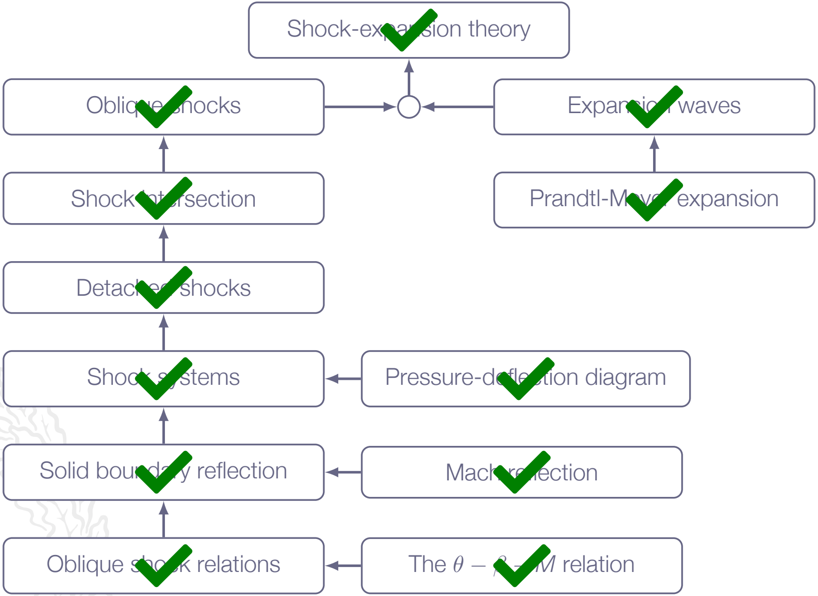




M	decrease
V	decrease
p	increase
ρ	increase
T	increase

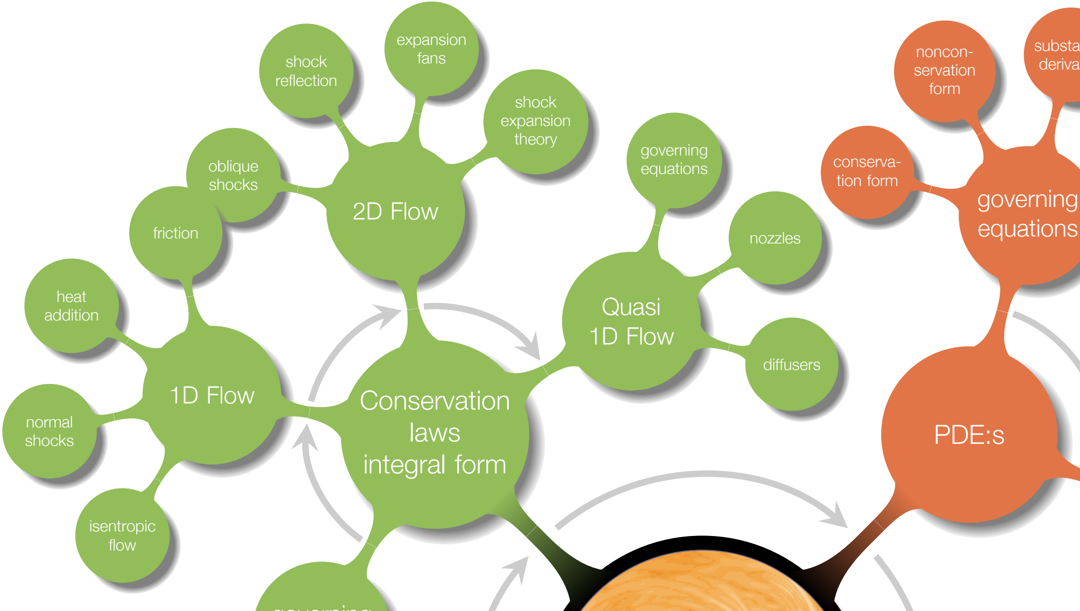
M	increase
V	increase
p	decrease
ρ	decrease
T	decrease

Roadmap - Oblique Shocks and Expansion Waves



Chapter 5
Quasi-One-Dimensional Flow

Overview


shock reflection
expansion fans
shock expansion theory
oblique shocks
2D Flow
friction
governing equations
nozzles
Quasi 1D Flow
diffusers
heat addition
1D Flow
normal shocks
isentropic flow
Conservation laws integral form
noncon-servation form
conserva-tion form
governing equations
PDE:s

Learning Outcomes

4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on

6 **Define** the special cases of calorically perfect gas, thermally perfect gas and real gas and **explain** the implication of each of these special cases

7 **Explain** why entropy is important for flow discontinuities

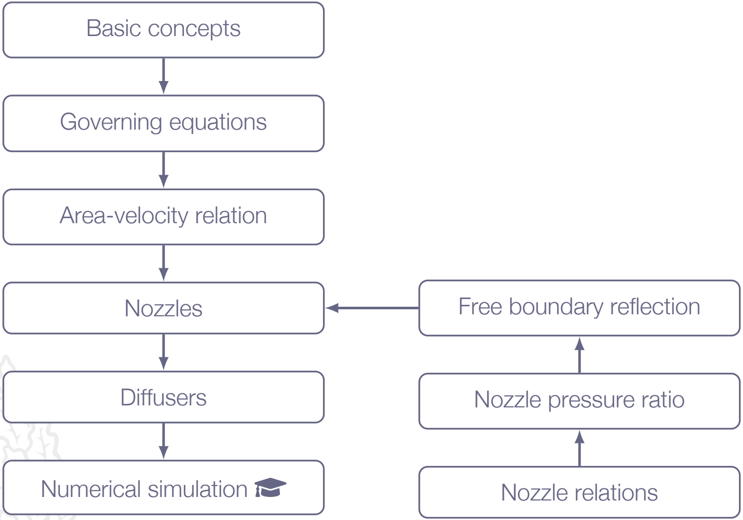
8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics

  a 1D isentropic flow*
  b normal shocks*
  i detached blunt body shocks, nozzle flows

9 **Solve** engineering problems involving the above-mentioned phenomena (8a-8k)

what does quasi-1D mean? either the flow is 1D or not, or?

Roadmap - Quasi-One-Dimensional Flow

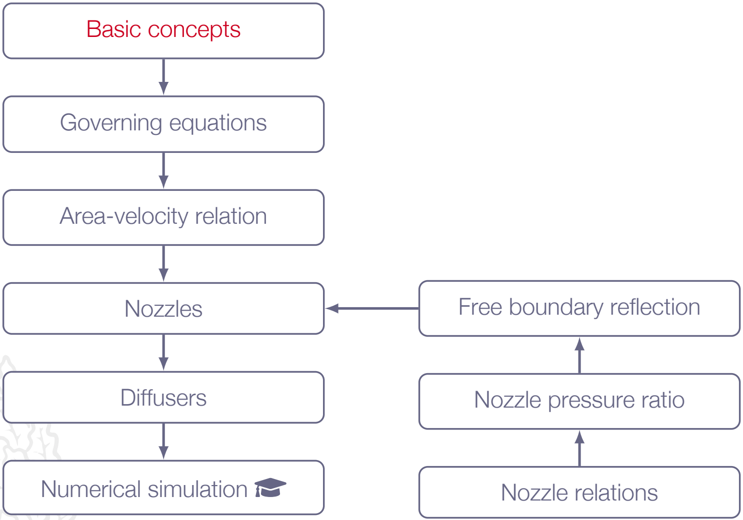


Motivation

By extending the one-dimensional theory to quasi-one-dimensional, we can study important applications such as nozzles and diffusers

Even though the flow in nozzles and diffusers are in essence three dimensional we will be able to establish important relations using the quasi-one-dimensional approach

Roadmap - Quasi-One-Dimensional Flow



Quasi-One-Dimensional Flow

Chapter 3

overall assumption

one-dimensional flow
steady state
constant cross-section area

applications

normal shock
1D flow with heat addition
1D flow with friction

Chapter 4

overall assumption

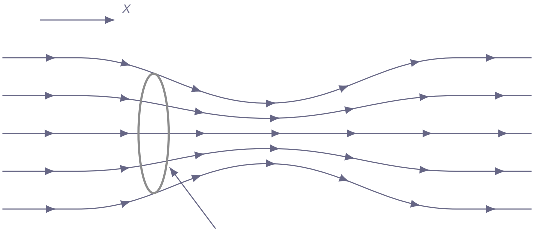
two-dimensional flow
steady state
uniform freestream

applications

oblique shocks
expansion fans
shock-expansion theory

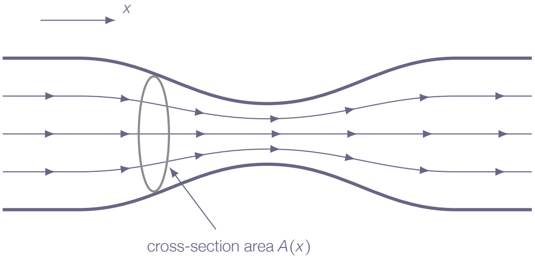
Quasi-One-Dimensional Flow

Extension of one-dimensional flow to allow **variations in streamtube area** (steady-state flow assumption still applied)

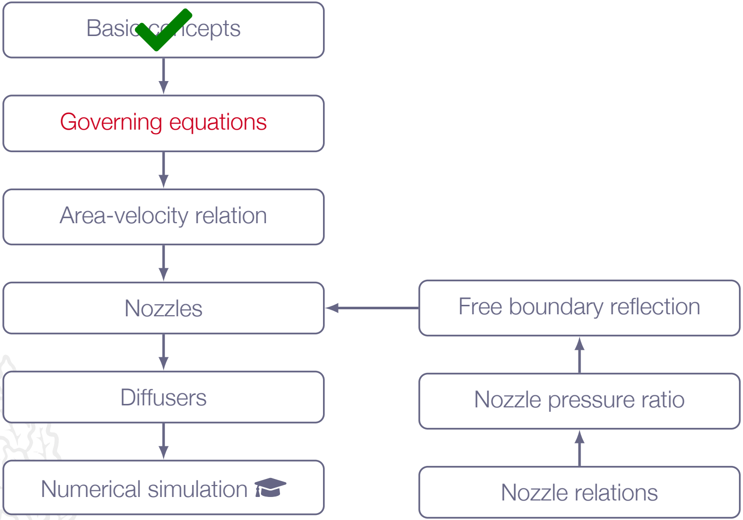


Quasi-One-Dimensional Flow

Example: tube with variable cross-section area



Roadmap - Quasi-One-Dimensional Flow

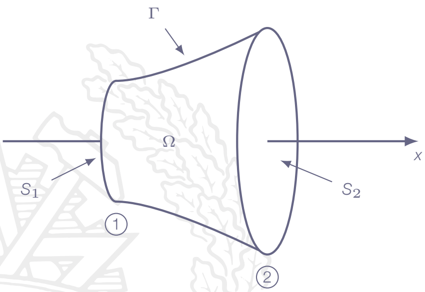


Chapter 5.2
Governing Equations

Governing Equations

Introduce **cross-section-averaged flow quantities** $\Rightarrow$
all quantities depend on x only

$$A = A(x),\ \rho = \rho(x),\ u = u(x),\ p = p(x),\ \ldots$$



Ω	control volume
S_1	left boundary (area A_1)
S_2	right boundary (area A_2)
Γ	perimeter boundary

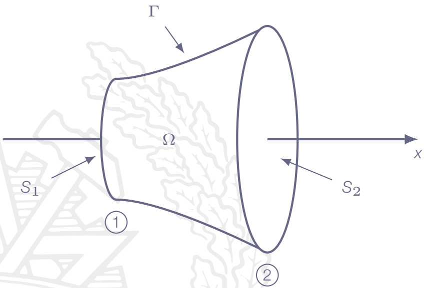
$$\partial\Omega = S_1 \cup \Gamma \cup S_2$$

Governing Equations - Assumptions

1. Inviscid flow (no boundary layers)

2. Steady-state flow (no unsteady effects)

3. No flow through Γ (control volume aligned with streamlines)



Governing Equations - Conservation of Mass

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho d\mathscr{V}}_{=0} + \underbrace{\oiint\limits_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS}_{-\rho_1 u_1 A_1+\rho_2 u_2 A_2} = 0$$

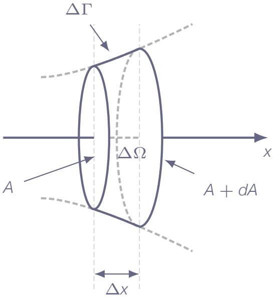
$$\boxed{\rho_1 u_1 A_1 = \rho_2 u_2 A_2}$$

Governing Equations - Conservation of Momentum

$$\underbrace{\frac{d}{dt}\iiint_{\Omega} \rho \mathbf{v} d\mathscr{V}}_{=0} + \oiint_{\partial\Omega} [\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v} + p\mathbf{n}]\, dS = 0$$

$$\oiint_{\partial\Omega} \rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v} dS = -\rho_1 u_1^2 A_1 + \rho_2 u_2^2 A_2$$

$$\oiint_{\partial\Omega} p\mathbf{n} dS = -p_1 A_1 + p_2 A_2 - \int_{A_1}^{A_2} p dA$$



$$(\rho_1 u_1^2 + p_1)A_1 + \int_{A_1}^{A_2} p dA = (\rho_2 u_2^2 + p_2)A_2$$

Governing Equations - Conservation of Energy

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho e_o d\mathscr{V}}_{=0} + \oiint\limits_{\partial\Omega}\left[\rho h_o(\mathbf{v}\cdot\mathbf{n})\right]dS = 0$$

which gives

$$\rho_1 u_1 A_1 h_{o_1} = \rho_2 u_2 A_2 h_{o_2}$$

from continuity we have that $\rho_1 u_1 A_1 = \rho_2 u_2 A_2 \Rightarrow$

$$\boxed{h_{o_1} = h_{o_2}}$$

Governing Equations - Summary

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

$$(\rho_1 u_1^2 + p_1)A_1 + \int_{A_1}^{A_2} p dA = (\rho_2 u_2^2 + p_2)A_2$$

$$h_{o_1} = h_{o_2}$$

Governing Equations - Differential Form

Continuity equation:

$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$ or $\rho u A = c$

where c is a constant $\Rightarrow$

$$d(\rho u A) = 0$$

Governing Equations - Differential Form

Momentum equation:

$$(\rho_1 u_1^2 + p_1)A_1 + \int_{A_1}^{A_2} p dA = (\rho_2 u_2^2 + p_2)A_2 \Rightarrow$$

$$d\left[(\rho u^2 + p)A\right] = p dA \Rightarrow$$

$$d(\rho u^2 A) + d(pA) = p dA \Rightarrow$$

$$u \underbrace{d(\rho u A)}_{=0} + \rho u A du + A dp + p dA = p dA \Rightarrow$$

$$\rho u A du + A dp = 0 \Rightarrow$$

$$\boxed{dp = -\rho u du} \quad \text{(Euler's equation)}$$

Governing Equations - Differential Form

Energy equation:

$$h_{o_1} = h_{o_2} \Rightarrow dh_o = 0$$

$$h_o = h + \frac{1}{2}u^2 \Rightarrow$$

$$\boxed{dh + udu = 0}$$

Governing Equations - Differential Form

Summary (valid for all gases):

$$d(\rho u A) = 0$$

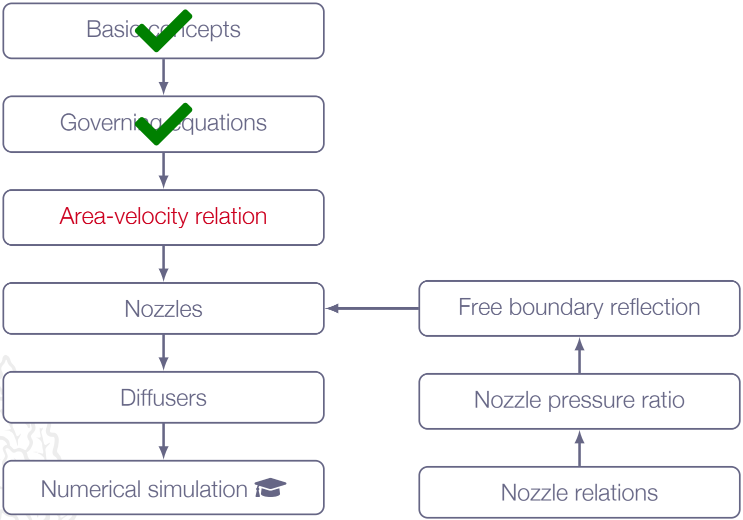
$$dp = -\rho u du$$

$$dh + u du = 0$$

Assumptions:

1. quasi-one-dimensional flow
2. inviscid flow
3. steady-state flow

Roadmap - Quasi-One-Dimensional Flow



Chapter 5.3
Area-Velocity Relation

Area-Velocity Relation

$$d(\rho uA) = 0 \Rightarrow uAd\rho + \rho Adu + \rho udA = 0$$

divide by ρuA gives

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

Euler's equation:

$$dp = -\rho udu \Rightarrow \frac{dp}{\rho} = \frac{dp}{d\rho}\frac{d\rho}{\rho} = -udu$$

Assuming adiabatic, reversible (isentropic) process and the definition of speed of sound gives

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s = a^2 \Rightarrow a^2\frac{d\rho}{\rho} = -udu \Rightarrow \frac{d\rho}{\rho} = -M^2\frac{du}{u}$$

Area-Velocity Relation

Now, inserting the expression for $\frac{d\rho}{\rho}$ in the rewritten continuity equation gives

$$(1 - M^2)\frac{du}{u} + \frac{dA}{A} = 0$$

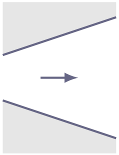
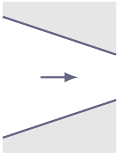
or

$$\frac{dA}{A} = (M^2 - 1)\frac{du}{u}$$

which is the **area-velocity relation**

The Area-Velocity Relation

$$\frac{dA}{A} = \frac{du}{u}(M^2 - 1)$$

		Subsonic $M < 1$	**Supersonic** $M > 1$
	$dA > 0$	subsonic diffuser $du < 0$ $dp > 0$	supersonic nozzle $du > 0$ $dp < 0$
	$dA < 0$	subsonic nozzle $du > 0$ $dp < 0$	supersonic diffuser $du < 0$ $dp > 0$

The Area-Velocity Relation

$$\frac{du}{u}(M^2 - 1) = \frac{dA}{A}$$

What happens when $M = 1$?

The Area-Velocity Relation

$$\frac{du}{u}(M^2-1)=\frac{dA}{A}$$

What happens when $M=1$?

$M=1$ when $dA=0$

The Area-Velocity Relation

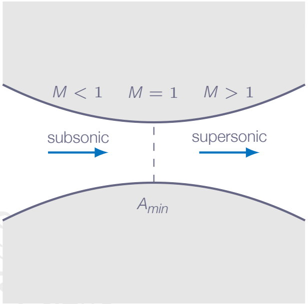
$$\frac{du}{u}(M^2-1)=\frac{dA}{A}$$

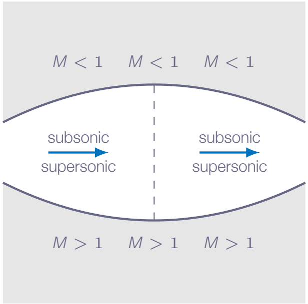
What happens when $M=1$?

$M=1$ when $dA=0$

maximum or minimum area

The Area-Velocity Relation





The Area-Velocity Relation

A converging-diverging nozzle is the **only possibility** to obtain supersonic flow!

A supersonic flow entering a convergent-divergent nozzle will slow down and, if the conditions are right, become sonic at the throat - hard to obtain a shock-free flow in this case

Area-Velocity Relation

$$M \rightarrow 0 \Rightarrow \frac{dA}{A} = -\frac{du}{u}$$

$$\frac{dA}{A} + \frac{du}{u} = 0 \Rightarrow$$

$$\frac{1}{Au}\left[udA + Adu\right] = 0 \Rightarrow$$

$$d(uA) = 0 \Rightarrow Au = c$$

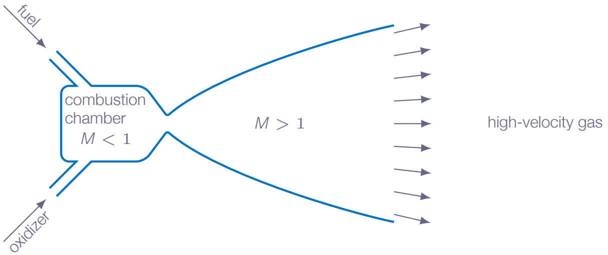
where c is a constant

Area-Velocity Relation

Note 1 The area-velocity relation is only valid for isentropic flow
not valid across a compression shock (due to entropy increase)

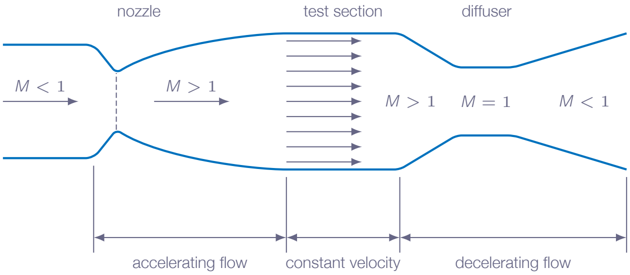
Note 2 The area-velocity relation is valid for all gases

Area-Velocity Relation Examples - Rocket Engine

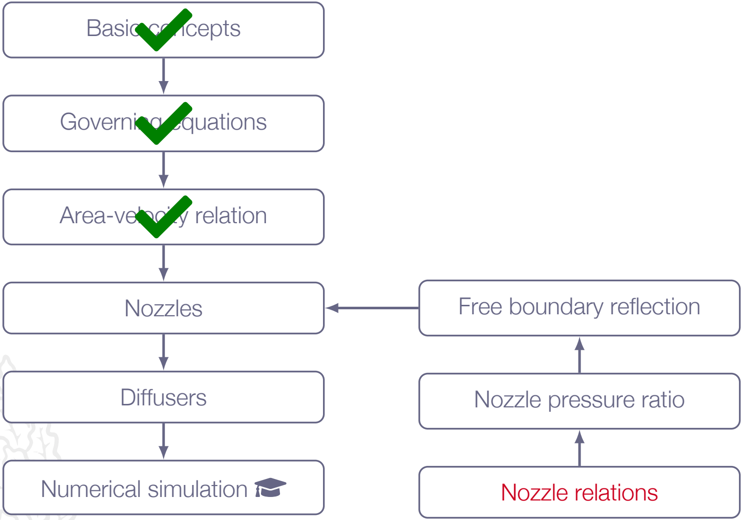


High-temperature, high-pressure gas in combustion chamber expand through the nozzle to very high velocities. Typical figures for a LH^2/LOx rocket engine: $p_o \sim 120$ [bar], $T_o \sim 3600$ [K], exit velocity ~ 4000 [m/s]

Area-Velocity Relation Examples - Wind Tunnel



Roadmap - Quasi-One-Dimensional Flow



Chapter 5.4
Nozzles

Nozzle Flow with Varying Pressure Ratio

time for rocket science!

Nozzle Flow - Relations

Calorically perfect gas assumed:

From Chapter 3:

$$\frac{T_o}{T} = \left(\frac{a_o}{a}\right)^2 = 1 + \frac{1}{2}(\gamma - 1)M^2$$

$$\frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{\frac{1}{\gamma-1}}$$

Nozzle Flow - Relations

Critical conditions:

$$\frac{T_o}{T^*} = \left(\frac{a_o}{a^*}\right)^2 = \frac{1}{2}(\gamma + 1)$$

$$\frac{p_o}{p^*} = \left(\frac{T_o}{T^*}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_o}{\rho^*} = \left(\frac{T_o}{T^*}\right)^{\frac{1}{\gamma-1}}$$

Nozzle Flow - Relations

$$M^{*2} = \frac{u^2}{a^{*2}} = \frac{u^2}{a^2}\frac{a^2}{a^{*2}} = \frac{u^2}{a^2}\frac{a^2}{a_o^2}\frac{a_o^2}{a^{*2}} \Rightarrow$$

$$\left.\begin{aligned} \frac{u^2}{a^2} &= M^2 \\ \frac{a^2}{a_o^2} &= \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-1} \\ \frac{a_o^2}{a^{*2}} &= \frac{1}{2}(\gamma + 1) \end{aligned}\right\} \Rightarrow M^{*2} = M^2 \frac{\frac{1}{2}(\gamma + 1)}{1 + \frac{1}{2}(\gamma - 1)M^2}$$

Nozzle Flow - Relations

For nozzle flow we have

$$\rho u A = c$$

where c is a constant and therefore

$$\rho^* u^* A^* = \rho u A$$

or, since at critical conditions $u^* = a^*$

$$\rho^* a^* A^* = \rho u A$$

which gives

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{a^*}{u} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{a^*}{u}$$

Nozzle Flow - Relations

$$\frac{A}{A^*} = \frac{\rho^*}{\rho_o} \frac{\rho_o}{\rho} \frac{a^*}{u}$$

$$\left.\begin{aligned} \frac{\rho^*}{\rho_o} &= \left(\frac{T_o}{T^*}\right)^{\frac{-1}{\gamma-1}} \\ \frac{\rho_o}{\rho} &= \left(\frac{T_o}{T}\right)^{\frac{1}{\gamma-1}} \\ \frac{a^*}{u} &= \frac{1}{M^*} \end{aligned}\right\} \Rightarrow \frac{A}{A^*} = \frac{\left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{\frac{1}{\gamma-1}}}{\left[\frac{1}{2}(\gamma + 1)\right]^{\frac{1}{\gamma-1}} M^*}$$

Nozzle Flow - Relations

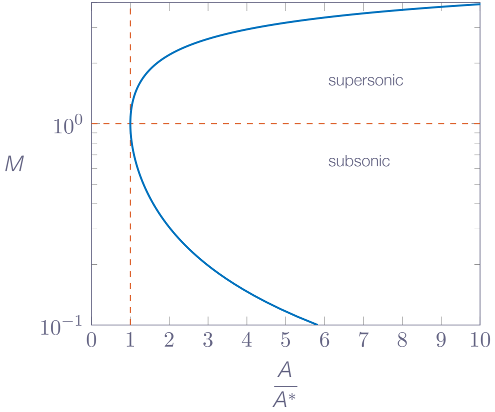
$$\left.\begin{aligned}\left(\frac{A}{A^*}\right)^2 &= \frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{2}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{2}{\gamma-1}} {M^*}^2} \\ {M^*}^2 &= M^2\frac{\frac{1}{2}(\gamma+1)}{1+\frac{1}{2}(\gamma-1)M^2}\end{aligned}\right\} \Rightarrow$$

$$\left(\frac{A}{A^*}\right)^2 = \frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{\gamma+1}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{\gamma+1}{\gamma-1}} M^2}$$

which is the **area-Mach-number relation**

The Area-Mach-Number Relation

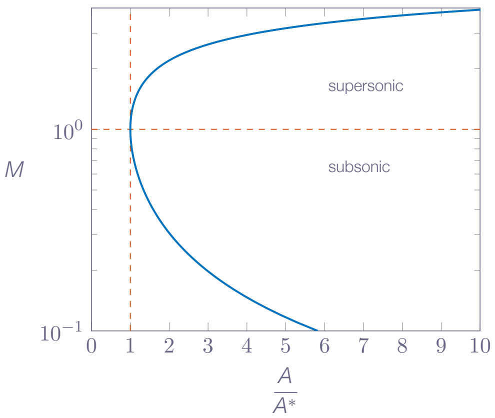
$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{(\gamma+1)/(\gamma-1)}$$



The Area-Mach-Number Relation

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{(\gamma+1)/(\gamma-1)}$$

Note! $\dfrac{A}{A^*} = \dfrac{\rho^* u^*}{\rho u}$



Area-Mach-Number Relation

Note 1 Critical conditions used here are those corresponding to **isentropic flow**. Do not confuse these with the conditions in the cases of one-dimensional flow with heat addition and friction

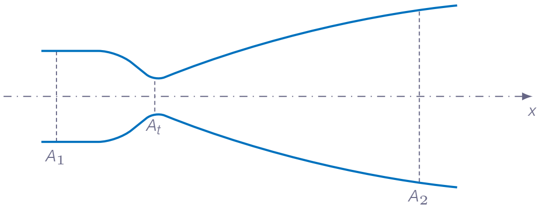
Note 2 For quasi-one-dimensional flow, assuming inviscid steady-state flow, both **total and critical conditions are constant along streamlines** unless shocks are present (then the flow is no longer isentropic)

Note 3 The derived area-Mach-number relation is **only valid for calorically perfect gas and for isentropic flow**. It is not valid across a compression shock

Nozzle Flow

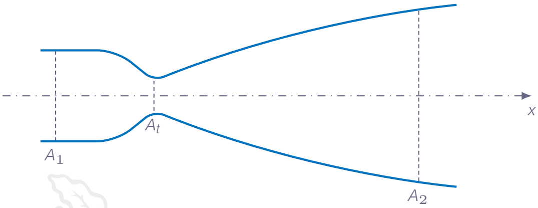
Assumptions:

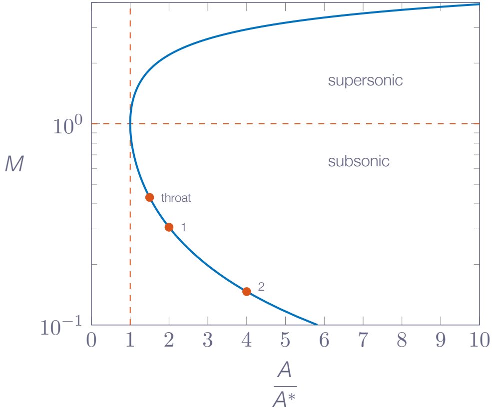
1. inviscid
2. steady-state
3. quasi-one-dimensional
4. calorically perfect gas



The Area-Mach-Number Relation

Sub-critical (non-choked) nozzle flow





$M < 1$ at nozzle throat

$A_t > A^*$

$M_1 < 1$

$M_2 < 1$

The Area-Mach-Number Relation

Subcritical nozzle flow (non-choked and subsonic $\Rightarrow$ isentropic):

A^* is constant throughout the nozzle ($A^* < A_t$)

M_1 given by the subsonic solution of

$$\left(\frac{A_1}{A^*}\right)^2 = \frac{1}{M_1^2}\left[\frac{2}{\gamma+1}(1+\frac{1}{2}(\gamma-1)M_1^2)\right]^{\frac{\gamma+1}{\gamma-1}}$$

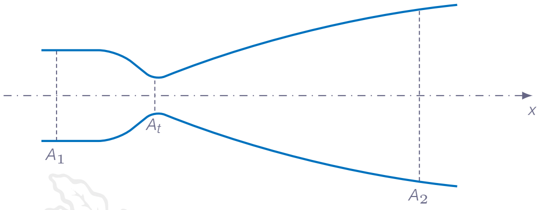
M_2 given by the subsonic solution of

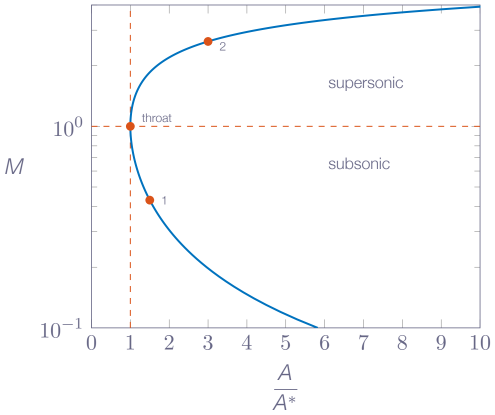
$$\left(\frac{A_2}{A^*}\right)^2 = \frac{1}{M_2^2}\left[\frac{2}{\gamma+1}(1+\frac{1}{2}(\gamma-1)M_2^2)\right]^{\frac{\gamma+1}{\gamma-1}}$$

M is uniquely determined everywhere in the nozzle, with subsonic flow both upstream and downstream of the throat

The Area-Mach-Number Relation

Critical (choked) nozzle flow





$M = 1$ at nozzle throat

$A_t = A^*$

$M_1 < 1$

$M_2 > 1$

The Area-Mach-Number Relation

Supercritical nozzle flow (choked flow without shocks $\Rightarrow$ isentropic):

A^* is constant throughout the nozzle ($A^* = A_t$)

M_1 given by the subsonic solution of

$$\left(\frac{A_1}{A^*}\right)^2 = \left(\frac{A_1}{A_t}\right)^2 = \frac{1}{M_1^2}\left[\frac{2}{\gamma+1}(1+\frac{1}{2}(\gamma-1)M_1^2)\right]^{\frac{\gamma+1}{\gamma-1}}$$

M_2 given by the supersonic solution of

$$\left(\frac{A_2}{A^*}\right)^2 = \left(\frac{A_2}{A_t}\right)^2 = \frac{1}{M_2^2}\left[\frac{2}{\gamma+1}(1+\frac{1}{2}(\gamma-1)M_2^2)\right]^{\frac{\gamma+1}{\gamma-1}}$$

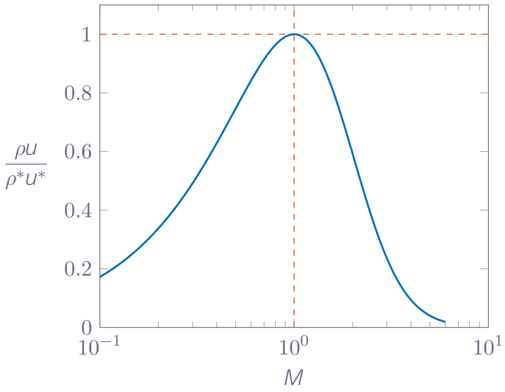
M is uniquely determined everywhere in the nozzle, with subsonic flow upstream of the throat and supersonic flow downstream of the throat

Nozzle Mass Flow

$$\rho u A = \rho^* A^* u^* \Rightarrow \frac{A^*}{A} = \frac{\rho u}{\rho^* u^*}$$

From the area-Mach-number relation

$$\frac{A^*}{A} = \begin{cases} < 1 & \text{if} \quad M < 1 \\ 1 & \text{if} \quad M = 1 \\ < 1 & \text{if} \quad M > 1 \end{cases}$$



The maximum possible massflow through a duct is achieved when its throat reaches sonic conditions

Nozzle Mass Flow

For a choked nozzle:

$$\dot{m} = \rho_1 u_1 A_1 = \rho^* u^* A^* = \rho_2 u_2 A_2$$

$$\left.\begin{aligned} \rho^* &= \frac{\rho^*}{\rho_o}\rho_o = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \frac{p_o}{RT_o} \\ a^* &= \frac{a^*}{a_o}a_o = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}} \sqrt{\gamma R T_o} \end{aligned}\right\} \Rightarrow$$

$$\boxed{\dot{m} = \frac{p_o A_t}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}}$$

Nozzle Mass Flow

$$\dot{m} = \frac{p_o A_t}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R} \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma+1}{\gamma-1}}}$$

The **maximum mass flow** that can be sustained through the nozzle

Valid for quasi-one-dimensional, inviscid, steady-state flow and calorically perfect gas

Note! The massflow formula is valid even if there are shocks present downstream of throat!

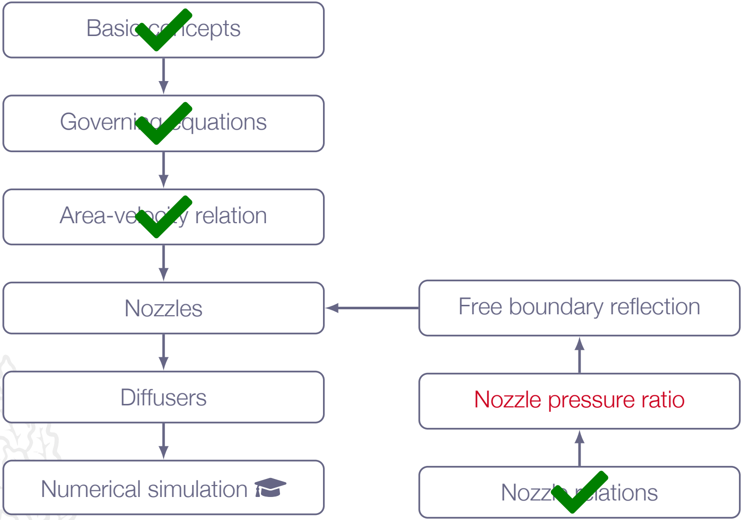
Nozzle Mass Flow

$$\dot{m} = \frac{p_o A_t}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

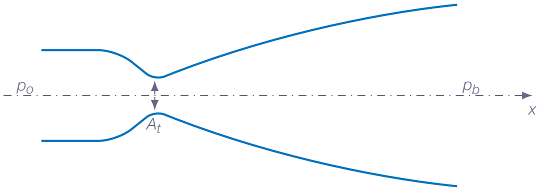
How can we increase mass flow through nozzle?

1. increase p_o
2. decrease T_o
3. increase A_t
4. decrease R
   (increase molecular weight, without changing γ)

Roadmap - Quasi-One-Dimensional Flow

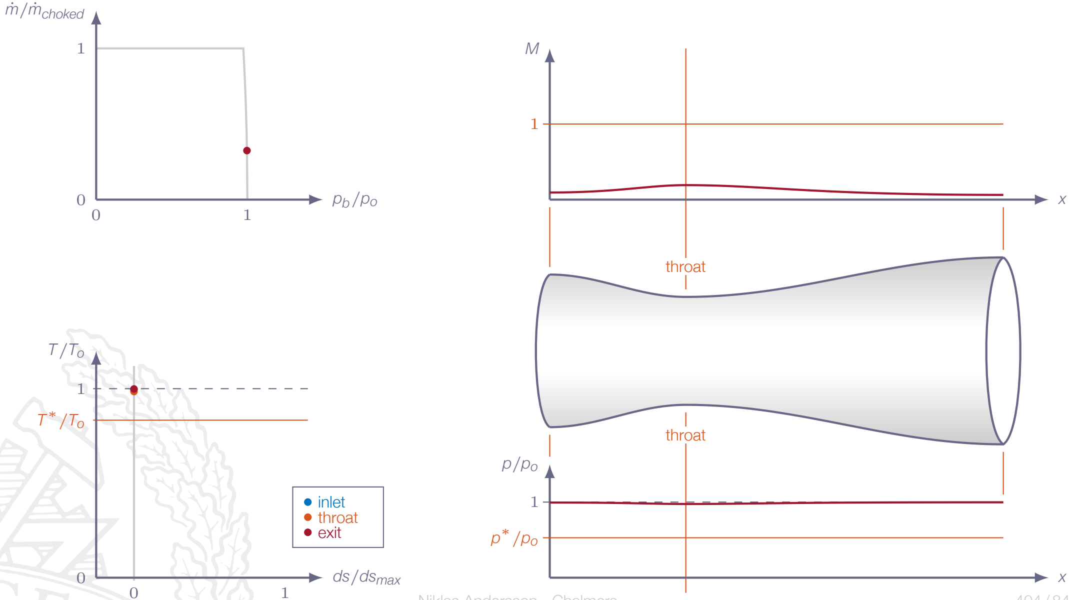


Nozzle Flow with Varying Pressure Ratio

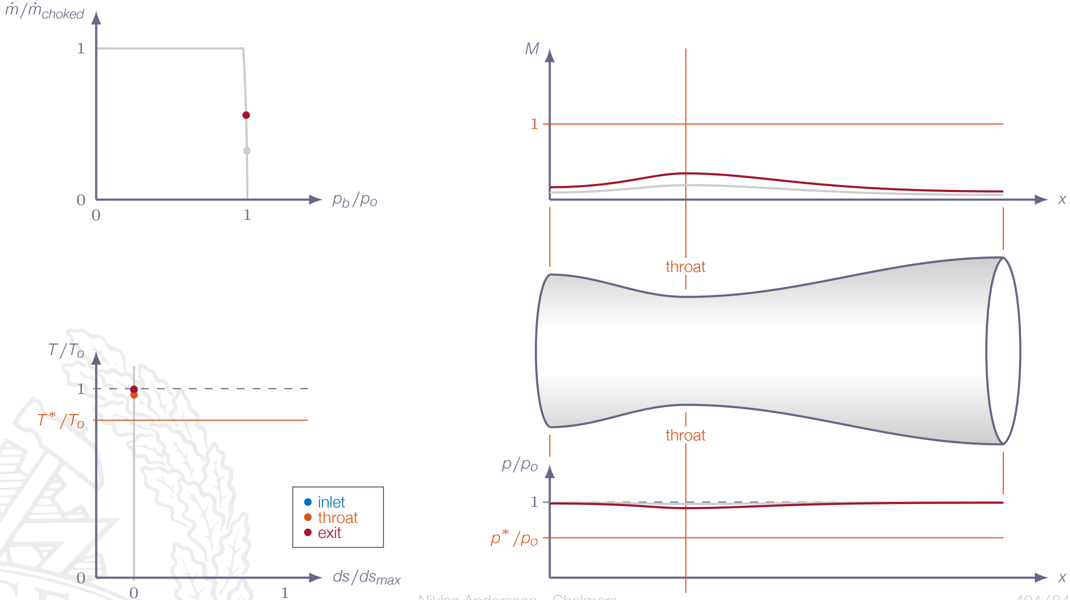


$A(x)$	area function
A_t	$\min\{A(x)\}$
p_o	inlet total pressure
p_b	outlet static pressure (ambient pressure)
p_o/p_b	pressure ratio

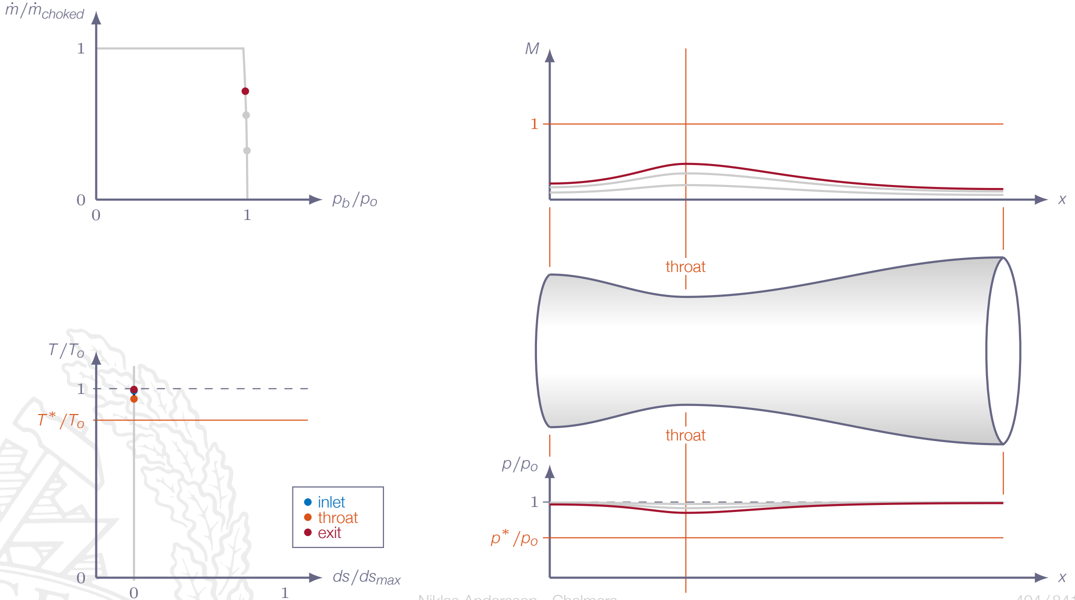
Nozzle Flow with Varying Pressure Ratio



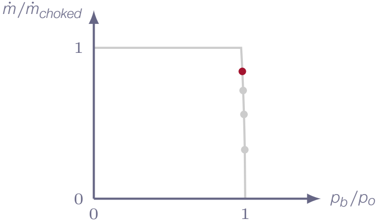
Nozzle Flow with Varying Pressure Ratio

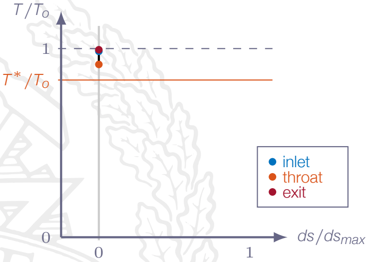


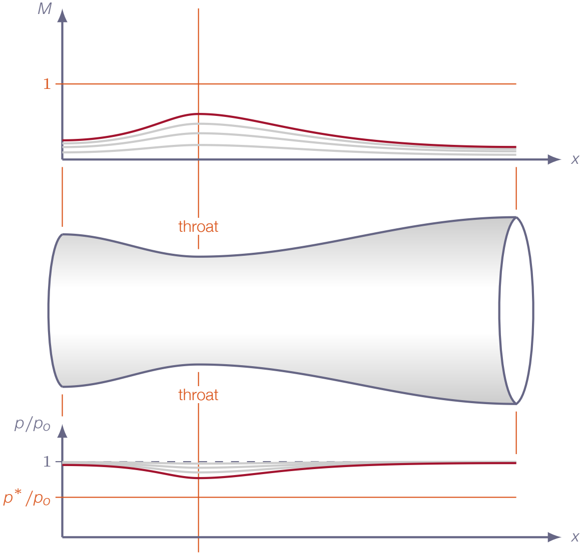
Nozzle Flow with Varying Pressure Ratio



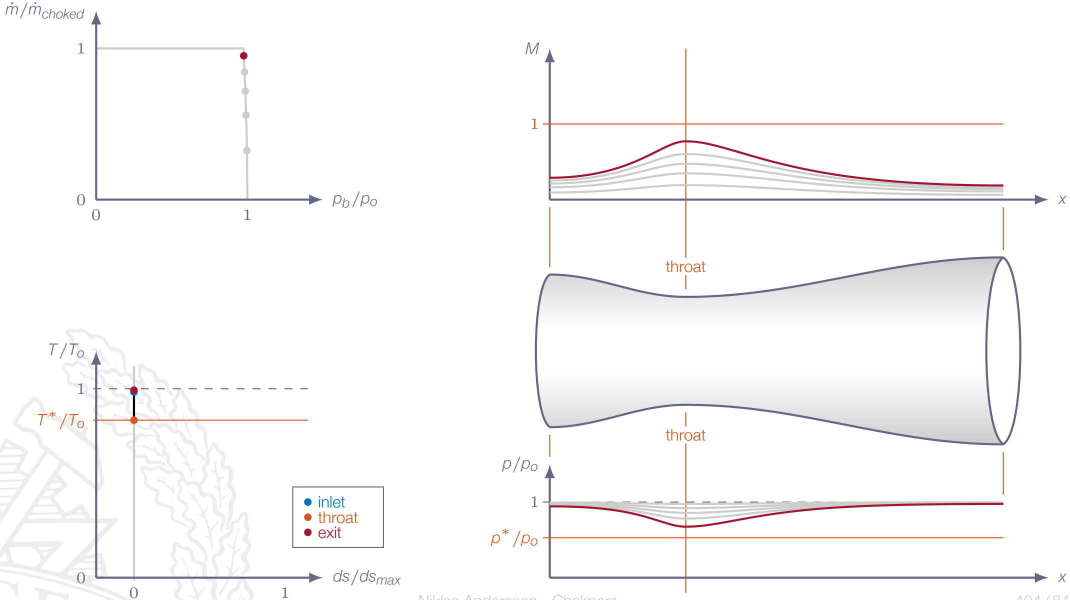
Nozzle Flow with Varying Pressure Ratio



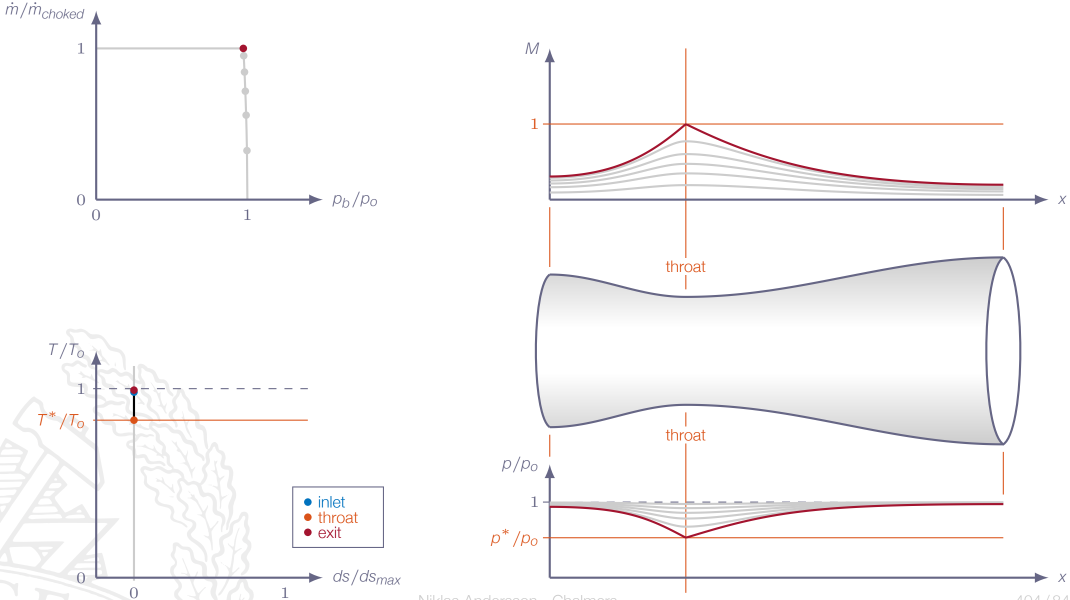




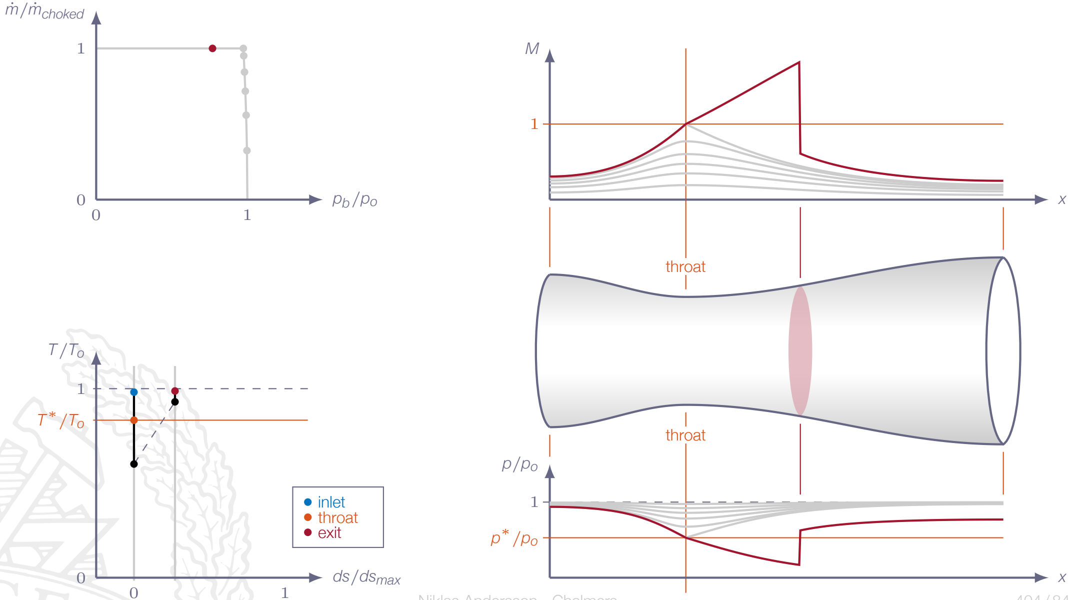
Nozzle Flow with Varying Pressure Ratio



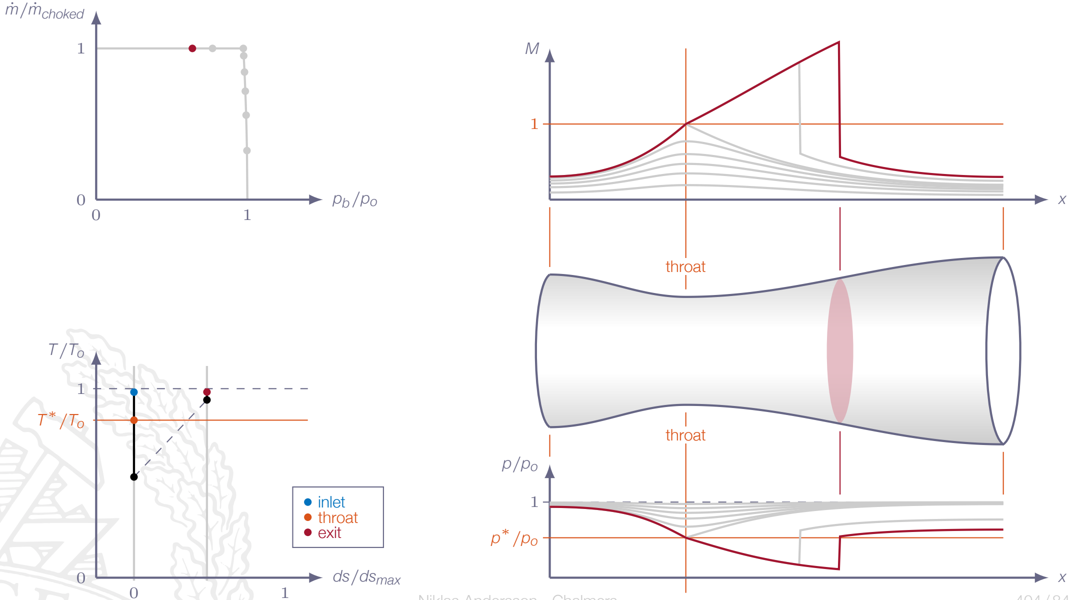
Nozzle Flow with Varying Pressure Ratio



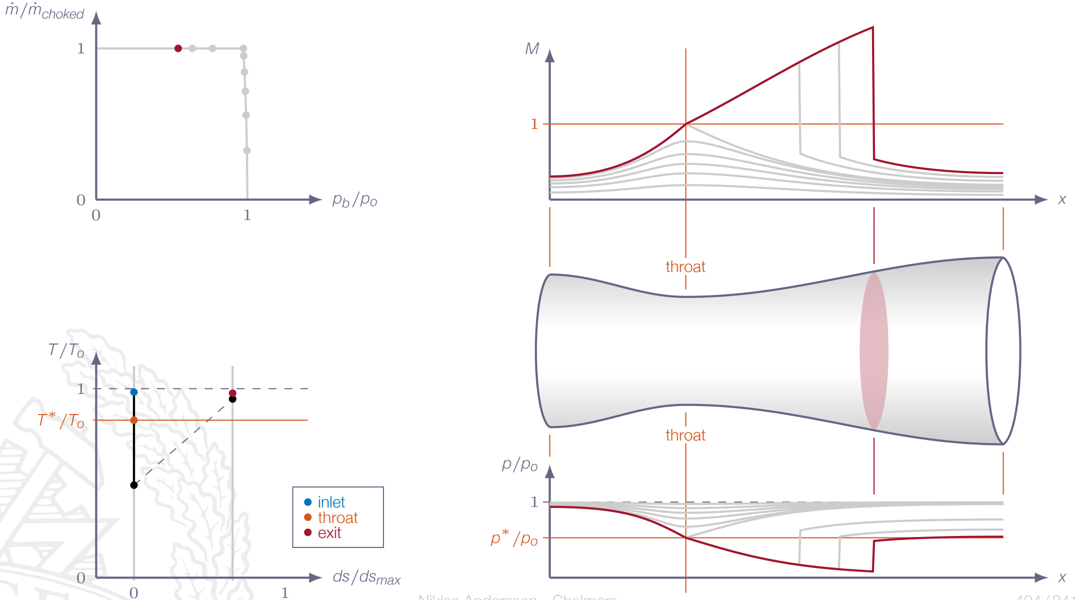
Nozzle Flow with Varying Pressure Ratio



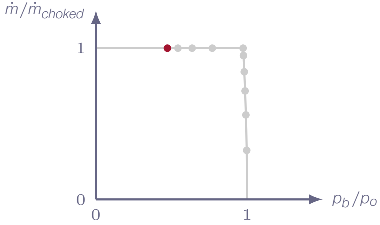
Nozzle Flow with Varying Pressure Ratio

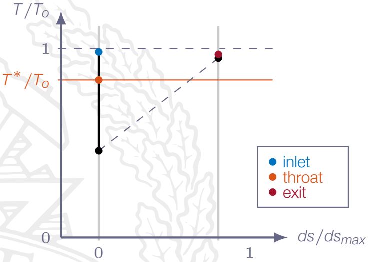


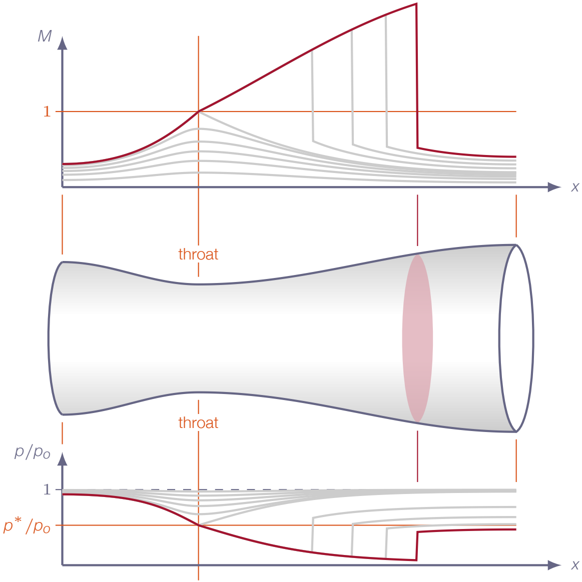
Nozzle Flow with Varying Pressure Ratio



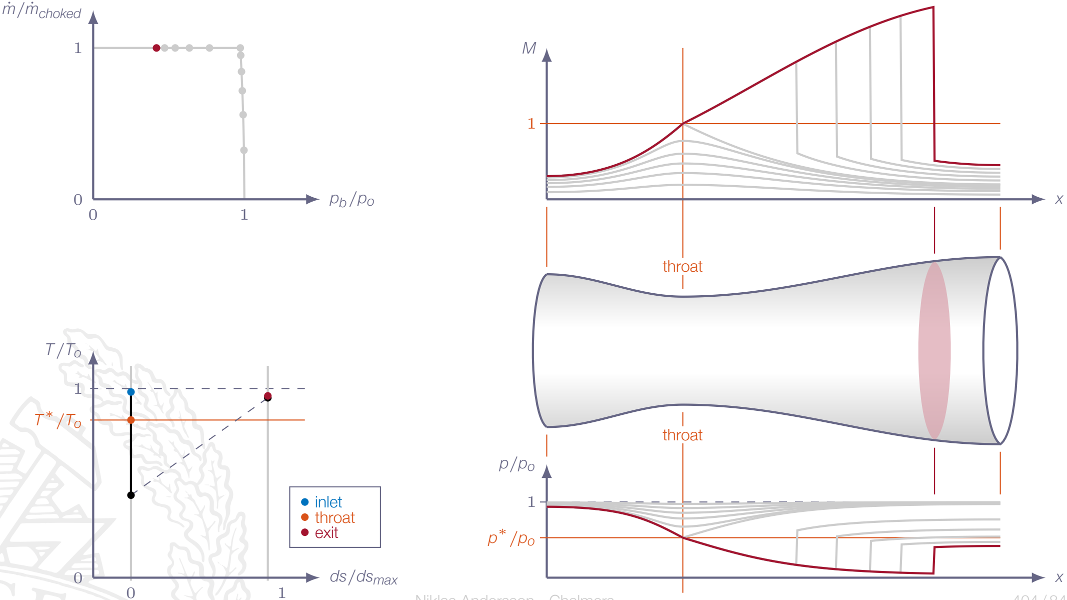
Nozzle Flow with Varying Pressure Ratio



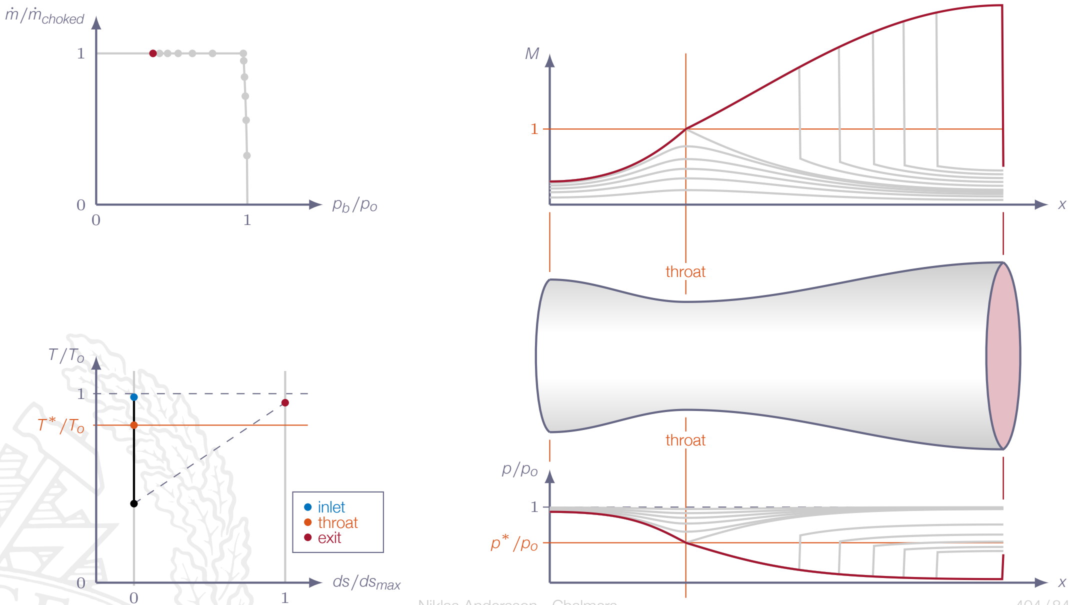




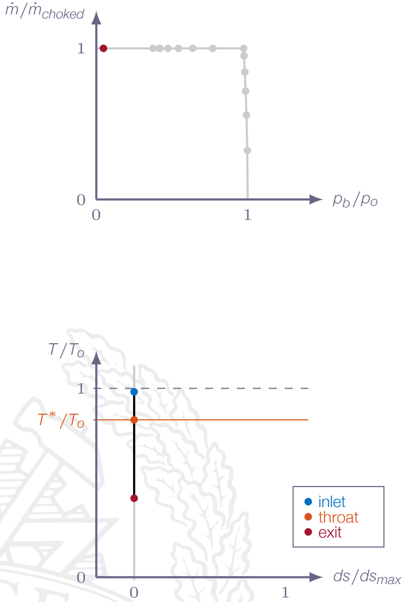
Nozzle Flow with Varying Pressure Ratio

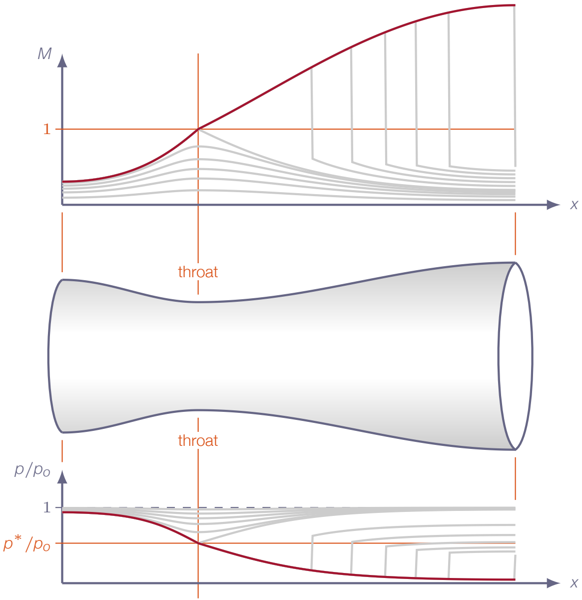


Nozzle Flow with Varying Pressure Ratio



Nozzle Flow with Varying Pressure Ratio

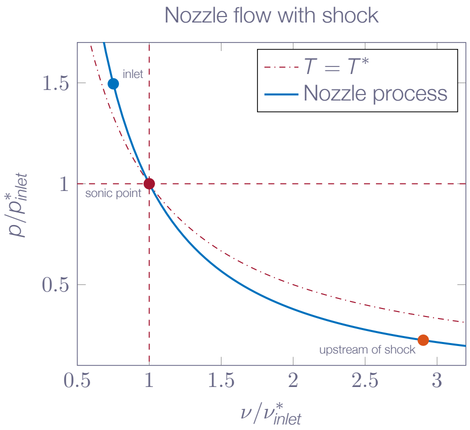




Nozzle Flow with Internal Shock

The nozzle flow process follows an isentrope up to the location of the internal normal shock

Sonic conditions at the nozzle throat

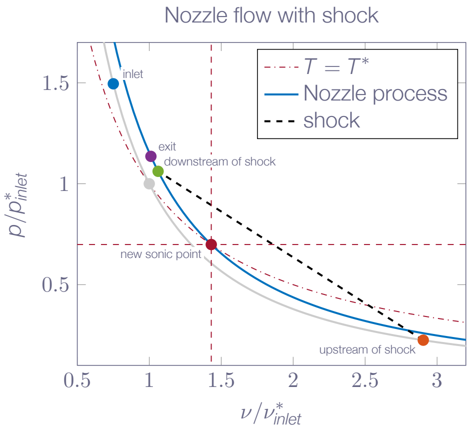


Nozzle Flow with Internal Shock

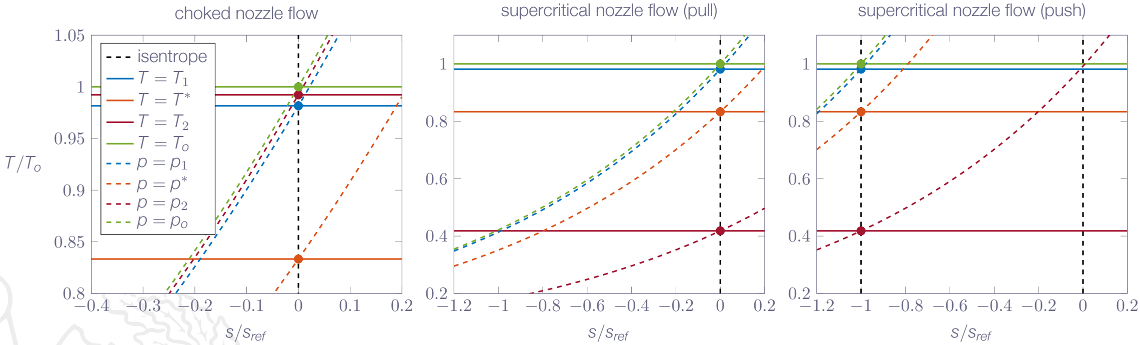
The normal shock moves the process line to another isentrope

T_o and thus T^* is not affected by the shock

p_o decreases over the shock which means that p^* decreases and ν^* increases



Nozzle Operation - Pull vs. Push

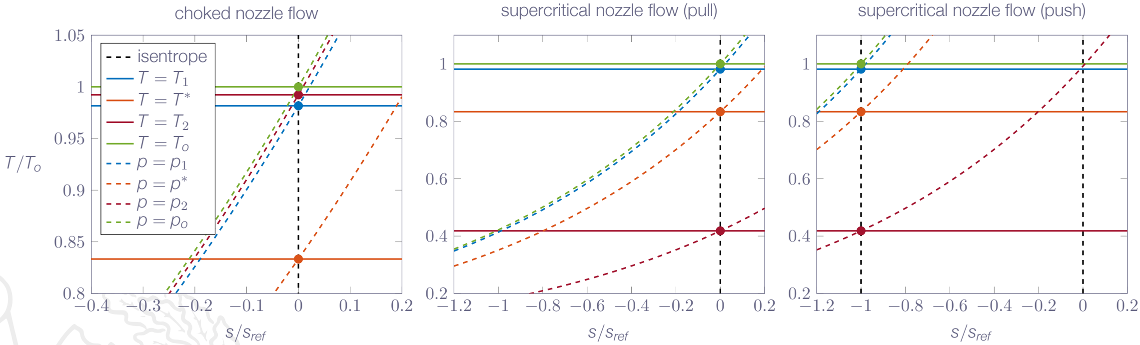


Nozzle Pressure Ratio $NPR = p_o/p_b$

Pull - increase *NPR* by reducing the back pressure (p_b)

Push - increase *NPR* by increasing the inlet total pressure (p_o)

Nozzle Operation - Pull vs. Push

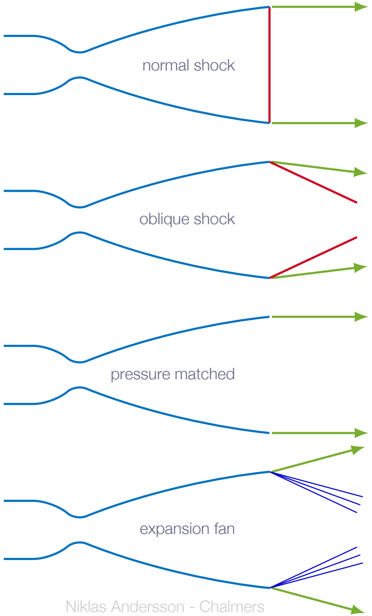


$$\left.\begin{aligned} p^*_{push} &> p^*_{pull} \\ T^*_{push} &= T^*_{pull} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \rho^*_{push} &> \rho^*_{pull} \\ a^*_{push} &= a^*_{pull} \end{aligned}\right.$$

$$\dot{m} = \rho^* a^* A^* \;\; (A^*_{push} = A^*_{pull})$$

$$\dot{m}_{push} > \dot{m}_{pull} = \dot{m}_{choked}$$

Nozzle Flow with Varying Pressure Ratio - Downstream Flow



$p_o/p_b = (p_o/p_b)_{ne}$
normal shock at nozzle exit

$(p_o/p_b)_{ne} < p_o/p_b < (p_o/p_b)_{sc}$
overexpanded nozzle flow

$p_o/p_b = (p_o/p_b)_{sc}$
pressure matched nozzle flow

$p_o/p_b > (p_o/p_b)_{sc}$
underexpanded nozzle flow

Nozzle Flow with Varying Pressure Ratio (Summary)

$(p_o/p_b) < (p_o/p_b)_{cr}$

- **subsonic, isentropic** flow throughout the nozzle
- the mass flow changes with p_b, *i.e.* the flow is not choked

$(p_o/p_b) = (p_o/p_b)_{cr}$

- **sonic** flow ($M = 1.0$) at the throat
- the flow will flip to the supersonic solution downstream of the throat, for an infinitesimal increase of (p_o/p_b)

$(p_o/p_b)_{cr} < (p_o/p_b) < (p_o/p_b)_{ne}$

- the flow is **choked** (fixed mass flow)
- a **normal shock** will appear downstream of the throat, with strength and position depending on (p_o/p_b)

Nozzle Flow with Varying Pressure Ratio (Summary)

$(p_o/p_b) = (p_o/p_b)_{ne}$

normal shock at the nozzle exit

supersonic, isentropic flow from throat to exit

$(p_o/p_b)_{ne} < (p_o/p_b) < (p_o/p_b)_{sc}$

overexpanded flow (supersonic, isentropic flow from throat to exit)

oblique shocks formed downstream of the nozzle exit

$(p_o/p_b) = (p_o/p_b)_{sc}$

supercritical flow (pressure matched)

supersonic, isentropic flow from the throat and downstream of the nozzle exit

$(p_o/p_b)_{sc} < (p_o/p_b)$

underexpanded flow (supersonic, isentropic flow from throat to exit)

expansion fans formed downstream of the nozzle exit

Nozzle Flow with Varying Pressure Ratio - Q1D Limitations

Quasi-one-dimensional theory

When the interior normal shock is ”pushed out” through the exit (by increasing (p_o/p_b), *i.e.* lowering the back pressure), it disappears completely.

The flow through the nozzle is then **shock free** (and thus also **isentropic** since we neglect viscosity).

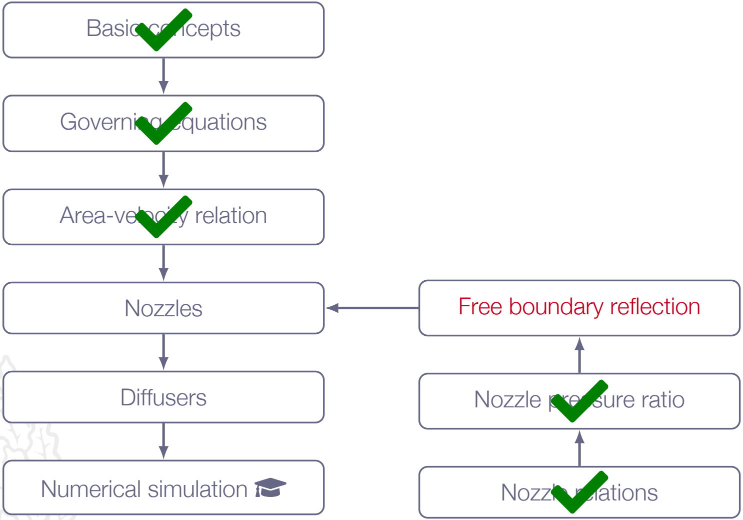
Three-dimensional nozzle flow

When the interior normal shock is ”pushed out” through the exit (by increasing (p_o/p_b)), an **oblique shock** is formed outside of the nozzle exit.

For the exact **supercritical** value of (p_o/p_b) this oblique shock disappears.

For (p_o/p_b) above the supercritical value an **expansion fan** is formed at the nozzle exit.

Roadmap - Quasi-One-Dimensional Flow

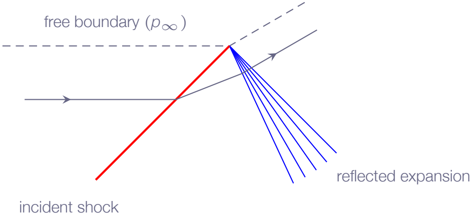


Chapter 5.6
Wave Reflection From a Free Boundary

Free-Boundary Reflection

Free boundary - shear layer, interface between different fluids, etc

Free-Boundary Reflection - Shock Reflection

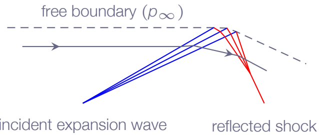


No discontinuity in pressure at the free boundary possible

Incident **shock reflects as expansion** waves at the free boundary

Reflection results in **net turning** of the flow

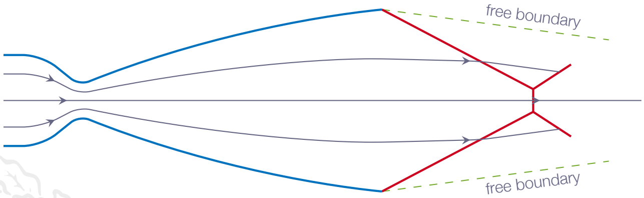
Free-Boundary Reflection - Expansion Wave Reflection



- No discontinuity in pressure at the free boundary possible
- Incident **expansion** waves **reflects as compression** waves at the free boundary
- Finite compression waves coalesces into a shock
- Reflection results in **net turning** of the flow

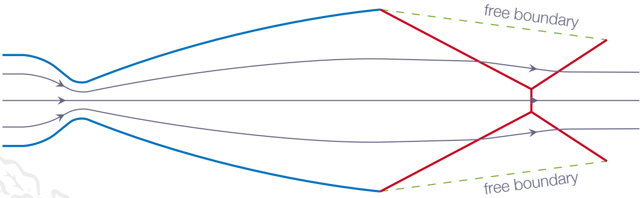
Free-Boundary Reflection - System of Reflections

overexpanded nozzle flow



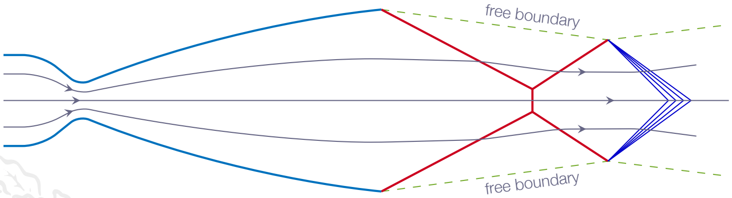
Free-Boundary Reflection - System of Reflections

shock reflection at jet centerline



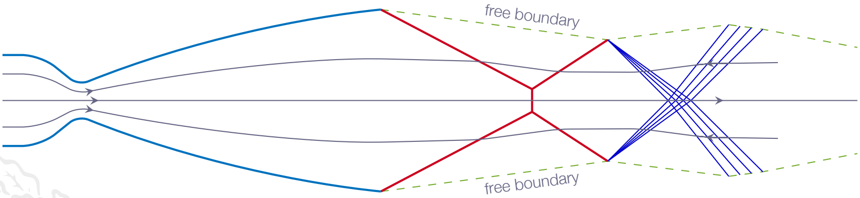
Free-Boundary Reflection - System of Reflections

shock reflection at free boundary



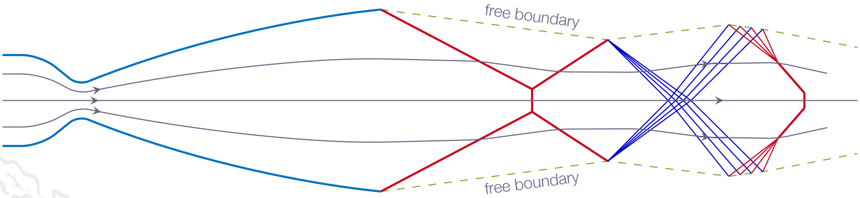
Free-Boundary Reflection - System of Reflections

expansion wave reflection at jet centerline



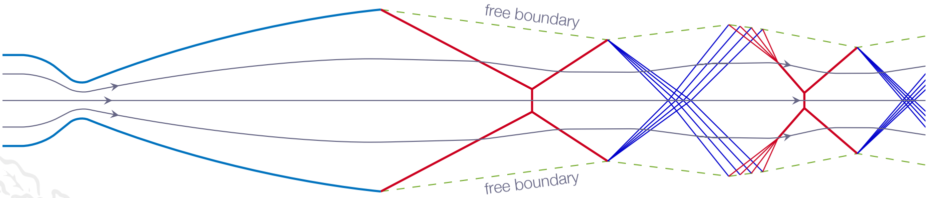
Free-Boundary Reflection - System of Reflections

expansion wave reflection at free boundary



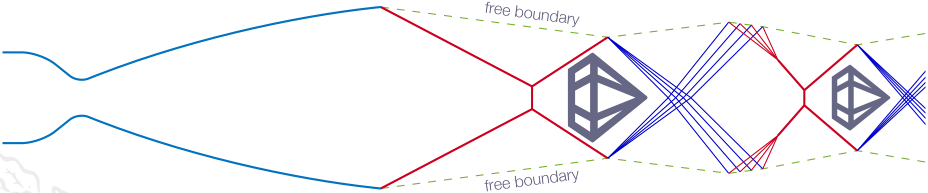
Free-Boundary Reflection - System of Reflections

repeated shock/expansion system



Free-Boundary Reflection - System of Reflections

shock diamonds



Free-Boundary Reflection - Summary

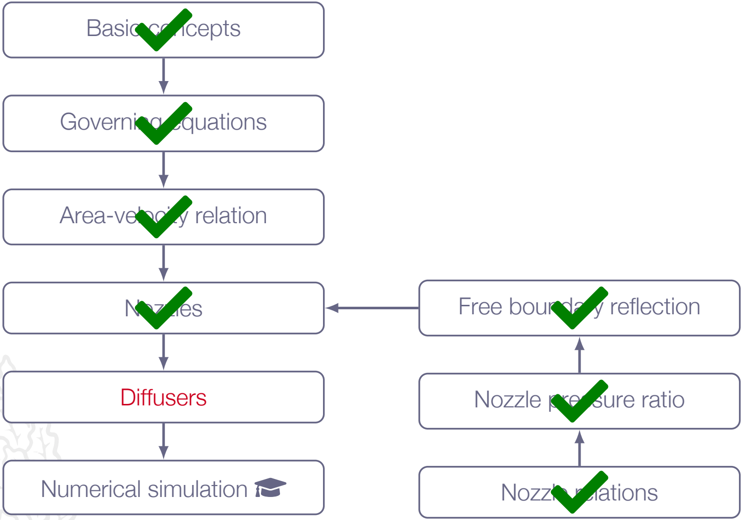
Solid-wall reflection

- Compression waves reflects as compression waves
- Expansion waves reflects as expansion waves

Free-boundary reflection

- Compression waves reflects as expansion waves
- Expansion waves reflects as compression waves

Roadmap - Quasi-One-Dimensional Flow

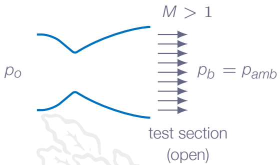


Chapter 5.5
Diffusers

Supersonic Wind Tunnel

wind tunnel with supersonic test section

open test section



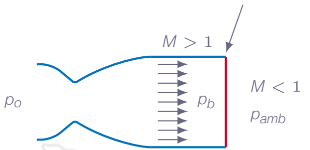
$p_o/p_b = (p_o/p_b)_{sc}$

$M = 3.0$ in test section $\Rightarrow p_o/p_b = 36.7$!!!

Supersonic Wind Tunnel

wind tunnel with supersonic test section

enclosed test section, normal shock at exit



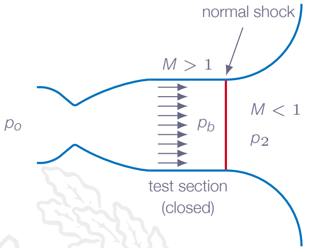
$p_o/p_{amb} = (p_o/p_b)(p_b/p_{amb}) < (p_o/p_b)_{sc}$

$M = 3.0$ in test section $\Rightarrow$
$p_o/p_{amb} = 36.7/10.33 = 3.55$

Supersonic Wind Tunnel

wind tunnel with supersonic test section

add subsonic diffuser after normal shock



$p_o/p_{amb} = (p_o/p_b)(p_b/p_2)(p_2/p_{o_2})$

$M = 3.0$ in test section $\Rightarrow$

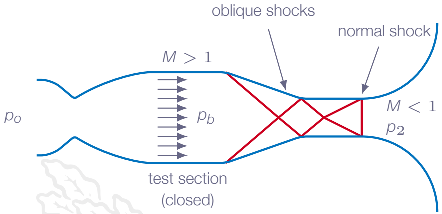
$p_o/p_{amb} = 36.7/10.33/1.17 = 3.04$

Note! this corresponds exactly to total pressure loss across normal shock

Supersonic Wind Tunnel

wind tunnel with supersonic test section

add supersonic diffuser before normal shock



$\left(p_{o_2} = p_{amb}\right)$

well-designed supersonic + subsonic diffuser $\Rightarrow$

1. decreased total pressure loss
2. decreased p_o and power to drive wind tunnel

Supersonic Wind Tunnel

Main problems:

1. **Complex 3D flow** in the diffuser section

    viscous effects

    complex systems of oblique shocks

    flow separation

    shock/boundary-layer interaction

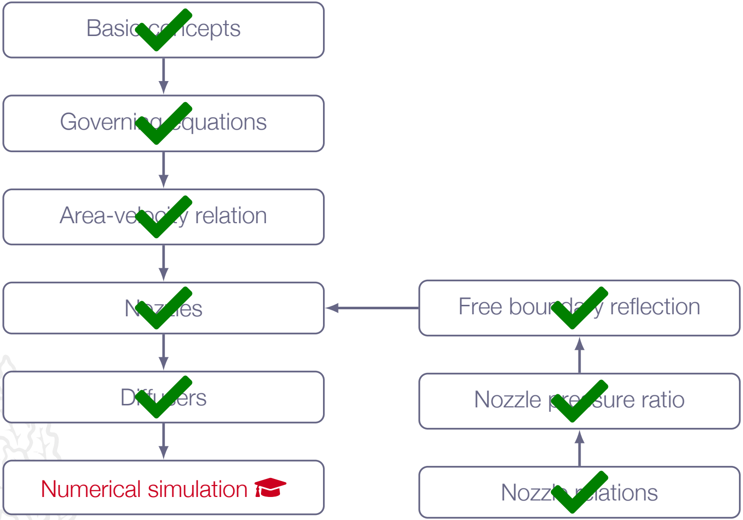
2. **Starting requirements**
   second throat must be significantly larger than first throat

    variable geometry diffuser

    second throat larger during startup procedure

    decreased second throat to optimum value after supersonic flow is established

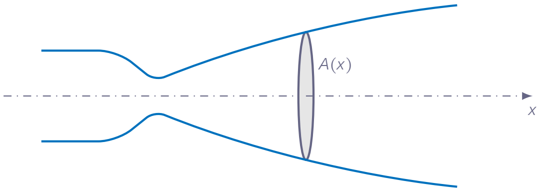
Roadmap - Quasi-One-Dimensional Flow



Quasi-One-Dimensional Euler Equations

Quasi-One-Dimensional Euler Equations

Example: choked flow through a convergent-divergent nozzle



Assumptions: inviscid, $Q = Q(x, t)$

Quasi-One-Dimensional Euler Equations

$$A(x)\frac{\partial}{\partial t}Q + \frac{\partial}{\partial x}[A(x)E] = A'(x)H$$

where $A(x)$ is the cross section area and

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho e_o \end{bmatrix}, \; E(Q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho h_o u \end{bmatrix}, \; H(Q) = \begin{bmatrix} 0 \\ p \\ 0 \end{bmatrix}$$

Numerical Approach

Discretization:
Finite-Volume Method (FVM) - Quasi-1D formulation

Numerical scheme:
third-order characteristic upwind scheme

Time stepping technique:
three-stage second-order Runge-Kutta explicit time marching

Boundary conditions:

left-end boundary:

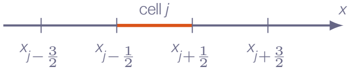
subsonic inflow
specify: inlet total temperature (T_o) and total pressure (p_o)

right-end boundary:

subsonic outflow
specify: outlet static pressure (p)

Finite-Volume Spatial Discretization

$$\left(\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}\right)$$



Integration over cell j gives:

$$\frac{1}{2}\left[A(x_{j-\frac{1}{2}}) + A(x_{j+\frac{1}{2}})\right]\Delta x_j \frac{d}{dt}\bar{Q}_j +$$
$$\left[A(x_{j+\frac{1}{2}})\hat{E}_{j+\frac{1}{2}} - A(x_{j-\frac{1}{2}})\hat{E}_{j-\frac{1}{2}}\right] =$$
$$\left[A(x_{j+\frac{1}{2}}) - A(x_{j-\frac{1}{2}})\right]\hat{H}_j$$

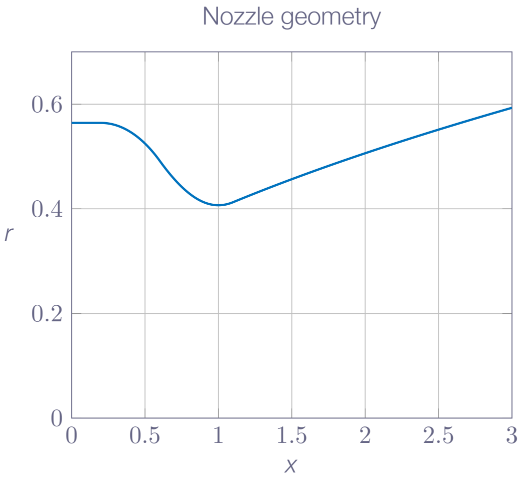
Finite-Volume Spatial Discretization

$$\bar{Q}_j = \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} QA(x)dx \right) \Big/ \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} A(x)dx \right)$$

$$\hat{E}_{j+\frac{1}{2}} \approx E\left(Q\left(x_{j+\frac{1}{2}}\right)\right)$$

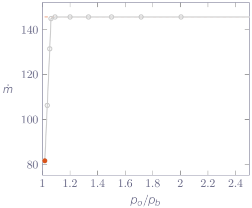
$$\hat{H}_j \approx \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} HA'(x)dx \right) \Big/ \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} A'(x)dx \right)$$

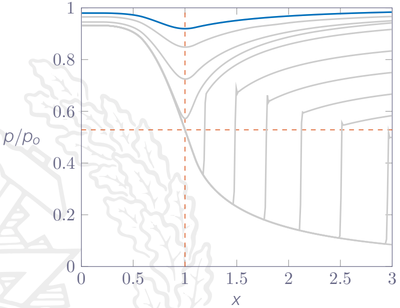
Nozzle Simulation - Back Pressure Sweep

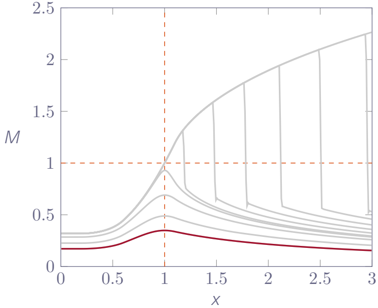


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	1.18 [bar]
p_o/p_b	1.02
$\dot{m}$	81.61 [kg/s]
M_{max}	0.35

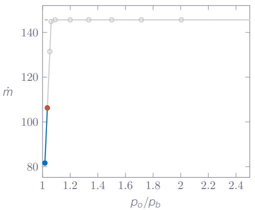


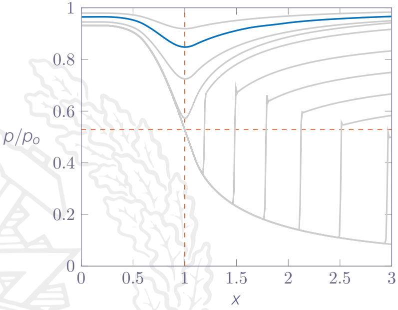


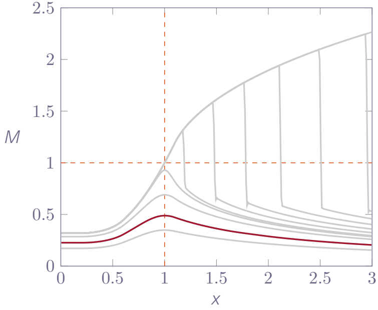


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	1.16 [bar]
p_o/p_b	1.03
$\dot{m}$	106.27 [kg/s]
M_{max}	0.49

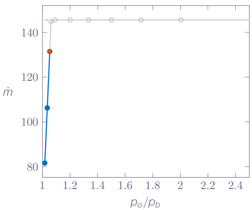


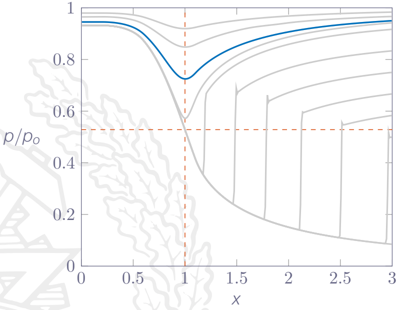


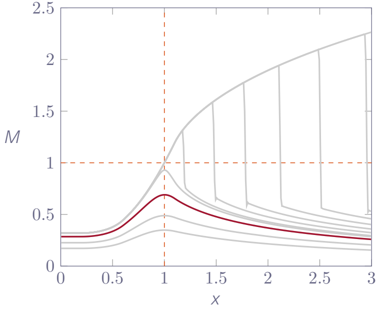


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	1.14 [bar]
p_o/p_b	1.05
$\dot{m}$	131.45 [kg/s]
M_{max}	0.69

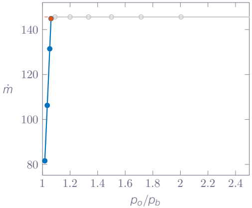


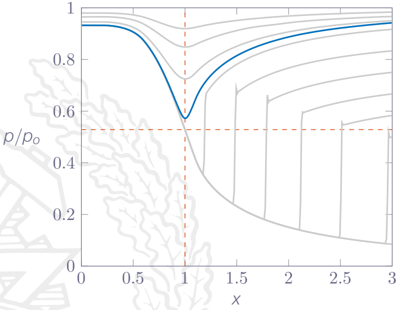


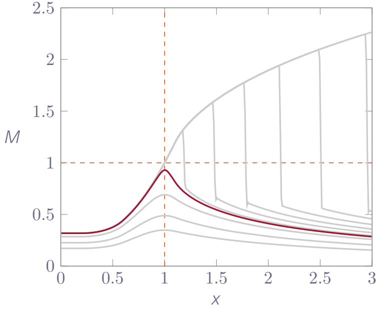


Nozzle Simulation - Back Pressure Sweep

p_o	1.20	[bar]
p_b	1.13	[bar]
p_o/p_b	1.06	
$\dot{m}$	144.88	[kg/s]
M_{max}	0.93	

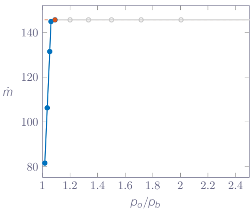


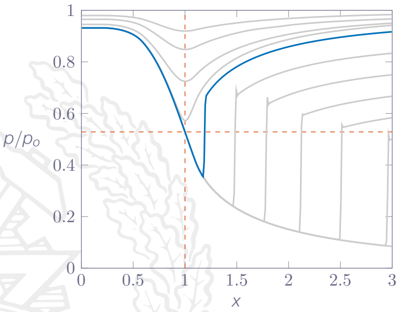


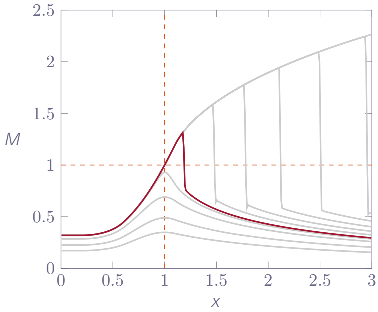


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	1.10 [bar]
p_o/p_b	1.09
$\dot{m}$	145.62 [kg/s]
M_{max}	1.31

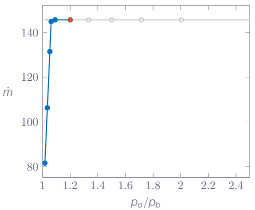


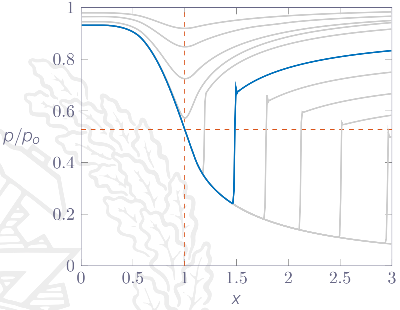


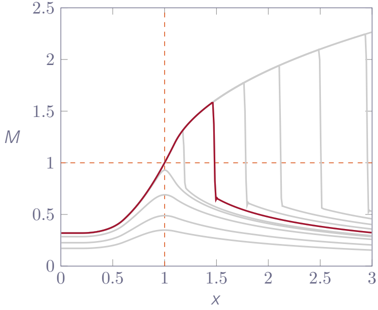


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	1.00 [bar]
p_o/p_b	1.20
$\dot{m}$	145.6 [kg/s]
M_{max}	1.58

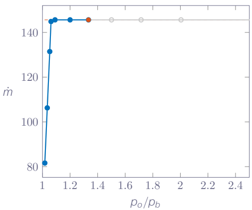


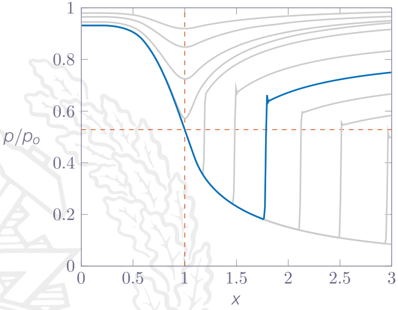


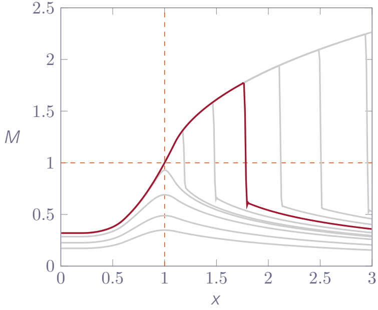


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	0.90 [bar]
p_o/p_b	1.33
$\dot{m}$	145.6 [kg/s]
M_{max}	1.77

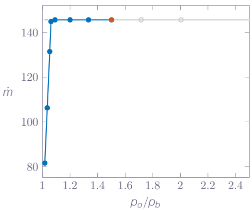


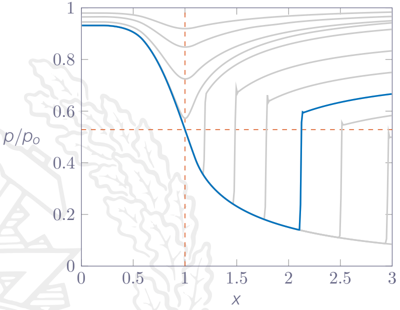


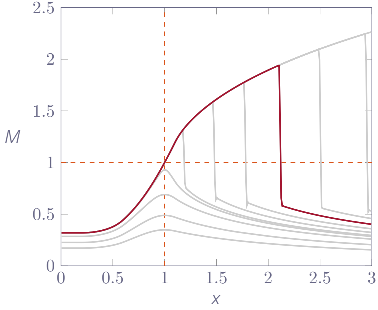


Nozzle Simulation - Back Pressure Sweep

p_o	1.20	[bar]
p_b	0.80	[bar]
p_o/p_b	1.50	
$\dot{m}$	145.6	[kg/s]
M_{max}	1.94	

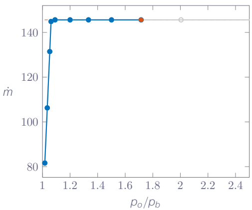


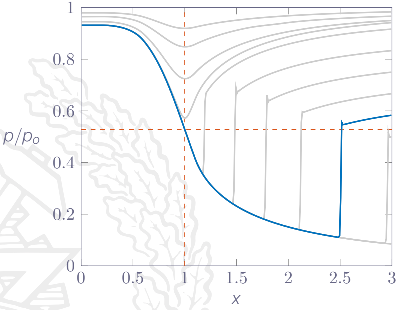


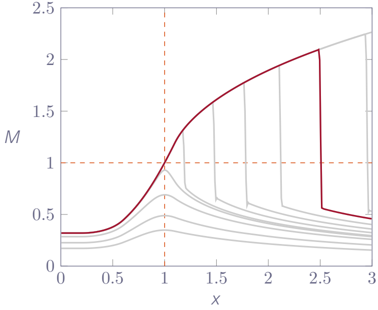


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	0.70 [bar]
p_o/p_b	1.71
$\dot{m}$	145.6 [kg/s]
M_{max}	2.10

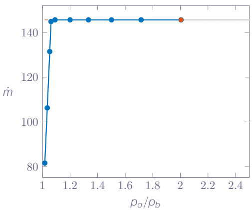


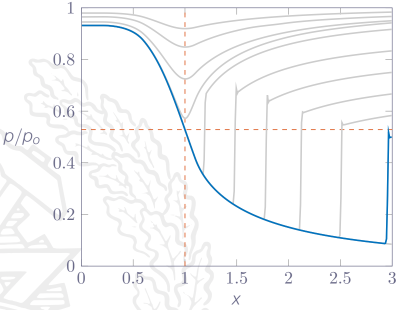


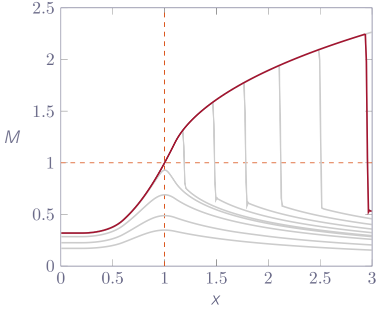


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	0.60 [bar]
p_o/p_b	2.00
$\dot{m}$	145.6 [kg/s]
M_{max}	2.24

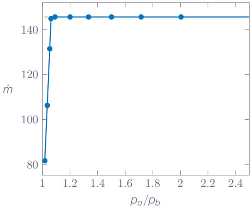


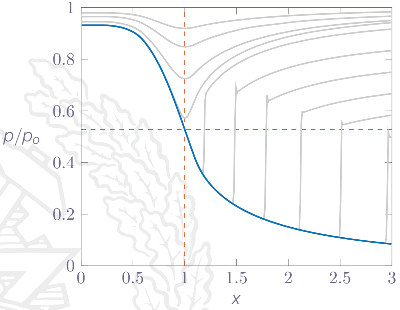


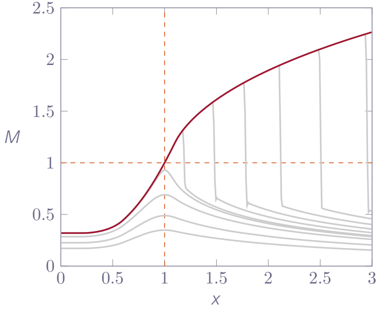


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	0.50 [bar]
p_o/p_b	11.8
$\dot{m}$	145.6 [kg/s]
M_{max}	2.26

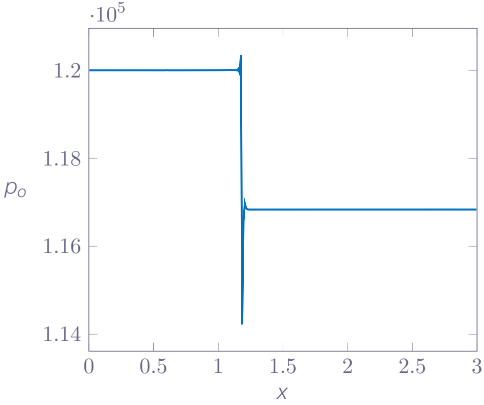






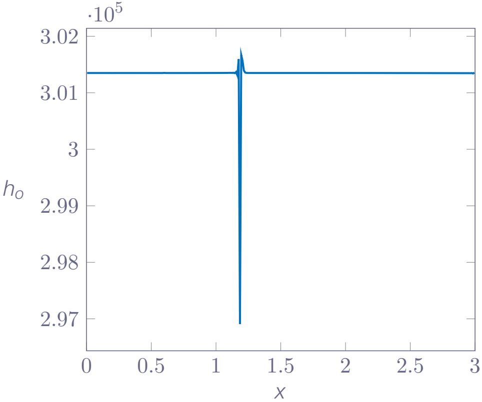
Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	1.10 [bar]
p_o/p_b	1.09
$\dot{m}$	145.62 [kg/s]
M_{max}	1.31

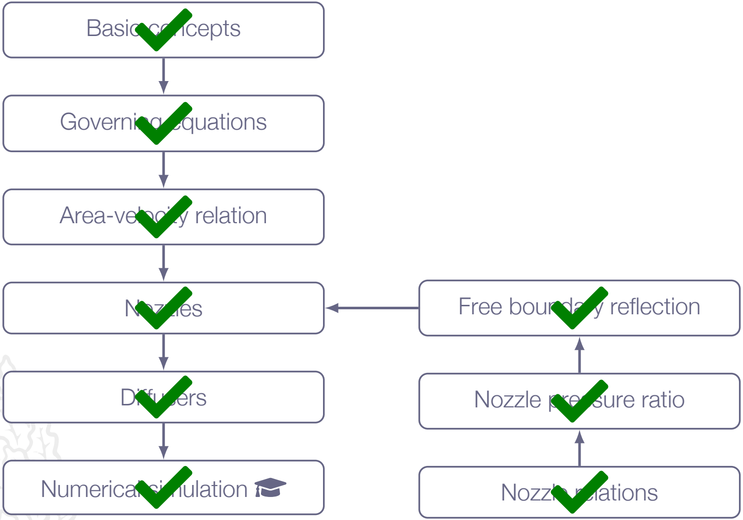


Nozzle Simulation - Back Pressure Sweep

p_o	1.20	[bar]
p_b	1.10	[bar]
p_o/p_b	1.09	
$\dot{m}$	145.62	[kg/s]
M_{max}	1.31	

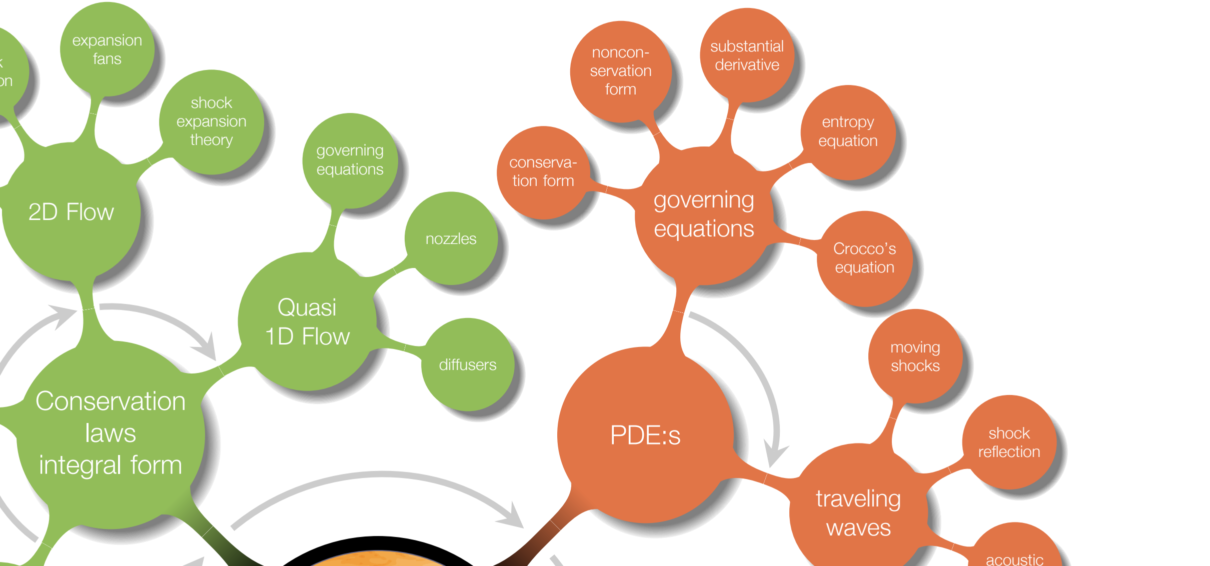


Roadmap - Quasi-One-Dimensional Flow



Chapter 6
Differential Conservation Equations for Inviscid Flows

Overview



expansion fans
shock expansion theory
2D Flow
governing equations
nozzles
Quasi 1D Flow
diffusers
Conservation laws integral form
noncon-servation form
substantial derivative
entropy equation
conserva-tion form
governing equations
Crocco's equation
PDE:s
moving shocks
shock reflection
traveling waves

Learning Outcomes

4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on

the governing equations for compressible flows on differential form - finally ...

Roadmap - Differential Equations for Inviscid Flows

Control volume formulations:

conservation of mass
conservation of momentum
conservation of energy

→ PDE:s on conservation form

The substantial derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

→ PDE:s on non-conservation form

↓

The entropy equation

↓

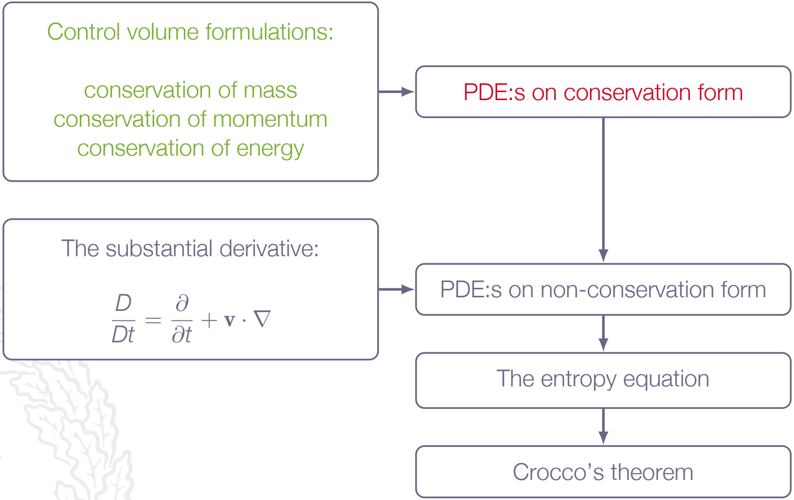
Crocco's theorem

Motivation

The differential form of the conservation equations is needed when analyzing unsteady problems

The differential form of the conservation equations forms the basis for multi-dimensional analysis and CFD

Roadmap - Differential Equations for Inviscid Flows



Chapter 6.2
Differential Equations in Conservation Form

Differential Equations in Conservation Form

Basic principle to derive PDE:s in conservation form:

1. Start with control volume formulation
2. Convert to volume integral via Gauss Theorem
3. Arbitrary control volume implies that integrand equals to zero everywhere

Continuity Equation - Conservation of Mass

Control volume formulation

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho d\mathscr{V}+\oiint\limits_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS=0$$

where Ω is a fixed control volume and thus $\frac{d}{dt}\iiint\limits_{\Omega}\rho d\mathscr{V}=\iiint\limits_{\Omega}\frac{\partial\rho}{\partial t}d\mathscr{V}$

Applying Gauss' Theorem on the surface integral gives

$$\oiint\limits_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS=\iiint\limits_{\Omega}\nabla\cdot(\rho\mathbf{v})d\mathscr{V}$$

Continuity Equation

Therefore

$$\iiint_{\Omega} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] d\mathscr{V} = 0$$

Ω is an arbitrary control volume, can be made infinitesimal and thus

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0}$$

which is the continuity equation on differential form

Momentum Equation - Conservation of Momentum

Control volume formulation

$$\frac{d}{dt}\iiint_{\Omega}\rho\mathbf{v}d\mathscr{V}+\oiint_{\partial\Omega}\left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}+p\mathbf{n}\right]dS=\iiint_{\Omega}\rho\mathbf{f}d\mathscr{V}$$

where Ω is a fixed control volume and thus $\frac{d}{dt}\iiint_{\Omega}\rho\mathbf{v}d\mathscr{V}=\iiint_{\Omega}\frac{\partial}{\partial t}(\rho\mathbf{v})d\mathscr{V}$

Applying Gauss' Theorem on the surface integrals gives

$$\oiint_{\partial\Omega}\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}dS=\iiint_{\Omega}\nabla\cdot(\rho\mathbf{v}\mathbf{v})d\mathscr{V}\ ;\ \oiint_{\partial\Omega}p\mathbf{n}dS=\iiint_{\Omega}\nabla p d\mathscr{V}$$

Momentum Equation

Therefore

$$\iiint_{\Omega} \left[\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla p - \rho \mathbf{f}\right] d\mathscr{V} = 0$$

Ω is an arbitrary control volume, can be made infinitesimal and thus

$$\boxed{\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla p = \rho \mathbf{f}}$$

which is the momentum equation on differential form

Momentum Equation

In cartesian form ($\mathbf{v} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$):

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho u \mathbf{v}) + \frac{\partial p}{\partial x} = \rho f_x$$
$$\frac{\partial}{\partial t}(\rho v) + \nabla \cdot (\rho v \mathbf{v}) + \frac{\partial p}{\partial y} = \rho f_y$$
$$\frac{\partial}{\partial t}(\rho w) + \nabla \cdot (\rho w \mathbf{v}) + \frac{\partial p}{\partial z} = \rho f_z$$

Momentum Equation

or expanded:

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) + \frac{\partial p}{\partial x} = \rho f_x$$
$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho vu) + \frac{\partial}{\partial y}(\rho vv) + \frac{\partial}{\partial z}(\rho vw) + \frac{\partial p}{\partial y} = \rho f_y$$
$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho wu) + \frac{\partial}{\partial y}(\rho wv) + \frac{\partial}{\partial z}(\rho ww) + \frac{\partial p}{\partial z} = \rho f_z$$

Energy Equation - Conservation of Energy

Control volume formulation

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho e_o d\mathscr{V} + \oiint\limits_{\partial\Omega}\rho h_o(\mathbf{v}\cdot\mathbf{n})dS = \iiint\limits_{\Omega}\rho\mathbf{f}\cdot\mathbf{v}d\mathscr{V}$$

where Ω is a fixed control volume and thus $\frac{d}{dt}\iiint\limits_{\Omega}\rho e_o d\mathscr{V} = \iiint\limits_{\Omega}\frac{\partial}{\partial t}(\rho e_o)d\mathscr{V}$

Applying Gauss' Theorem on the surface integral gives

$$\oiint\limits_{\partial\Omega}\rho h_o(\mathbf{v}\cdot\mathbf{n})dS = \iiint\limits_{\Omega}\nabla\cdot(\rho h_o\mathbf{v})d\mathscr{V}$$

Energy Equation

Therefore

$$\iiint\limits_{\Omega} \left[\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho h_o \mathbf{v}) - \rho(\mathbf{f} \cdot \mathbf{v}) \right] d\mathscr{V} = 0$$

Ω is an arbitrary control volume, can be made infinitesimal and thus

$$\boxed{\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho h_o \mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v})}$$

which is the energy equation on differential form

Partial Differential Equations in Conservation Form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla p = \rho \mathbf{f}$$
$$\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho h_o \mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v})$$

These equations are referred to as PDE:s on conservation form since they stem directly from the integral conservation equations applied to a fixed control volume

Roadmap - Differential Equations for Inviscid Flows

Control volume formulations:

conservation of mass
conservation of momentum
conservation of energy

The substantial derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

PDE:s on conservation form

PDE:s on non-conservation form

The entropy equation

Crocco's theorem

The Substantial Derivative

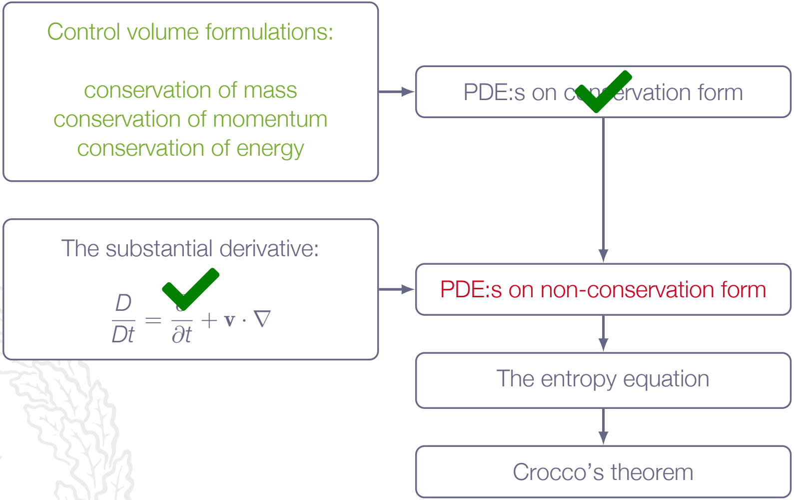
Introducing the substantial derivative operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

”... the time rate of change of any quantity associated with a particular moving fluid element is given by the substantial derivative ...”

”... the properties of the fluid element are changing as it moves past a point in a flow because the flowfield itself may be fluctuating with time (the local derivative) and because the fluid element is simply on its way to another point in the flowfield where the properties are different (the convective derivative) ...”

Roadmap - Differential Equations for Inviscid Flows



Chapter 6.4 Differential Equations in Non-Conservation Form

Non-Conservation Form of the Continuity Equation

Applying the **substantial derivative** operator to density gives

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho(\nabla \cdot \mathbf{v}) = 0 \Rightarrow$$

$$\boxed{\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0}$$

Non-Conservation Form of the Continuity Equation

$$\boxed{\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0}$$

"... the mass of a fluid element made up of a fixed set of particles (molecules or atoms) is constant as the fluid element moves through space ..."

Non-Conservation Form of the Momentum Equation

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v} + p\mathbf{I}) = \rho \mathbf{f} \Rightarrow$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v}(\nabla \cdot \rho \mathbf{v}) + \nabla p = \rho \mathbf{f} \Rightarrow$$

$$\underbrace{\rho \left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right]}_{=\frac{D\mathbf{v}}{Dt}} + \mathbf{v} \underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v}\right]}_{=0} + \nabla p = \rho \mathbf{f}$$

$$\boxed{\frac{D\mathbf{v}}{Dt} + \frac{1}{\rho}\nabla p = \mathbf{f}}$$

Non-Conservation Form of the Energy Equation

$$\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho h_o \mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v}) + \rho \dot{q}$$

$$h_o = e_o + \frac{p}{\rho} \Rightarrow$$

$$\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho e_o \mathbf{v}) + \nabla \cdot (p\mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v}) + \rho \dot{q} \Rightarrow$$

$$\rho \frac{\partial e_o}{\partial t} + e_o \frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \nabla e_o + e_o \nabla \cdot (\rho \mathbf{v}) + \nabla \cdot (p\mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v}) + \rho \dot{q} \Rightarrow$$

$$\rho \underbrace{\left[\frac{\partial e_o}{\partial t} + \mathbf{v} \cdot \nabla e_o\right]}_{=\frac{De_o}{Dt}} + e_o \underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right]}_{=0} + \nabla \cdot (p\mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v}) + \rho \dot{q}$$

Non-Conservation Form of the Energy Equation

$$\rho\frac{De_o}{Dt}+\nabla\cdot(p+\mathbf{v})=\rho\mathbf{f}\cdot\mathbf{v}+\rho\dot{q}$$

$$e_o=e+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\Rightarrow$$

$$\rho\frac{De}{Dt}+\rho\mathbf{v}\cdot\frac{D\mathbf{v}}{Dt}+\nabla\cdot(p\mathbf{v})=\rho\mathbf{f}\cdot\mathbf{v}+\rho\dot{q}$$

Using the momentum equation, $\left(\frac{D\mathbf{v}}{Dt}+\frac{1}{\rho}\nabla p=\mathbf{f}\right)$, gives

$$\rho\frac{De}{Dt}-\mathbf{v}\cdot\nabla p+\rho\mathbf{f}\cdot\mathbf{v}+\mathbf{v}\cdot\nabla p+p(\nabla\cdot\mathbf{v})=\rho\mathbf{f}\cdot\mathbf{v}+\rho\dot{q}\Rightarrow$$

$$\boxed{\frac{De}{Dt}+\frac{p}{\rho}(\nabla\cdot\mathbf{v})=\dot{q}}$$

Non-Conservation Form of the Energy Equation

$$\frac{De}{Dt} + \frac{p}{\rho}(\nabla \cdot \mathbf{v}) = \dot{q}$$

From the continuity equation we get

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0 \Rightarrow \nabla \cdot \mathbf{v} = -\frac{1}{\rho}\frac{D\rho}{Dt} \Rightarrow$$

$$\frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = \dot{q} \Rightarrow \frac{De}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = \dot{q}$$

$$\boxed{\frac{De}{Dt} = \dot{q} - p\frac{D\nu}{Dt}}$$

where $\nu = 1/\rho$

Non-Conservation Form of the Energy Equation

Compare with first law of thermodynamics: $de = \delta q - \delta W$

$$\boxed{\frac{De}{Dt} = \dot{q} - p\frac{D\nu}{Dt}}$$

Non-Conservation Form of the Energy Equation

If we instead express the energy equation in terms of enthalpy:

$$\frac{De}{Dt} = \dot{q} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right) \Rightarrow \frac{De}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = \dot{q}$$

$$h = e + \frac{p}{\rho} \Rightarrow \frac{Dh}{Dt} = \frac{De}{Dt} + \frac{1}{\rho}\frac{Dp}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) \Rightarrow$$

$$\boxed{\frac{Dh}{Dt} = \dot{q} + \frac{1}{\rho}\frac{Dp}{Dt}}$$

Non-Conservation Form of the Energy Equation

and total enthalpy ...

$$h_o = h + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} \Rightarrow \frac{Dh_o}{Dt} = \frac{Dh}{Dt} + \mathbf{v}\cdot\frac{D\mathbf{v}}{Dt}$$

From the momentum equation we get

$$\rho\frac{D\mathbf{v}}{Dt} + \nabla p = \rho\mathbf{f} \Rightarrow \frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{f} \Rightarrow$$

$$\frac{Dh_o}{Dt} = \underbrace{\frac{Dh}{Dt}}_{\dot{q}+\frac{1}{\rho}\frac{Dp}{Dt}} - \frac{1}{\rho}\mathbf{v}\cdot\nabla p + \mathbf{f}\cdot\mathbf{v} = \dot{q} + \frac{1}{\rho}\left[\frac{Dp}{Dt} - \mathbf{v}\cdot\nabla p\right] + \mathbf{f}\cdot\mathbf{v}$$

Non-Conservation Form of the Energy Equation

$$\frac{Dh_o}{Dt} = \dot{q} + \frac{1}{\rho}\left[\frac{Dp}{Dt} - \mathbf{v}\cdot\nabla p\right] + \mathbf{f}\cdot\mathbf{v}$$

Now, expanding the substantial derivative $\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p$ gives

$$\boxed{\frac{Dh_o}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial t} + \dot{q} + \mathbf{f}\cdot\mathbf{v}}$$

Let's examine the above relation ...

Non-Conservation Form of the Energy Equation

$$\frac{Dh_o}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial t} + \dot{q} + \mathbf{f}\cdot\mathbf{v}$$

The total enthalpy of a moving fluid element in an inviscid flow can change due to

1. unsteady flow: $\partial p/\partial t \neq 0$
2. heat transfer: $\dot{q} \neq 0$
3. body forces: $\mathbf{f}\cdot\mathbf{v} \neq 0$

Non-Conservation Form of the Energy Equation

Adiabatic flow without body forces $\Rightarrow$

$$\frac{Dh_o}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial t}$$

Steady-state adiabatic flow without body forces $\Rightarrow$

$$\frac{Dh_o}{Dt} = 0$$

h_o is constant along streamlines!

Additional Form of the Energy Equation

Start from

$$\frac{De}{Dt} = \dot{q} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

Calorically perfect gas:

$$e = C_v T \;;\; C_v = \frac{R}{\gamma - 1} \;;\; p = \rho R T \;;\; \gamma, R = const$$

$$\frac{De}{Dt} = C_v\frac{DT}{Dt} = \frac{R}{\gamma - 1}\frac{D}{Dt}\left(\frac{p}{\rho R}\right) = \frac{1}{\gamma - 1}\frac{D}{Dt}\left(\frac{p}{\rho}\right) \Rightarrow \frac{1}{\gamma - 1}\frac{D}{Dt}\left(\frac{p}{\rho}\right) = \dot{q} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

Additional Form of the Energy Equation

$$\frac{1}{\gamma-1}\frac{D}{Dt}\left(\frac{p}{\rho}\right)=\dot{q}-p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\Rightarrow$$

$$\frac{1}{\gamma-1}\left[p\frac{D}{Dt}\left(\frac{1}{\rho}\right)+\left(\frac{1}{\rho}\right)\frac{Dp}{Dt}\right]=\dot{q}-p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

$$p\frac{D}{Dt}\left(\frac{1}{\rho}\right)+\left(\frac{1}{\rho}\right)\frac{Dp}{Dt}=(\gamma-1)\dot{q}-(\gamma-1)p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

$$\gamma p\frac{D}{Dt}\left(\frac{1}{\rho}\right)+\left(\frac{1}{\rho}\right)\frac{Dp}{Dt}=(\gamma-1)\dot{q}$$

Additional Form of the Energy Equation

Continuity:

$$\frac{D\rho}{Dt} = -\rho(\nabla \cdot \mathbf{v}) \Rightarrow \frac{D}{Dt}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\frac{D\rho}{Dt} = \frac{1}{\rho}(\nabla \cdot \mathbf{v}) \Rightarrow$$

$$\frac{\gamma p}{\rho}(\nabla \cdot \mathbf{v}) + \left(\frac{1}{\rho}\right)\frac{Dp}{Dt} = (\gamma - 1)\dot{q}$$

Additional Form of the Energy Equation

$$\frac{Dp}{Dt} + \gamma p(\nabla \cdot \mathbf{v}) = (\gamma - 1)\rho\dot{q}$$

Adiabatic flow (no added heat):

$$\boxed{\frac{Dp}{Dt} + \gamma p(\nabla \cdot \mathbf{v}) = 0}$$

Non-conservation form (calorically perfect gas)

Conservation Form

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial G}{\partial z} = 0$$

where $Q(x, y, z, t)$, $E(x, y, z, t)$, ... may be scalar or vector fields

Example: the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

If an equation **cannot** be written in this form, it is said to be in **non-conservation form**

Euler Equations - Conservation Form

Continuity, momentum and energy equations in Cartesian coordinates, velocity components u, v, w (no body forces, no added heat)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u u + p) + \frac{\partial}{\partial y}(\rho u v) + \frac{\partial}{\partial z}(\rho u w) = 0$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v u) + \frac{\partial}{\partial y}(\rho v v + p) + \frac{\partial}{\partial z}(\rho v w) = 0$$

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho w u) + \frac{\partial}{\partial y}(\rho w v) + \frac{\partial}{\partial z}(\rho w w + p) = 0$$

$$\frac{\partial}{\partial t}(\rho e_o) + \frac{\partial}{\partial x}(\rho h_o u) + \frac{\partial}{\partial y}(\rho h_o v) + \frac{\partial}{\partial z}(\rho h_o w) = 0$$

Euler Equations - Non-Conservation Form

Continuity, momentum and energy equations in Cartesian coordinates, velocity components u, v, w (no body forces, no added heat), calorically perfect gas

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} + \gamma p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

Conservation and Non-Conservation Form

The governing equations on non-conservation form are not, although the name might give that impression, less physically accurate than the equations on conservation form. The nomenclature comes from CFD where the equations on conservation form are preferred.

Using the conservation form as a basis for a Finite-Volume Method (FVM) solver ensures conservation of mass, momentum and energy.

Conservation and Non-Conservation Form

Conservative equations are equations that directly stems from **conservation of flow quantities** over a control volume

The equations on **non-conservation form** are derived from the corresponding equations on conservation form using the **chain rule** for derivatives

Thus the equations on non-conservation form do not stem directly from a conservation law - **but aren't the two formulations still equivalent?**

Only for continuous solutions! The chain rule can only be used for continuous fields

Conservation and Non-Conservation Form

Conservation forms are useful for:

1. Numerical methods for compressible flow
2. Theoretical understanding of non-linear waves (shocks etc)
3. Provide link between integral forms (control volume formulations) and PDE:s

Non-conservation forms are useful for:

1. Theoretical understanding of behavior of numerical methods
2. Theoretical understanding of boundary conditions
3. Analysis of linear waves (aero-acoustics)

Roadmap - Differential Equations for Inviscid Flows

Control volume formulations:

conservation of mass
conservation of momentum
conservation of energy

→ PDE:s on conservation form ✓

The substantial derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$ ✓

→ PDE:s on non-conservation form ✓

↓

The entropy equation

↓

Crocco's theorem

Chapter 6.5
The Entropy Equation

The Entropy Equation

From the first and second law of thermodynamics we have

$$\frac{De}{Dt} = T\frac{Ds}{Dt} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

which is called the entropy equation

The Entropy Equation

Compare the entropy equation

$$\frac{De}{Dt} = T\frac{Ds}{Dt} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

with the energy equation (inviscid flow):

$$\frac{De}{Dt} = \dot{q} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

we see that

$$T\frac{Ds}{Dt} = \dot{q}$$

The Entropy Equation

If $\dot{q} = 0$ (adiabatic flow) then

$$\frac{Ds}{Dt} = 0$$

i.e., entropy is constant for moving fluid element

Furthermore, if the flow is steady we have

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + (\mathbf{v} \cdot \nabla)s = (\mathbf{v} \cdot \nabla)s = 0$$

i.e., entropy is constant along streamlines

Roadmap - Differential Equations for Inviscid Flows

Control volume formulations:

conservation of mass
conservation of momentum
conservation of energy

→ PDE:s on conservation form ✓

The substantial derivative: ✓

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

→ PDE:s on non-conservation form ✓

↓

The entropy equation ✓

↓

Crocco's theorem

Chapter 6.6
Crocco's Theorem

Crocco's Theorem

"... a relation between gradients of total enthalpy, gradients of entropy, and flow rotation ..."

Crocco's Theorem

Momentum equation (no body forces)

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p$$

Writing out the substantial derivative gives

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p \Rightarrow \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho}\nabla p$$

First and second law of thermodynamics (energy equation)

$$dh = Tds + \frac{1}{\rho}dp$$

Replace differentials with a gradient operator

$$\nabla h = T\nabla s + \frac{1}{\rho}\nabla p \Rightarrow T\nabla s = \nabla h - \frac{1}{\rho}\nabla p$$

Crocco's Theorem

With pressure derivative from the momentum equation inserted in the energy equation we get

$$T\nabla s = \nabla h + \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$$

$$h = h_o - \frac{1}{2}\mathbf{v} \cdot \mathbf{v} \Rightarrow \nabla h = \nabla h_o - \nabla(\frac{1}{2}\mathbf{v} \cdot \mathbf{v})$$

$$\nabla(\frac{1}{2}\mathbf{v} \cdot \mathbf{v}) = \mathbf{v} \times (\nabla \times \mathbf{v}) + \mathbf{v} \cdot \nabla \mathbf{v}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$\mathbf{A} = \mathbf{B} = \mathbf{v} \Rightarrow \nabla(\mathbf{v} \cdot \mathbf{v}) = 2[\mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{v})]$$

Crocco's Theorem

$$T\nabla s = \nabla h_o - \mathbf{v} \times (\nabla \times \mathbf{v}) - \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$$

$$\boxed{T\nabla s = \nabla h_o + \frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v})}$$

Note! $\nabla \times \mathbf{v}$ is the vorticity of the fluid

the rotational motion of the fluid is described by the angular velocity $\omega = \frac{1}{2}(\nabla \times \mathbf{v})$

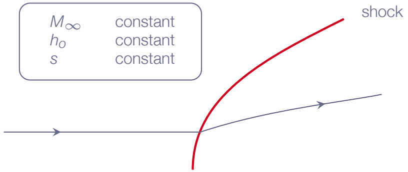
Crocco's Theorem

$$T\nabla s = \nabla h_o + \frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v})$$

"... when a steady flow field has gradients of total enthalpy and/or entropy Crocco's theorem dramatically shows that it is ***rotational*** *..."*

Crocco's Theorem - Example

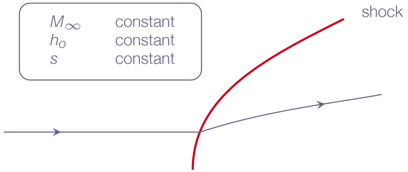
Curved stationary shock (steady-state flow)



1. s is constant upstream of shock
2. jump in s across shock depends on local shock angle
3. s will vary from streamline to streamline downstream of shock
4. $\nabla s \neq 0$ downstream of shock

Crocco's Theorem - Example

Curved stationary shock (steady-state flow)



Total enthalpy upstream of shock

h_o is constant along streamlines
h_o is uniform

Total enthalpy downstream of shock

h_o is uniform

$$\nabla h_o = 0$$

Crocco's Theorem - Example

Crocco's equation for steady-state flow:

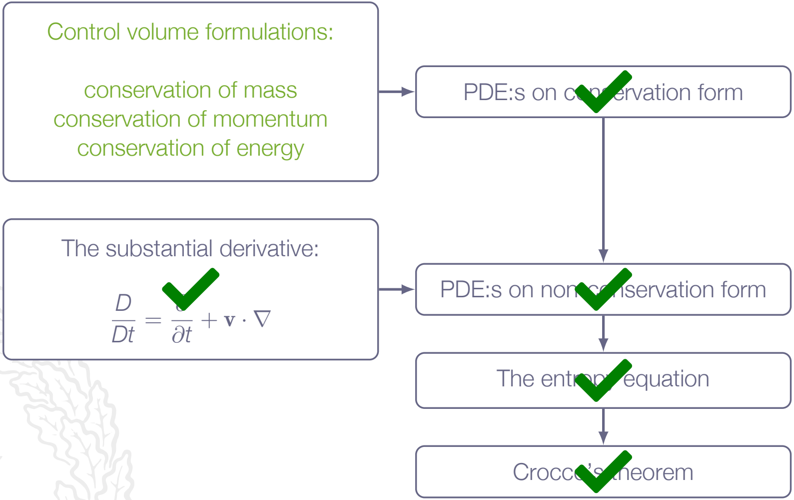
$$T\nabla s = \nabla h_o - \mathbf{v} \times (\nabla \times \mathbf{v})$$

$\mathbf{v} \times (\nabla \times \mathbf{v}) \neq 0$ downstream of a curved shock

the rotation $\nabla \times \mathbf{v} \neq 0$ downstream of a curved shock

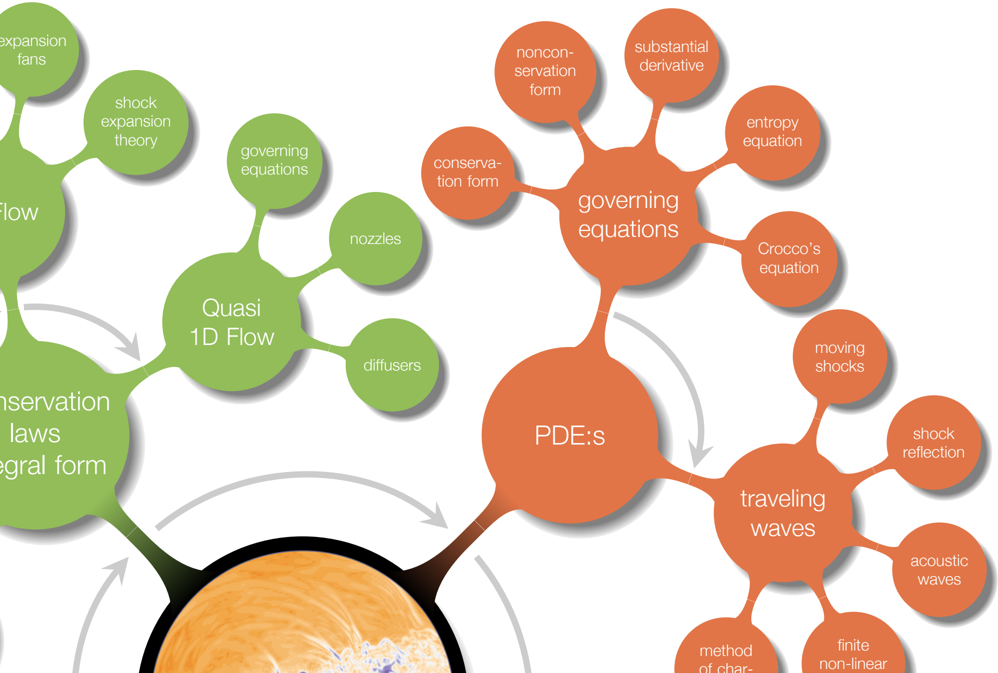
Explains why it is difficult to solve such problems by analytic means!

Roadmap - Differential Equations for Inviscid Flows



Chapter 7
Unsteady Wave Motion

Overview



expansion fans
shock expansion theory
flow
governing equations
nozzles
Quasi 1D Flow
diffusers
nservation laws egral form
noncon-servation form
substantial derivative
entropy equation
conserva-tion form
governing equations
Crocco's equation
PDE:s
moving shocks
shock reflection
traveling waves
acoustic waves
method of char-
finite non-linear

Learning Outcomes

3 **Describe** typical engineering flow situations in which compressibility effects are more or less predominant (e.g. Mach number regimes for steady-state flows)

4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on

8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics

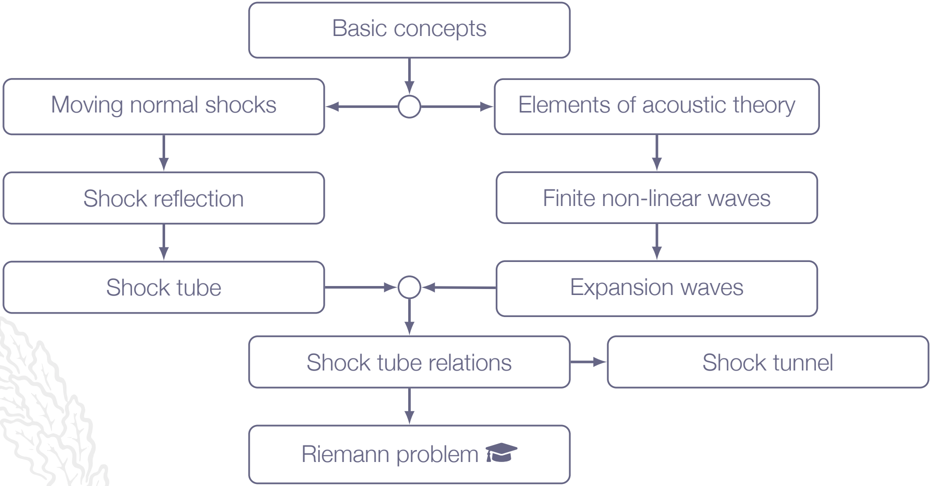
- a 1D isentropic flow*
- b normal shocks*
- j unsteady waves and discontinuities in 1D
- k basic acoustics

9 **Solve** engineering problems involving the above-mentioned phenomena (8a-8k)

11 **Explain** how the equations for aero-acoustics and classical acoustics are derived as limiting cases of the compressible flow equations

moving normal shocks - frame of reference seems to be the key here?!

Roadmap - Unsteady Wave Motion



Motivation

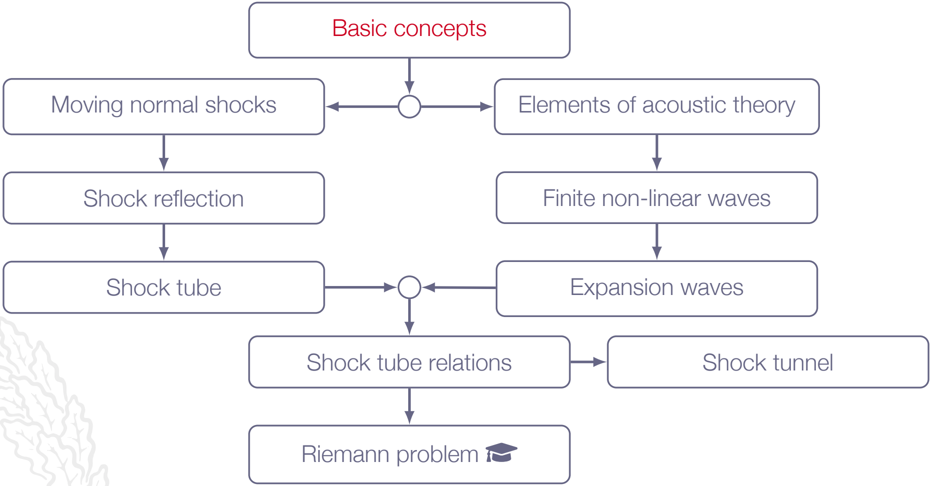
Most practical flows are unsteady

Traveling waves appears in many real-life situations and is an important topic within compressible flows

We will study unsteady flows in one dimension in order to reduce complexity and focus on the physical effects introduced by the unsteadiness

Throughout this section, we will study an application called the shock tube, which is a rather rare application but it lets us study unsteady waves in one dimension and it includes all physical principles introduced in chapter 7

Roadmap - Unsteady Wave Motion



Unsteady Wave Motion

inertial frames!

Physical laws are the same for both frame of references

Shock characteristics are the same for both observers (shape, strength, etc)

Recall - the Hugonoit relation does not include velocities, only static thermodynamic quantities that are independent of reference frame

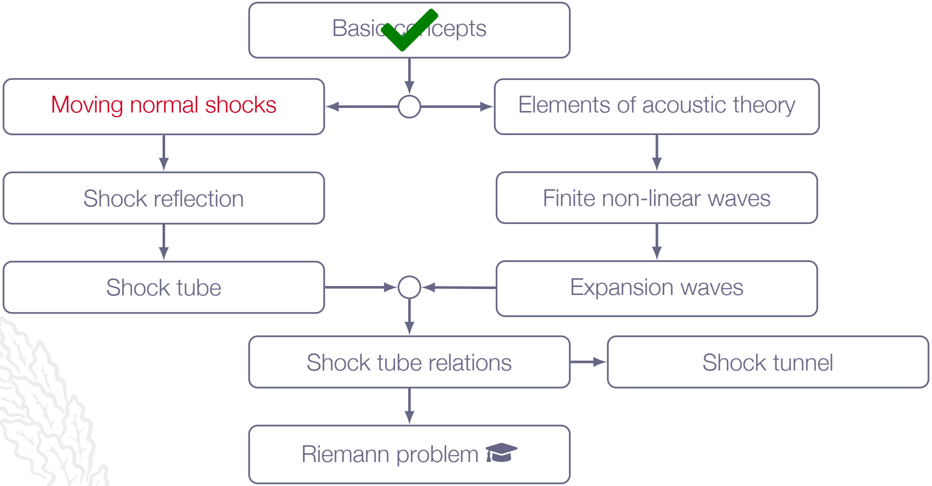
Unsteady Wave Motion

Is there a connection with stationary shock waves?

Answer: Yes!

Locally, in a **moving frame of reference**, the shock may be viewed as a **stationary normal shock**

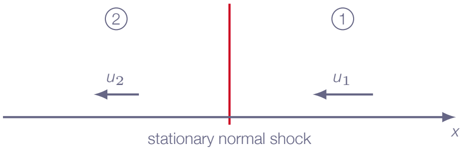
Roadmap - Unsteady Wave Motion



Chapter 7.2
Moving Normal Shock Waves

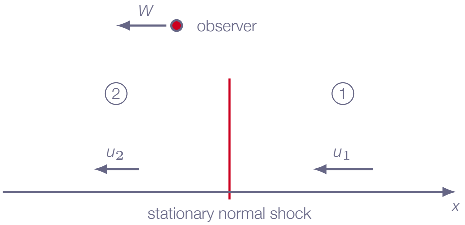
Moving Normal Shock Waves

Chapter 3: stationary normal shock



$u_1 > a_1$	(supersonic flow)
$u_2 < a_2$	(subsonic flow)
$p_2 > p_1$	(sudden compression)
$s_2 > s_1$	(shock loss)

Moving Normal Shock Waves



Introduce observer moving to the left with speed W

- if W is constant the observer is still in an inertial system (all physical laws are unchanged)

The observer sees a normal shock moving to the right with speed W

- gas velocity ahead of shock: $u_1' = W - u_1$
- gas velocity behind shock: $u_2' = W - u_2$

Moving Normal Shock Waves

Now, let $W = u_1 \Rightarrow$

$$u'_1 = 0$$

$$u'_2 = u_1 - u_2 > 0$$

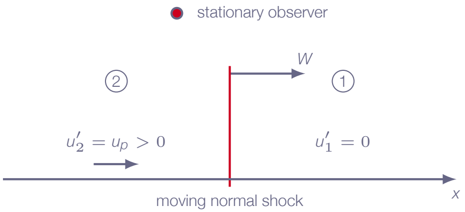
The observer now sees the shock traveling to the right with speed $W = u_1$ into a stagnant gas, leaving a compressed gas ($p_2 > p_1$) with velocity $u'_2 > 0$ behind it

Introducing u_p:

$$u_p = u'_2 = u_1 - u_2$$

Moving Normal Shock Waves

Analogy:



Case 1

stationary normal shock

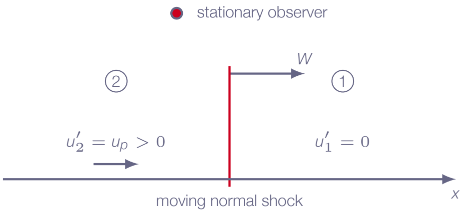
observer moving with velocity W

Case 2

normal shock moving with velocity W

stationary observer

Moving Normal Shock Waves - Governing Equations



For stationary normal shocks we have:

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

With $(u_1 = W)$ and $(u_2 = W - u_p)$ we get:

$$\rho_1 W = \rho_2 (W - u_p)$$

$$\rho_1 W^2 + p_1 = \rho_2 (W - u_p)^2 + p_2$$

$$h_1 + \frac{1}{2}W^2 = h_2 + \frac{1}{2}(W - u_p)^2$$

Moving Normal Shock Waves - Relations

Starting from the governing equations

$$\rho_1 W = \rho_2 (W - u_p)$$

$$\rho_1 W^2 + p_1 = \rho_2 (W - u_p)^2 + p_2$$

$$h_1 + \frac{1}{2} W^2 = h_2 + \frac{1}{2} (W - u_p)^2$$

and using $h = e + \dfrac{p}{\rho}$

it is possible to show that

$$e_2 - e_1 = \frac{p_1 + p_2}{2} \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right)$$

Moving Normal Shock Waves - Relations

$$e_2 - e_1 = \frac{p_1 + p_2}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

same Hugoniot equation as for stationary normal shock

This means that we will have same shock strength, *i.e.* same discontinuities in density, velocity, pressure, etc

Moving Normal Shock Waves - Relations

Starting from the Hugoniot equation one can show that

$$\frac{\rho_2}{\rho_1} = \frac{1 + \dfrac{\gamma+1}{\gamma-1}\left(\dfrac{p_2}{p_1}\right)}{\dfrac{\gamma+1}{\gamma-1} + \dfrac{p_2}{p_1}}$$

and

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\left[\frac{\dfrac{\gamma+1}{\gamma-1} + \dfrac{p_2}{p_1}}{1 + \dfrac{\gamma+1}{\gamma-1}\left(\dfrac{p_2}{p_1}\right)}\right]$$

Moving Normal Shock Waves - Relations

For calorically perfect gas and stationary normal shock:

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_s^2 - 1)$$

same as eq. (3.57) in Anderson with $M_1 = M_s$

where

$$M_s = \frac{W}{a_1}$$

M_s is simply the speed of the shock, traveling into the stagnant gas, normalized by the speed of sound in the gas ahead of the shock

Note!

- $M_s > 1$, otherwise there is no shock!
- **shocks always moves faster than sound** - no warning before it hits you ☺

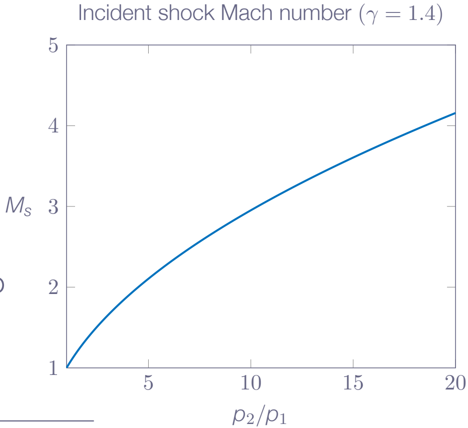
Moving Normal Shock Waves - Relations

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_s^2 - 1)$$

Re-arrange $\Rightarrow$

$$M_s = \sqrt{\frac{\gamma+1}{2\gamma}\left(\frac{p_2}{p_1} - 1\right) + 1}$$

shock speed directly linked to pressure ratio



$$M_s = \frac{W}{a_1} \Rightarrow W = a_1 M_s = a_1\sqrt{\frac{\gamma+1}{2\gamma}\left(\frac{p_2}{p_1} - 1\right) + 1}$$

Moving Normal Shock Waves - Induced Flow Velocity

From the continuity equation we get:

$$u_p = W\left(1 - \frac{\rho_1}{\rho_2}\right) > 0$$

After some derivation we obtain:

$$u_p = \frac{a_1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left[\frac{\dfrac{2\gamma}{\gamma+1}}{\dfrac{p_2}{p_1} + \dfrac{\gamma-1}{\gamma+1}}\right]^{1/2}$$

Moving Normal Shock Waves - Induced Flow Mach Number

$$M_p = \frac{u_p}{a_2} = \frac{u_p}{a_1}\frac{a_1}{a_2} = \frac{u_p}{a_1}\sqrt{\frac{T_1}{T_2}}$$

inserting u_p/a_1 and T_1/T_2 from relations on previous slides we get:

$$M_p = \frac{1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left[\frac{\frac{2\gamma}{\gamma+1}}{\frac{\gamma-1}{\gamma+1} + \frac{p_2}{p_1}}\right]^{1/2}\left[\frac{1 + \left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right)}{\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right) + \left(\frac{p_2}{p_1}\right)^2}\right]^{1/2}$$

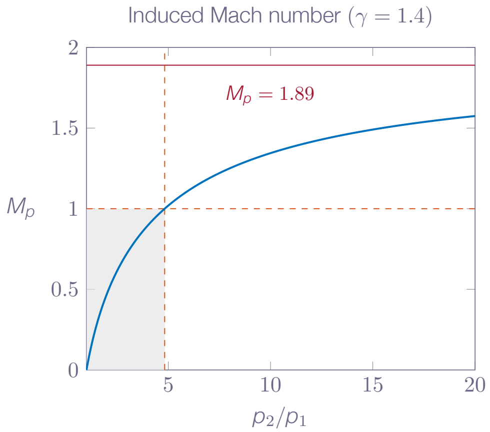
Moving Normal Shock Waves - Induced Flow Mach Number

Note!

$$\lim_{\frac{p_2}{p_1}\to\infty} M_p \to \sqrt{\frac{2}{\gamma(\gamma-1)}}$$

for air ($\gamma = 1.4$)

$$\lim_{\frac{p_2}{p_1}\to\infty} M_p \to 1.89$$



Moving Normal Shock Waves - Example

Moving normal shock with $p_2/p_1 = 10$

$(p_1 = 1.0\ bar,\ T_1 = 300\ K,\ \gamma = 1.4)$

$\Rightarrow M_s = 2.95$ and $W = 1024.2\ m/s$

The shock is advancing with almost **three times** the speed of sound!

Behind the shock the induced velocity is $u_p = 756.2\ m/s \Rightarrow$ supersonic flow $(a_2 = 562.1\ m/s)$

May be calculated by formulas 7.13, 7.16, 7.10, 7.11 or by using Table A.2 for stationary normal shock ($u_1 = W$, $u_2 = W - u_p$)

Moving Normal Shock Waves - Total Enthalpy

Note! $h_{o_1} \neq h_{o_2}$

constant total enthalpy is **only valid for stationary shocks!**

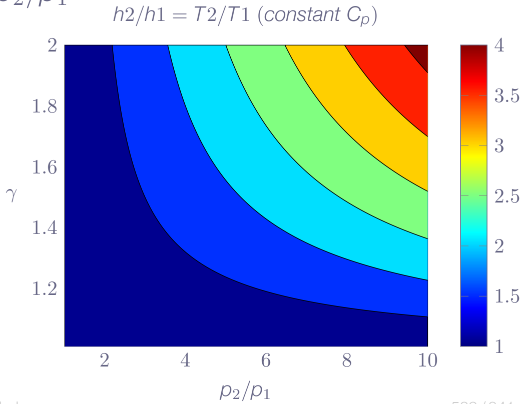
shock is uniquely defined by pressure ratio p_2/p_1

$$u_1 = 0$$

$$h_{o_1} = h_1 + \frac{1}{2}u_1^2 = h_1$$

$$h_{o_2} = h_2 + \frac{1}{2}u_2^2$$

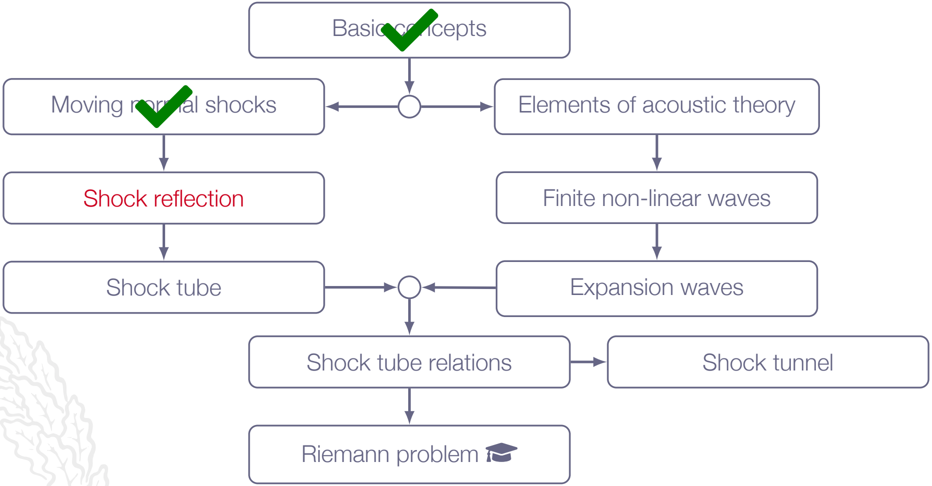
$$h_2 > h_1 \Rightarrow h_{o_2} > h_{o_1}$$



Moving Normal Shock Waves - Total Enthalpy

Gas/Vapor	Ratio of specific heats (γ)	Gas constant R
Acetylene	1.23	319
Air (standard)	1.40	287
Ammonia	1.31	530
Argon	1.67	208
Benzene	1.12	100
Butane	1.09	143
Carbon Dioxide	1.29	189
Carbon Disulphide	1.21	120
Carbon Monoxide	1.40	297
Chlorine	1.34	120
Ethane	1.19	276
Ethylene	1.24	296
Helium	1.67	2080
Hydrogen	1.41	4120
Hydrogen chloride	1.41	230
Methane	1.30	518
Natural Gas (Methane)	1.27	500
Nitric oxide	1.39	277
Nitrogen	1.40	297
Nitrous oxide	1.27	180
Oxygen	1.40	260
Propane	1.13	189
Steam (water)	1.32	462
Sulphur dioxide	1.29	130

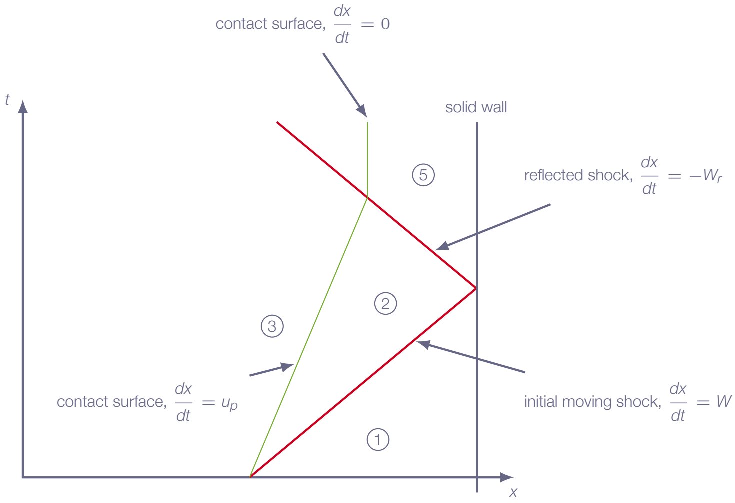
Roadmap - Unsteady Wave Motion



Chapter 7.3
Reflected Shock Wave

what happens when a moving shock approaches a wall?

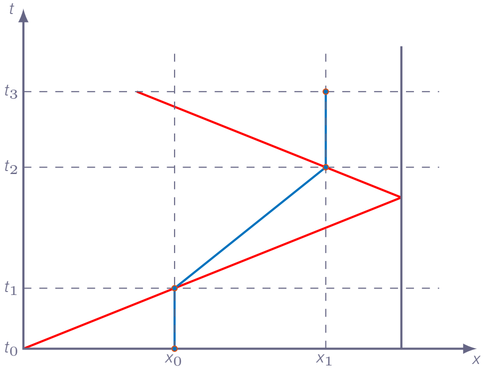
Shock Reflection



Shock Reflection - Particle Path

A fluid particle located at x_0 at time t_0 (a location ahead of the shock) will be affected by the moving shock and follow the blue path

time	location	velocity
t_0	x_0	0
t_1	x_0	u_p
t_2	x_1	u_p
t_3	x_1	0



Shock Reflection Relations

In the frame of reference of the reflected shock we have

velocity ahead of shock: $W_r + u_p$

velocity behind shock: W_r

where W_r is the velocity of the reflected shock and u_p is the induced flow velocity behind the incident shock

Shock Reflection Relations

Continuity:

$$\rho_2(W_r + u_p) = \rho_5 W_r$$

Momentum:

$$p_2 + \rho_2(W_r + u_p)^2 = p_5 + \rho_5 W_r^2$$

Energy:

$$h_2 + \frac{1}{2}(W_r + u_p)^2 = h_5 + \frac{1}{2}W_r^2$$

Shock Reflection Relations

Reflected shock is determined such that $u_5 = 0$

$$\frac{M_r}{M_r^2 - 1} = \frac{M_s}{M_s^2 - 1}\sqrt{1 + \frac{2(\gamma - 1)}{(\gamma + 1)^2}(M_s^2 - 1)\left(\gamma + \frac{1}{M_s^2}\right)}$$

where

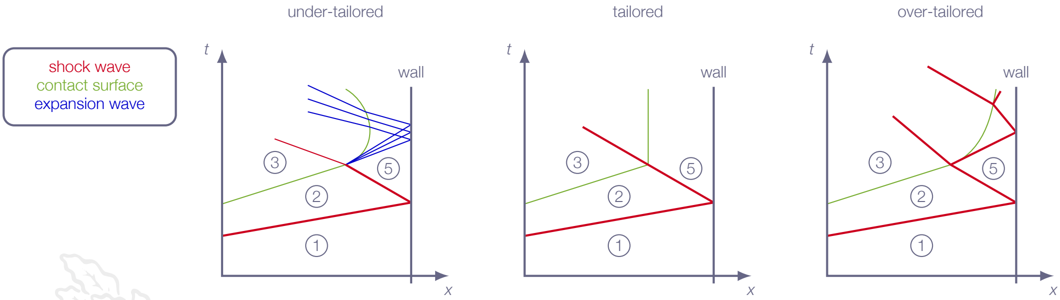
$$M_r = \frac{W_r + u_p}{a_2}$$

Tailored v.s. Non-Tailored Shock Reflection

The time duration of condition 5 is determined by what happens after interaction between reflected shock and contact discontinuity

For special choice of initial conditions (tailored case), this interaction is negligible, thus prolonging the duration of condition 5

Tailored v.s. Non-Tailored Shock Reflection



Under-tailored conditions:

Mach number of incident wave lower than in tailored conditions

Over-tailored conditions:

Mach number of incident wave higher than in tailored conditions

Shock Reflection - Example

Shock reflection in shock tube ($\gamma = 1.4$)
(Example 7.1 in Anderson)

Given data

p_2/p_1	10.0	
T_2/T_1	2.623	
p_1	1.0	bar
T_1	300.0	K

Calculated data

M_s	2.95
M_r	2.09
p_5/p_2	4.978
T_5/T_2	1.77

$$p_5 = \left(\frac{p_5}{p_2}\right)\left(\frac{p_2}{p_1}\right)p_1 = 49.78$$

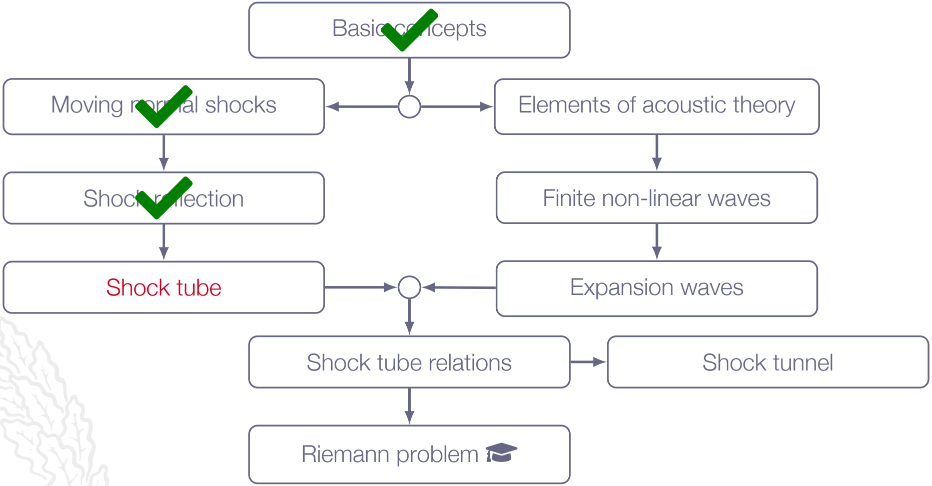
$$T_5 = \left(\frac{T_5}{T_2}\right)\left(\frac{T_2}{T_1}\right)T_1 = 1393$$

Shock Reflection - Shock Tube

Very high pressure and temperature conditions in a specified location with very high precision (p_5, T_5)

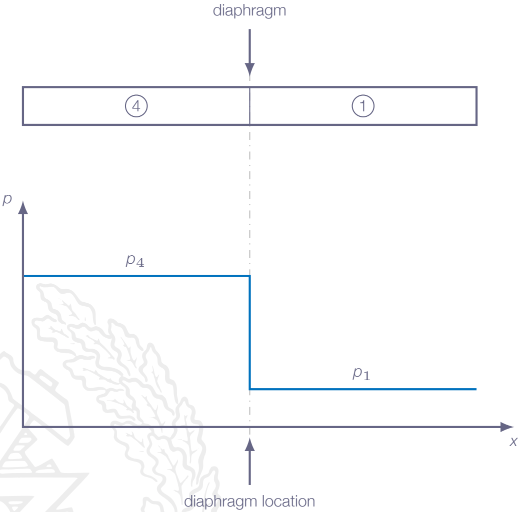
- measurements of thermodynamic properties of various gases at extreme conditions, *e.g.* dissociation energies, molecular relaxation times, etc.
- measurements of chemical reaction properties of various gas mixtures at extreme conditions

Roadmap - Unsteady Wave Motion



The Shock Tube

Shock Tube

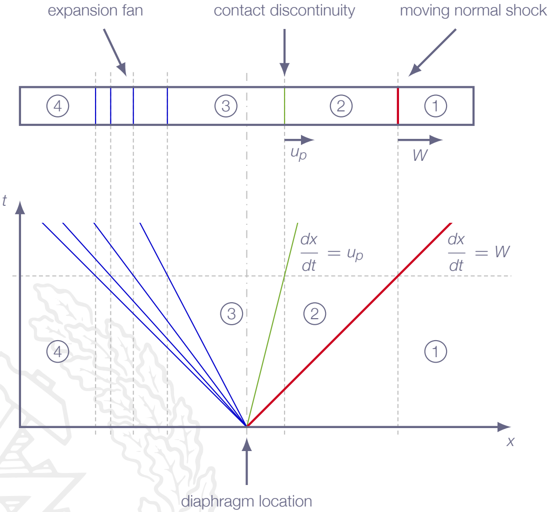


tube with closed ends
diaphragm inside, separating two different constant states
(could also be two different gases)

if diaphragm is removed suddenly (by inducing a breakdown) the two states come into contact and a flow develops

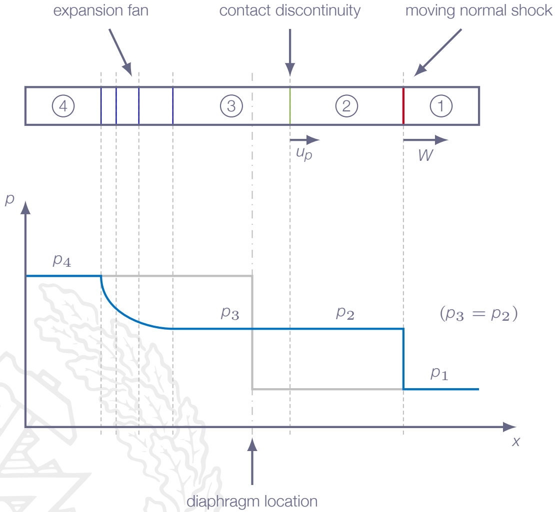
assume that $p_4 > p_1$:
state 4 is "driver" section
state 1 is "driven" section

Shock Tube



flow at some time after diaphragm breakdown

Shock Tube



flow at some time after diaphragm breakdown

Shock Tube - Basic Principles

As the diaphragm is removed, a pressure discontinuity is generated

The only process that can generate a pressure **discontinuity** in the gas is a **shock**

In chapter 3 we learned that the velocity upstream of the shock **must be supersonic**

Since the gas is standing still when the shock tube is started, **the shock must move** in order to establish the required **relative velocity**

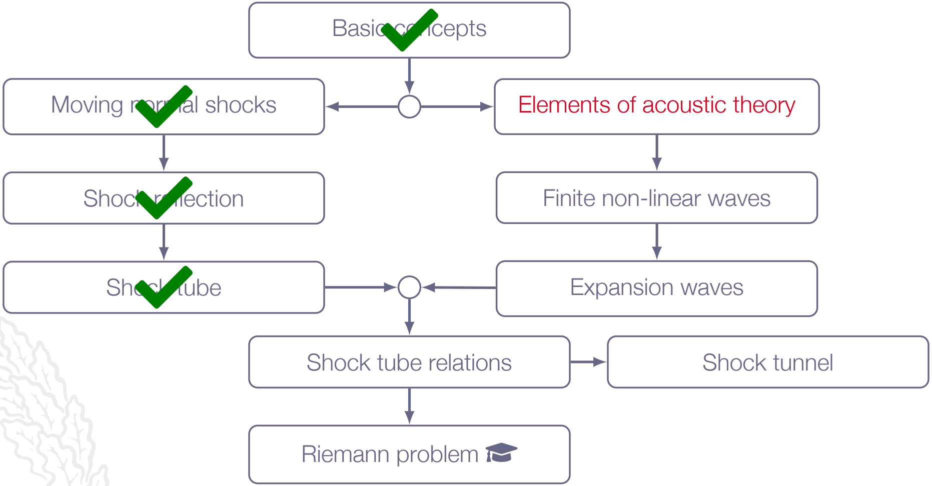
The shock must move in to the gas with the lower pressure

Shock Tube - Basic Principles

By using light gases for the **driver section** (*e.g.* He) and heavier gases for the **driven section** (*e.g.* air) the pressure p_4 required for a specific p_2/p_1 ratio is significantly reduced

If T_4/T_1 is increased, the pressure p_4 required for a specific p_2/p_1 is also reduced

Roadmap - Unsteady Wave Motion



Chapter 7.5
Elements of Acoustic Theory

Sound Waves - Sound Pressure Level

sound wave	L_p [dB]	Δp [Pa]
Weakest audible sound wave	0	2.83×10^{-5}
Loud sound wave	91	1.00×10^{0}
Amplified music	120	2.80×10^{1}
Jet engine @ 30 m	130	9.00×10^{1}
Threshold of pain	140	2.83×10^{2}
Military jet @ 30 m	150	8.90×10^{2}

Example (Loud sound wave):

$\Delta p \sim$ 1 Pa (91 dB) gives $\Delta\rho \sim 8.5 \times 10^{-6}$ kg/m^3 and $\Delta u \sim 2.4 \times 10^{-3}$ m/s

Elements of Acoustic Theory

PDE:s for conservation of mass and momentum derived in Chapter 6:

	conservation form	non-conservation form
mass	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$	$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0$
momentum	$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v} + p\mathbf{I}) = 0$	$\rho \frac{D\mathbf{v}}{Dt} + \nabla p = 0$

Elements of Acoustic Theory

For adiabatic inviscid flow we also have the entropy equation as

$$\frac{Ds}{Dt} = 0$$

Assume one-dimensional flow

$$\left.\begin{aligned} \rho &= \rho(x,t) \\ \mathbf{v} &= u(x,t)\mathbf{e}_x \\ p &= p(x,t) \\ &\ldots \end{aligned}\right\} \Rightarrow$$

continuity	$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$
momentum	$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0$
s=constant	

Elements of Acoustic Theory

continuity $$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

momentum $$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0$$

s=constant

More unknowns than equations $\Rightarrow$ the equation system can not be solved

Can $\frac{\partial p}{\partial x}$ be expressed in terms of density?

Leading question; it is possible so let's do just that ...

Elements of Acoustic Theory

From Chapter 1: any thermodynamic state variable is uniquely defined by any two other state variables

$$p = p(\rho, s) \Rightarrow dp = \left(\frac{\partial p}{\partial \rho}\right)_s d\rho + \left(\frac{\partial p}{\partial s}\right)_\rho ds$$

s=constant gives

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_s d\rho = a^2 d\rho$$

$$\Rightarrow \begin{cases} \dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial x} + \rho\dfrac{\partial u}{\partial x} = 0 \\[2ex] \rho\dfrac{\partial u}{\partial t} + \rho u\dfrac{\partial u}{\partial x} + a^2\dfrac{\partial \rho}{\partial x} = 0 \end{cases}$$

Elements of Acoustic Theory

Assume **small perturbations** around stagnant reference condition:

$$\rho = \rho_\infty + \Delta\rho \qquad p = p_\infty + \Delta p \qquad T = T_\infty + \Delta T \qquad u = u_\infty + \Delta u = \{u_\infty = 0\} = \Delta u$$

where ρ_∞, p_∞, and T_∞ are constant

Now, insert $\rho = (\rho_\infty + \Delta\rho)$ and $u = \Delta u$ in the continuity and momentum equations (derivatives of ρ_∞ are zero)

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \Delta u \dfrac{\partial}{\partial x}(\Delta\rho) + (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial x}(\Delta u) = 0 \\ (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial t}(\Delta u) + (\rho_\infty + \Delta\rho)\Delta u \dfrac{\partial}{\partial x}(\Delta u) + a^2 \dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Elements of Acoustic Theory

Assume **small perturbations** around stagnant reference condition:

$$\rho = \rho_\infty + \Delta\rho \qquad p = p_\infty + \Delta p \qquad T = T_\infty + \Delta T \qquad u = u_\infty + \Delta u = \{u_\infty = 0\} = \Delta u$$

where ρ_∞, p_∞, and T_∞ are constant

Now, insert $\rho = (\rho_\infty + \Delta\rho)$ and $u = \Delta u$ in the continuity and momentum equations (derivatives of ρ_∞ are zero)

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \Delta u \dfrac{\partial}{\partial x}(\Delta\rho) + (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial x}(\Delta u) = 0 \\ (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial t}(\Delta u) + (\rho_\infty + \Delta\rho)\Delta u \dfrac{\partial}{\partial x}(\Delta u) + a^2 \dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Elements of Acoustic Theory

Speed of sound is a thermodynamic state variable $\Rightarrow a^2 = a^2(\rho, s)$. With entropy constant $\Rightarrow a^2 = a^2(\rho)$

Taylor expansion around a_∞ with $(\Delta\rho = \rho - \rho_\infty)$ gives

$$a^2 = a_\infty^2 + \left(\frac{\partial}{\partial \rho}(a^2)\right)_\infty \Delta\rho + \frac{1}{2}\left(\frac{\partial^2}{\partial \rho^2}(a^2)\right)_\infty (\Delta\rho)^2 + ...$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \Delta u \dfrac{\partial}{\partial x}(\Delta\rho) + (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial x}(\Delta u) = 0 \\ (\rho_\infty + \Delta\rho)\dfrac{\partial}{\partial t}(\Delta u) + (\rho_\infty + \Delta\rho)\Delta u \dfrac{\partial}{\partial x}(\Delta u) + \left[a_\infty^2 + \left(\dfrac{\partial}{\partial \rho}(a^2)\right)_\infty \Delta\rho + \ldots\right]\dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Elements of Acoustic Theory - Acoustic Equations

Since $\Delta\rho$ and Δu are assumed to be small ($\Delta\rho \ll \rho_\infty$, $\Delta u \ll a$)

1. products of perturbations can be neglected
2. higher-order terms in the Taylor expansion can be neglected

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t}(\Delta\rho) + \rho_\infty \dfrac{\partial}{\partial x}(\Delta u) = 0 \\ \\ \rho_\infty \dfrac{\partial}{\partial t}(\Delta u) + a_\infty^2 \dfrac{\partial}{\partial x}(\Delta\rho) = 0 \end{cases}$$

Note! The assumption is only valid for small perturbations (sound waves)

This type of derivation is based on linearization, *i.e.* the acoustic equations are **linear**

Elements of Acoustic Theory - Acoustic Equations

Acoustic equations:

> *"... describe the motion of gas induced by the passage of a sound wave ..."*

Elements of Acoustic Theory - Wave Equation

Combining linearized continuity and the momentum equations we get

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = a_\infty^2 \frac{\partial^2}{\partial x^2}(\Delta\rho)$$

(combine the time derivative of the continuity eqn. and the divergence of the momentum eqn.)

General solution:

$$\Delta\rho(x,t) = F(x - a_\infty t) + G(x + a_\infty t)$$

$F(x - a_\infty t)$: wave traveling in positive x-direction with speed a_∞

$G(x + a_\infty t)$: wave traveling in negative x-direction with speed a_∞

F and G may be arbitrary functions

Wave shape is determined by functions F and G

Elements of Acoustic Theory - Wave Equation

Spatial and temporal derivatives of F are obtained according to

$$\begin{cases} \dfrac{\partial F}{\partial t} = \dfrac{\partial F}{\partial (x - a_\infty t)} \dfrac{\partial (x - a_\infty t)}{\partial t} = -a_\infty F' \\[2ex] \dfrac{\partial F}{\partial x} = \dfrac{\partial F}{\partial (x - a_\infty t)} \dfrac{\partial (x - a_\infty t)}{\partial x} = F' \end{cases}$$

spatial and temporal derivatives of G can of course be obtained in the same way...

Elements of Acoustic Theory - Wave Equation

with $\Delta\rho(x,t) = F(x - a_\infty t) + G(x + a_\infty t)$ and the derivatives of F and G we get

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = a_\infty^2 F'' + a_\infty^2 G''$$

and

$$\frac{\partial^2}{\partial x^2}(\Delta\rho) = F'' + G''$$

which gives

$$\frac{\partial^2}{\partial t^2}(\Delta\rho) - a_\infty^2 \frac{\partial^2}{\partial x^2}(\Delta\rho) = 0$$

i.e., the proposed solution fulfils the wave equation

Elements of Acoustic Theory - Wave Equation

F and G may be arbitrary functions, assume $G = 0$

$$\Delta\rho(x,t) = F(x - a_\infty t)$$

If $\Delta\rho$ is constant (constant wave amplitude), $(x - a_\infty t)$ must be a constant which implies

$$x = a_\infty t + c$$

where c is a constant

$$\frac{dx}{dt} = a_\infty$$

Elements of Acoustic Theory - Wave Equation

Let's try to find a relation between $\Delta\rho$ and Δu

$\Delta\rho(x,t) = F(x - a_\infty t)$ (wave in positive x direction) gives:

$$\frac{\partial}{\partial t}(\Delta\rho) = -a_\infty F' \quad \text{and} \quad \frac{\partial}{\partial x}(\Delta\rho) = F'$$

$$\underbrace{\frac{\partial}{\partial t}(\Delta\rho)}_{-a_\infty F'} + a_\infty \underbrace{\frac{\partial}{\partial x}(\Delta\rho)}_{F'} = 0$$

or

$$\frac{\partial}{\partial x}(\Delta\rho) = -\frac{1}{a_\infty}\frac{\partial}{\partial t}(\Delta\rho)$$

Elements of Acoustic Theory - Wave Equation

Linearized momentum equation:

$$\rho_\infty \frac{\partial}{\partial t}(\Delta u) = -a_\infty^2 \frac{\partial}{\partial x}(\Delta \rho) \Rightarrow$$

$$\frac{\partial}{\partial t}(\Delta u) = -\frac{a_\infty^2}{\rho_\infty}\frac{\partial}{\partial x}(\Delta \rho) = \left\{\frac{\partial}{\partial x}(\Delta \rho) = -\frac{1}{a_\infty}\frac{\partial}{\partial t}(\Delta \rho)\right\} = \frac{a_\infty}{\rho_\infty}\frac{\partial}{\partial t}(\Delta \rho)$$

$$\frac{\partial}{\partial t}\left(\Delta u - \frac{a_\infty}{\rho_\infty}\Delta \rho\right) = 0 \Rightarrow \Delta u - \frac{a_\infty}{\rho_\infty}\Delta \rho = const$$

In undisturbed gas $\Delta u = \Delta \rho = 0$ which implies that the constant must be zero and thus

$$\boxed{\Delta u = \frac{a_\infty}{\rho_\infty}\Delta \rho}$$

Elements of Acoustic Theory - Wave Equation

Similarly, for $\Delta\rho(x,t) = G(x + a_\infty t)$ (wave in negative x direction) we obtain:

$$\boxed{\Delta u = -\frac{a_\infty}{\rho_\infty}\Delta\rho}$$

Also, since $\Delta p = a_\infty^2 \Delta\rho$ we get:

Right going wave (+x direction) $\Delta u = \frac{a_\infty}{\rho_\infty}\Delta\rho = \frac{1}{a_\infty\rho_\infty}\Delta p$

Left going wave (-x direction) $\Delta u = -\frac{a_\infty}{\rho_\infty}\Delta\rho = -\frac{1}{a_\infty\rho_\infty}\Delta p$

Elements of Acoustic Theory - Wave Equation

Δu denotes **induced mass motion** and is positive in the positive x-direction

$$\Delta u = \pm \frac{a_\infty \Delta \rho}{\rho_\infty} = \pm \frac{\Delta p}{a_\infty \rho_\infty}$$

condensation (the part of the sound wave where $\Delta \rho > 0$):
Δu is always in the **same** direction as the wave motion

rarefaction (the part of the sound wave where $\Delta \rho < 0$):
Δu is always in the direction **opposite** to the wave motion

Elements of Acoustic Theory - Wave Equation *Summary*

Combining **linearized** continuity and the momentum equations we get

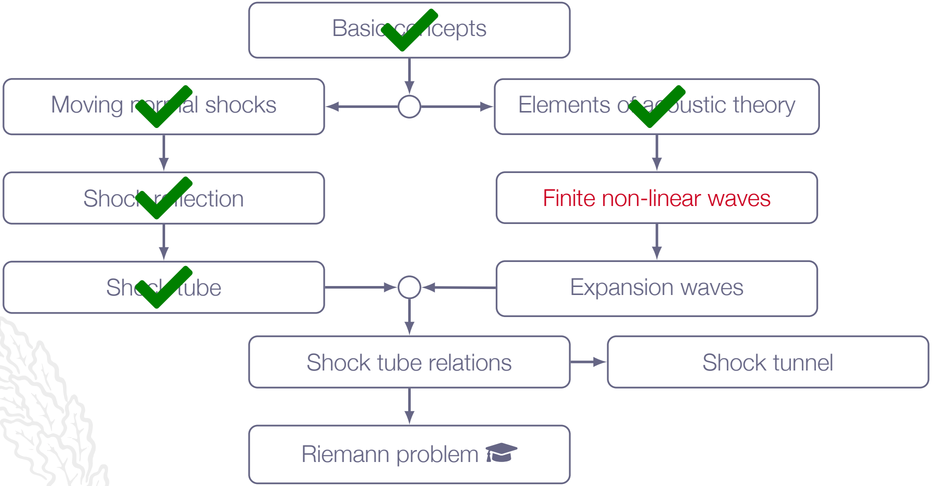
$$\frac{\partial^2}{\partial t^2}(\Delta\rho) = a_\infty^2 \frac{\partial^2}{\partial x^2}(\Delta\rho)$$

Due to the assumptions made, the **equation is not exact**

More and more accurate as the perturbations becomes smaller and smaller

So, how should we describe waves with larger amplitudes?

Roadmap - Unsteady Wave Motion



Chapter 7.6
Finite (Non-Linear) Waves

Finite (Non-Linear) Waves

When $\Delta\rho$, Δu, Δp, ... Become large, the **linearized acoustic equations become poor approximations**

Non-linear equations must be used

One-dimensional non-linear continuity and momentum equations:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

Finite (Non-Linear) Waves

We still assume isentropic flow, $ds = 0$

$$\frac{\partial \rho}{\partial t} = \left(\frac{\partial \rho}{\partial p}\right)_s \frac{\partial p}{\partial t} = \frac{1}{a^2}\frac{\partial p}{\partial t} \qquad \frac{\partial \rho}{\partial x} = \left(\frac{\partial \rho}{\partial p}\right)_s \frac{\partial p}{\partial x} = \frac{1}{a^2}\frac{\partial p}{\partial x}$$

Inserted in the continuity equation this gives:

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho a^2 \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

Finite (Non-Linear) Waves

Add $1/(\rho a)$ times the continuity equation to the momentum equation:

$$\left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right] + \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right] = 0$$

If we instead **subtract** $1/(\rho a)$ times the continuity equation from the momentum equation, we get:

$$\left[\frac{\partial u}{\partial t} + (u-a)\frac{\partial u}{\partial x}\right] - \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u-a)\frac{\partial p}{\partial x}\right] = 0$$

Finite (Non-Linear) Waves

Since $u = u(x, t)$, we have:

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}\frac{dx}{dt}dt$$

Let $\frac{dx}{dt} = u + a$ gives

$$du = \left[\frac{\partial u}{\partial t} + (u + a)\frac{\partial u}{\partial x}\right]dt$$

Interpretation: change of u in the direction of line $\frac{dx}{dt} = u + a$

Finite (Non-Linear) Waves

In the same way we get:

$$dp = \frac{\partial p}{\partial t}dt + \frac{\partial p}{\partial x}\frac{dx}{dt}dt$$

and thus

$$dp = \left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right]dt$$

Interpretation: change of p in the direction of line $\dfrac{dx}{dt} = u + a$

Finite (Non-Linear) Waves

Now, if we combine

$$\left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right] + \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right] = 0$$

$$du = \left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right] dt$$

$$dp = \left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right] dt$$

we get

$$\boxed{\frac{du}{dt} + \frac{1}{\rho a}\frac{dp}{dt} = 0}$$

Characteristic Lines

Thus, along a line $dx = (u + a)dt$ we have

$$du + \frac{dp}{\rho a} = 0$$

In the same way we get along a line where $dx = (u - a)dt$

$$du - \frac{dp}{\rho a} = 0$$

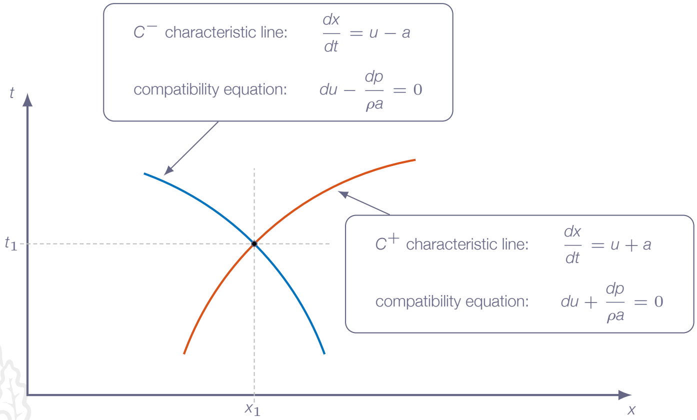
Characteristic Lines

We have found a path through a point (x, t) along which the governing partial differential equations reduces to ordinary differential equations

These paths or lines are called **characteristic lines**

The C^+ and C^- characteristic lines are physically the paths of **right- and left-running acoustic waves** in the *xt*-plane

Characteristic Lines



Characteristic Lines - Summary

$$\frac{du}{dt} + \frac{1}{\rho a}\frac{dp}{dt} = 0 \quad \text{along } C^+ \text{ characteristic}$$

$$\frac{du}{dt} - \frac{1}{\rho a}\frac{dp}{dt} = 0 \quad \text{along } C^- \text{ characteristic}$$

$$du + \frac{dp}{\rho a} = 0 \quad \text{along } C^+ \text{ characteristic}$$

$$du - \frac{dp}{\rho a} = 0 \quad \text{along } C^- \text{ characteristic}$$

Riemann Invariants

Integration gives:

$$J^+ = u + \int \frac{dp}{\rho a} = \text{constant along } C^+ \text{ characteristic}$$

$$J^- = u - \int \frac{dp}{\rho a} = \text{constant along } C^- \text{ characteristic}$$

We need to rewrite $\dfrac{dp}{\rho a}$ to be able to perform the integrations

Riemann Invariants

For an isentropic processes the **isentropic relations** give:

$$p = c_1 T^{\gamma/(\gamma-1)} = c_2 a^{2\gamma/(\gamma-1)}$$

where c_1 and c_2 are constants and thus

$$dp = c_2 \left(\frac{2\gamma}{\gamma-1}\right) a^{[2\gamma/(\gamma-1)-1]} da$$

Assume **calorically perfect gas**: $a^2 = \dfrac{\gamma p}{\rho} \Rightarrow \rho = \dfrac{\gamma p}{a^2}$

with $p = c_2 a^{2\gamma/(\gamma-1)}$ we get $\rho = c_2 \gamma a^{[2\gamma/(\gamma-1)-2]}$

Riemann Invariants

$$J^{+} = u + \int \frac{dp}{\rho a} = u + \int \frac{c_2 \left(\frac{2\gamma}{\gamma-1}\right) a^{[2\gamma/(\gamma-1)-1]}}{c_2 \gamma a^{[2\gamma/(\gamma-1)-1]}} da = u + \int \frac{2da}{\gamma - 1}$$

$$J^{+} = u + \frac{2a}{\gamma - 1}$$

$$J^{-} = u - \frac{2a}{\gamma - 1}$$

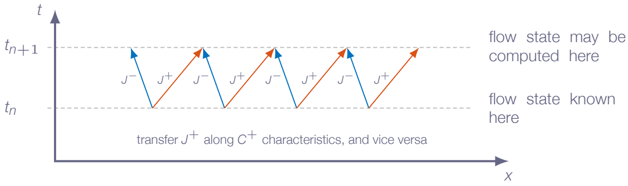
Riemann Invariants

If J^+ and J^- are known at some point $(x,\ t)$, then

$$\begin{cases} J^+ + J^- = 2u \\ \\ J^+ - J^- = \dfrac{4a}{\gamma - 1} \end{cases} \Rightarrow \begin{cases} u = \dfrac{1}{2}(J^+ + J^-) \\ \\ a = \dfrac{\gamma - 1}{4}(J^+ - J^-) \end{cases}$$

With the Riemann invariants known, the flow state is uniquely defined!

Method of Characteristics



Summary

Acoustic waves

1. $\Delta\rho$, Δu, etc - **very small**
2. All parts of the wave propagate with the same **velocity** a_∞
3. The **wave shape** stays the **same**
4. The flow is governed by **linear relations**

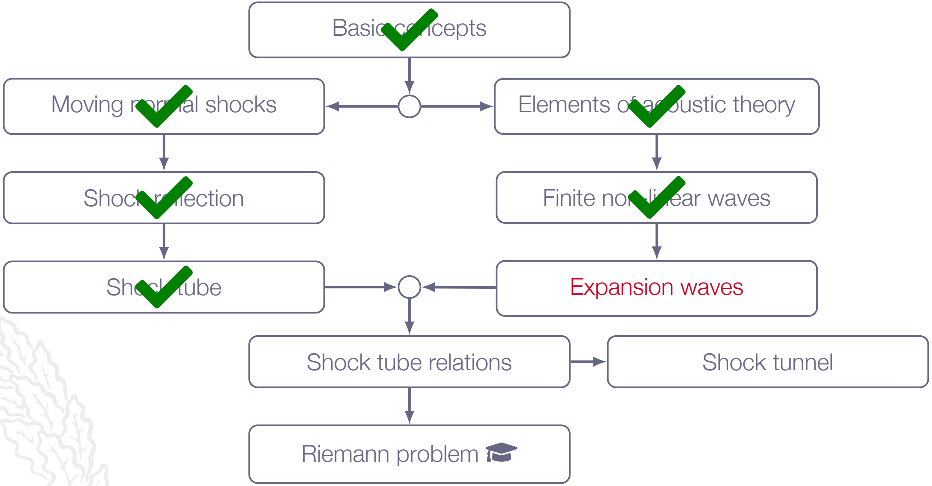
Finite (non-linear) waves

1. $\Delta\rho$, Δu, etc - can be **large**
2. Each local part of the wave propagates at the **local velocity** $(u + a)$
3. The wave **shape changes** with time
4. The flow is governed by **non-linear relations**

Summary

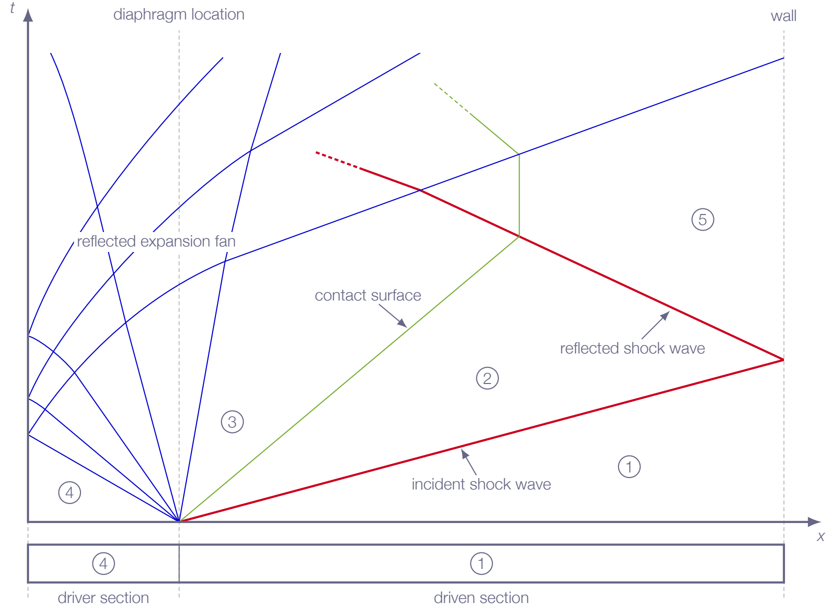
the method of characteristics is a central element in classic compressible flow theory

Roadmap - Unsteady Wave Motion



Chapter 7.7
Incident and Reflected Expansion Waves

Expansion Waves



Expansion Waves

Properties of a left-running expansion wave

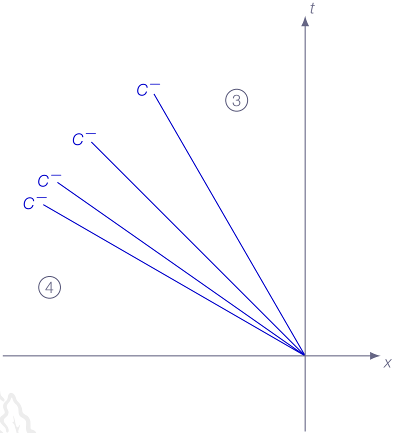
1. All flow properties are constant along C^- characteristics
2. The wave **head** is propagating **into region 4** (high pressure)
3. The wave **tail** defines the **limit of region 3** (lower pressure)
4. Regions 3 and 4 are assumed to be **constant states**

For calorically perfect gas:

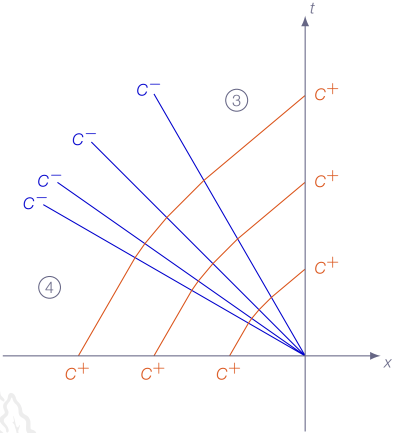
$$J^+ = u + \frac{2a}{\gamma - 1}$$ is constant along C^+ lines

$$J^- = u - \frac{2a}{\gamma - 1}$$ is constant along C^- lines

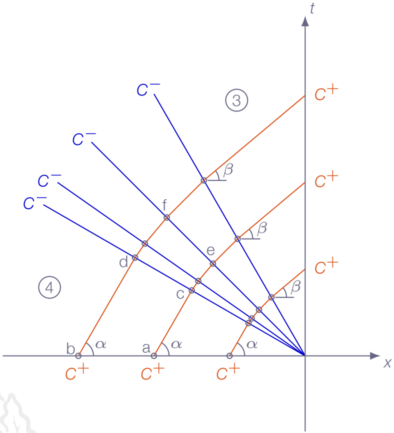
Expansion Waves



Expansion Waves



Expansion Waves



constant flow properties in region 4: $J_a^+ = J_b^+$

J^+ invariants constant along C^+ characteristics:

$$J_a^+ = J_c^+ = J_e^+$$

$$J_b^+ = J_d^+ = J_f^+$$

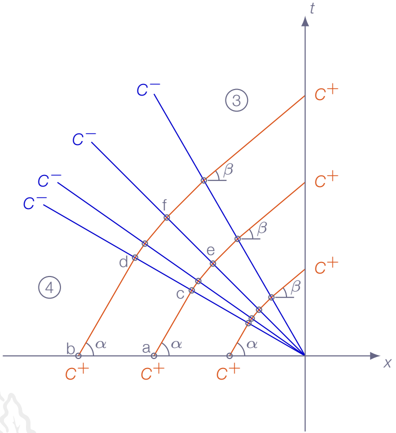
since $J_a^+ = J_b^+$ this also implies $J_e^+ = J_f^+$

J^- invariants constant along C^- characteristics:

$$J_c^- = J_d^-$$

$$J_e^- = J_f^-$$

Expansion Waves



constant flow properties in region 4: $J_a^+ = J_b^+$

J^+ invariants constant along C^+ characteristics:

$$J_a^+ = J_c^+ = J_e^+$$

$$J_b^+ = J_d^+ = J_f^+$$

since $J_a^+ = J_b^+$ this also implies $J_e^+ = J_f^+$

J^- invariants constant along C^- characteristics:

$$J_c^- = J_d^-$$

$$J_e^- = J_f^-$$

$$u_e = \frac{1}{2}(J_e^+ + J_e^-),\ u_f = \frac{1}{2}(J_f^+ + J_f^-),\ \Rightarrow u_e = u_f$$

$$a_e = \frac{\gamma - 1}{4}(J_e^+ - J_e^-),\ a_f = \frac{\gamma - 1}{4}(J_f^+ - J_f^-),\ \Rightarrow a_e = a_f$$

Expansion Waves

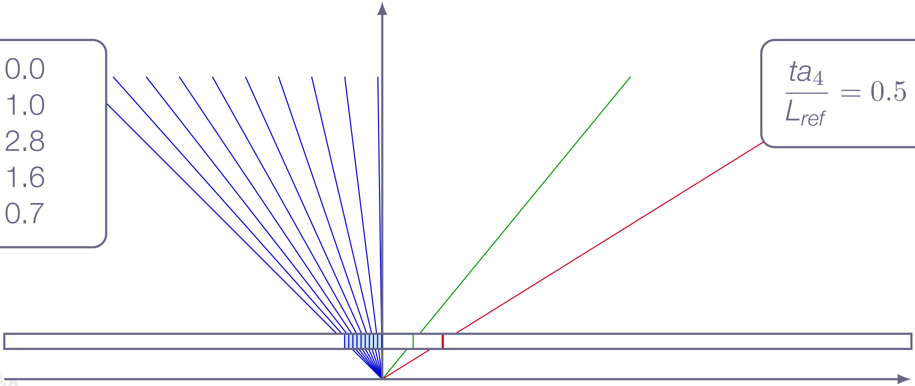
Along each C^- line u and a are **constants** which means that

$$\frac{dx}{dt} = u - a = const$$

C^- characteristics are **straight lines** in xt-space

Expansion Waves - Shock Tube

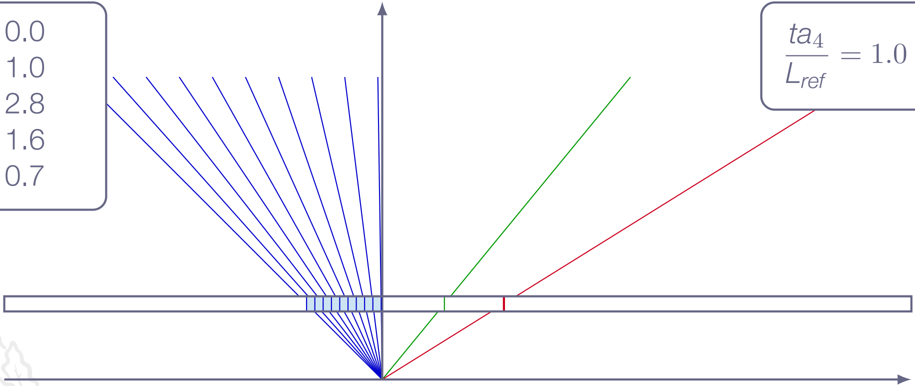
p_4/p_1	10.0
T_4/T_1	1.0
p_2/p_1	2.8
M_s	1.6
M_p	0.7



$$\frac{ta_4}{L_{ref}} = 0.5$$

Expansion Waves - Shock Tube

p_4/p_1	10.0
T_4/T_1	1.0
p_2/p_1	2.8
M_s	1.6
M_p	0.7

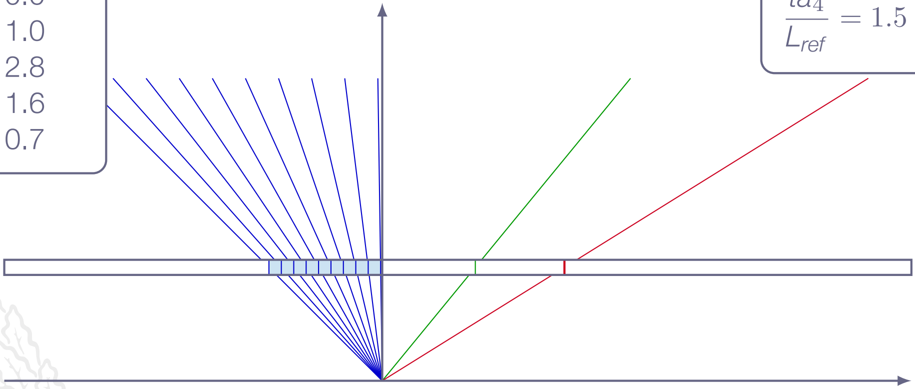


$$\frac{ta_4}{L_{ref}} = 1.0$$

Expansion Waves - Shock Tube

p_4/p_1	10.0
T_4/T_1	1.0
p_2/p_1	2.8
M_s	1.6
M_p	0.7

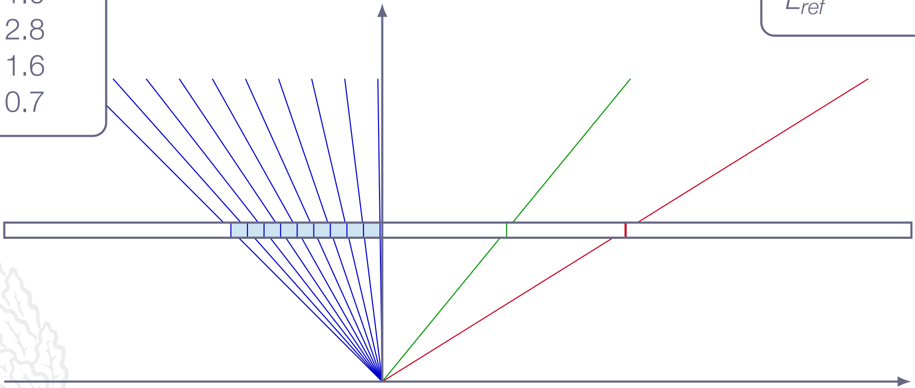
$$\frac{ta_4}{L_{ref}} = 1.5$$



Expansion Waves - Shock Tube

p_4/p_1	10.0
T_4/T_1	1.0
p_2/p_1	2.8
M_s	1.6
M_p	0.7

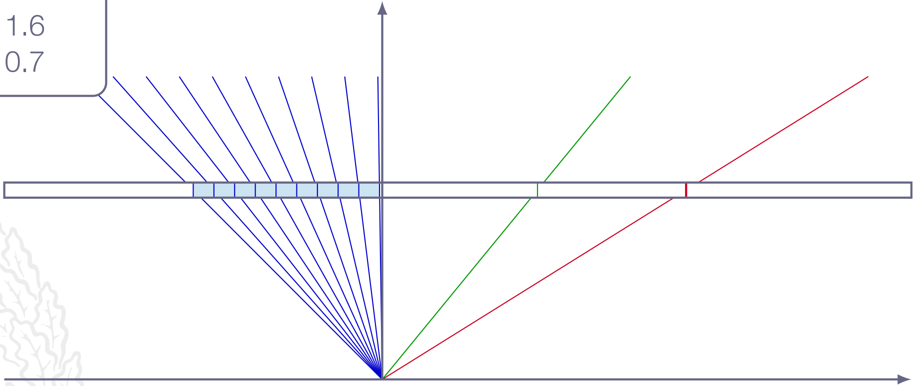
$$\frac{ta_4}{L_{ref}} = 2.0$$



Expansion Waves - Shock Tube

p_4/p_1	10.0
T_4/T_1	1.0
p_2/p_1	2.8
M_s	1.6
M_p	0.7

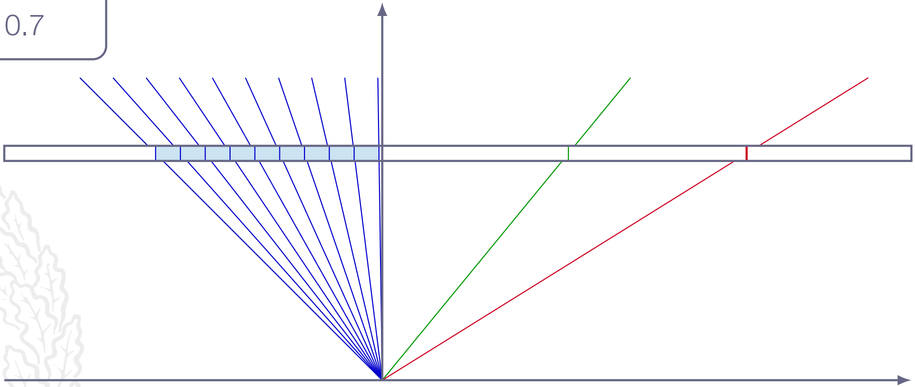
$$\frac{ta_4}{L_{ref}} = 2.5$$



Expansion Waves - Shock Tube

p_4/p_1	10.0
T_4/T_1	1.0
p_2/p_1	2.8
M_s	1.6
M_p	0.7

$$\frac{ta_4}{L_{ref}} = 3.0$$



Shock Tube Expansion Waves - Summary

The start and end conditions are the same for all C^+ lines

J^+ invariants have the same value for all C^+ characteristics

C^- characteristics are straight lines in xt-space

Simple expansion waves centered at $(x, t) = (0, 0)$

Expansion Waves

In a left-running expansion fan:

J^+ is constant throughout expansion fan, which implies:

$$u + \frac{2a}{\gamma - 1} = u_4 + \frac{2a_4}{\gamma - 1} = u_3 + \frac{2a_3}{\gamma - 1}$$

J^- is constant along C^- lines, but varies from one line to the next, which means that

$$u - \frac{2a}{\gamma - 1}$$

is constant along each C^- line

Expansion Waves

Since $u_4 = 0$ we obtain:

$$u + \frac{2a}{\gamma - 1} = u_4 + \frac{2a_4}{\gamma - 1} = \frac{2a_4}{\gamma - 1} \Rightarrow$$

$$\frac{a}{a_4} = 1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}$$

with $a = \sqrt{\gamma R T}$ we get

$$\frac{T}{T_4} = \left[1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}\right]^2$$

Expansion Wave Relations

Isentropic flow $\Rightarrow$ we can use the isentropic relations

complete description in terms of u/a_4

$$\frac{T}{T_4} = \left[1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}\right]^2$$

$$\frac{p}{p_4} = \left[1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}\right]^{\frac{2\gamma}{\gamma-1}}$$

$$\frac{\rho}{\rho_4} = \left[1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4}\right]^{\frac{2}{\gamma-1}}$$

Expansion Wave Relations

Since C^- characteristics are straight lines, we have:

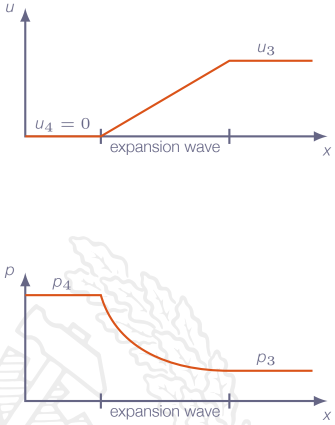
$$\frac{dx}{dt} = u - a \Rightarrow x = (u - a)t$$

$$\frac{a}{a_4} = 1 - \frac{1}{2}(\gamma - 1)\frac{u}{a_4} \Rightarrow a = a_4 - \frac{1}{2}(\gamma - 1)u \Rightarrow$$

$$x = \left[u - a_4 + \frac{1}{2}(\gamma - 1)u\right]t = \left[\frac{1}{2}(\gamma - 1)u - a_4\right]t \Rightarrow$$

$$u = \frac{2}{\gamma + 1}\left[a_4 + \frac{x}{t}\right]$$

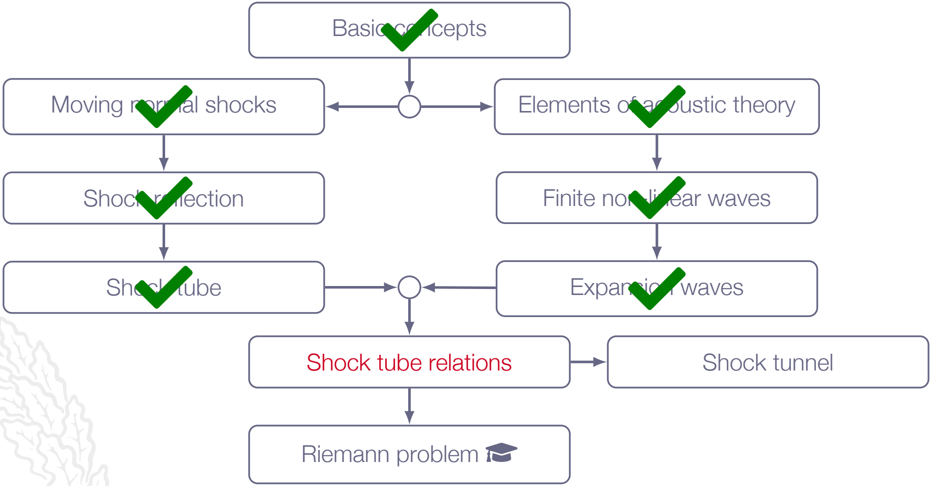
Expansion Wave Relations



Expansion wave head is advancing to the left with speed a_4 into the stagnant gas

Expansion wave tail is advancing with speed $u_3 - a_3$, which may be positive or negative, depending on the initial states

Roadmap - Unsteady Wave Motion



Chapter 7.8
Shock Tube Relations

Shock Tube Relations

$$u_p = u_2 = \frac{a_1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left[\frac{\dfrac{2\gamma_1}{\gamma_1+1}}{\dfrac{p_2}{p_1} + \dfrac{\gamma_1-1}{\gamma_1+1}}\right]^{1/2}$$

$$\frac{p_3}{p_4} = \left[1 - \frac{\gamma_4-1}{2}\left(\frac{u_3}{a_4}\right)\right]^{2\gamma_4/(\gamma_4-1)}$$

solving for u_3 gives

$$u_3 = \frac{2a_4}{\gamma_4-1}\left[1 - \left(\frac{p_3}{p_4}\right)^{(\gamma_4-1)/(2\gamma_4)}\right]$$

Shock Tube Relations

But, $p_3 = p_2$ and $u_3 = u_2$ (no change in velocity and pressure over contact discontinuity)

$$\Rightarrow u_2 = \frac{2a_4}{\gamma_4 - 1}\left[1 - \left(\frac{p_2}{p_4}\right)^{(\gamma_4-1)/(2\gamma_4)}\right]$$

We have now two expressions for u_2 which gives us

$$\frac{a_1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left[\frac{\dfrac{2\gamma_1}{\gamma_1+1}}{\dfrac{p_2}{p_1} + \dfrac{\gamma_1-1}{\gamma_1+1}}\right]^{1/2} = \frac{2a_4}{\gamma_4 - 1}\left[1 - \left(\frac{p_2}{p_4}\right)^{(\gamma_4-1)/(2\gamma_4)}\right]$$

Shock Tube Relations

Rearranging gives:

$$\frac{p_4}{p_1} = \frac{p_2}{p_1}\left\{1 - \frac{(\gamma_4 - 1)(a_1/a_4)(p_2/p_1 - 1)}{\sqrt{2\gamma_1\left[2\gamma_1 + (\gamma_1 + 1)(p_2/p_1 - 1)\right]}}\right\}^{-2\gamma_4/(\gamma_4 - 1)}$$

p_2/p_1 as implicit function of p_4/p_1

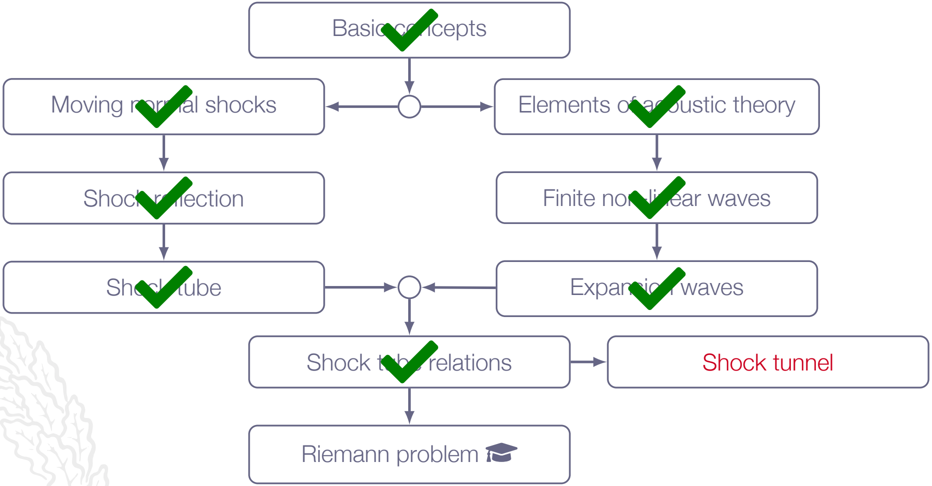
for a given p_4/p_1, p_2/p_1 will increase with decreased a_1/a_4

$$a = \sqrt{\gamma R T} = \sqrt{\gamma (R_u/M) T}$$

the speed of sound in a light gas is higher than in a heavy gas

- driver gas: low molecular weight, high temperature
- driven gas: high molecular weight, low temperature

Roadmap - Unsteady Wave Motion



Shock Tunnel

Addition of a convergent-divergent nozzle to a shock tube configuration

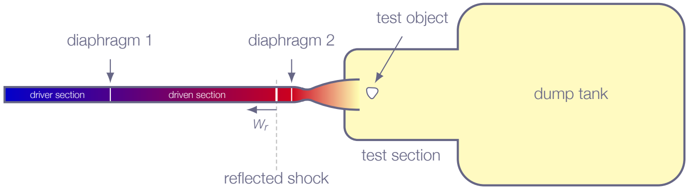
Capable of producing flow conditions which are close to those during the reentry of a space vehicles into the earth's atmosphere

- high-enthalpy, hypersonic flows (short time)
- real gas effects

Example - Aachen TH2:

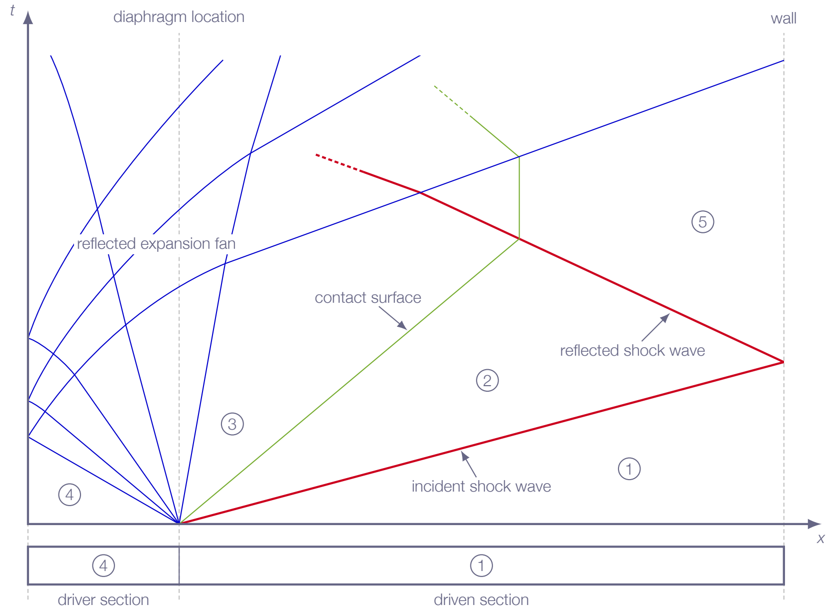
- velocities up to 4 km/s
- stagnation temperatures of several thousand degrees

Shock Tunnel



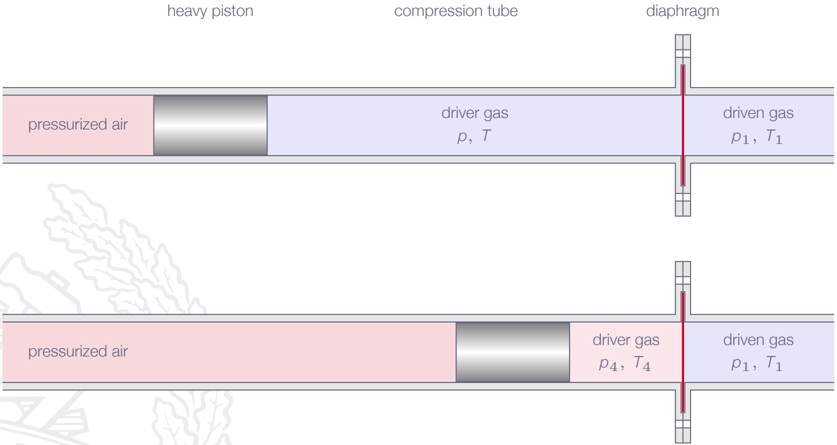
1. High pressure in region 4 (driver section)
   - diaphragm 1 burst
   - primary shock generated
2. Primary shock reaches end of shock tube
   - shock reflection
3. High pressure in region 5
   - diaphragm 2 burst
   - nozzle flow initiated
   - hypersonic flow in test section

Shock Tunnel

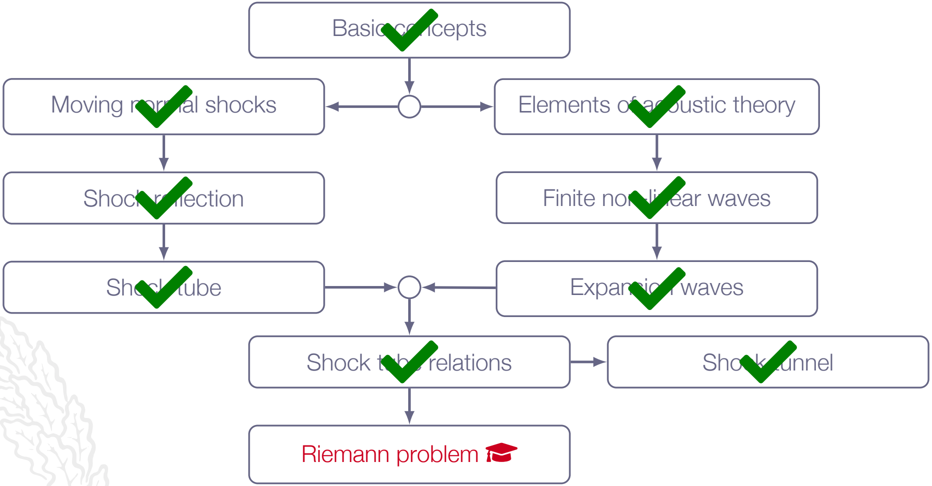


Shock Tunnel

By adding a compression tube to the shock tube a very high p_4 and T_4 may be achieved for any gas in a fairly simple manner



Roadmap - Unsteady Wave Motion



Riemann Problem

The shock tube problem is a special case of the general **Riemann Problem**

> *"... A Riemann problem, named after Bernhard Riemann, consists of an initial value problem composed by a conservation equation together with piece-wise constant data having a single discontinuity ..."*
>
> Wikipedia

Riemann Problem

May show that solutions to the shock tube problem have the general form:

$$
\begin{aligned}
p &= p(x/t) \\
\rho &= \rho(x/t) \\
u &= u(x/t) \\
T &= T(x/t) \\
a &= a(x/t)
\end{aligned}
$$

where $x = 0$ denotes the position of the initial jump between states 1 and 4

Riemann Problem - Shock Tube Simulation

Numerical method:

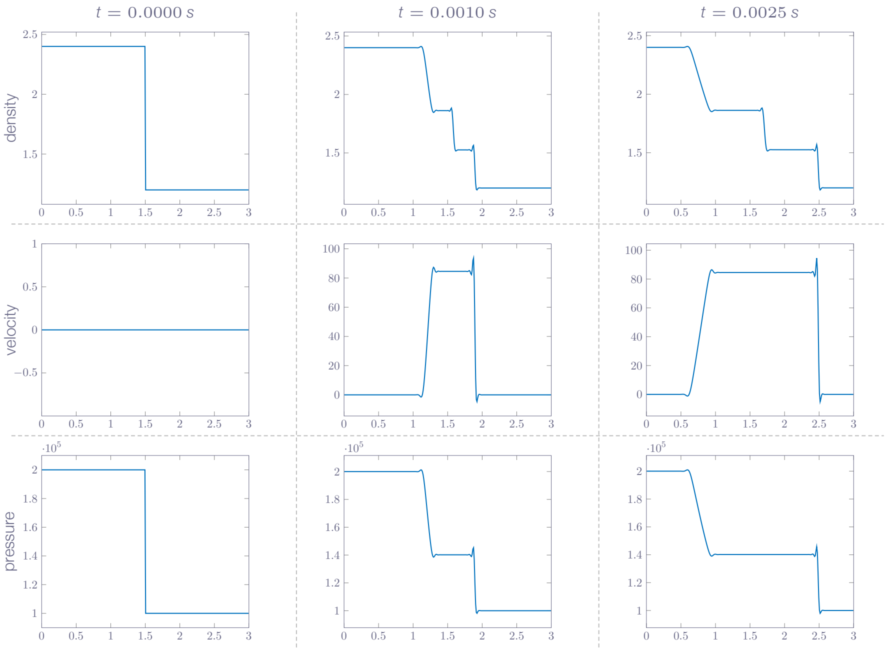
- Finite-Volume Method (FVM) solver
- three-stage Runge-Kutta time stepping
- third-order characteristic upwinding scheme
- local artificial damping

Left side conditions (state 4):

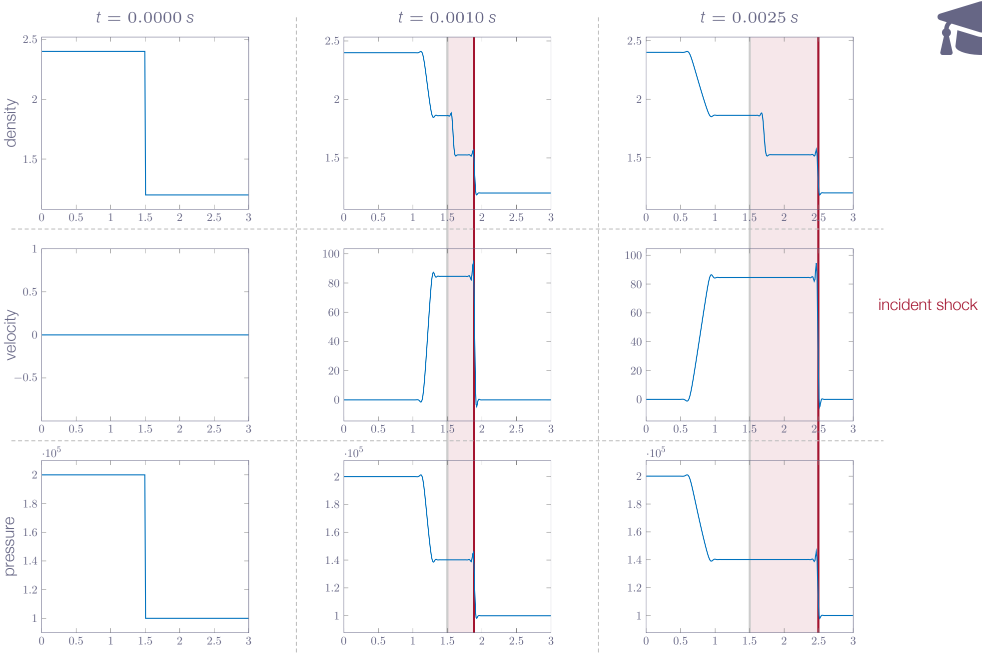
$$\rho = 2.4\ kg/m^3$$
$$u = 0.0\ m/s$$
$$p = 2.0\ bar$$

Right side conditions (state 1):

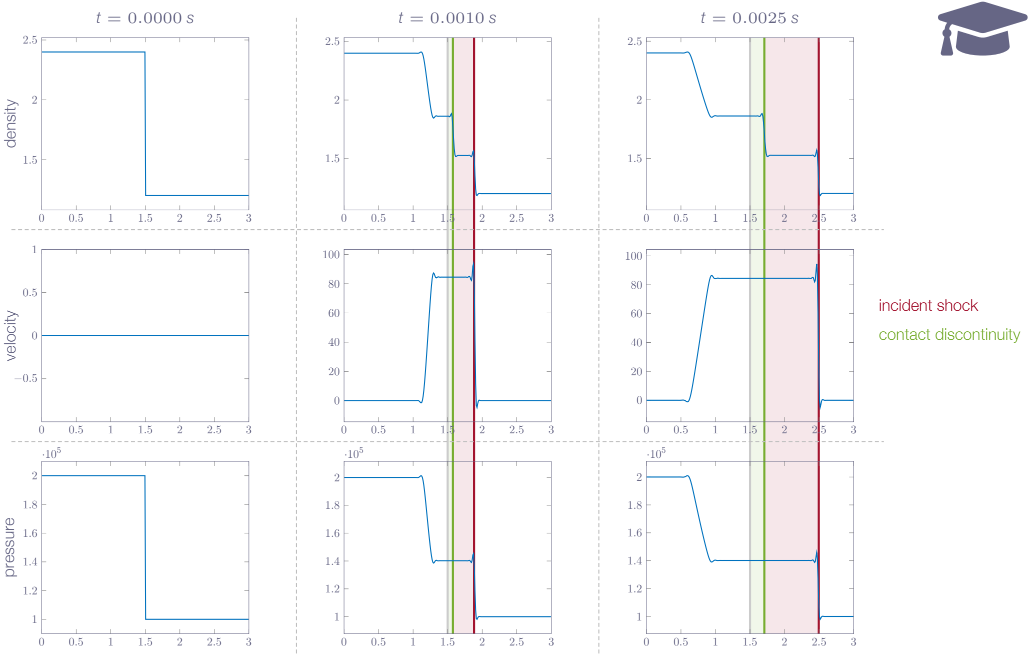
$$\rho = 1.2\ kg/m^3$$
$$u = 0.0\ m/s$$
$$p = 1.0\ bar$$



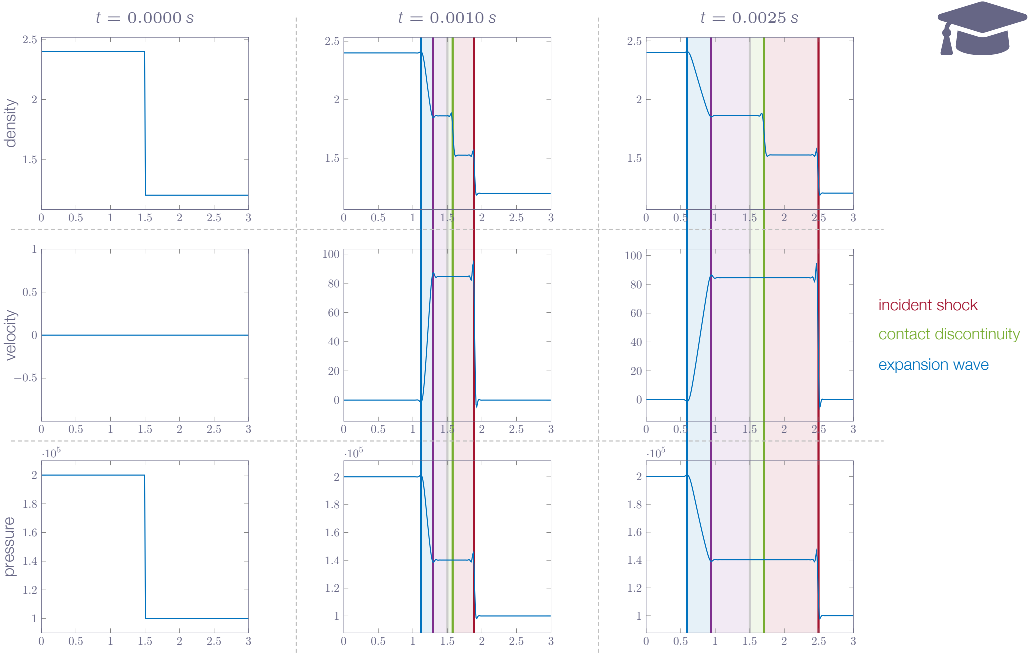
$t = 0.0000\,s$
$t = 0.0010\,s$
$t = 0.0025\,s$
density
velocity
pressure
$\cdot 10^5$
$\cdot 10^5$
$\cdot 10^5$



t = 0.0000 s
t = 0.0010 s
t = 0.0025 s
density
velocity
pressure
·10^5
incident shock

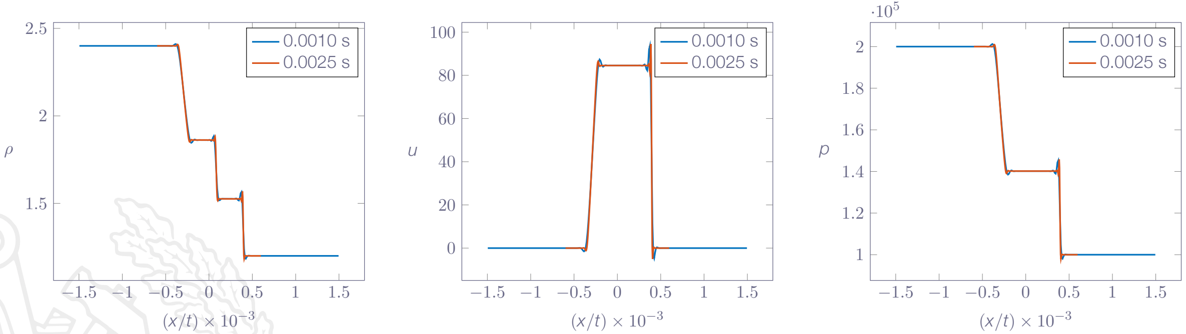


t = 0.0000 s
t = 0.0010 s
t = 0.0025 s
density
velocity
pressure
·10^5
incident shock
contact discontinuity



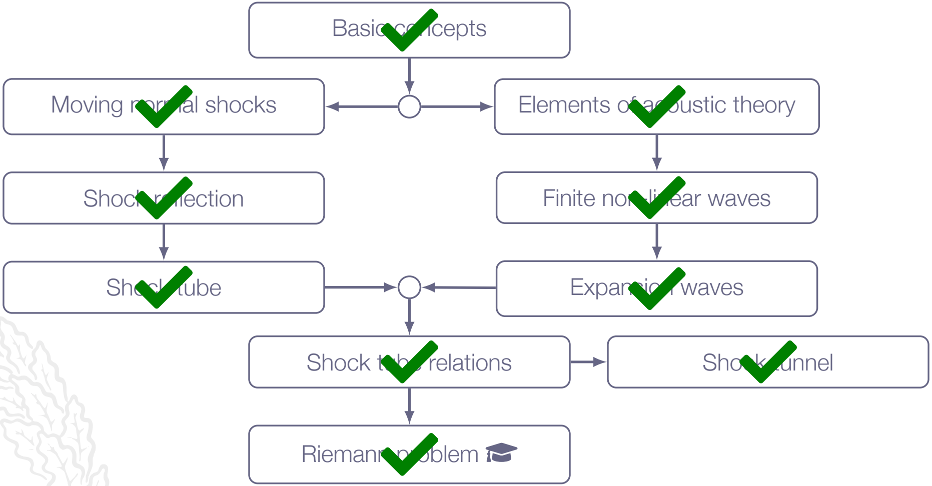
$t = 0.0000\ s$
$t = 0.0010\ s$
$t = 0.0025\ s$
density
velocity
pressure
$\cdot 10^5$
incident shock
contact discontinuity
expansion wave

Riemann Problem - Shock Tube Simulation



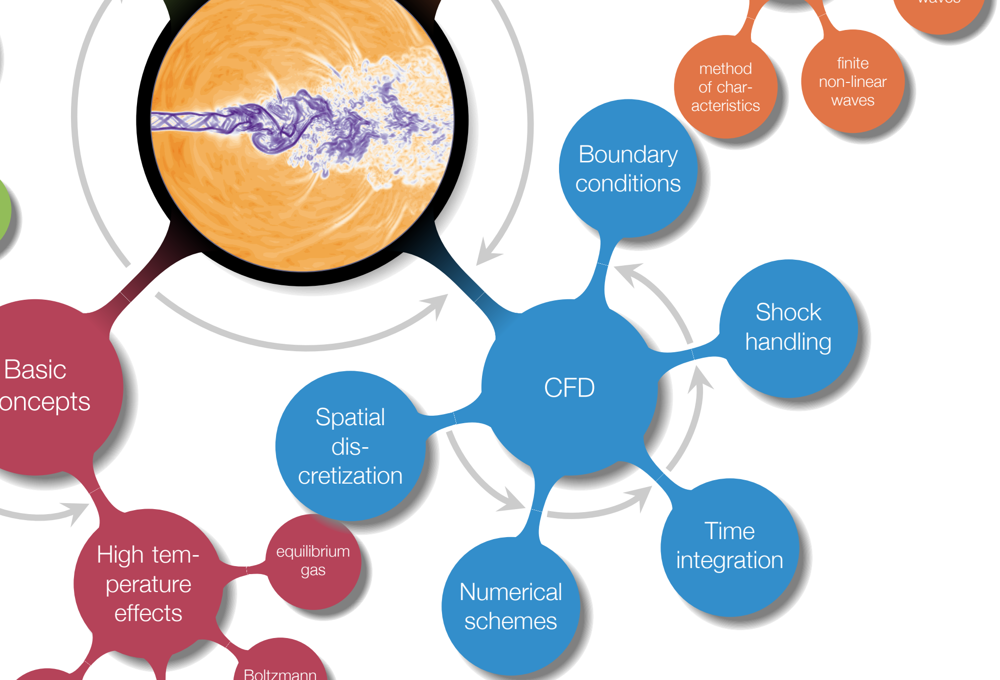
The solution can be made self similar by plotting the flow field variables as function of x/t

Roadmap - Unsteady Wave Motion



Chapter 12
The Time-Marching Technique

Overview



method of char-acteristics
finite non-linear waves
Boundary conditions
Shock handling
CFD
Spatial dis-cretization
Time integration
Numerical schemes
Basic
oncepts
High tem-perature effects
equilibrium gas
Boltzmann

Learning Outcomes

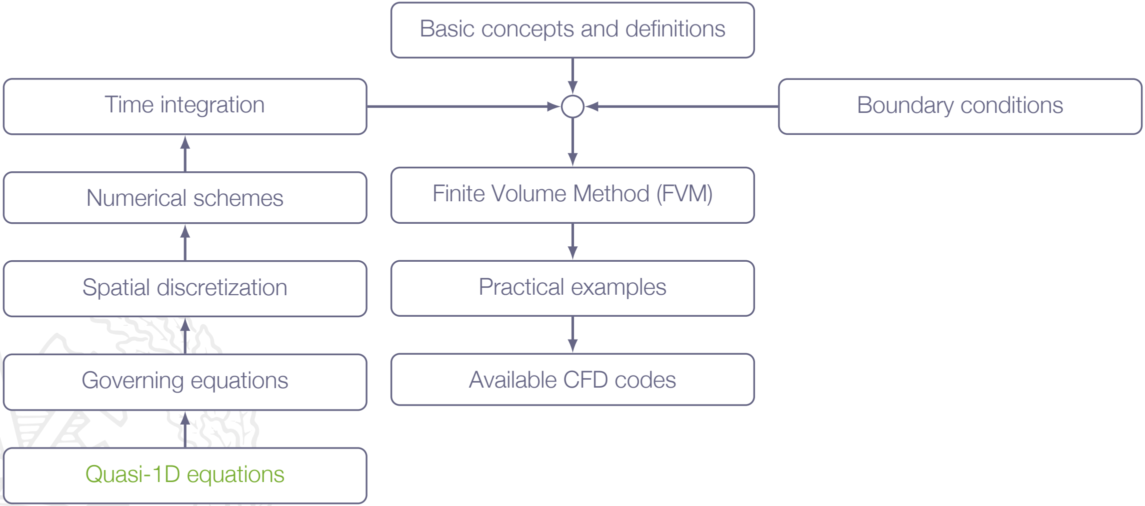
12 **Explain** the main principles behind a modern Finite Volume CFD code and such concepts as explicit/implicit time stepping, CFL number, conservation, handling of compression shocks, and boundary conditions

14 **Analyze** and **verify** the quality of the numerical solution

15 **Explain** the limitations in fluid flow simulation software

time for CFD!

Roadmap - The Time-Marching Technique



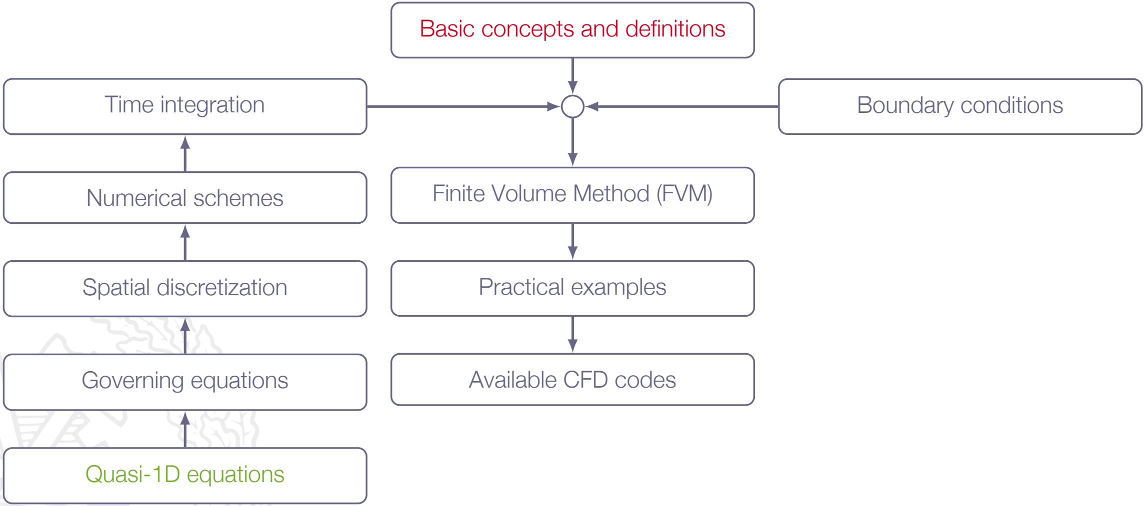
Motivation

Computational Fluid Dynamics (CFD) is the backbone of all practical engineering compressible flow analysis

As an engineer doing numerical compressible flow analyzes it is extremely important to have knowledge about the fundamental numerical principles and their **limitations**

Going through the material covered in this section will not make you understand all the details but you will get a feeling, which is a good start

Roadmap - The Time-Marching Technique



The Time-Marching Technique

The problems that we like to investigate numerically within the field of compressible flows can be categorized as

steady-state compressible flows

unsteady compressible flows

The **Time-marching technique** is a solver framework that addresses both problem categories

The Time-Marching Technique

Steady-state problems:

1. define simple initial solution
2. apply specified boundary conditions
3. march in time until steady-state solution is reached

Unsteady problems:

1. apply specified initial solution
2. apply specified boundary conditions
3. march in time for specified total time to reach a desired unsteady solution

   establish fully developed flow before initiating data sampling

The Time-Marching Technique

The time-marching approach is a good alternative for simulating flows where there are both supersonic and subsonic regions

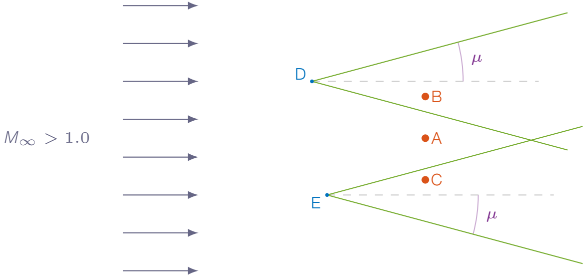
supersonic/hyperbolic:

- perturbations propagate in preferred directions
- zone of influence/zone of dependence
- PDEs can be transformed into ODEs

subsonic/elliptic:

- perturbations propagate in all directions

Zone of Influence and Zone of Dependence



A, B and C at the same axial position in the flow

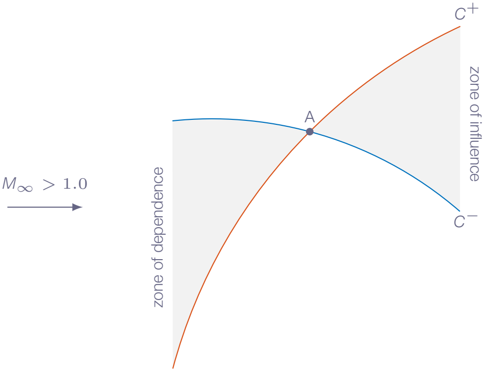
D and E are located upstream of A, B and C

Mach waves generated at D will affect the flow in B but not in A and C

Mach waves generated at E will affect the flow in C but not in A and B

The flow in A is unaffected by the both D and E

Zone of Influence and Zone of Dependence



The **zone of dependence** for point A and the **zone of influence** of point A are defined by C^+ and C^- characteristic lines

Characterization of CFD Methods

Density-based	**Pressure-based**
Fully coupled	**Segregated**
Structured	**Unstructured**
Explicit	**Implicit**

Characterization of CFD Methods

Approach taken in this presentation

Approach taken in this presentation	
Density-based	Pressure-based
Fully coupled	Segregated
Structured	Unstructured
Explicit	Implicit

Characterization of CFD Methods - Equations

Density-based

solve for density in the continuity equation

suitable for transonic/supersonic flows

Pressure-based

the continuity and momentum equations are combined to form a pressure correction equation

suitable for subsonic/transonic flows

Characterization of CFD Methods - Solver Approach

Fully coupled

all equations (continuity, momentum, energy, ...) are solved simultaneously

suitable for transonic/supersonic flows

Segregated

the governing equations are solved in sequence

suitable for subsonic flows

Characterization of CFD Methods - Time Stepping

Explicit

- short time steps

\+ very stable

Implicit

\+ longer time steps possible

Characterization of CFD Methods - Time Stepping

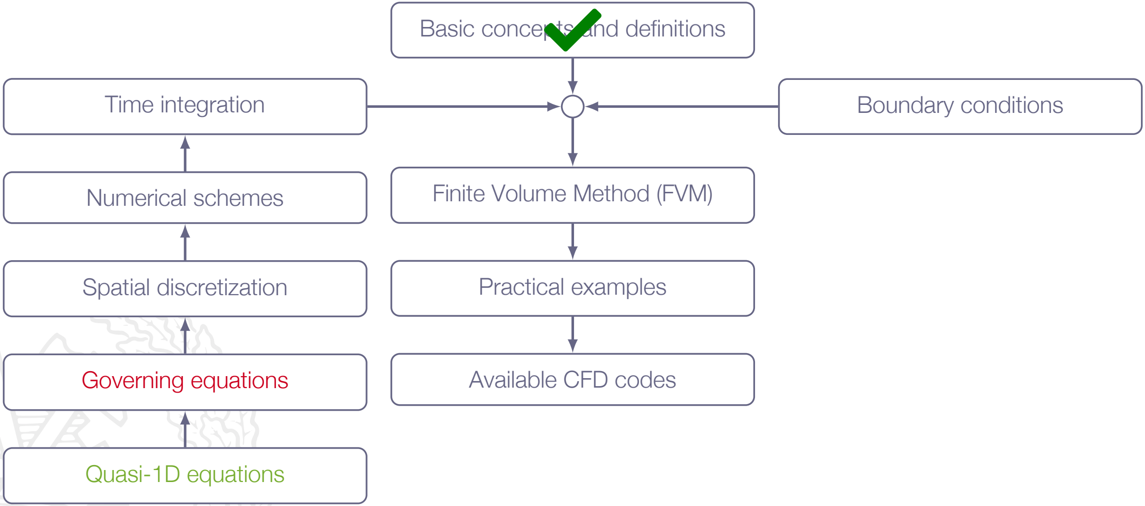
Explicit Time Stepping

Implicit Time Stepping

In general implicit solvers are more efficient than explicit solvers

For high-supersonic flows, explicit solvers may very well outperform implicit solvers

Roadmap - The Time-Marching Technique

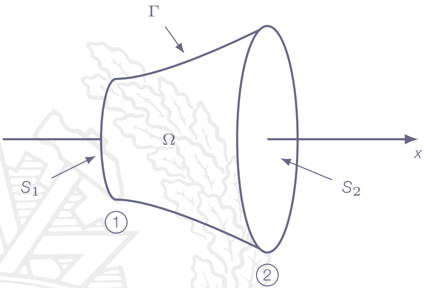


Governing Equations

Quasi-One-Dimensional Flow - Conceptual Idea

Introduce **cross-section-averaged flow quantities** $\Rightarrow$
all quantities depend on x only

$$A = A(x),\ \rho = \rho(x),\ u = u(x),\ p = p(x),\ \dots$$



Ω	control volume
S_1	left boundary (area A_1)
S_2	right boundary (area A_2)
Γ	perimeter boundary

$$\partial\Omega = S_1 \cup \Gamma \cup S_2$$

Governing equations (general form):

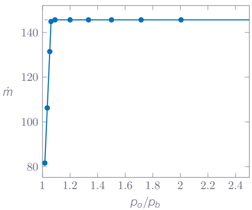
$$\frac{d}{dt}\iiint_{\Omega} \rho d\mathscr{V} + \oint\!\!\!\oint_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS = 0$$

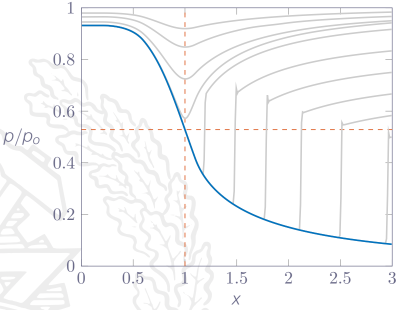
$$\frac{d}{dt}\iiint_{\Omega} \rho u d\mathscr{V} + \oint\!\!\!\oint_{\partial\Omega} \left[\rho(\mathbf{v} \cdot \mathbf{n})u + p(\mathbf{n} \cdot \mathbf{e}_x)\right] dS = 0$$

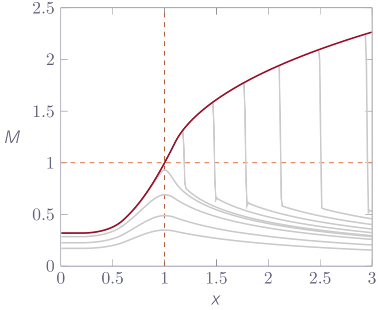
$$\frac{d}{dt}\iiint_{\Omega} \rho e_o d\mathscr{V} + \oint\!\!\!\oint_{\partial\Omega} \rho h_o(\mathbf{v} \cdot \mathbf{n}) dS = 0$$

Quasi-One-Dimensional Flow - Example: Nozzle Flow

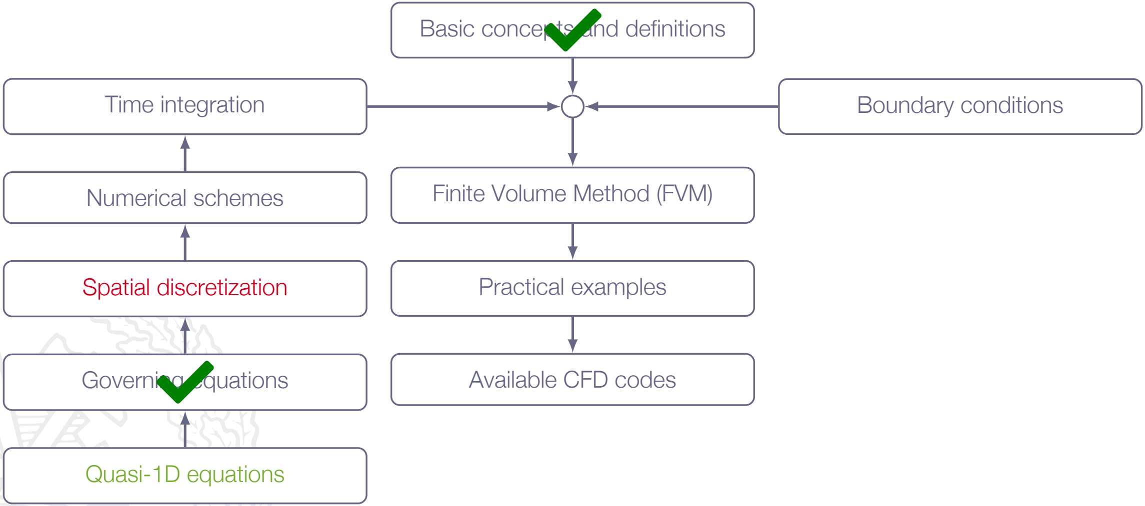
p_o	1.20 [bar]
p_b	0.50 [bar]
p_o/p_b	11.8
$\dot{m}$	145.6 [kg/s]
M_{max}	2.26







Roadmap - The Time-Marching Technique



Spatial Discretization

Spatial Discretization

Discretization in space and time:

Method of Lines (a very common approach):

1. discretize in space $\Rightarrow$ system of ordinary differential equations (ODEs)
2. discretize in time $\Rightarrow$ time-stepping scheme for system of ODEs

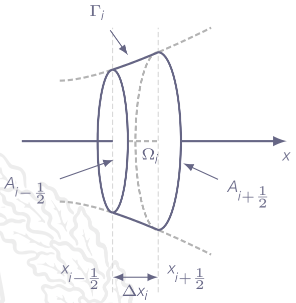
Spatial discretization techniques:

FDM Finite-Difference Method

FVM **Finite-Volume Method**

FEM Finite-Element Method

Let's look at a small tube segment with length Δx



Streamtube with area $A(x)$

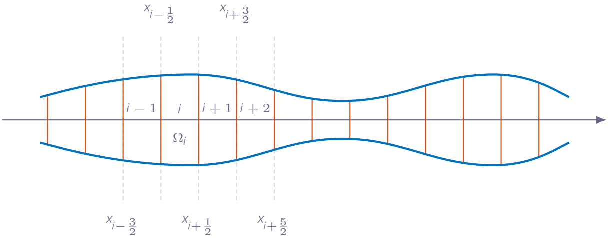
$$A_{i-\frac{1}{2}} = A(x_{i-\frac{1}{2}})$$

$$A_{i+\frac{1}{2}} = A(x_{i+\frac{1}{2}})$$

$$\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$$

Ω_i - control volume enclosed by $A_{i-\frac{1}{2}}$, $A_{i+\frac{1}{2}}$, and Γ_i

$\Rightarrow$ **spatial discretization**

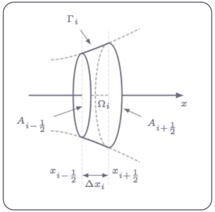


Integer indices: control volumes or **cells**

Fractional indices: interfaces between control volumes or **cell faces**

Apply control volume formulations for mass, momentum, energy to control volume Ω_i

Quasi-One-Dimensional Flow - Spatial Discretization



cell-averaged quantity

face-averaged quantity

Conservation of mass:

$$\underbrace{\frac{d}{dt}\iiint_{\Omega_i}\rho d\mathscr{V}}_{VOL_i \frac{d}{dt}\bar{\rho}_i} + \underbrace{\iint_{x_{i-\frac{1}{2}}}\rho\mathbf{v}\cdot\mathbf{n}dS}_{-\overline{(\rho u)}_{i-\frac{1}{2}}A_{i-\frac{1}{2}}} + \underbrace{\iint_{x_{i+\frac{1}{2}}}\rho\mathbf{v}\cdot\mathbf{n}dS}_{\overline{(\rho u)}_{i+\frac{1}{2}}A_{i+\frac{1}{2}}} + \underbrace{\iint_{\Gamma_i}\rho\mathbf{v}\cdot\mathbf{n}dS}_{0} = 0$$

where

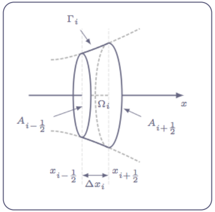
$$VOL_i = \iiint_{\Omega_i} d\mathscr{V} \qquad \overline{(\rho u)}_{i-\frac{1}{2}} = \frac{1}{A_{i-\frac{1}{2}}}\iint_{x_{i-\frac{1}{2}}}\rho u dS$$

$$\bar{\rho}_i = \frac{1}{VOL_i}\iiint_{\Omega_i}\rho d\mathscr{V} \qquad \overline{(\rho u)}_{i+\frac{1}{2}} = \frac{1}{A_{i+\frac{1}{2}}}\iint_{x_{i+\frac{1}{2}}}\rho u dS$$

CFLOW

Quasi-One-Dimensional Flow - Spatial Discretization

cell-averaged quantity
face-averaged quantity
source term



Conservation of momentum:

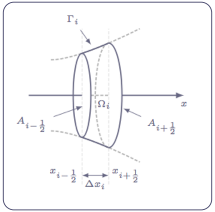
$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega_i} \rho u d\mathscr{V}}_{VOL_i \frac{d}{dt}\overline{(\rho u)}_i} + \underbrace{\iint\limits_{x_{i-\frac{1}{2}}} \left[\rho(\mathbf{v}\cdot\mathbf{n})u + p(\mathbf{n}\cdot\mathbf{e}_x)\right] dS}_{-\overline{(\rho u^2+p)}_{i-\frac{1}{2}}A_{i-\frac{1}{2}}} +$$

$$+ \underbrace{\iint\limits_{x_{i+\frac{1}{2}}} \left[\rho(\mathbf{v}\cdot\mathbf{n})u + p(\mathbf{n}\cdot\mathbf{e}_x)\right] dS}_{\overline{(\rho u^2+p)}_{i+\frac{1}{2}}A_{i+\frac{1}{2}}} + \underbrace{\iint\limits_{\Gamma_i} \left[\rho(\mathbf{v}\cdot\mathbf{n})u + p(\mathbf{n}\cdot\mathbf{e}_x)\right] dS}_{-\iint_{\Gamma_i} p dA} = 0$$

Quasi-One-Dimensional Flow - Spatial Discretization

cell-averaged quantity

face-averaged quantity

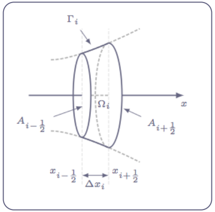


Conservation of energy:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega_i} \rho e_o d\mathscr{V}}_{VOL_i \frac{d}{dt}\overline{(\rho e_o)}_i} + \underbrace{\iint\limits_{x_{i-\frac{1}{2}}} \rho h_o(\mathbf{v}\cdot\mathbf{n})dS}_{-\overline{(\rho u h_o)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}}} +$$

$$+ \underbrace{\iint\limits_{x_{i+\frac{1}{2}}} \rho h_o(\mathbf{v}\cdot\mathbf{n})dS}_{\overline{(\rho u h_o)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}}} + \underbrace{\iint\limits_{\Gamma_i} \rho h_o(\mathbf{v}\cdot\mathbf{n})dS}_{0} = 0$$

Quasi-One-Dimensional Flow - Spatial Discretization



Lower order term due to varying stream tube area:

$$\iint\limits_{\Gamma_i} p dA \approx \bar{p}_i \left(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}} \right)$$

where $\bar{p}_i$ is **calculated from cell-averaged quantities** (DOFs) $\left\{ \bar{\rho}, \overline{(\rho u)}, \overline{(\rho e_o)} \right\}_i$ as

$$\bar{p}_i = (\gamma - 1) \left(\overline{(\rho e_o)}_i - \frac{1}{2} \bar{\rho}_i \bar{u}_i^2 \right), \ \bar{u}_i = \frac{\overline{(\rho u)}_i}{\bar{\rho}_i}$$

cell-averaged quantity
face-averaged quantity
source term

$$VOL_i \frac{d}{dt} \bar{\rho}_i - \overline{(\rho u)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} + \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} = 0$$

$$VOL_i \frac{d}{dt} \overline{(\rho u)}_i - \overline{(\rho u^2 + p)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} + \overline{(\rho u^2 + p)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} = \bar{p}_i \left(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}} \right)$$

$$VOL_i \frac{d}{dt} \overline{(\rho e_o)}_i - \overline{(\rho u h_o)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} + \overline{(\rho u h_o)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} = 0$$

Application of these equations to all cells $i \in \{1, 2, ..., N\}$ of the computational domain results in a system of ODEs

Spatial Discretization - Summary

Steps to achieve spatial discretization:

1. Choose primary variables (degrees of freedom)
2. Approximate all other quantities in terms of the primary variables

$\Rightarrow$ **System of ordinary differential equations** (ODEs)

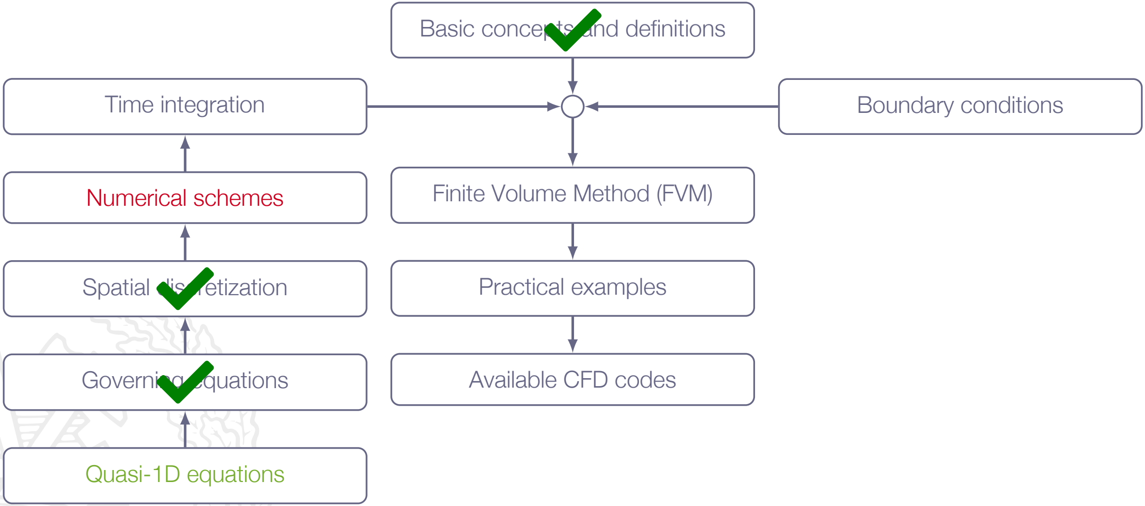
Degrees of freedom:

Choose $\left\{\bar{\rho}, \overline{(\rho u)}, \overline{(\rho e_o)}\right\}_i$ in all control volumes Ω_i, $i \in \{1, 2, ..., N\}$ as degrees of freedom, or primary variables

Note that these are **cell-averaged quantities**

What about the face values?

Roadmap - The Time-Marching Technique



Numerical Schemes

Flux Term Approximation

$$\left\{\begin{array}{c}\overline{(\rho u)}\\ \\ \overline{(\rho u^2+p)}\\ \\ \overline{(\rho u h_o)}\end{array}\right\}_{i+\frac{1}{2}} = f\left(\left\{\begin{array}{c}\overline{\rho}\\ \\ \overline{(\rho u)}\\ \\ \overline{(\rho e_o)}\end{array}\right\}_{i},\left\{\begin{array}{c}\overline{\rho}\\ \\ \overline{(\rho u)}\\ \\ \overline{(\rho e_o)}\end{array}\right\}_{i+1},\ldots\right)$$

cell face values

cell-averaged values

Simple example:

$$\overline{(\rho u)}_{i+\frac{1}{2}} \approx \frac{1}{2}\left[\overline{(\rho u)}_i + \overline{(\rho u)}_{i+1}\right]$$

Flux Term Approximation

More complex approximations usually needed

High-order schemes:

- increased accuracy
- more cell values involved (*wider flux molecule*)
- boundary conditions more difficult to implement

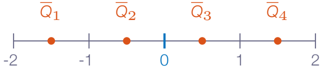
Optimized numerical dissipation:

- upwind type of flux scheme

Shock handling:

- non-linear treatment needed (*e.g.* TVD schemes)
- artificial damping

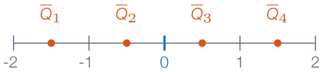
Flux Term Approximation



$$Q(x) = A + Bx + Cx^2 + Dx^3$$

Assume constant area: $A(x) = 1.0$

Flux Term Approximation

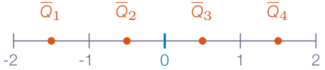


$$\overline{Q}_1 = \frac{1}{VOL_1}\int_{-2}^{-1} Q(x)dx$$

$$VOL_1 = A_1\Delta x_1 = \{A_1 = 1.0,\ \Delta x_1 = 1.0\} = 1.0$$

$$\Rightarrow \overline{Q}_1 = \int_{-2}^{-1} Q(x)dx$$

Flux Term Approximation



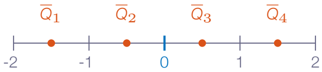
$$\overline{Q}_1 = \int_{-2}^{-1} Q(x)dx = \left[Ax + \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 + \frac{1}{4}Dx^4\right]_{-2}^{-1}$$

$$\overline{Q}_2 = \int_{-1}^{0} Q(x)dx = \left[Ax + \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 + \frac{1}{4}Dx^4\right]_{-1}^{0}$$

$$\overline{Q}_3 = \int_{0}^{1} Q(x)dx = \left[Ax + \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 + \frac{1}{4}Dx^4\right]_{0}^{1}$$

$$\overline{Q}_4 = \int_{1}^{2} Q(x)dx = \left[Ax + \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 + \frac{1}{4}Dx^4\right]_{1}^{2}$$

Flux Term Approximation



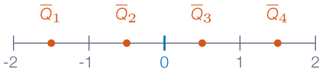
$$\overline{Q}_1 = A - \frac{3}{2}B + \frac{7}{3}C - \frac{15}{4}D$$

$$\overline{Q}_2 = A - \frac{1}{2}B + \frac{1}{3}C - \frac{1}{4}D$$

$$\overline{Q}_3 = A + \frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D$$

$$\overline{Q}_4 = A + \frac{3}{2}B + \frac{7}{3}C + \frac{15}{4}D$$

Flux Term Approximation



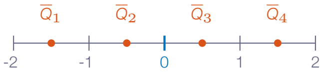
$$A = \frac{1}{12}\left[-\overline{Q}_1 + 7\overline{Q}_2 + 7\overline{Q}_3 - \overline{Q}_4\right]$$

$$B = \frac{1}{12}\left[\overline{Q}_1 - 15\overline{Q}_2 + 15\overline{Q}_3 - \overline{Q}_4\right]$$

$$C = \frac{1}{4}\left[\overline{Q}_1 - \overline{Q}_2 - \overline{Q}_3 + \overline{Q}_4\right]$$

$$D = \frac{1}{6}\left[-\overline{Q}_1 + 3\overline{Q}_2 - 3\overline{Q}_3 + \overline{Q}_4\right]$$

Flux Term Approximation



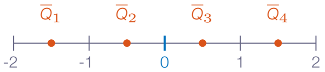
$$Q_0 = Q(0) + \delta Q'''(0) \Rightarrow Q_0 = A + 6\delta D$$

$\delta = 0 \Rightarrow$ fourth-order central scheme

$\delta = 1/12 \Rightarrow$ third-order upwind scheme

$\delta = 1/96 \Rightarrow$ third-order low-dissipation upwind scheme

Flux Term Approximation



$$Q_0 = A + 6\delta D = \{\delta = 1/12\} = -\frac{1}{6}\overline{Q}_1 + \frac{5}{6}\overline{Q}_2 + \frac{1}{3}\overline{Q}_3$$

$$Q_{0_{left}} = -\frac{1}{6}\overline{Q}_1 + \frac{5}{6}\overline{Q}_2 + \frac{1}{3}\overline{Q}_3$$

$$Q_{0_{right}} = -\frac{1}{6}\overline{Q}_4 + \frac{5}{6}\overline{Q}_3 + \frac{1}{3}\overline{Q}_2$$

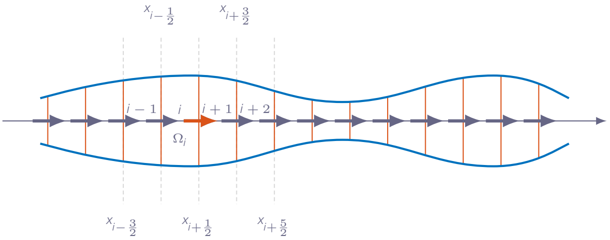
method of characteristics used in order to decide whether left- or right-upwinded flow quantities should be used

Flux Term Approximation

High-order numerical schemes:

- low numerical dissipation (smearing due to amplitudes errors)
- low dispersion errors (wiggles due to phase errors)

Conservative Scheme



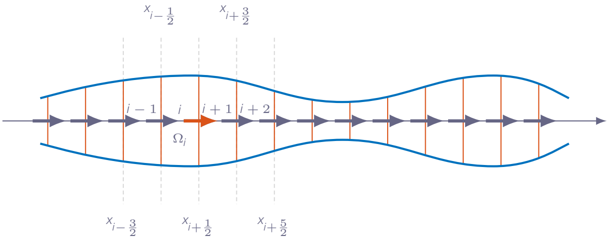
mass conservation:

cell (i): $$VOL_i \frac{d}{dt}\bar{\rho}_i + \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} - \overline{(\rho u)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} = 0$$

cell $(i+1)$: $$VOL_{i+1} \frac{d}{dt}\bar{\rho}_{i+1} + \overline{(\rho u)}_{i+\frac{3}{2}} A_{i+\frac{3}{2}} - \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} = 0$$

(similarly for momentum and energy conservation)

Conservative Scheme



mass conservation:

cell (i): $$VOL_i \frac{d}{dt}\bar{\rho}_i + \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} - \overline{(\rho u)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} = 0$$

cell $(i+1)$: $$VOL_{i+1} \frac{d}{dt}\bar{\rho}_{i+1} + \overline{(\rho u)}_{i+\frac{3}{2}} A_{i+\frac{3}{2}} - \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} = 0$$

(similarly for momentum and energy conservation)

Conservative scheme

"The flux leaving one control volume equals the flux entering neighbouring control volume"

Conservation of for mass, momentum and energy is crucial for the correct prediction of shocks*

* correct prediction of shocks:
strength
position
velocity

Shock Capturing

Jameson shock detector:

$$\nu_{i+\frac{1}{2}} = max\left\{\nu_i, \nu_{i+1}\right\}$$

where ν_i is a scaled pressure derivative

$$\nu_i = \frac{|p_{i+1} - 2p_i + p_{i-1}|}{p_{i+1} + 2p_i + p_{i-1}}$$

For a smooth pressure field ν $\mathcal{O}(\Delta x^2)$ and near a shock ν $\mathcal{O}(1)$

Artificial damping term (α is a user-defined constant):

$$\alpha\left(|u| + c\right)_{i+\frac{1}{2}} \nu_{i+\frac{1}{2}} A_{i+\frac{1}{2}} \left(Q_{i+1} - Q_i\right)$$

Density Discontinuities

Jameson-type detector:

$$\nu_{i+\frac{1}{2}} = max\left\{\nu_i, \nu_{i+1}\right\}$$

where ν_i is a scaled density derivative

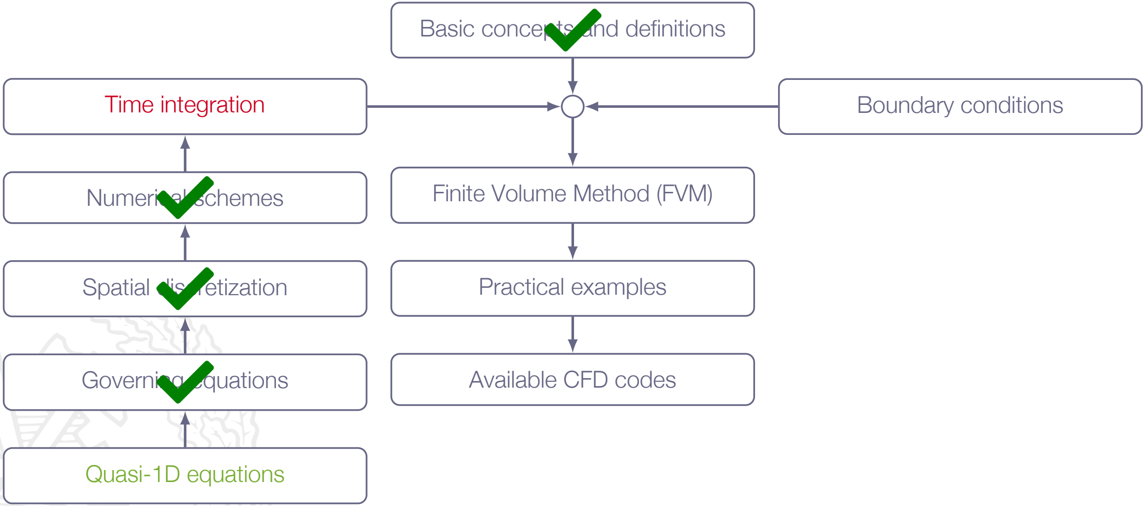
$$\nu_i = \frac{|\rho_{i+1} - 2\rho_i + \rho_{i-1}|}{\rho_{i+1} + 2\rho_i + \rho_{i-1}}$$

For a smooth density field ν $\mathcal{O}(\Delta x^2)$ and near a density discontinuity ν $\mathcal{O}(1)$

Artificial damping term (β is a user-defined constant):

$$\beta \left|u\right|_{i+\frac{1}{2}} \nu_{i+\frac{1}{2}} A_{i+\frac{1}{2}} \left(Q_{i+1} - Q_i\right)$$

Roadmap - The Time-Marching Technique



Time Stepping

Quasi-One-Dimensional Flow - Spatial Discretization

cell-averaged quantity
face-averaged quantity
source term

$$VOL_i \frac{d}{dt} \bar{\rho}_i - \overline{(\rho u)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} + \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} = 0$$

$$VOL_i \frac{d}{dt} \overline{(\rho u)}_i - \overline{(\rho u^2 + p)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} + \overline{(\rho u^2 + p)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} = \bar{p}_i \left(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}} \right)$$

$$VOL_i \frac{d}{dt} \overline{(\rho e_o)}_i - \overline{(\rho u h_o)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} + \overline{(\rho u h_o)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} = 0$$

Application of these equations to all cells $i \in \{1, 2, ..., N\}$ of the computational domain results in a system of ODEs

CFLOW

cell-averaged quantity
face-averaged quantity
source term

$$VOL_i \frac{d}{dt}\bar{\rho}_i = \overline{(\rho u)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} - \overline{(\rho u)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}}$$

$$VOL_i \frac{d}{dt}\overline{(\rho u)}_i = \overline{(\rho u^2 + p)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} - \overline{(\rho u^2 + p)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}} + \bar{p}_i \left(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}\right)$$

$$VOL_i \frac{d}{dt}\overline{(\rho e_o)}_i = \overline{(\rho u h_o)}_{i-\frac{1}{2}} A_{i-\frac{1}{2}} - \overline{(\rho u h_o)}_{i+\frac{1}{2}} A_{i+\frac{1}{2}}$$

Quasi-One-Dimensional Flow - Spatial Discretization

cell-averaged quantity
face-averaged quantity
source term

$$\frac{d}{dt}\bar{\rho}_i = \frac{1}{VOL_i}\left[\overline{(\rho u)}_{i-\frac{1}{2}}A_{i-\frac{1}{2}} - \overline{(\rho u)}_{i+\frac{1}{2}}A_{i+\frac{1}{2}}\right]$$

$$\frac{d}{dt}\overline{(\rho u)}_i = \frac{1}{VOL_i}\left[\overline{(\rho u^2+p)}_{i-\frac{1}{2}}A_{i-\frac{1}{2}} - \overline{(\rho u^2+p)}_{i+\frac{1}{2}}A_{i+\frac{1}{2}} + \bar{p}_i\left(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}\right)\right]$$

$$\frac{d}{dt}\overline{(\rho e_o)}_i = \frac{1}{VOL_i}\left[\overline{(\rho u h_o)}_{i-\frac{1}{2}}A_{i-\frac{1}{2}} - \overline{(\rho u h_o)}_{i+\frac{1}{2}}A_{i+\frac{1}{2}}\right]$$

$$\frac{d}{dt}\overline{\mathbf{Q}}_i = \mathbf{F}(\overline{\mathbf{Q}}_i) \text{ where } \overline{\mathbf{Q}}_i = [\overline{\rho},\ \overline{\rho u},\ \overline{\rho e_o}]_i,\ i \in \{1 : NCells\}$$

Time Stepping

The system of ODEs obtained from the spatial discretization in vector notation

$$\frac{d}{dt}\mathbf{Q} = \mathbf{F}(\mathbf{Q})$$

$\mathbf{Q}$ is a vector containing all DOFs in all cells

$\mathbf{F}(\mathbf{Q})$ is the **time derivative** of $\mathbf{Q}$ resulting from above mentioned **flux approximations** - *non-linear vector-valued function*

Three-stage Runge-Kutta - *one example of many*:

- **Explicit** time-marching scheme
- **Second-order** accurate

Time Stepping - Three-stage Runge-Kutta

$$\frac{d}{dt}\mathbf{Q} = \mathbf{F}(\mathbf{Q})$$

Let $\mathbf{Q}^n = \mathbf{Q}(t_n)$ and $\mathbf{Q}^{n+1} = \mathbf{Q}(t_{n+1})$

t_n is the current time level and t_{n+1} is the next time level
$\Delta t = t_{n+1} - t_n$ is the solver time step

Algorithm:

1. $\mathbf{Q}^{*} = \mathbf{Q}^n + \Delta t \mathbf{F}(\mathbf{Q}^n)$
2. $\mathbf{Q}^{**} = \mathbf{Q}^n + \frac{1}{2}\Delta t \mathbf{F}(\mathbf{Q}^n) + \frac{1}{2}\Delta t \mathbf{F}(\mathbf{Q}^{*})$
3. $\mathbf{Q}^{n+1} = \mathbf{Q}^n + \frac{1}{2}\Delta t \mathbf{F}(\mathbf{Q}^n) + \frac{1}{2}\Delta t \mathbf{F}(\mathbf{Q}^{**})$

Time Stepping - Three-stage Runge-Kutta

```
void RungeKutta::fwd(Domain *dom){
  G3DCopy(dom->cons,cons0);

  /* Runge-Kutta step 1 */

  dom->update();
  if(!G3DMode::constdt){LocalTimeStep(dom);}
  dcons->evaluate(dom);
  G3DWAXPY(dom->cons,1.0,dcons,cons0);
  G3DAXPBY(cons0,0.5,0.5,dom->cons);

  /* Runge-Kutta step 2 */

  dom->update();
  dcons->evaluate(dom);
  G3DWAXPY(dom->cons,0.5,dcons,cons0);

  /* Runge-Kutta step 3 */

  dom->update();
  dcons->evaluate(dom);
  G3DWAXPY(dom->cons,0.5,dcons,cons0);
}
```

Time Stepping - Explicit Schemes

Properties of explicit time-stepping schemes:

+ **Easy to implement** in computer codes
+ **Efficient execution** on most computers
+ Easy to adapt for **parallel execution** on distributed memory systems (*e.g.* Linux clusters)

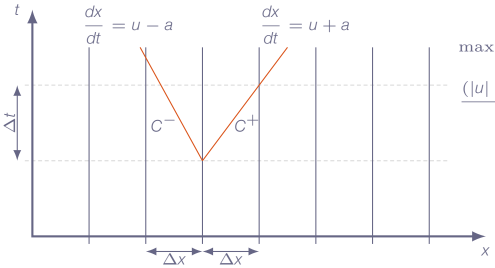
- **Time step limitation** (CFL number)
- Convergence to steady-state **often slow** (there are, however, some remedies for this)

Courant-Friedrich-Levy **(CFL)** number - *one-dimensional case*:

$$CFL_i = \frac{\Delta t(|u_i| + a_i)}{\Delta x_i} \leq 1$$

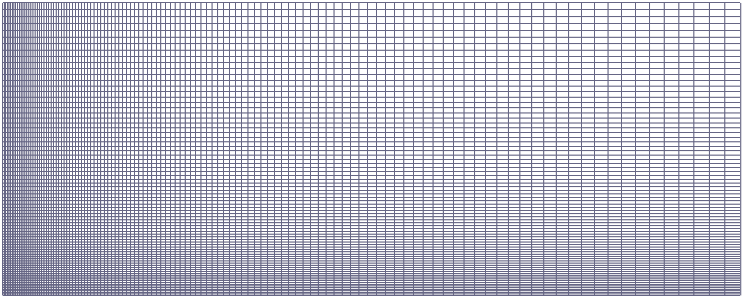
Interpretation: The fastest characteristic (C^+ or C^-) must not travel longer than Δx during one time step

Time Stepping - Explicit Schemes



$$\max(|u-a|, |u+a|)\Delta t = (|u|+a)\Delta t \leq \Delta x \Rightarrow$$

$$\frac{(|u|+a)\Delta t}{\Delta x} = CFL \leq 1$$



Steady-state problems:

local time stepping

each cell has an individual time step

Δt_i maximum allowed value based on CFL criteria

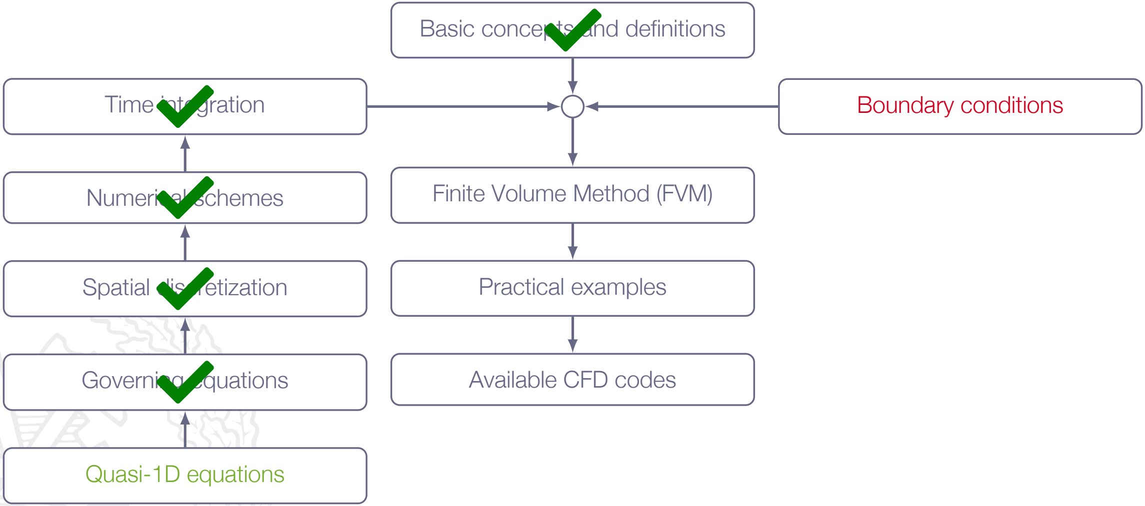
Unsteady problems:

time accurate

all cells have the same time step

$\Delta t_i = \min\{\Delta t_1, ..., \Delta t_N\}$

Roadmap - The Time-Marching Technique



Boundary Conditions

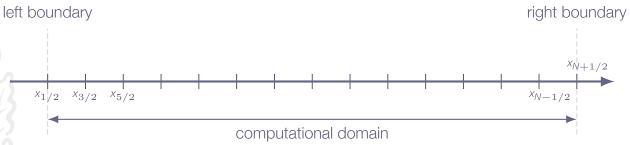
Boundary Conditions

Boundary conditions are very important for numerical simulation of compressible flows

Main reason: both **flow** and **acoustics** involved!

Example 1:
Finite-volume CFD code for Quasi-1D compressible flow (Time-marching procedure)

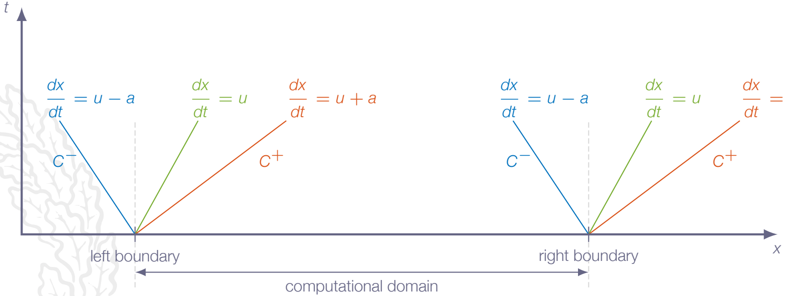
What boundary conditions should be applied at the left and right ends?



Boundary Conditions

three characteristics:

1. C^+
2. C^-
3. advection



Boundary Conditions

C^+ and C^- characteristics describe the transport of **isentropic pressure waves** (often referred to as **acoustics**)

The advection characteristic simply describes the **transport** of certain quantities **with the fluid itself** (for example **entropy**)

In one space dimension and time, these three characteristics, together with the quantities that are known to be constant along them, give a **complete description** of the time evolution of the flow

We can use the characteristics as a guide to tell us what information that should be specifed at the boundaries

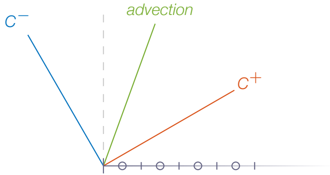
Left Boundary - Subsonic Inflow

we have three PDEs, and are solving for three unknowns

Subsonic inflow: $0 < u < a$

$$u - a < 0$$
$$u > 0$$
$$u + a > 0$$

one outgoing characteristic
two ingoing characteristics



Two variables should be **specified** at the boundary

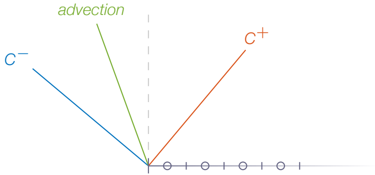
The third variable must be left free

Left Boundary - Subsonic Outflow

we have three PDEs, and are solving for three unknowns

Subsonic outflow: $-a < u < 0$

$$u - a < 0$$
$$u < 0$$
$$u + a > 0$$



two outgoing characteristics
one ingoing characteristic

One variable should be **specified** at the boundary

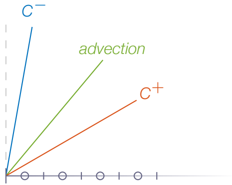
The second and third variables must be left free

Left Boundary - Supersonic Inflow

we have three PDEs, and are solving for three unknowns

Supersonic inflow: $u > a$

$$u - a > 0$$
$$u > 0$$
$$u + a > 0$$



no outgoing characteristics
three ingoing characteristics

All three variables should be **specified** at the boundary

No variables must be left free

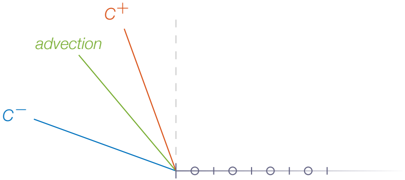
Left Boundary - Supersonic Outflow

we have three PDEs, and are solving for three unknowns

Supersonic outflow: $u < -a$

$$u - a < 0$$
$$u < 0$$
$$u + a < 0$$

three outgoing characteristics
no ingoing characteristics



No variables should be **specified** at the boundary

All variables must be left free

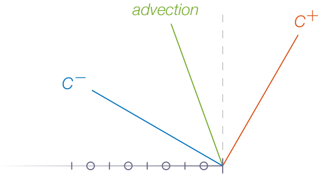
Right Boundary - Subsonic Inflow

we have three PDEs, and are solving for three unknowns

Subsonic inflow: $-a < u < 0$

$$u - a < 0$$
$$u < 0$$
$$u + a > 0$$

two ingoing characteristics
one outgoing characteristic



Two variables should be **specified** at the boundary

The third variables must be left free

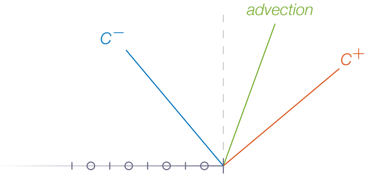
Right Boundary - Subsonic Outflow

we have three PDEs, and are solving for three unknowns

Subsonic outflow: $0 < u < a$

$$u - a < 0$$
$$u > 0$$
$$u + a > 0$$

one ingoing characteristic
two outgoing characteristics



One variable should be **specified** at the boundary

The second and third variables must be left free

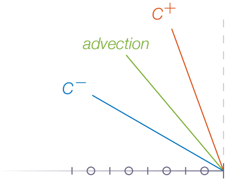
Right Boundary - Supersonic Inflow

we have three PDEs, and are solving for three unknowns

Supersonic inflow: $u < -a$

$$u - a < 0$$
$$u < 0$$
$$u + a < 0$$

three ingoing characteristics
no outgoing characteristics



All three variables should be **specified** at the boundary

No variables must be left free

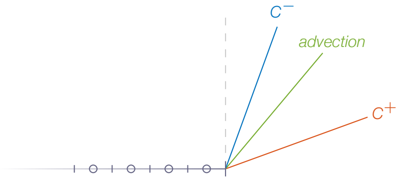
Right Boundary - Supersonic Outflow

we have three PDEs, and are solving for three unknowns

Supersonic outflow: $u > a$

$$u - a > 0$$
$$u > 0$$
$$u + a > 0$$

no ingoing characteristics
three outgoing characteristics



No variables should be **specified** at the boundary

All three variables must be left free

1D Boundary Conditions (Summary)

Characteristic		1D subsonic inflow (left)	1D subsonic inflow (right)
advection	$\mathbf{v}\cdot\mathbf{n}$	$(u,0,0)\cdot(-1,0,0)=-u<0$	$(-u,0,0)\cdot(1,0,0)=-u<0$
C^-	$\mathbf{v}\cdot\mathbf{n}-a$	$-u-a<0$	$-u-a<0$
C^+	$\mathbf{v}\cdot\mathbf{n}+a$	$-u+a>0$	$-u+a>0$
Characteristic		**1D subsonic outflow (left)**	**1D subsonic outflow (right)**
advection	$\mathbf{v}\cdot\mathbf{n}$	$(-u,0,0)\cdot(-1,0,0)=u>0$	$(u,0,0)\cdot(1,0,0)=u>0$
C^-	$\mathbf{v}\cdot\mathbf{n}-a$	$u-a<0$	$u-a<0$
C^+	$\mathbf{v}\cdot\mathbf{n}+a$	$u+a>0$	$u+a>0$
Characteristic		**1D supersonic inflow (left)**	**1D supersonic inflow (right)**
advection	$\mathbf{v}\cdot\mathbf{n}$	$(u,0,0)\cdot(-1,0,0)=-u<0$	$(-u,0,0)\cdot(1,0,0)=-u<0$
C^-	$\mathbf{v}\cdot\mathbf{n}-a$	$-u-a<0$	$-u-a<0$
C^+	$\mathbf{v}\cdot\mathbf{n}+a$	$-u+a<0$	$-u+a<0$
Characteristic		**1D supersonic outflow (left)**	**1D supersonic outflow (right)**
advection	$\mathbf{v}\cdot\mathbf{n}$	$(-u,0,0)\cdot(-1,0,0)=u>0$	$(u,0,0)\cdot(1,0,0)=u>0$
C^-	$\mathbf{v}\cdot\mathbf{n}-a$	$u-a>0$	$u-a>0$
C^+	$\mathbf{v}\cdot\mathbf{n}+a$	$u+a>0$	$u+a>0$

Subsonic Inflow (Left Boundary) - Example

Subsonic inflow: we should specify two variables

Alt	specified variable 1	specified variable 2	well-posed	non-reflective
1	p_o	T_o	X	
2	ρu	T_o	X	
3	s	J^+	X	X

well posed:

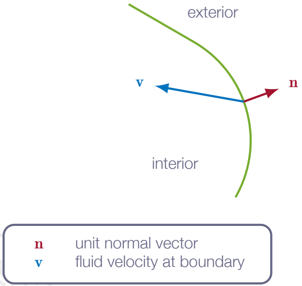
1. the problem has a solution
2. the solution is unique
3. the solution's behaviour changes continuously with initial conditions

Subsonic Outflow (Left Boundary) - Example

Subsonic outflow: we should specify one variable

Alt	specified variable	well-posed	non-reflective
1	p	X	
2	ρu	X	
3	J^+	X	X

Subsonic Inflow 2D/3D



Subsonic inflow

Assumption:

$-a < \mathbf{v} \cdot \mathbf{n} < 0$

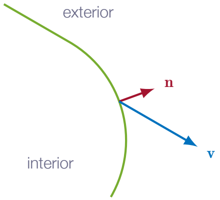
Four ingoing characteristics

One outgoing characteristic

Specify four variables at the boundary:

p_o, T_o, and flow direction (two angles)

Subsonic Outflow 2D/3D



$\mathbf{n}$	unit normal vector
$\mathbf{v}$	fluid velocity at boundary

Subsonic outflow

Assumption:

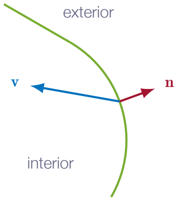
$0 < \mathbf{v} \cdot \mathbf{n} < a$

One ingoing characteristics

Four outgoing characteristic

Specify one variables at the boundary: static pressure

Supersonic Inflow 2D/3D



$\mathbf{n}$	unit normal vector
$\mathbf{v}$	fluid velocity at boundary

Supersonic inflow

Assumption:

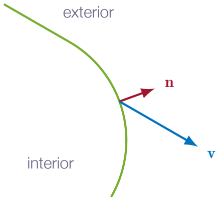
$\mathbf{v} \cdot \mathbf{n} < -a$

Five ingoing characteristics

No outgoing characteristics

Specify five variables at the boundary: solver variables

Supersonic Outflow 2D/3D



$\mathbf{n}$	unit normal vector
$\mathbf{v}$	fluid velocity at boundary

Supersonic outflow

Assumption:

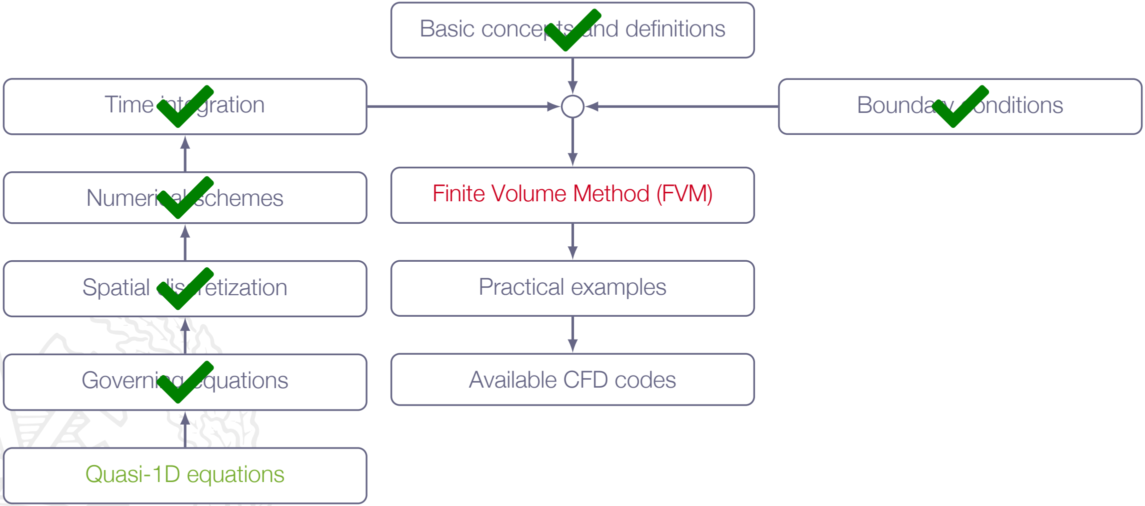
$\mathbf{v} \cdot \mathbf{n} > a$

No ingoing characteristics

Five outgoing characteristics

No variables specified at the boundary

Roadmap - The Time-Marching Technique



Explicit Finite-Volume Method - Summary

CFLOW

The described numerical approach can be categorized as

- **Density-based**
- **Fully coupled**
- **Structured**
- **Explicit**

with the following features

- **High-order convective scheme**
- **Shock handling (artificial damping)**

Explicit Finite-Volume Method - Summary

Spatial discretization:

Control volume formulations of conservation equations are applied to the cells of the discretized domain

Cell-averaged flow quantities ($\overline{\rho}, \overline{\rho u}, \overline{\rho e_o}$) are chosen as degrees of freedom

Flux terms are **approximated** in terms of the chosen degrees of freedom

high-order, upwind type of flux approximation is used for optimum results

A **fully conservative** scheme is obtained

the flux leaving one cell is identical to the flux entering the neighboring cell

The result of the spatial discretization is a system of ODEs

Explicit Finite-Volume Method - Summary

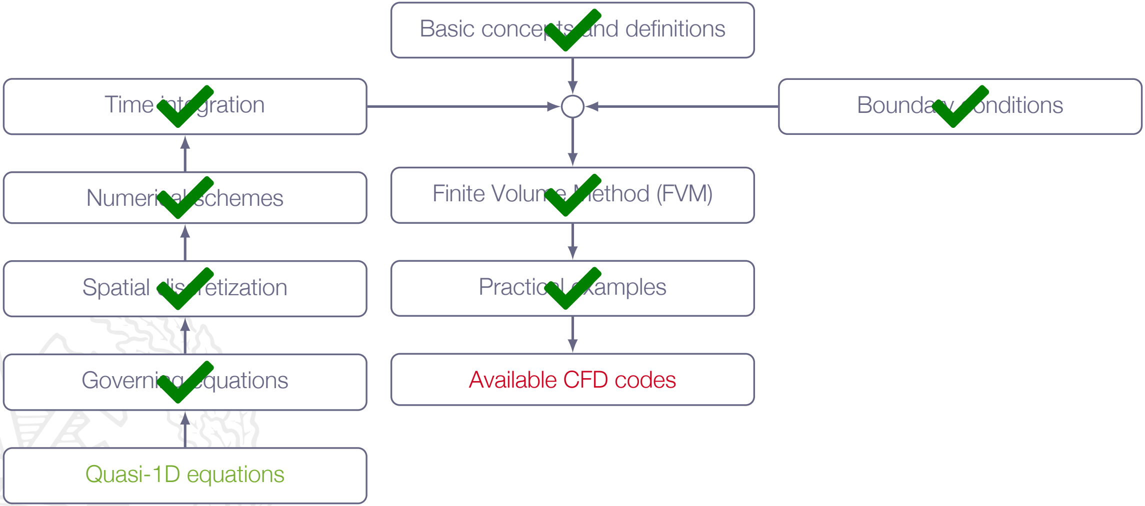
Time marching:

Three-stage, second-order accurate Runge-Kutta scheme

Explicit time-stepping

Time step length **limited by the** *CFL* **condition** ($CFL \leq 1$)

Roadmap - The Time-Marching Technique



Available CFD Codes

CFD Codes

List of free and commercial CFD codes:

`http://www.cfd-online.com/Wiki/Codes`

- Free codes are in general unsupported and poorly documented
- Commercial codes are often claimed to be suitable for all types of flows
  The reality is that the user must make sure of this!

CFD Codes - General Guidlines

Simulation of high-speed and/or unsteady compressible flows:

Use correct solver options

otherwise you may obtain completely wrong solution!

1. coupled solver
2. equation of state
3. energy equation included

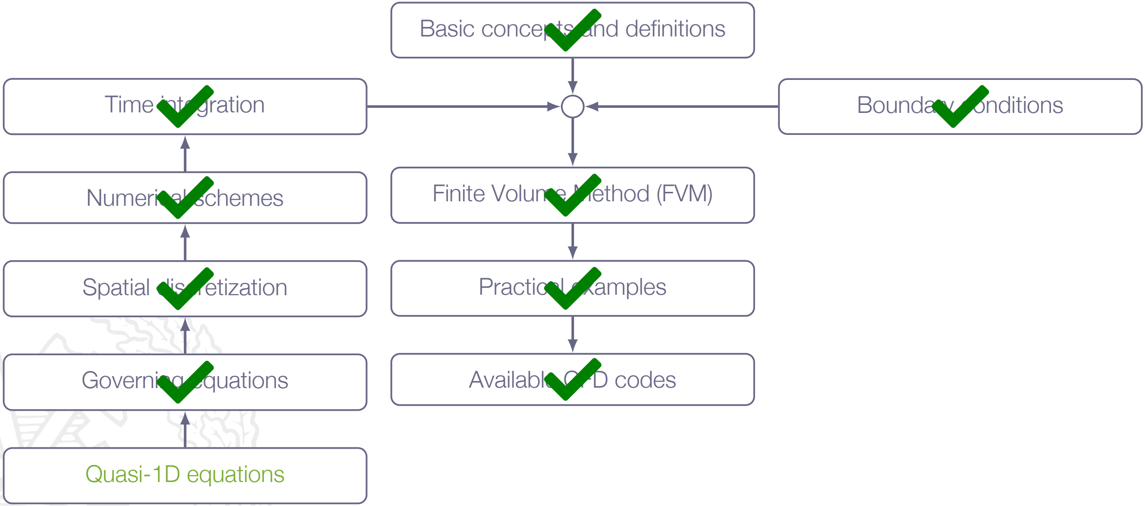
Use a high-quality grid

a poor grid will either not give you any solution at all (no convergence) or at best a very inaccurate solution!

ANSYS-FLUENT®/STAR-CCM+® - Typical Experiences

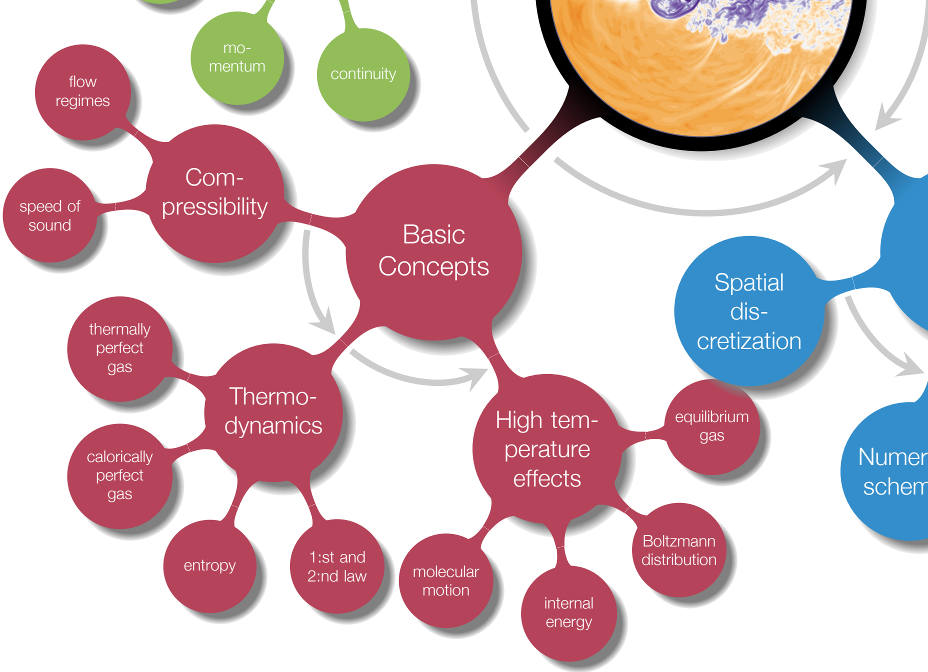
1. Very robust solvers - will almost always give you a solution
2. Accuracy of solution depends a lot on **grid quality**
3. **Shocks** are generally **smeared** more than in specialized codes
4. Solver is generally very **efficient** for **steady-state** problems
5. Solver is less efficient for truly unsteady problems, where both flow and acoustics must be resolved accurately

Roadmap - The Time-Marching Technique



Chapter 16
Properties of High-Temperature Gases

Overview



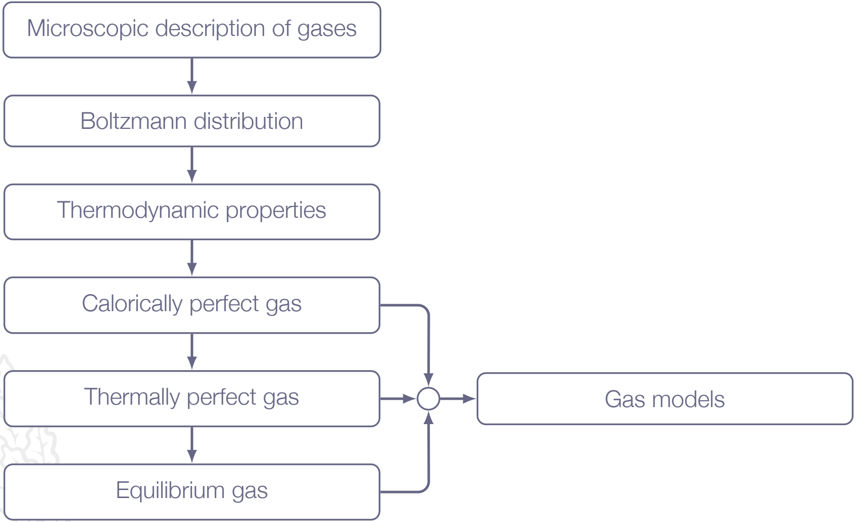
mo-
mentum
continuity
flow
regimes
Com-
pressibility
speed of
sound
Basic
Concepts
Spatial
dis-
cretization
thermally
perfect
gas
Thermo-
dynamics
High tem-
perature
effects
equilibrium
gas
calorically
perfect
gas
entropy
1:st and
2:nd law
molecular
motion
internal
energy
Boltzmann
distribution

Learning Outcomes

6 **Define** the special cases of calorically perfect gas, thermally perfect gas and real gas and **explain** the implication of each of these special cases

A deep dive into the theory behind the definitions of calorically perfect gas, thermally perfect gas, and other models

Roadmap - High-Temperature Gases



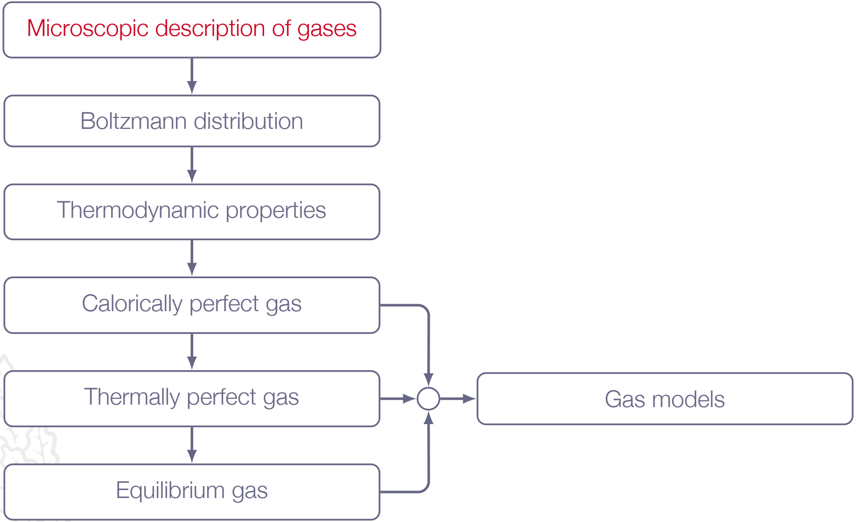
Motivation

Explosions and combustion are two examples of cases where high-temperature effects must be taken into account

The temperature does not have to be extremely high in order for temperature effects to appear, 600 K is enough

In this section you will learn what happens in a gas on a molecular level when the temperature increases and what implications that has on applicability of physical models

Roadmap - High-Temperature Gases



Chapter 16.2
Microscopic Description of Gases

Microscopic Description of Gases

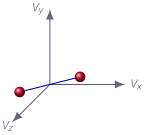
Hard to make measurements

Accurate, reliable theoretical models needed

Available models do work quite well

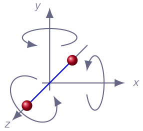
Molecular Energy

Translation



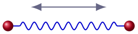
Translational kinetic energy
thermal degrees of freedom: 3

Rotation



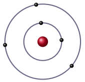
Rotational kinetic energy
thermal degrees of freedom:
0 for monoatomic gases
2 for diatomic gases
2 for linear polyatomic gases
3 for non-linear polyatomic gases

Vibration

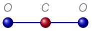


Vibrational energy
(kinetic energy + potential energy)
thermal degrees of freedom: 2

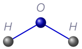
Electronic energy



Electronic energy of electrons in orbit
(kinetic energy + potential energy)



CO_2
linear polyatomic molecule



H_2O
non-linear polyatomic molecule

Molecular Energy

The energy for one molecule can be described by

$$\varepsilon' = \varepsilon'_{trans} + \varepsilon'_{rot} + \varepsilon'_{vib} + \varepsilon'_{el}$$

Results of quantum mechanics have shown that **energy is quantized** *i.e.* energy can **exist only at specific discrete values**

Energy is not continuous! Might seem unintuitive

Molecular Energy

The lowest quantum numbers defines the **zero-point energy** for each mode

$\varepsilon'_{0_{rot}} = 0$

$\varepsilon'_{0_{trans}} > 0$ (very small but finite)

At **absolute zero**, molecules still moves but not much. The rotational energy is, however, exactly zero.

$$\varepsilon_{j_{trans}} = \varepsilon'_{j_{trans}} - \varepsilon'_{0_{trans}}$$

$$\varepsilon_{l_{vib}} = \varepsilon'_{l_{vib}} - \varepsilon'_{0_{vib}}$$

$$\varepsilon_{k_{rot}} = \varepsilon'_{k_{rot}}$$

$$\varepsilon_{m_{el}} = \varepsilon'_{m_{el}} - \varepsilon'_{0_{el}}$$

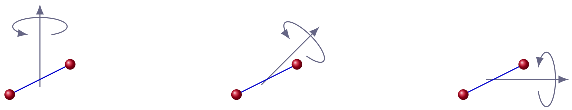
Molecular Energy

Thus the total energy of a specific molecule may be expressed as

$$\varepsilon_i' = \varepsilon_{j_{trans}} + \varepsilon_{k_{rot}} + \varepsilon_{l_{vib}} + \varepsilon_{m_{el}} + \varepsilon_o'$$

Note! since ε_i' is the sum of individually quantized energy levels, ε_i' itself is also quantized

Energy States - Example



three cases with the **same rotational energy**

different direction of angular momentum

quantum mechanics ⇒ different **distinguishable states**

a finite number of possible **degenerate states** g_j at each energy level j

Macrostates and Microstates

Macrostate:

molecules collide and exchange energy $\Rightarrow$ the number of molecules at each energy level j (the macrostate or the N_j distribution) will change over time

some macrostates are more probable than other

most probable macrostates (energy distributions) $\Rightarrow$ **thermodynamic equilibrium**

Microstate:

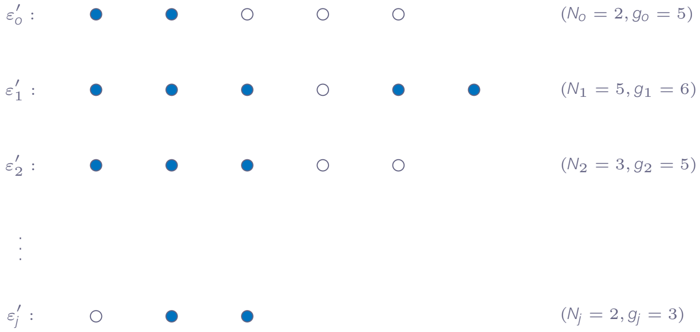
different microstates constitute the same number of molecules in each energy level (same macrostate) but molecules are in **different degenerate states**

the **most probable macrostate** is the one with the **most possible microstates** $\Rightarrow$ possible to find the most probable macrostate by counting microstates

Macrostates and Microstates

ε_j	molecular energy at energy level j
N_j	the number of molecules at energy level j
g_j	the number of possible degenerate states at energy level j

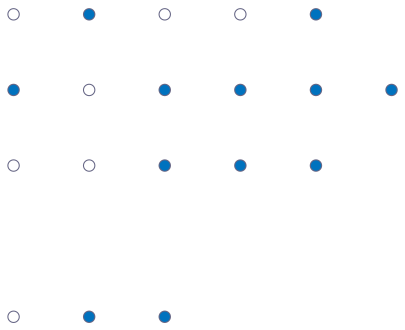
Macrostate I   Microstate I



Macrostates and Microstates

ε_j	molecular energy at energy level j
N_j	the number of molecules at energy level j
g_j	the number of possible degenerate states at energy level j

Macrostate I Microstate II



ε'_o : $(N_o = 2, g_o = 5)$

ε'_1 : $(N_1 = 5, g_1 = 6)$

ε'_2 : $(N_2 = 3, g_2 = 5)$

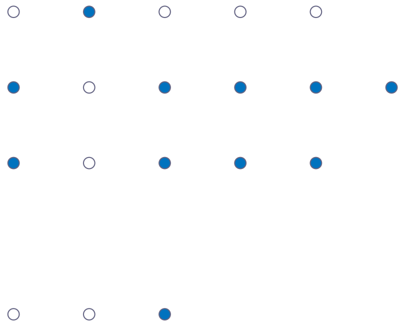
$\vdots$

ε'_j : $(N_j = 2, g_j = 3)$

Macrostates and Microstates

ε_j	molecular energy at energy level j
N_j	the number of molecules at energy level j
g_j	the number of possible degenerate states at energy level j

Macrostate II Microstate I



ε'_o : $(N_o = 1, g_o = 5)$

ε'_1 : $(N_1 = 5, g_1 = 6)$

ε'_2 : $(N_2 = 4, g_2 = 5)$

$\vdots$

ε'_j : $(N_j = 1, g_j = 3)$

Macrostates and Microstates

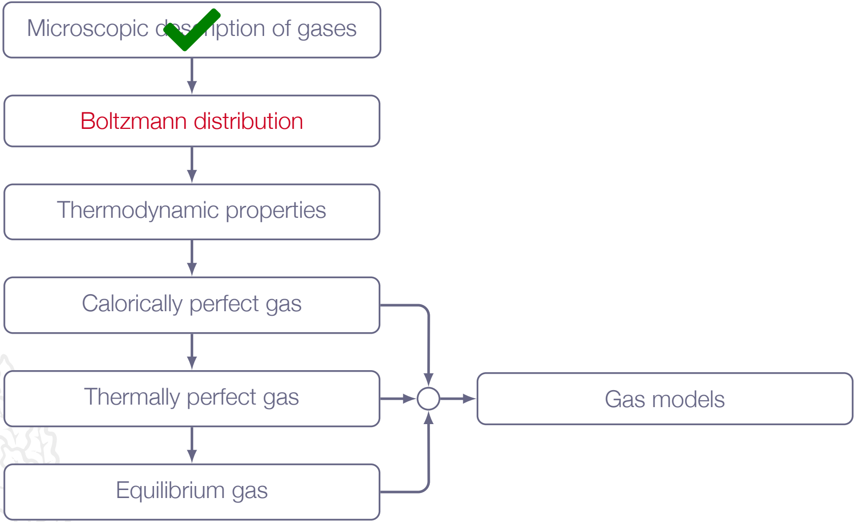
$$N = \sum_j N_j$$

N is the total number of molecules and N_j is the number of molecules at energy level j

$$E = \sum_j \varepsilon_j' N_j$$

E is the total energy and ε_j' is the energy per molecule at energy level j

Roadmap - High-Temperature Gases



Chapter 16.5
The Limiting Case: Boltzmann Distribution

Boltzmann Distribution

The Boltzmann distribution:

$$N_j^* = N\frac{g_j \mathrm{e}^{-\varepsilon_j/kT}}{Q}$$

where $Q = f(T, V)$ is the state sum defined as

$$Q \equiv \sum_j g_j \mathrm{e}^{-\varepsilon_j/kT}$$

g_j is the number of **degenerate states**, ε_j is the **energy above zero-level** $(\varepsilon_j = \varepsilon_j' - \varepsilon_o)$, and k is the Boltzmann constant

Boltzmann Distribution

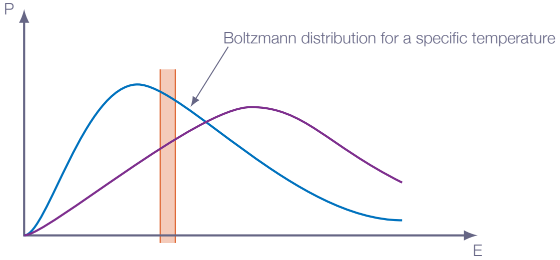
The Boltzmann distribution:

$$N_j^* = N\frac{g_j \mathrm{e}^{-\varepsilon_j/kT}}{Q}$$

For molecules or atoms of a given species, quantum mechanics says that a set of well-defined energy levels ε_j exists, over which the molecules or atoms can be distributed at any given instant, and that each energy level has a certain number of energy states, g_j.

For a system of N molecules or atoms at a given T and V, N_j^* are the number of molecules or atoms in each energy level ε_j when the system is in thermodynamic equilibrium.

Boltzmann Distribution

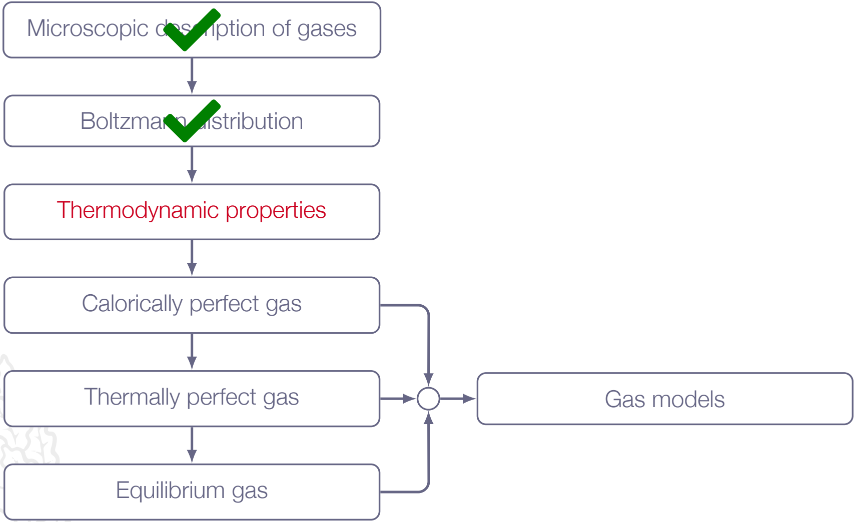


Boltzmann distribution: describes the **probability** (P) of population of an **energy level** with the energy (E)

At temperatures above $\sim$ 5K, molecules are distributed over many energy levels, and therefore the states are generally **sparsely populated** ($N_j \ll g_j$)

Higher energy levels become more populated as temperature increases

Roadmap - High-Temperature Gases



Chapter 16.6 - 16.8
Evaluation of Gas Thermodynamic Properties

Internal Energy

The internal energy is calculated as

$$E = NkT^2 \left(\frac{\partial \ln Q}{\partial T} \right)_V$$

The internal energy per unit mass is obtained as

$$e = \frac{E}{M} = \frac{NkT^2}{Nm} \left(\frac{\partial \ln Q}{\partial T} \right)_V = \left\{ \frac{k}{m} = R \right\} = RT^2 \left(\frac{\partial \ln Q}{\partial T} \right)_V$$

k is the Boltzmann constant, m is the molecular weight, R is the gas constant, and Q is the state sum

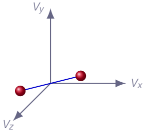
Internal Energy

$$e = RT^2 \left(\frac{\partial \ln Q}{\partial T} \right)_V$$

$$Q \equiv \sum_j g_j \mathrm{e}^{-\varepsilon_j / kT}$$

In order to be able to calculate the internal energy of a gas at a given temperature, we need an estimate of the state sum Q

Internal Energy - Translation



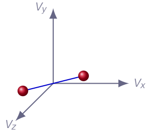
$$\varepsilon'_{trans} = \frac{h^2}{8m}\left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} + \frac{n_3^2}{a_3^2}\right)$$

$n_1 - n_3$	quantum numbers (1,2,3,...)
$a_1 - a_3$	linear dimensions that describes the size of the system
h	Planck's constant
m	mass of the individual molecule

$$\Rightarrow \cdots \Rightarrow$$

$$Q_{trans} = \left(\frac{2\pi mkT}{h^2}\right)^{3/2} V$$

Internal Energy - Translation



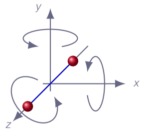
$$Q_{trans} = \left(\frac{2\pi mkT}{h^2}\right)^{3/2} V$$

$$\ln Q_{trans} = \frac{3}{2}\ln T + \frac{3}{2}\ln\frac{2\pi mk}{h^2} + \ln V \Rightarrow$$

$$\left(\frac{\partial \ln Q_{trans}}{\partial T}\right)_V = \frac{3}{2}\frac{1}{T} \Rightarrow$$

$$e_{trans} = RT^2\left(\frac{\partial \ln Q_{trans}}{\partial T}\right)_V = RT^2\frac{3}{2T} = \frac{3}{2}RT$$

Internal Energy - Rotation



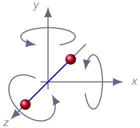
$$\varepsilon'_{rot} = \frac{h^2}{8\pi^2 I} J(J+1)$$

J	rotational quantum number (0,1,2,...)
I	moment of inertia (tabulated for common molecules)
h	Planck's constant

$$\Rightarrow \cdots \Rightarrow$$

$$Q_{rot} = \frac{8\pi^2 IkT}{h^2}$$

Internal Energy - Rotation



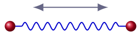
$$Q_{rot} = \frac{8\pi^2 IkT}{h^2}$$

$$\ln Q_{rot} = \ln T + \ln \frac{8\pi^2 Ik}{h^2} \Rightarrow$$

$$\left(\frac{\partial \ln Q_{rot}}{\partial T}\right)_V = \frac{1}{T} \Rightarrow$$

$$e_{rot} = RT^2 \left(\frac{\partial \ln Q_{rot}}{\partial T}\right)_V = RT^2 \frac{1}{T} = RT$$

Internal Energy - Vibration



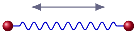
$$\varepsilon'_{vib} = h\nu \left(n + \frac{1}{2} \right)$$

n	vibrational quantum number (0,1,2,...)
ν	fundamental vibrational frequency (tabulated for common molecules)
h	Planck's constant

$$\Rightarrow \cdots \Rightarrow$$

$$Q_{vib} = \frac{1}{1 - e^{-h\nu/kT}}$$

Internal Energy - Vibration



$$Q_{vib} = \frac{1}{1 - e^{-h\nu/kT}}$$

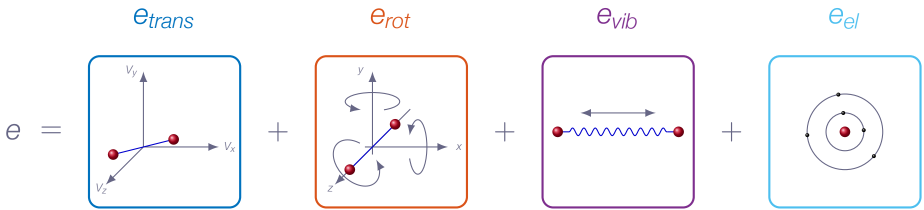
$$\ln Q_{vib} = -\ln(1 - e^{-h\nu/kT}) \Rightarrow$$

$$\left(\frac{\partial \ln Q_{vib}}{\partial T}\right)_V = \frac{h\nu/kT^2}{e^{h\nu/kT} - 1} \Rightarrow$$

$$e_{vib} = RT^2 \left(\frac{\partial \ln Q_{vib}}{\partial T}\right)_V = RT^2 \frac{h\nu/kT^2}{e^{h\nu/kT} - 1} = \frac{h\nu/kT}{e^{h\nu/kT} - 1} RT$$

$$\lim_{T \to \infty} \frac{h\nu/kT}{e^{h\nu/kT} - 1} = 1 \Rightarrow e_{vib} \leq RT$$

Specific Heat



$$e = \frac{3}{2}RT + RT + \frac{h\nu/kT}{e^{h\nu/kT-1}}RT + e_{el}$$

From before, we know that the specific heat is defined as follows:

$$C_v \equiv \left(\frac{\partial e}{\partial T}\right)_V$$

Specific Heat

$$e = e_{trans} + e_{rot} + e_{vib} + e_{el} = \frac{3}{2}RT + RT + \frac{h\nu/kT}{e^{h\nu/kT-1}}RT + e_{el}$$

For molecules with only translational and rotational energy

$$e = \frac{3}{2}RT + RT = \frac{5}{2}RT \Rightarrow C_v \equiv \left(\frac{\partial e}{\partial T}\right)_V = \frac{5}{2}R$$

$$C_p = C_v + R = \frac{7}{2}R$$

$$\gamma = \frac{C_p}{C_v} = \frac{7}{5} = 1.4$$

Specific Heat

$$e = e_{trans} + e_{rot} + e_{vib} + e_{el} = \frac{3}{2}RT + RT + \frac{h\nu/kT}{e^{h\nu/kT-1}}RT + e_{el}$$

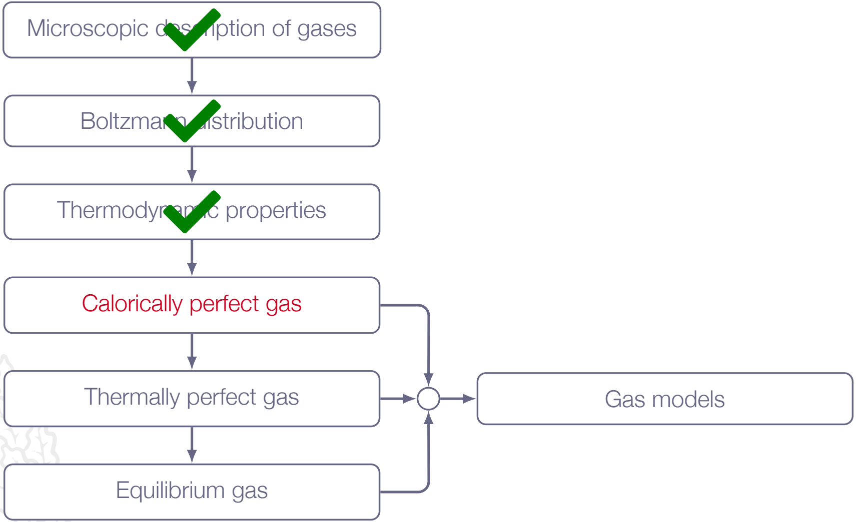
For mono-atomic gases with only translational and (rotational) energy

$$e = \frac{3}{2}RT + 0 \Rightarrow C_v \equiv \left(\frac{\partial e}{\partial T}\right)_V = \frac{3}{2}R$$

$$C_p = C_v + R = \frac{5}{2}R$$

$$\gamma = \frac{C_p}{C_v} = \frac{5}{3} = 1\frac{2}{3} \simeq 1.67$$

Roadmap - High-Temperature Gases



Calorically Perfect Gas

$$e = e_{trans} + e_{rot} + e_{vib} + e_{el} = \frac{3}{2}RT + RT + \frac{h\nu/kT}{e^{h\nu/kT-1}}RT + e_{el}$$

In general, only translational and rotational modes of molecular excitation

Translational and rotational energy levels are sparsely populated, according to Boltzmann distribution (the Boltzmann limit)

Vibrational energy levels are practically unpopulated (except for the zero level)

Calorically Perfect Gas

$$e = e_{trans} + e_{rot} + e_{vib} + e_{el} = \frac{3}{2}RT + RT + \frac{h\nu/kT}{e^{h\nu/kT-1}}RT + e_{el}$$

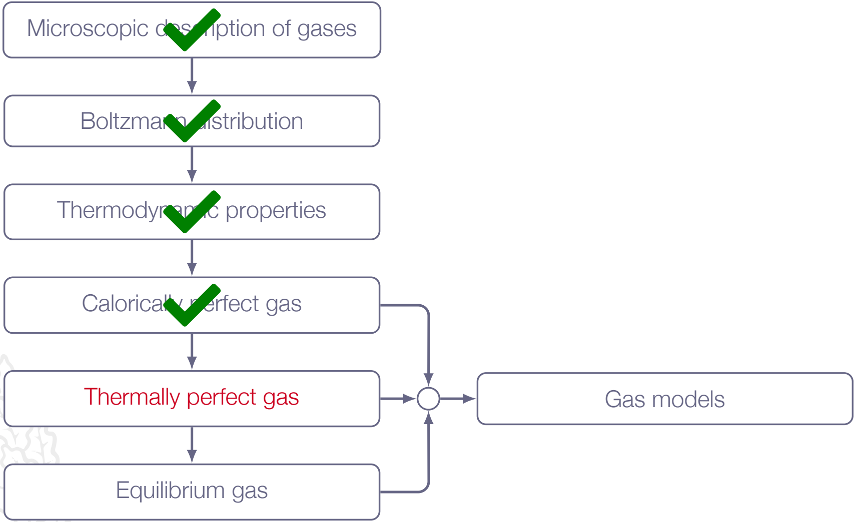
Characteristic values of γ for each type of molecule, *e.g.* mono-atomic gas, di-atomic gas, tri-atomic gas, etc

He, Ar, Ne, ... - mono-atomic gases ($\gamma = 5/3$)

H_2, O_2, N_2, ... - di-atomic gases ($\gamma = 7/5$)

H_2O (gaseous), CO_2, ... - tri-atomic gases ($\gamma < 7/5$)

Roadmap - High-Temperature Gases



Thermally Perfect Gas

$$e = e_{trans} + e_{rot} + e_{vib} + e_{el} = \frac{3}{2}RT + RT + \frac{h\nu/kT}{e^{h\nu/kT-1}}RT + e_{el}$$

In general, only translational, rotational and vibrational modes of molecular excitation

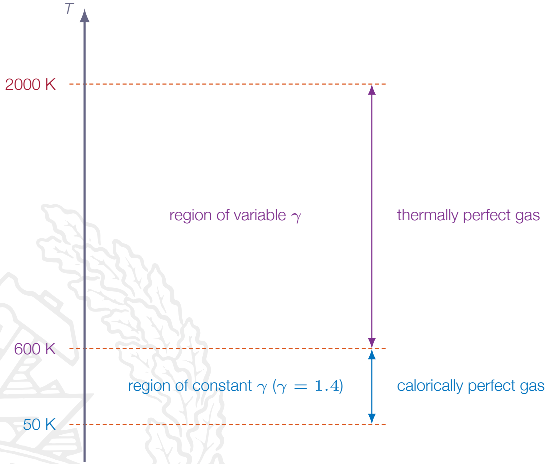
Translational and rotational energy levels are sparsely populated, according to Boltzmann distribution (the **Boltzmann limit**)

The population of the vibrational energy levels **approaches the Boltzmann limit** as temperature increases

Temperature dependent values of γ for all types of molecules except mono-atomic (no vibrational modes possible)

High-Temperature Effects

Example: properties of air



Thermally perfect gas:
e and *h* are non-linear functions of *T*

the temperature range represents standard atmospheric pressure (lower pressure gives lower temperatures)

High-Temperature Effects

$$e = e_{trans} + e_{rot} + e_{vib} + e_{el} = \frac{3}{2}RT + RT + \frac{h\nu/kT}{e^{h\nu/kT-1}}RT + e_{el}$$

For cases where the vibrational energy is not negligible (*at high temperatures*)

$$\lim_{T\to\infty} e_{vib} = RT \Rightarrow C_v = \frac{7}{2}R$$

However, chemical reactions and ionization will take place long before that ...

Translational and rotational energy fully excited above $\sim$5 K

Vibrational energy is non-negligible above 600 K

Chemical reactions begin to occur above $\sim$2000 K

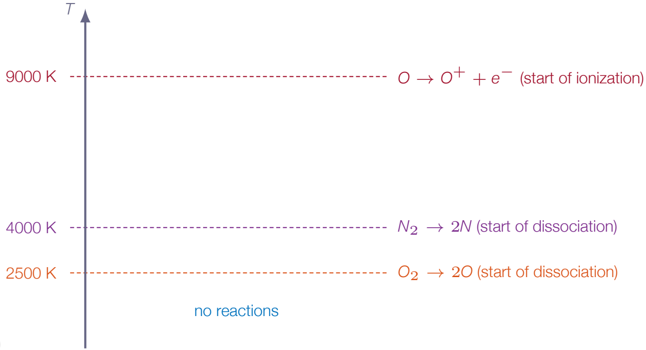
High-Temperature Effects

As temperature increase further vibrational energy becomes less important

Why is that so?

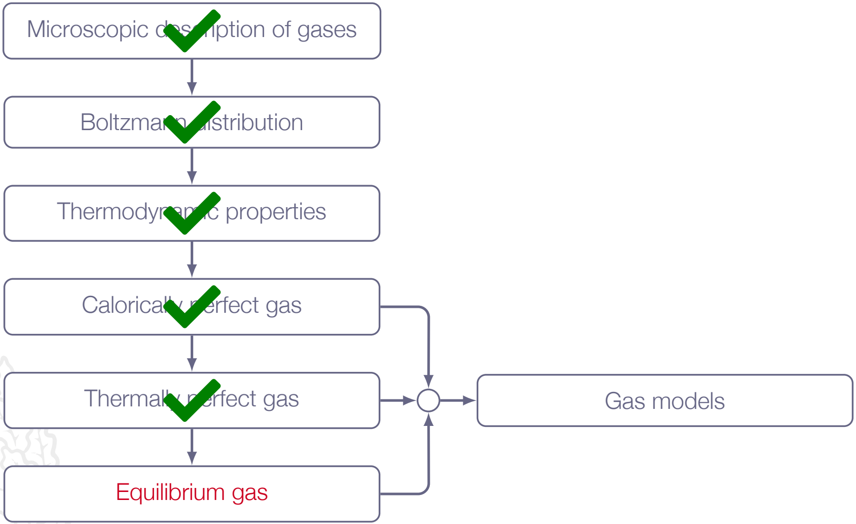
High-Temperature Effects

Example: properties of air (continued)



With increasing temperature, the gas becomes more and more mono-atomic which means that vibrational modes becomes less important

Roadmap - High-Temperature Gases



Equilibrium Gas

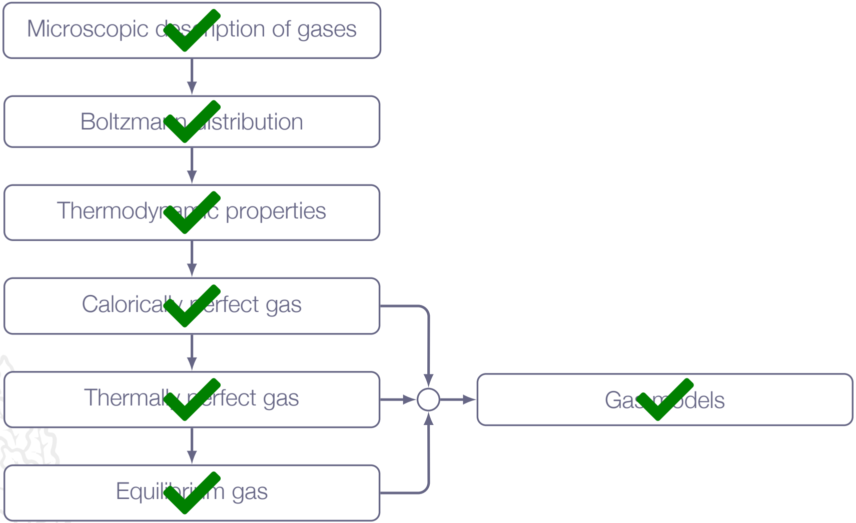
For temperatures $T > 2500K$

Air may be described as being in **thermodynamic** and **chemical equilibrium** (Equilibrium Gas)

- reaction rates (time scales) low compared to flow time scales
- reactions in both directions (example: $O_2 \rightleftharpoons 2O$)

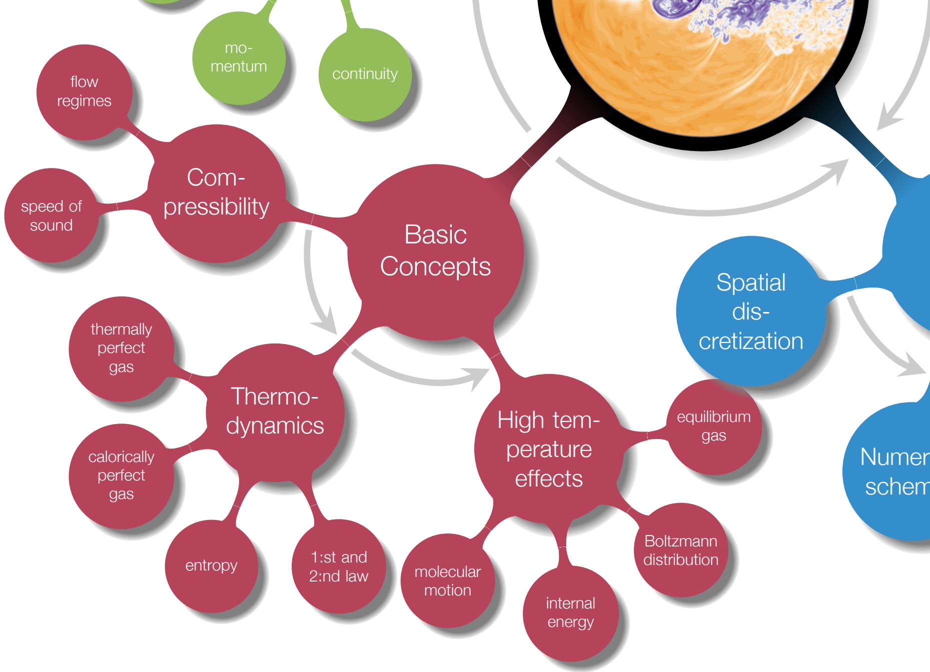
Tables must be used (Equilibrium Air Data) or special functions which have been made to fit the tabulated data

Roadmap - High-Temperature Gases



Chapter 17
High-Temperature Flows: Basic Examples

Overview



mo-mentum
continuity
flow regimes
Com-pressibility
speed of sound
Basic Concepts
Spatial dis-cretization
thermally perfect gas
Thermo-dynamics
calorically perfect gas
entropy
1:st and 2:nd law
High tem-perature effects
equilibrium gas
Boltzmann distribution
molecular motion
internal energy

Learning Outcomes

6 **Define** the special cases of calorically perfect gas, thermally perfect gas and real gas and **explain** the implication of each of these special cases

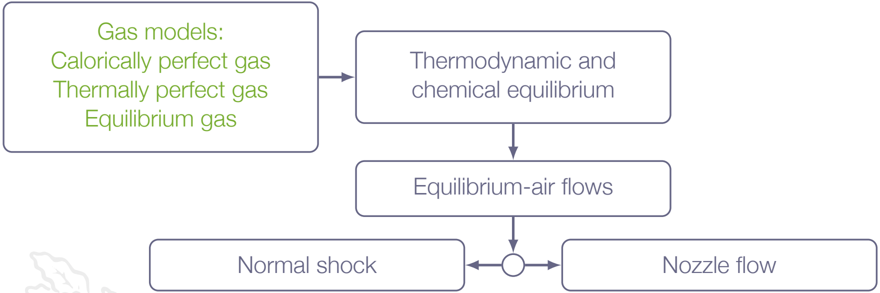
8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics

   b normal shocks*

   i detached blunt body shocks, nozzle flows

How does increased temperature affect a compressible flow?

Roadmap - High Temperature Effects



Motivation

High-temperature effects can be rather dramatic

We will examine a couple of flow situations where the temperature is high enough to effect the flow properties significantly in order to get e feeling for high-temperature flows

Properties of High - Temperature Gases

Applications:

- Rocket nozzle flows
- Reentry vehicles
- Shock tubes / Shock tunnels
- Internal combustion engines
- Gasturbines

Properties of High - Temperature Gases

Example: Reentry vehicle

Mach number: 32.5

Gas: air

Temperature: $T_\infty = 283$

Properties of High - Temperature Gases

Example: Reentry vehicle

Assume calorically perfect gas

Normal shock relations gives
$T/T_\infty = 206$

$T_\infty = 283 \Rightarrow T = 58\,300$ K

Properties of High - Temperature Gases

Example: Reentry vehicle

Assume calorically perfect gas

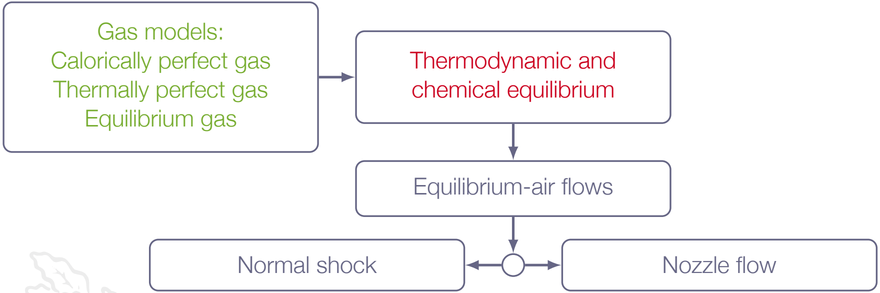
Normal shock relations gives
$T/T_\infty = 206$

$T_\infty = 283 \Rightarrow T = 58\,300$ K

A more correct value is $T = 11\,600$ K

Something is fishy here!

Roadmap - High Temperature Effects



Chapter 17.1
Thermodynamic and Chemical Equilibrium

Thermodynamic Equilibrium

Molecules are distributed among their possible energy states according to the **Boltzmann distribution** (which is a **statistical equilibrium**) for the given temperature of the gas

- extremely fast process (time and length scales of the molecular processes)
- much faster than flow time scales in general (not true inside shocks)

Thermodynamic Equilibrium

Global thermodynamic equilibrium:

”*true thermodynamic equilibrium*”

there are no gradients of p, T, ρ (or flow velocity, species concentrations, ...)

Local thermodynamic equilibrium:

gradients can be neglected locally

this requirement is fulfilled in most cases (hard not to get)

Chemical Equilibrium

Composition of gas (species concentrations) is **fixed in time**

- forward and backward rates of all chemical reactions are equal
- zero net reaction rates
- chemical reactions may be either slow or fast in comparison to flow time scale depending on the case studied

Chemical Equilibrium

Global chemical equilibrium:

there are no gradients of species concentrations

together with global thermodynamic equilibrium $\Rightarrow$
all gradients are zero

Local chemical equilibrium

gradients of species concentrations can be neglected locally

not always true - depends on reaction rates and flow time scales

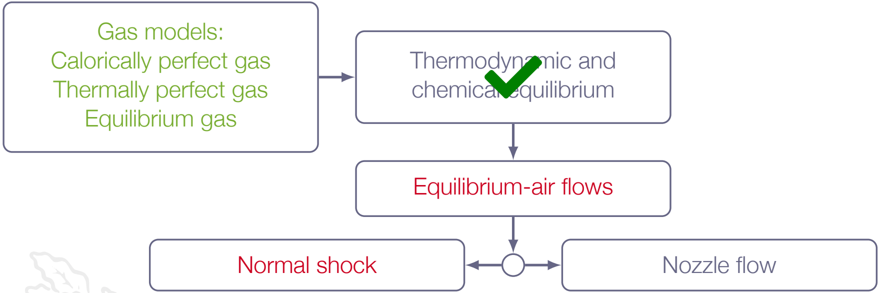
Thermodynamic and Chemical Equilibrium

Most common cases:

	Thermodynamic Equilibrium	Chemical Equilibrium	Gas Model
1	local thermodynamic equilibrium	local chemical equilibrium	equilibrium gas
2	local thermodynamic equilibrium	chemical non-equilibrium	finite rate chemistry
3	local thermodynamic equilibrium	frozen composition	frozen flow
4	thermodynamic non-equilibrium	frozen composition	vibrationally frozen flow

length and time scales of flow decreases from 1 to 4

Roadmap - High Temperature Effects



Chapter 17.2
Equilibrium Normal Shock Wave Flows

Equilibrium Normal Shock Wave Flows

Question:

Is the equilibrium gas assumption OK for normal shocks?

Answer:

for **hypersonic** flows with very **little ionization** in the shock region, it is a fair approximation

not perfect, since the assumption of **local thermodynamic** and **chemical equilibrium** is not really true around the shock

however, it gives a significant improvement compared to the calorically perfect gas assumption

Equilibrium Normal Shock Wave Flows

Basic relations (for all gases), stationary normal shock:

$$\begin{cases} \rho_1 u_1 = \rho_2 u_2 \\ \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \\ h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2 \end{cases}$$

For equilibrium gas we have:

$$\begin{cases} \rho = \rho(p, h) \\ T = T(\rho, h) \end{cases}$$

(we are free to choose any two states as independent variables)

Equilibrium Normal Shock Wave Flows

Assume that ρ_1, u_1, p_1, T_1, and h_1 are known

$$u_2 = \frac{\rho_1 u_1}{\rho_2} \Rightarrow \rho_1 u_1^2 + p_1 = \rho_2 \left(\frac{\rho_1}{\rho_2} u_1\right)^2 + p_2 \Rightarrow$$

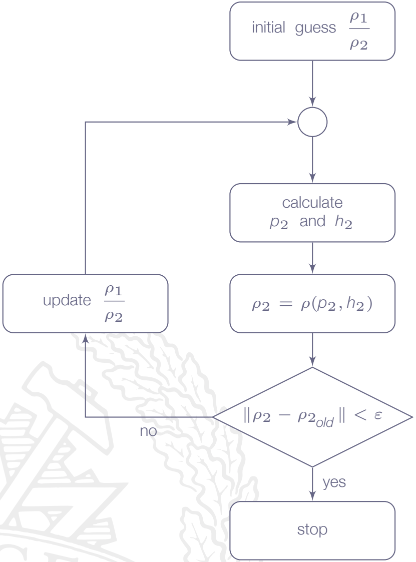
$$p_2 = p_1 + \rho_1 u_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right)$$

Also

$$h_1 + \frac{1}{2} u_1^2 = h_2 + \frac{1}{2} \left(\frac{\rho_1}{\rho_2} u_1\right)^2 \Rightarrow$$

$$h_2 = h_1 + \frac{1}{2} u_1^2 \left(1 - \left(\frac{\rho_1}{\rho_2}\right)^2\right)$$

Equilibrium Normal Shock Wave Flows



when converged:

$$\left.\begin{aligned} \rho_2 &= \rho(p_2, h_2) \\ T_2 &= T(\rho_2, h_2) \end{aligned}\right\} \Rightarrow$$

ρ_2, u_2, p_2, T_2, h_2 known

Equilibrium Air - Normal Shock

Tables of thermodynamic properties for different conditions are available

For a very strong shock case ($M_1 = 32$), the table below shows results for equilibrium air

	calorically perfect gas ($\gamma = 1.4$)	equilibrium air
p_2/p_1	1233	1387
ρ_2/ρ_1	5.97	15.19
h_2/h_1	206.35	212.80
T_2/T_1	206.35	41.64

Equilibrium Air - Normal Shock

Analysis:

Pressure ratio is comparable

Density ratio differs by factor of 2.5

Temperature ratio differs by factor of 5

Equilibrium Air - Normal Shock

Explanation:

Using equilibrium gas means that vibration, dissociation and chemical reactions are accounted for

The chemical reactions taking place in the shock region lead to an **absorption of energy** into chemical energy

- drastically reducing the temperature downstream of the shock
- this also explains the difference in density after the shock

Equilibrium Air - Normal Shock

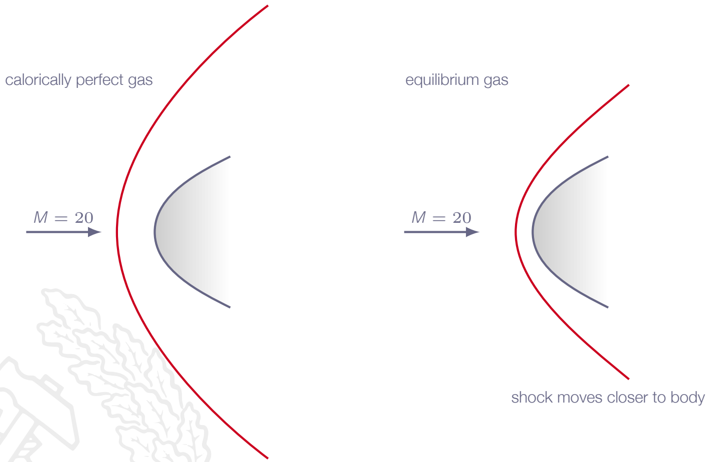
Additional notes:

1. For a normal shock in an **equilibrium gas**, the pressure ratio, density ratio, enthalpy ratio, temperature ratio, etc all depend on **three upstream variables**, *e.g.* u_1, p_1, T_1

2. For a normal shock in a **thermally perfect gas**, the pressure ratio, density ratio, enthalpy ratio, temperature ratio, etc all depend on **two upstream variables**, *e.g.* M_1, T_1

3. For a normal shock in a **calorically perfect gas**, the pressure ratio, density ratio, enthalpy ratio, temperature ratio, etc all depend on **one upstream variable**, *e.g.* M_1

Equilibrium Gas - Detached Shock



What's the reason for the difference in predicted shock position?

Equilibrium Gas - Detached Shock

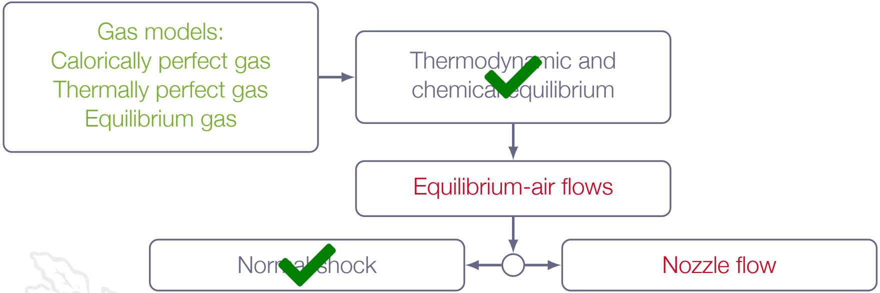
Calorically perfect gas:

all energy ends up in translation and rotation $\Rightarrow$ increased temperature

Equilibrium gas:

energy is **absorbed by reactions** $\Rightarrow$ does not contribute to the increase of gas temperature

Roadmap - High Temperature Effects



Chapter 17.3
Equilibrium Quasi-One-Dimensional Nozzle Flows

Equilibrium Quasi-1D Nozzle Flows

For a chemically reacting gas at high temperature:

1. Assuming inviscid and adiabatic flow, is the flow isentropic?

2. Can we use the area-velocity relation?

3. Is the area-Mach-number relation valid?

Equilibrium Quasi-1D Nozzle Flows - Isentropic Flow

First question:

Is a flow of a chemically reacting gas isentropic (*assuming inviscid and adiabatic flow*)?

entropy equation: $Tds = dh - \nu dp$

momentum equation: $dp = -\rho u du$

energy equation: $dh + udu = 0$

Note! The momentum and energy equations are the inviscid adiabatic quasi-1D equations on differential form (*valid for all gases*).

Equilibrium Quasi-1D Nozzle Flows - Isentropic Flow

momentum equation: $dp = -\rho udu \Rightarrow udu = -\dfrac{dp}{\rho} = -\nu dp$

energy equation: $dh + udu = 0 \Rightarrow dh = -udu$

entropy equation: $Tds = dh - \nu dp = -udu + udu = 0 \Rightarrow ds = 0$

Isentropic flow!

Equilibrium Quasi-1D Nozzle Flows - Area-Velocity Relation

Second question:

Can we use the area-velocity relation for a chemically reacting gas?

The area-velocity relation was derived from the quasi-1D formulation of the governing equations assuming isentropic flow

continuity equation: $d(\rho uA) = 0$

momentum equation: $dp = -\rho udu$

energy equation: $dh + udu = 0$

Equilibrium Quasi-1D Nozzle Flows - The Area-Velocity Relation

No assumption about the gas is made in the derivation, which means that we can use the area-velocity relation for a flow a of chemically reacting gas

$$\frac{dA}{A} = (M^2 - 1)\frac{du}{u}$$

$M = 1$ at nozzle throat still holds

Equilibrium Quasi-1D Nozzle Flows - The Area-Mach Relation

Third question:

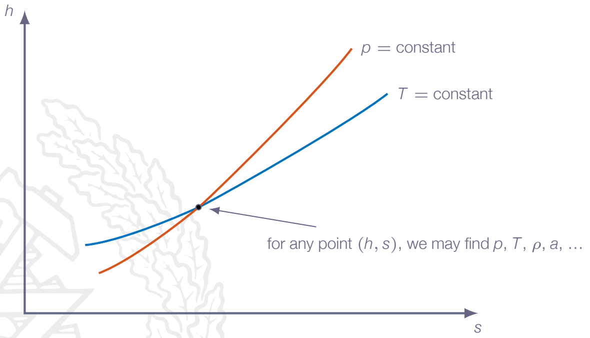
Is the area-Mach number relation valid for a chemically reacting gas?

In the derivation of the **area-Mach number relation**, calorically perfect gas is assumed and thus the relation is **not valid for a chemically reacting gas**

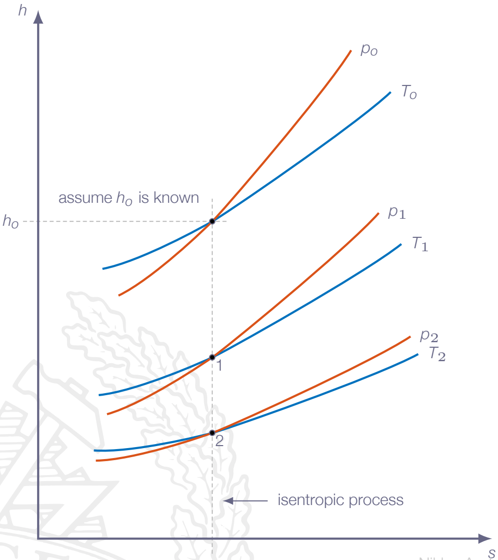
Equilibrium Quasi-1D Nozzle Flows

For general gas mixture in thermodynamic and chemical equilibrium, we may find tables or graphs describing relations between state variables.

Example: Mollier diagram



Equilibrium Quasi-1D Nozzle Flows



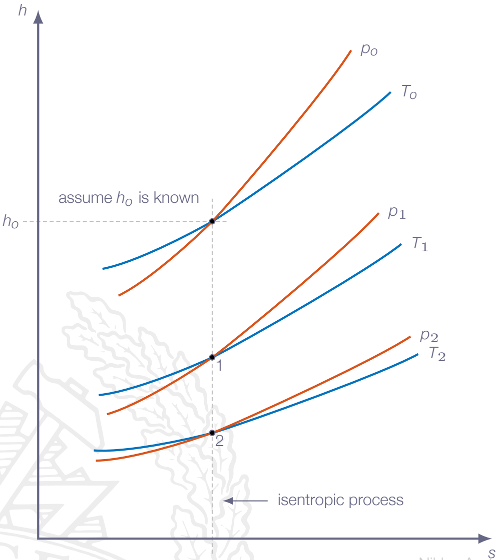
The energy equation for steady-state inviscid adiabatic nozzle flow:

$$dh_o = 0 \Rightarrow$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2 = h_o$$

where h_o is the reservoir enthalpy.

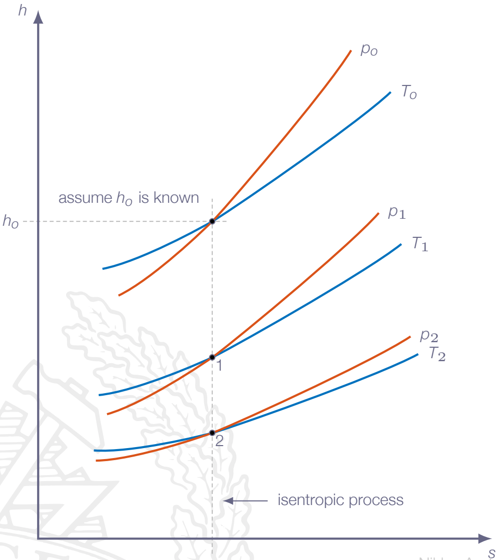
Equilibrium Quasi-1D Nozzle Flows



The velocity at point 1 can be obtained as:

$$\frac{1}{2}u_1^2 = h_o - h_1 \Rightarrow u_1 = \sqrt{2(h_o - h_1)}$$

Equilibrium Quasi-1D Nozzle Flows

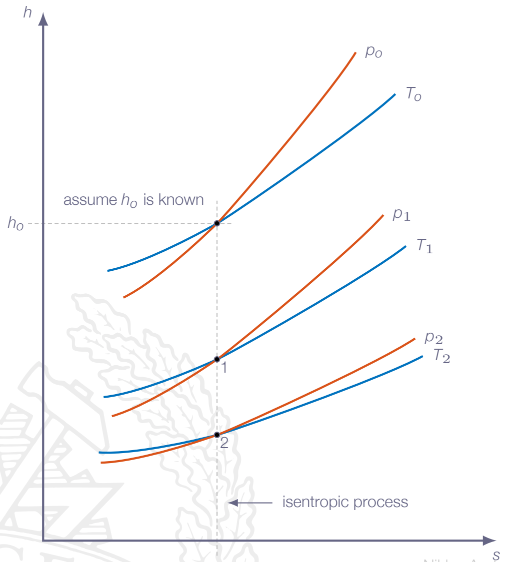


At any point along the isentropic line

$$\frac{1}{2}u^2 = h_o - h \Rightarrow u = \sqrt{2(h_o - h)}$$

T, p, ρ, a are given by the diagram

Equilibrium Quasi-1D Nozzle Flows



The continuity equation gives $\rho u A = const$

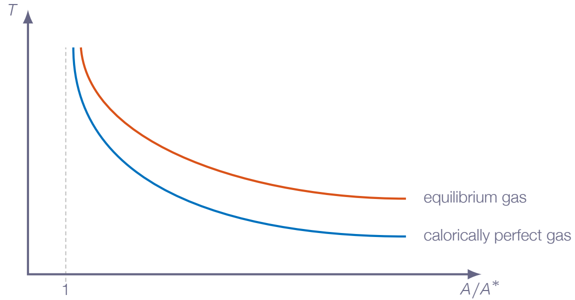
$$\rho u A = \rho^* a^* A^* \Rightarrow \frac{A}{A^*} = \frac{\rho^* a^*}{\rho u}$$

Thus, A/A^* may be computed for any point along isentropic line

Equilibrium Quasi-1D Nozzle Flows

Equilibrium gas gives higher T and more thrust than calorically perfect gas

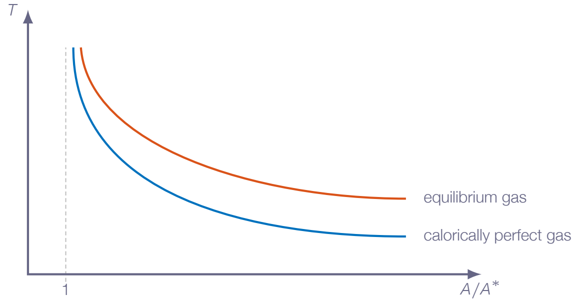
During the expansion chemical **energy is released** due to shifts in the equilibrium composition



Equilibrium Quasi-1D Nozzle Flows

Equilibrium gas gives higher T and more thrust than calorically perfect gas

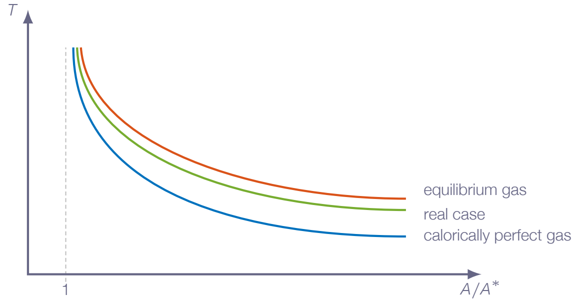
During the expansion chemical **energy is released** due to shifts in the equilibrium composition



Chemical and vibrational energy transferred to **translation** and **rotation** $\Rightarrow$ increased temperature

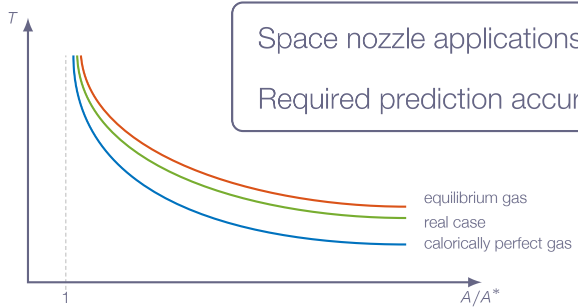
Equilibrium Quasi-1D Nozzle Flows - Reacting Mixture

Real nozzle flow with reacting gas mixture:



Equilibrium Quasi-1D Nozzle Flows - Reacting Mixture

Real nozzle flow with reacting gas mixture:



Space nozzle applications: $u_e \approx 4000$ m/s

Required prediction accuracy 5 m/s

Large Nozzles

High T_o, high p_o, high reactivity

very **fast chemical reactions**

local thermodynamic and chemical equilibrium

Large Nozzles

Real case is close to **equilibrium gas** results

Example: Ariane 5 launcher, main engine (Vulcain 2)

Chemical reactions: $H_2 + O_2 \rightarrow H_2O$ (in principle), but many different radicals and reactions involved (at least 10 species and 20 reactions)

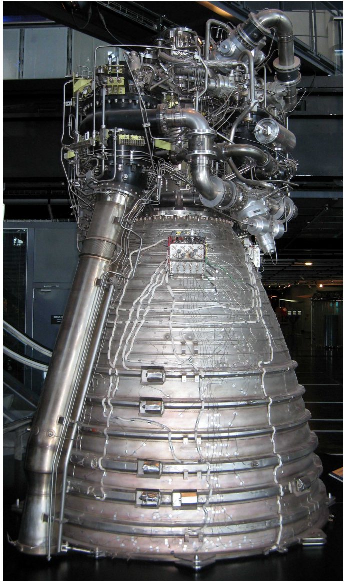
Nozzle inlet conditions:

$T_o \sim 3600\ K$

$p_o \sim 120\ bar$

Length scale $\sim$ a few meters

Gas mixture is quite close to equilibrium conditions all the way through the expansion



Small Nozzles

Low T_o, low p_o, lower reactivity

Real case is close to frozen flow results

Example:

Small rockets on satellites (for maneuvering, orbital adjustments, etc)

Small Nozzle With High-Speed Flow

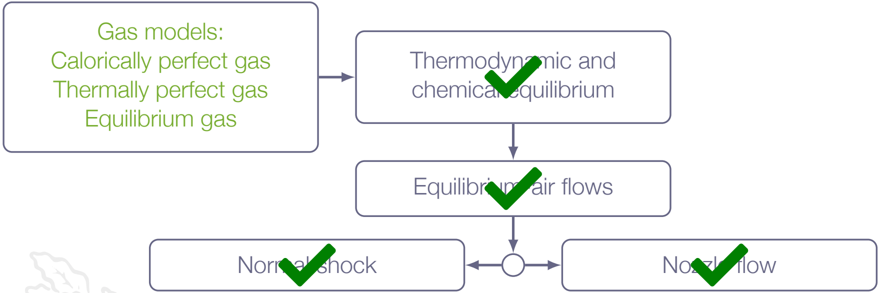
High-speed flows (short flow time scales) $\Rightarrow$ **thermodynamic non-equilibrium**

Very slow (or no) chemical reactions $\Rightarrow$ **frozen composition**

The residence time is to short for the vibrational energy of the molecules to change $\Rightarrow$ **Vibrationally frozen flow**

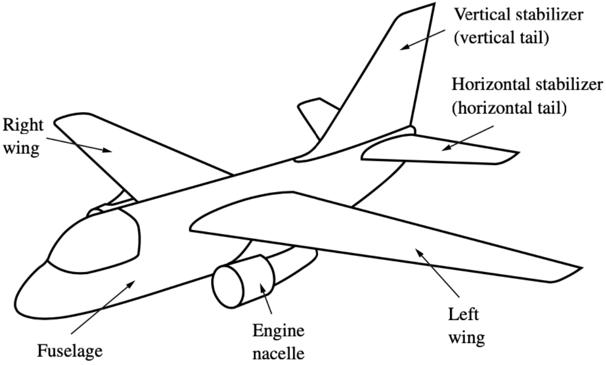
Only translational and rotational energy $\Rightarrow$ **Calorically perfect gas!**

Roadmap - High Temperature Effects

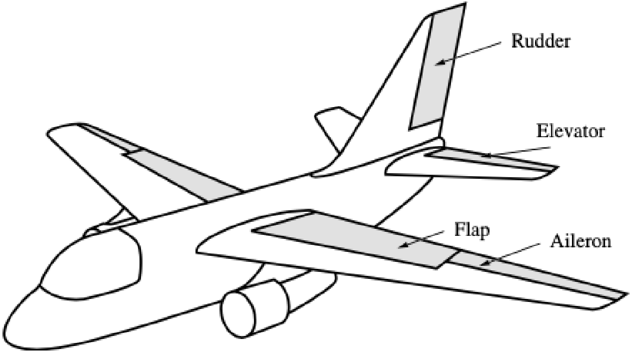


Aircraft Aerodynamics

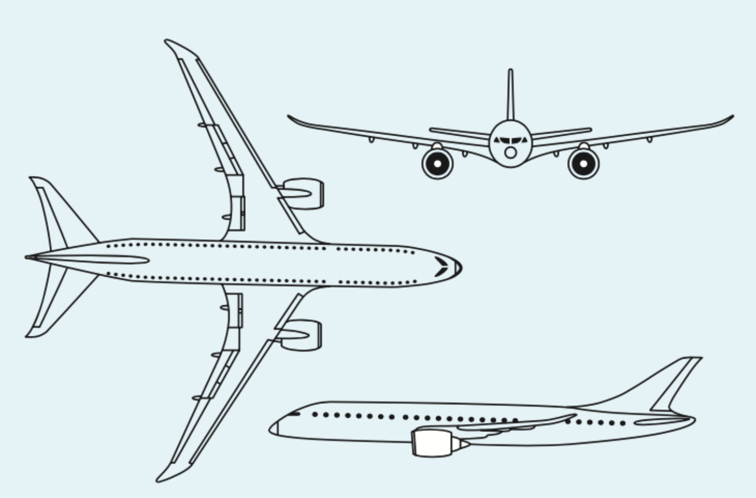
Control Surfaces



Control Surfaces



Control Surfaces

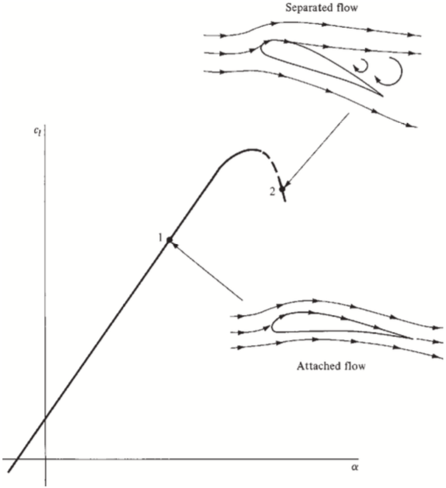


Lift and Drag

$$C_L = \frac{F_L}{\frac{1}{2}\rho U_\infty^2 A_p}$$

$$C_D = \frac{F_D}{\frac{1}{2}\rho U_\infty^2 A_p}$$

where A_p is the planform area



Lift and Drag

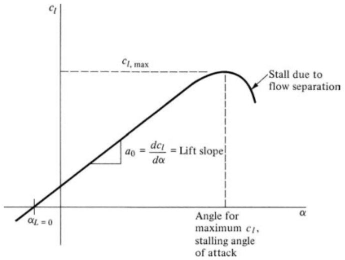
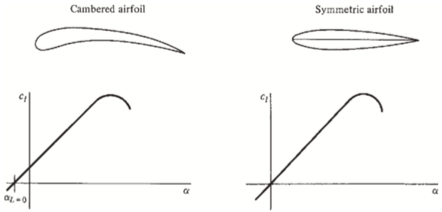
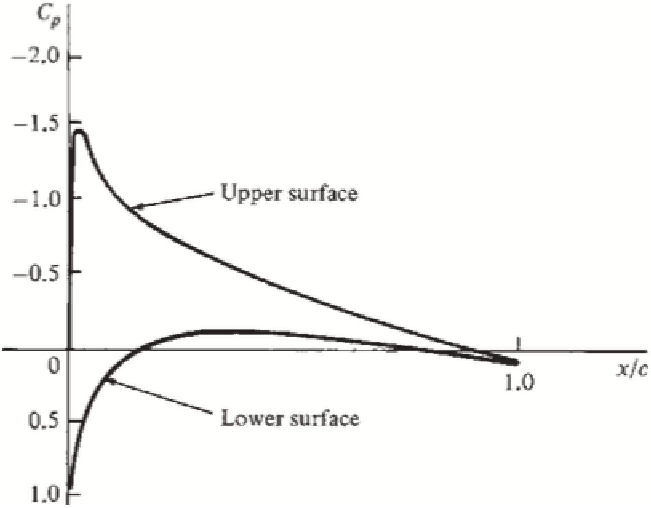


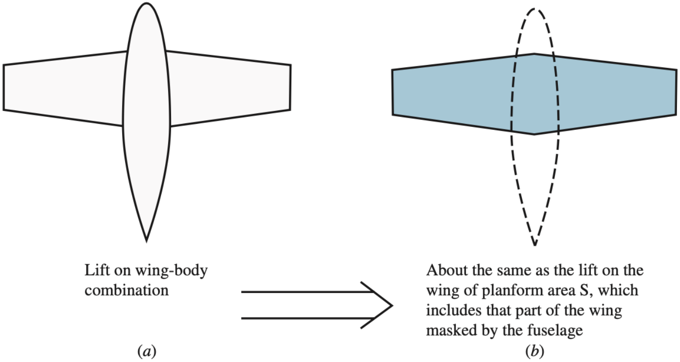
Figure 5.6 Sketch of a typical lift curve.



Lift and Drag - Pressure Coefficient



Lift and Drag - Fuselage Lift



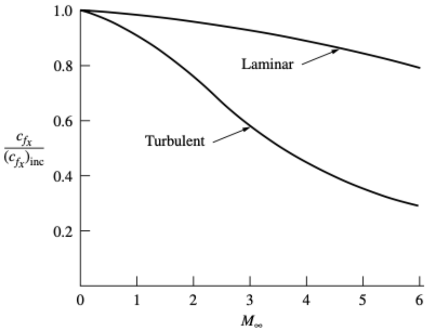
Lift and Drag

$$D = D_{pressure} + D_{friction} + D_{wave}$$

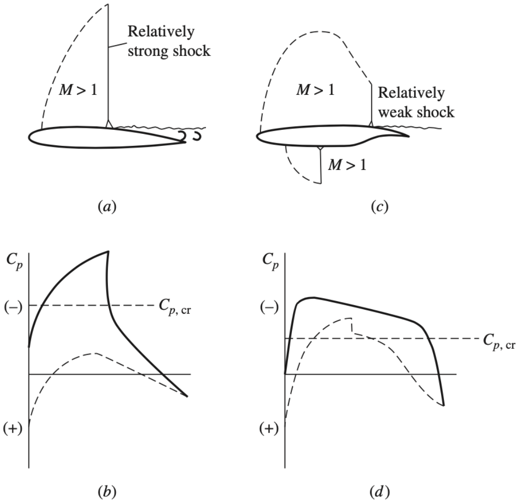
Friction Drag

laminar flow: $C_f = \dfrac{f_1(M_\infty)}{\sqrt{Re_x}}$

turbulent flow: $C_f = \dfrac{f_2(M_\infty)}{Re_x^{0.2}}$



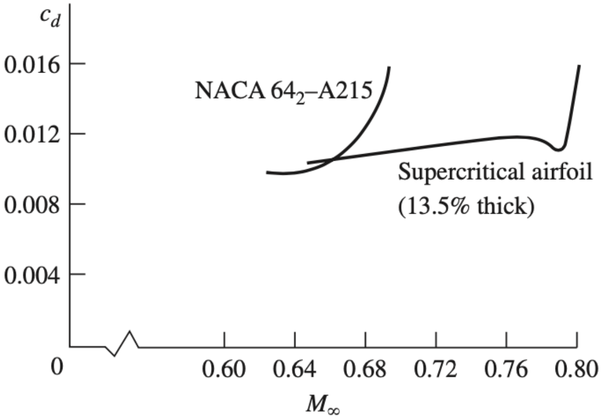
Wave Drag - The Supercritical Airfoil



NACA 64_2–A215 airfoil
$M_\infty = 0.69$

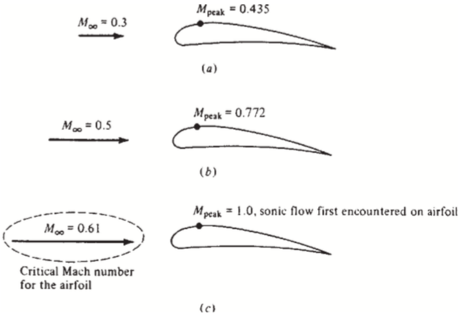
Supercritical airfoil (13.5% thick)
$M_\infty = 0.79$

Wave Drag - The Supercritical Airfoil

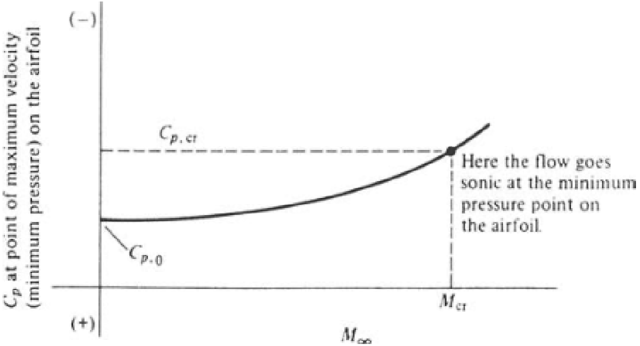


Critical Mach Number

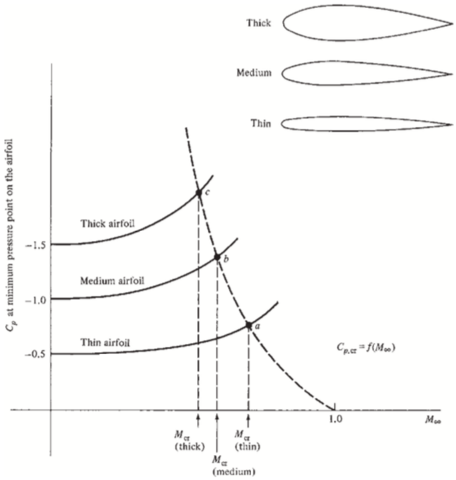
The critical Mach number is the lowest freestream Mach number for which the flow will accelerate to sonic conditions over the wing



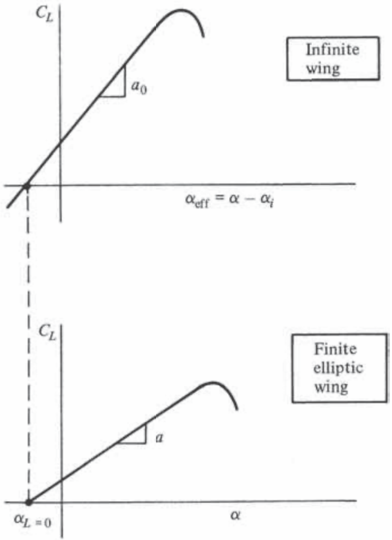
Critical Pressure Coefficient



Critical Pressure Coefficient



Finite Wing Span

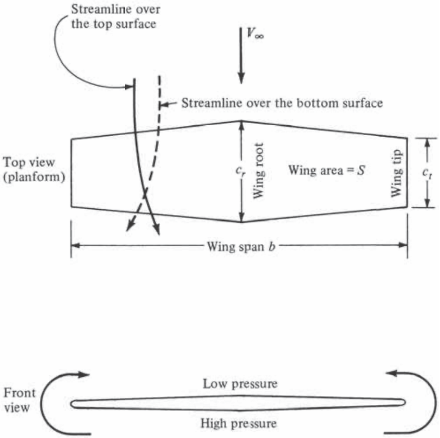


Induced Drag

The higher pressure on the lower side of the wing leads to a flow leakage over the wing tip

The flow below the wing has a velocity component towards the wing tip

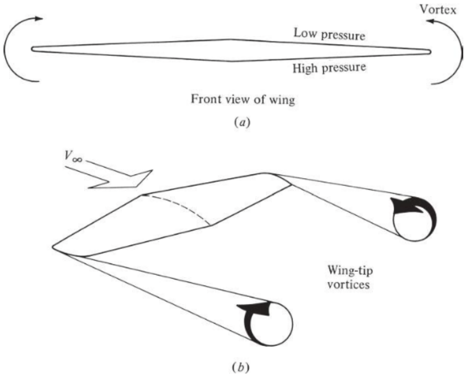
The flow over the wing has a velocity component towards the fuselage



Induced Drag

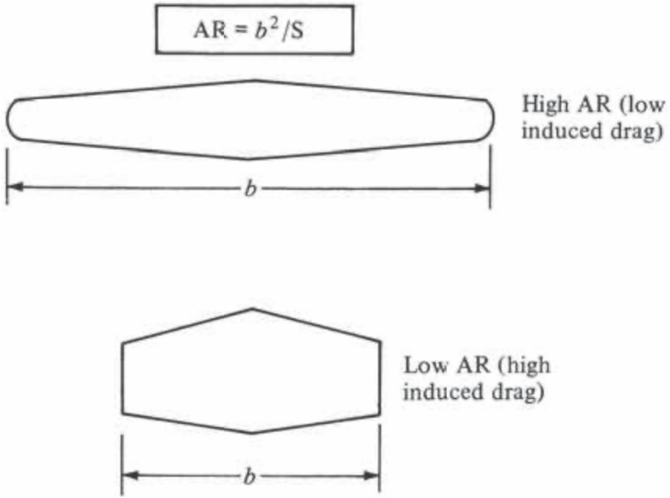
The flow from high pressure regions to low pressure regions forms a vortex at the wing tip

A net downwash flow is induced leading to a reduction of lift



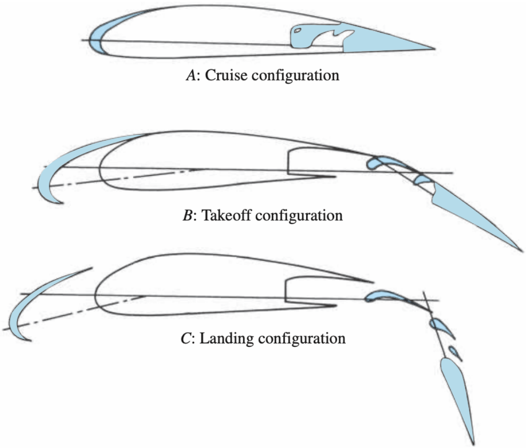
Induced Drag - Downwash

Induced Drag

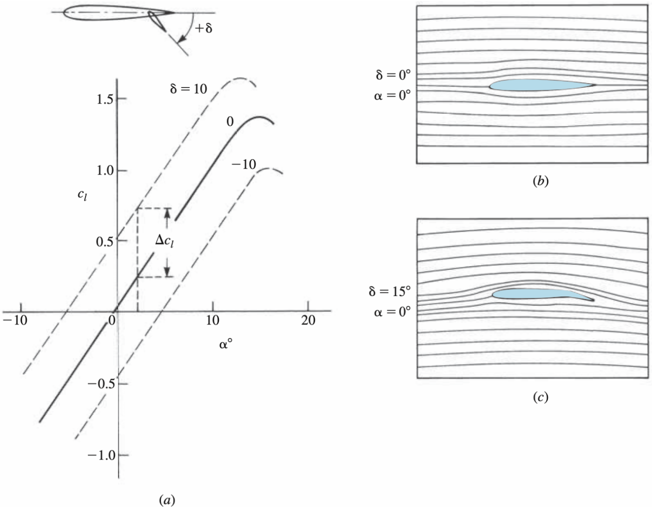


Induced Drag - Winglets

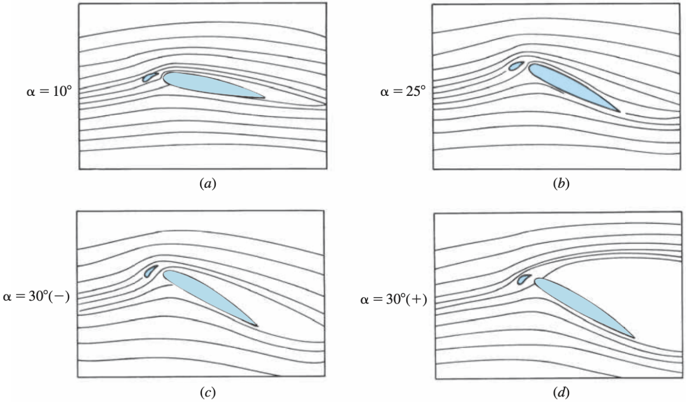
High-Lift Devices



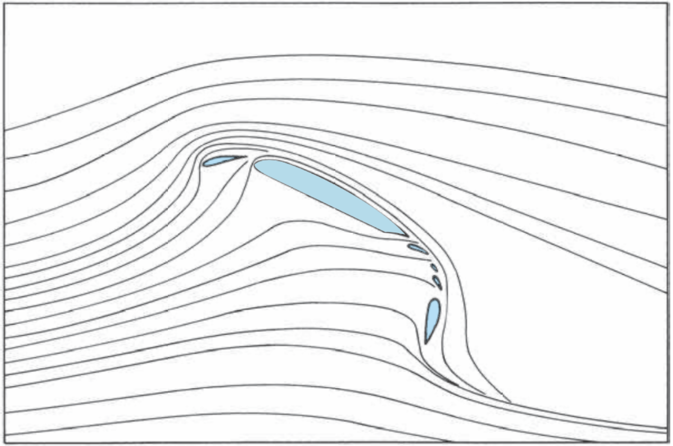
High-Lift Devices - Flaps



High-Lift Devices - Slats



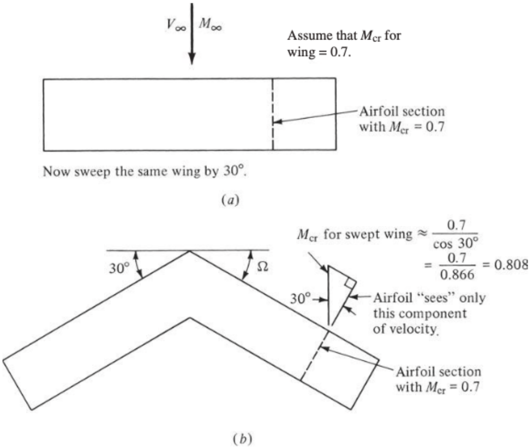
High-Lift Devices



$\alpha = 25°$

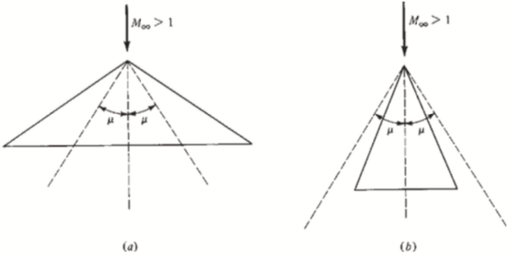
Swept Wings - Subsonic Aircraft

- The wing profile ”sees” a flow with the Mach number normal to the leading edge
- Increases the critical freestream Mach number
- Possible to operate at higher Mach number with lower drag
- Comes with the price of lower lift



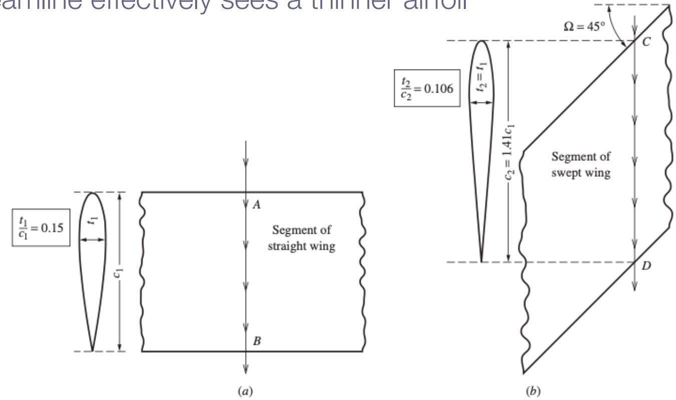
Swept Wings - Supersonic Aircraft

- If the wing is within the Mach angle cone, the trailing-edge-normal flow is subsonic



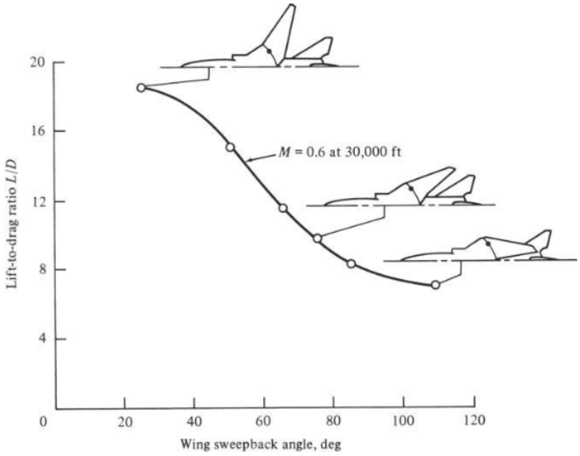
Swept Wings

- A swept wing leads to a longer coord in the flow direction
- With a swept wing, a streamline effectively sees a thinner airfoil

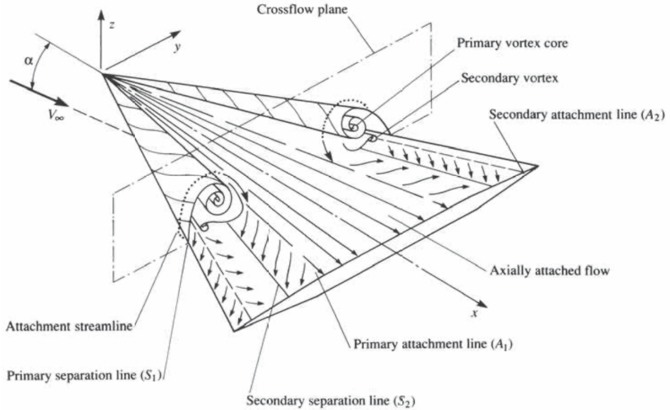


Swept Wings

Wing sweep reduces drag but there is also a significant reduction of lift

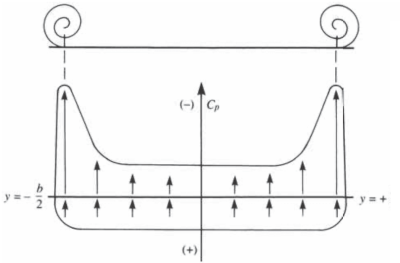


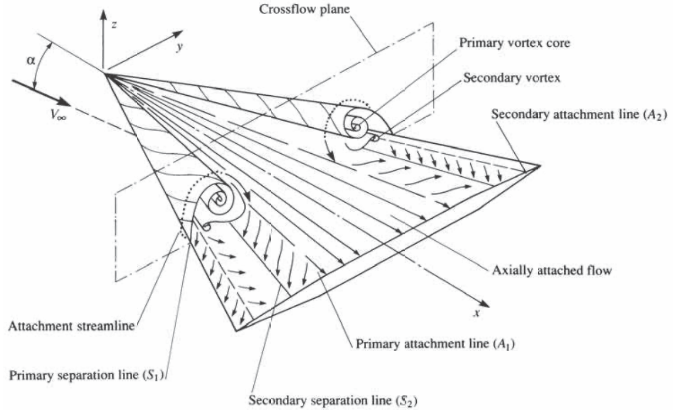
The Delta Wing



leakage of flow from high-pressure regions to low-pressure regions leads to the formation of vortices on the upper side of the wing

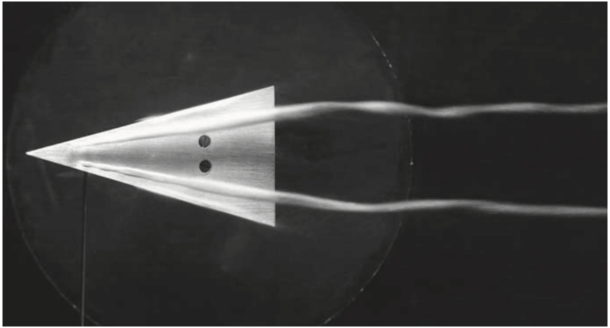
The Delta Wing





The vortical structures on the upper side of the wing reduces the pressure and increases lift

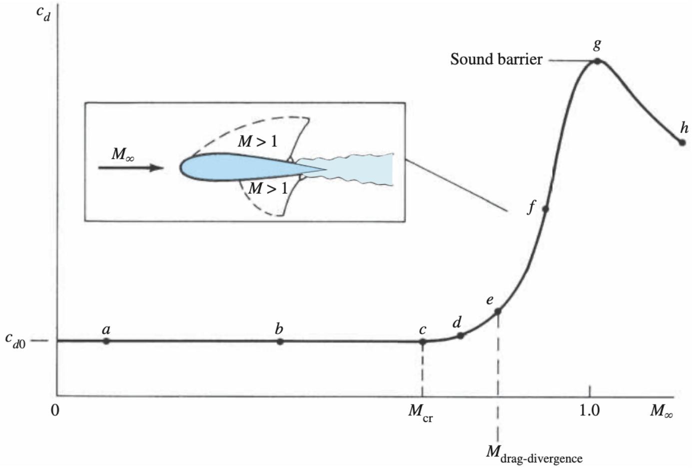
The Delta Wing



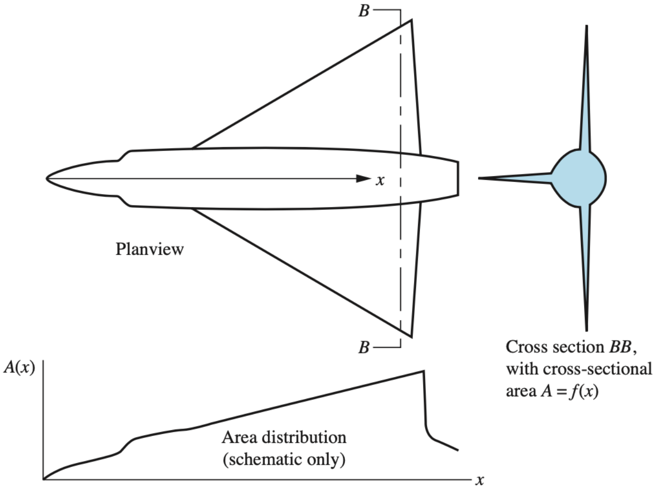
Visualization of vortex structures over a delta wing in a water tunnel experiment

The Delta Wing

The Sound Barrier

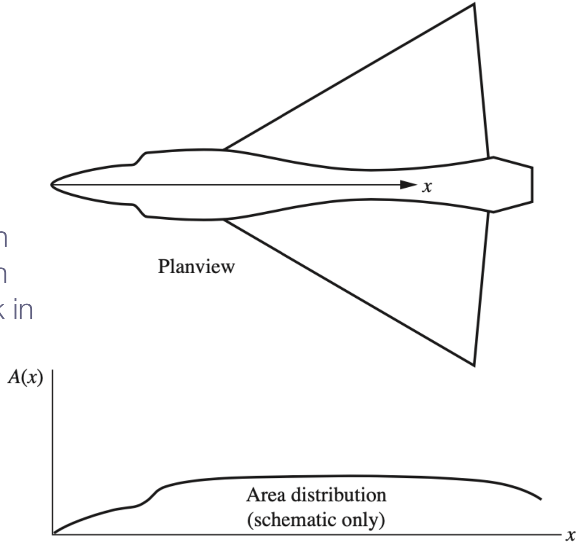


Area Rule

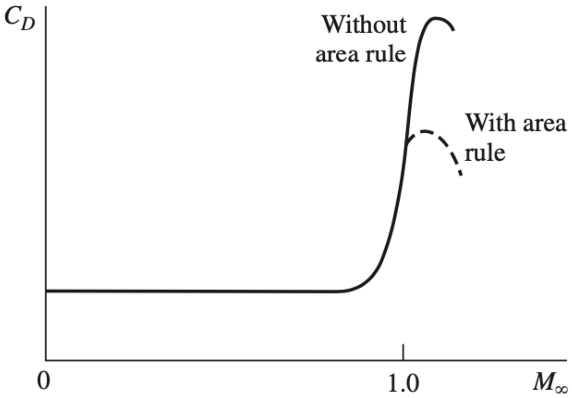

B
x
Planview
B
A(x)
Area distribution
(schematic only)
x
Cross section BB,
with cross-sectional
area A = f(x)

Area Rule

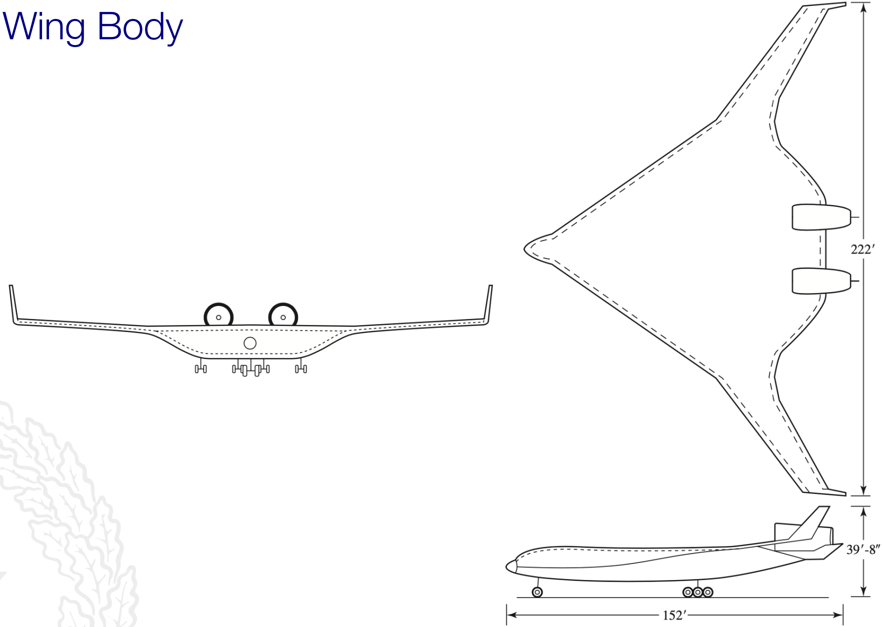
Designing the whole aircraft such that the variation in cross-section area is smooth reduces the peak in drag near Mach 1



Area Rule



Blended Wing Body



Blended Wing Body