Compressible Flow - TME085

Lecture 9

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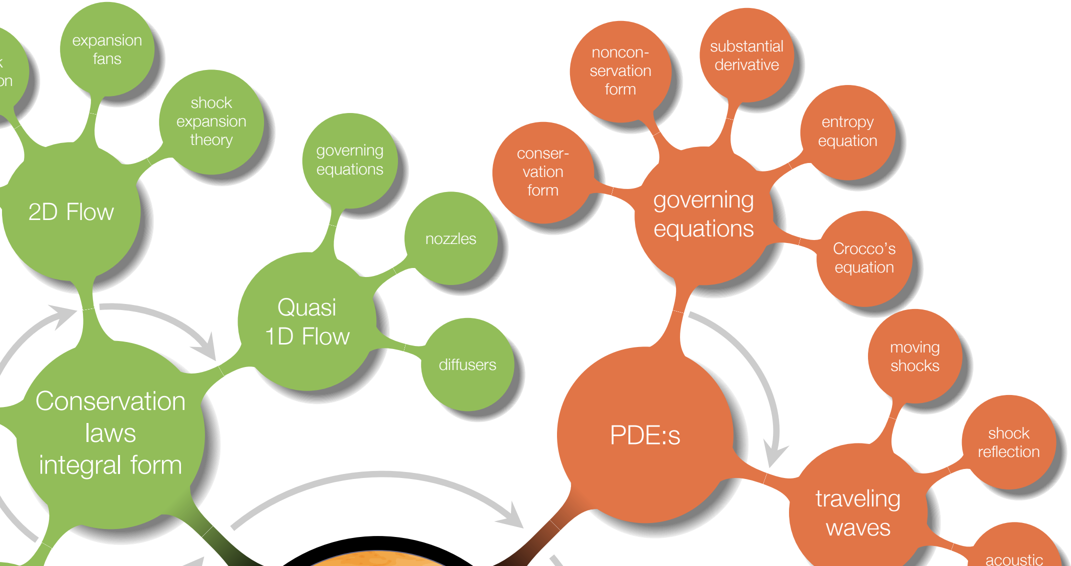
Chapter 6
Differential Conservation Equations for Inviscid Flows

Overview



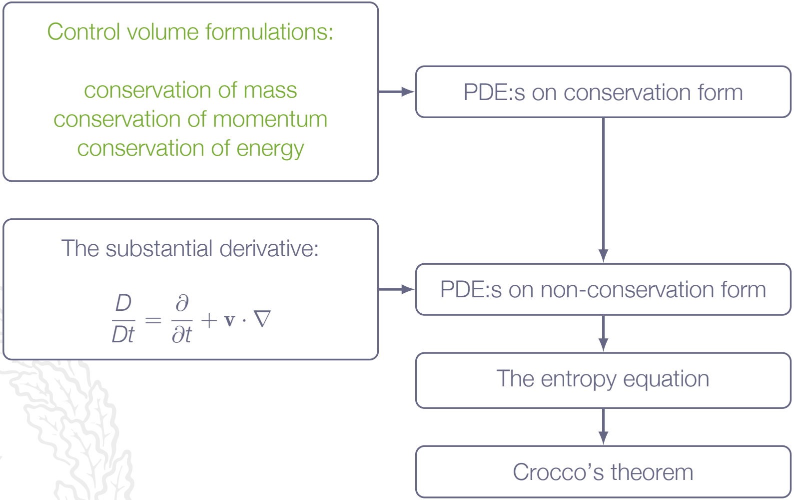
expansion fans
shock expansion theory
2D Flow
governing equations
nozzles
Quasi 1D Flow
diffusers
Conservation laws integral form
noncon-servation form
substantial derivative
entropy equation
conser-vation form
governing equations
Crocco's equation
PDE:s
moving shocks
shock reflection
traveling waves
acoustic

Learning Outcomes

4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on

the governing equations for compressible flows on differential form - finally ...

Roadmap - Differential Equations for Inviscid Flows

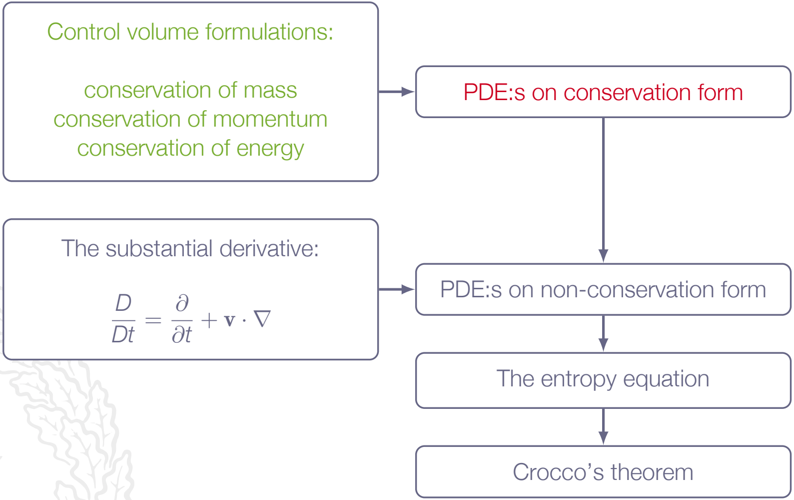


Motivation

The differential form of the conservation equations is needed when analyzing unsteady problems

The differential form of the conservation equations forms the basis for multi-dimensional analysis and CFD

Roadmap - Differential Equations for Inviscid Flows



Chapter 6.2 Differential Equations in Conservation Form

Differential Equations in Conservation Form

Basic principle to derive PDE:s in conservation form:

- Start with control volume formulation
- Convert to volume integral via Gauss Theorem
- Arbitrary control volume implies that integrand equals to zero everywhere

Continuity Equation - Conservation of Mass

Control volume formulation

$$\frac{d}{dt}\iiint_{\Omega}\rho d\mathscr{V} + \oint\!\!\oint_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS = 0$$

where Ω is a fixed control volume and thus $\frac{d}{dt}\iiint_{\Omega}\rho d\mathscr{V} = \iiint_{\Omega}\frac{\partial\rho}{\partial t}d\mathscr{V}$

Applying Gauss' Theorem on the surface integral gives

$$\oint\!\!\oint_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS = \iiint_{\Omega}\nabla\cdot(\rho\mathbf{v})d\mathscr{V}$$

Continuity Equation

Therefore

$$\iiint_{\Omega} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] d\mathscr{V} = 0$$

Ω is an arbitrary control volume, can be made infinitesimally small and thus

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0}$$

which is the continuity equation

Momentum Equation - Conservation of Momentum

Control volume formulation

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho\mathbf{v}d\mathscr{V}+\oiint\limits_{\partial\Omega}\left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}+p\mathbf{n}\right]dS=\iiint\limits_{\Omega}\rho\mathbf{f}d\mathscr{V}$$

where Ω is a fixed control volume and thus $\displaystyle\frac{d}{dt}\iiint\limits_{\Omega}\rho\mathbf{v}d\mathscr{V}=\iiint\limits_{\Omega}\frac{\partial}{\partial t}(\rho\mathbf{v})d\mathscr{V}$

Applying Gauss' Theorem on the surface integrals gives

$$\oiint\limits_{\partial\Omega}\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}dS=\iiint\limits_{\Omega}\nabla\cdot(\rho\mathbf{v}\mathbf{v})d\mathscr{V}\ ;\ \oiint\limits_{\partial\Omega}p\mathbf{n}dS=\iiint\limits_{\Omega}\nabla p d\mathscr{V}$$

Momentum Equation

Therefore

$$\iiint_{\Omega} \left[\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) + \nabla p - \rho\mathbf{f}\right] d\mathscr{V} = 0$$

Ω is an arbitrary control volume, can be made infinitesimally small and thus

$$\boxed{\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) + \nabla p = \rho\mathbf{f}}$$

which is the momentum equation

Momentum Equation

In cartesian form ($\mathbf{v} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$):

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho u \mathbf{v}) + \frac{\partial p}{\partial x} = \rho f_x$$
$$\frac{\partial}{\partial t}(\rho v) + \nabla \cdot (\rho v \mathbf{v}) + \frac{\partial p}{\partial y} = \rho f_y$$
$$\frac{\partial}{\partial t}(\rho w) + \nabla \cdot (\rho w \mathbf{v}) + \frac{\partial p}{\partial z} = \rho f_z$$

Momentum Equation

or expanded:

$$
\begin{aligned}
\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) + \frac{\partial p}{\partial x} = \rho f_x \\
\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho vu) + \frac{\partial}{\partial y}(\rho vv) + \frac{\partial}{\partial z}(\rho vw) + \frac{\partial p}{\partial y} = \rho f_y \\
\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho wu) + \frac{\partial}{\partial y}(\rho wv) + \frac{\partial}{\partial z}(\rho ww) + \frac{\partial p}{\partial z} = \rho f_z
\end{aligned}
$$

Momentum Equation

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla p = \rho \mathbf{f}$$

$$\begin{bmatrix} (\rho uu + p) & \rho uv & \rho uw \\ \rho vu & (\rho vv + p) & \rho vw \\ \rho wu & \rho wv & (\rho ww + p) \end{bmatrix} = \rho \mathbf{v}\mathbf{v} + p\mathbf{I}$$

$$\boxed{\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v} + p\mathbf{I}) = \rho \mathbf{f}}$$

Energy Equation - Conservation of Energy

Control volume formulation

$$\frac{d}{dt}\iiint_{\Omega} \rho e_o d\mathscr{V} + \oiint_{\partial\Omega} \rho h_o(\mathbf{v}\cdot\mathbf{n})dS = \iiint_{\Omega} \rho\mathbf{f}\cdot\mathbf{v}d\mathscr{V}$$

where Ω is a fixed control volume and thus $\frac{d}{dt}\iiint_{\Omega} \rho e_o d\mathscr{V} = \iiint_{\Omega} \frac{\partial}{\partial t}(\rho e_o)d\mathscr{V}$

Applying Gauss' Theorem on the surface integral gives

$$\oiint_{\partial\Omega} \rho h_o(\mathbf{v}\cdot\mathbf{n})dS = \iiint_{\Omega} \nabla\cdot(\rho h_o\mathbf{v})d\mathscr{V}$$

Energy Equation

Therefore

$$\iiint_{\Omega} \left[\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho h_o \mathbf{v}) - \rho(\mathbf{f} \cdot \mathbf{v}) \right] d\mathscr{V} = 0$$

Ω is an arbitrary control volume, can be made infinitesimally small and thus

$$\boxed{\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho h_o \mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v})}$$

which is the energy equation

Partial Differential Equations in Conservation Form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla p = \rho \mathbf{f}$$

$$\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho h_o \mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v})$$

These equations are referred to as PDE:s on conservation form since they stem directly from the integral conservation equations applied to a fixed control volume

Roadmap - Differential Equations for Inviscid Flows

Control volume formulations:

conservation of mass
conservation of momentum
conservation of energy

The substantial derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

PDE:s on conservation form ✓

PDE:s on non-conservation form

The entropy equation

Crocco's theorem

The Substantial Derivative

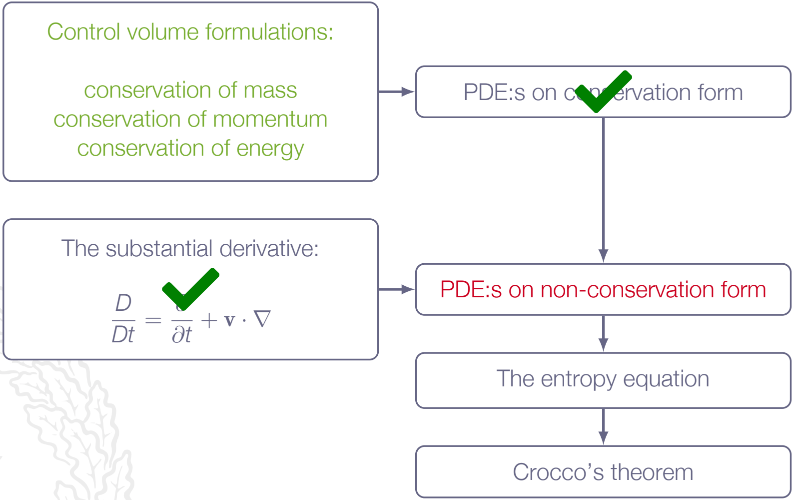
Introducing the substantial derivative operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

"... the time rate of change of any quantity associated with a particular moving fluid element is given by the substantial derivative ..."

"... the properties of the fluid element are changing as it moves past a point in a flow because the flowfield itself may be fluctuating with time (the local derivative) and because the fluid element is simply on its way to another point in the flowfield where the properties are different (the convective derivative) ..."

Roadmap - Differential Equations for Inviscid Flows



Chapter 6.4
Differential Equations in Non-Conservation Form

Non-Conservation Form of the Continuity Equation

Applying the substantial derivative operator to density gives

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho (\nabla \cdot \mathbf{v}) = 0 \Rightarrow$$

$$\boxed{\frac{D\rho}{Dt} + \rho (\nabla \cdot \mathbf{v}) = 0}$$

Non-Conservation Form of the Continuity Equation

$$\boxed{\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0}$$

"... the mass of a fluid element made up of a fixed set of particles (molecules or atoms) is constant as the fluid element moves through space ..."

Non-Conservation Form of the Momentum Equation

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v} + p\mathbf{I}) = \rho\mathbf{f} \Rightarrow$$

$$\rho\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial\rho}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} + \mathbf{v}(\nabla\cdot\rho\mathbf{v}) + \nabla p = \rho\mathbf{f} \Rightarrow$$

$$\underbrace{\rho\left[\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right]}_{=\frac{D\mathbf{v}}{Dt}} + \mathbf{v}\underbrace{\left[\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{v}\right]}_{=0} + \nabla p = \rho\mathbf{f}$$

$$\boxed{\frac{D\mathbf{v}}{Dt} + \frac{1}{\rho}\nabla p = \mathbf{f}}$$

Non-Conservation Form of the Energy Equation

$$\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho h_o \mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v}) + \rho \dot{q}$$

$$h_o = e_o + \frac{p}{\rho} \Rightarrow$$

$$\frac{\partial}{\partial t}(\rho e_o) + \nabla \cdot (\rho e_o \mathbf{v}) + \nabla \cdot (p \mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v}) + \rho \dot{q} \Rightarrow$$

$$\rho \frac{\partial e_o}{\partial t} + e_o \frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \nabla e_o + e_o \nabla \cdot (\rho \mathbf{v}) + \nabla \cdot (p \mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v}) + \rho \dot{q} \Rightarrow$$

$$\underbrace{\rho \left[\frac{\partial e_o}{\partial t} + \mathbf{v} \cdot \nabla e_o\right]}_{=\frac{De_o}{Dt}} + e_o \underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right]}_{=0} + \nabla \cdot (p \mathbf{v}) = \rho(\mathbf{f} \cdot \mathbf{v}) + \rho \dot{q}$$

Non-Conservation Form of the Energy Equation

$$\rho \frac{De_o}{Dt} + \nabla \cdot (p + \mathbf{v}) = \rho \mathbf{f} \cdot \mathbf{v} + \rho \dot{q}$$

$$e_o = e + \frac{1}{2}\mathbf{v} \cdot \mathbf{v} \Rightarrow$$

$$\rho \frac{De}{Dt} + \rho \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} + \nabla \cdot (p\mathbf{v}) = \rho \mathbf{f} \cdot \mathbf{v} + \rho \dot{q}$$

Using the momentum equation, $\left(\frac{D\mathbf{v}}{Dt} + \frac{1}{\rho}\nabla p = \mathbf{f}\right)$, gives

$$\rho \frac{De}{Dt} - \mathbf{v} \cdot \nabla p + \rho \mathbf{f} \cdot \mathbf{v} + \mathbf{v} \cdot \nabla p + p(\nabla \cdot \mathbf{v}) = \rho \mathbf{f} \cdot \mathbf{v} + \rho \dot{q} \Rightarrow$$

$$\boxed{\frac{De}{Dt} + \frac{p}{\rho}(\nabla \cdot \mathbf{v}) = \dot{q}}$$

Non-Conservation Form of the Energy Equation

$$\frac{De}{Dt} + \frac{p}{\rho}(\nabla \cdot \mathbf{v}) = \dot{q}$$

From the continuity equation we get

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0 \Rightarrow \nabla \cdot \mathbf{v} = -\frac{1}{\rho}\frac{D\rho}{Dt} \Rightarrow$$

$$\frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = \dot{q} \Rightarrow \frac{De}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = \dot{q}$$

$$\boxed{\frac{De}{Dt} = \dot{q} - p\frac{D\nu}{Dt}}$$

where $\nu = 1/\rho$

Non-Conservation Form of the Energy Equation

Compare with first law of thermodynamics: $de = \delta q - \delta W$

$$\boxed{\frac{De}{Dt} = \dot{q} - p\frac{D\nu}{Dt}}$$

Non-Conservation Form of the Energy Equation

If we instead express the energy equation in terms of enthalpy:

$$\frac{De}{Dt} = \dot{q} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right) \Rightarrow \frac{De}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = \dot{q}$$

$$h = e + \frac{p}{\rho} \Rightarrow \frac{Dh}{Dt} = \frac{De}{Dt} + \frac{1}{\rho}\frac{Dp}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) \Rightarrow$$

$$\boxed{\frac{Dh}{Dt} = \dot{q} + \frac{1}{\rho}\frac{Dp}{Dt}}$$

Non-Conservation Form of the Energy Equation

and total enthalpy ...

$$h_o = h + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} \Rightarrow \frac{Dh_o}{Dt} = \frac{Dh}{Dt} + \mathbf{v}\cdot\frac{D\mathbf{v}}{Dt}$$

From the momentum equation we get

$$\rho\frac{D\mathbf{v}}{Dt} + \nabla p = \rho\mathbf{f} \Rightarrow \frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{f} \Rightarrow$$

$$\frac{Dh_o}{Dt} = \underbrace{\frac{Dh}{Dt}}_{\dot{q}+\frac{1}{\rho}\frac{Dp}{Dt}} - \frac{1}{\rho}\mathbf{v}\cdot\nabla p + \mathbf{f}\cdot\mathbf{v} = \dot{q} + \frac{1}{\rho}\left[\frac{Dp}{Dt} - \mathbf{v}\cdot\nabla p\right] + \mathbf{f}\cdot\mathbf{v}$$

Non-Conservation Form of the Energy Equation

$$\frac{Dh_o}{Dt} = \dot{q} + \frac{1}{\rho}\left[\frac{Dp}{Dt} - \mathbf{v}\cdot\nabla p\right] + \mathbf{f}\cdot\mathbf{v}$$

Now, expanding the substantial derivative $\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p$ gives

$$\boxed{\frac{Dh_o}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial t} + \dot{q} + \mathbf{f}\cdot\mathbf{v}}$$

Let's examine the above relation ...

Non-Conservation Form of the Energy Equation

$$\frac{Dh_o}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial t} + \dot{q} + \mathbf{f} \cdot \mathbf{v}$$

The total enthalpy of a moving fluid element in an inviscid flow can change due to

- unsteady flow: $\partial p/\partial t \neq 0$
- heat transfer: $\dot{q} \neq 0$
- body forces: $\mathbf{f} \cdot \mathbf{v} \neq 0$

Non-Conservation Form of the Energy Equation

Adiabatic flow without body forces $\Rightarrow$

$$\frac{Dh_o}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial t}$$

Steady-state adiabatic flow without body forces $\Rightarrow$

$$\frac{Dh_o}{Dt} = 0$$

h_o is constant along streamlines!

Additional Form of the Energy Equation

Start from

$$\frac{De}{Dt} = \dot{q} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

Calorically perfect gas:

$$e = C_v T \ ; \ C_v = \frac{R}{\gamma - 1} \ ; \ p = \rho R T \ ; \ \gamma, R = const$$

$$\frac{De}{Dt} = C_v\frac{DT}{Dt} = \frac{R}{\gamma - 1}\frac{D}{Dt}\left(\frac{p}{\rho R}\right) = \frac{1}{\gamma - 1}\frac{D}{Dt}\left(\frac{p}{\rho}\right) \Rightarrow \frac{1}{\gamma - 1}\frac{D}{Dt}\left(\frac{p}{\rho}\right) = \dot{q} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

Additional Form of the Energy Equation

$$\frac{1}{\gamma-1}\frac{D}{Dt}\left(\frac{p}{\rho}\right)=\dot{q}-p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\Rightarrow$$

$$\frac{1}{\gamma-1}\left[p\frac{D}{Dt}\left(\frac{1}{\rho}\right)+\left(\frac{1}{\rho}\right)\frac{Dp}{Dt}\right]=\dot{q}-p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

$$p\frac{D}{Dt}\left(\frac{1}{\rho}\right)+\left(\frac{1}{\rho}\right)\frac{Dp}{Dt}=(\gamma-1)\dot{q}-(\gamma-1)p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

$$\gamma p\frac{D}{Dt}\left(\frac{1}{\rho}\right)+\left(\frac{1}{\rho}\right)\frac{Dp}{Dt}=(\gamma-1)\dot{q}$$

Additional Form of the Energy Equation

Continuity:

$$\frac{D\rho}{Dt} = -\rho(\nabla \cdot \mathbf{v}) \Rightarrow \frac{D}{Dt}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\frac{D\rho}{Dt} = \frac{1}{\rho}(\nabla \cdot \mathbf{v}) \Rightarrow$$

$$\frac{\gamma p}{\rho}(\nabla \cdot \mathbf{v}) + \left(\frac{1}{\rho}\right)\frac{Dp}{Dt} = (\gamma - 1)\dot{q}$$

Additional Form of the Energy Equation

$$\frac{Dp}{Dt} + \gamma p(\nabla \cdot \mathbf{v}) = (\gamma - 1)\rho\dot{q}$$

Adiabatic flow (no added heat):

$$\boxed{\frac{Dp}{Dt} + \gamma p(\nabla \cdot \mathbf{v}) = 0}$$

Non-conservation form (calorically perfect gas)

Conservation Form

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial G}{\partial z} = 0$$

where $Q(x, y, z, t)$, $E(x, y, z, t)$, ... may be scalar or vector fields

Example: the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

If an equation cannot be written in this form, it is said to be in non-conservation form

Euler Equations - Conservation Form

Continuity, momentum and energy equations in Cartesian coordinates, velocity components u, v, w (no body forces, no added heat)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u u + p) + \frac{\partial}{\partial y}(\rho u v) + \frac{\partial}{\partial z}(\rho u w) = 0$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v u) + \frac{\partial}{\partial y}(\rho v v + p) + \frac{\partial}{\partial z}(\rho v w) = 0$$

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho w u) + \frac{\partial}{\partial y}(\rho w v) + \frac{\partial}{\partial z}(\rho w w + p) = 0$$

$$\frac{\partial}{\partial t}(\rho e_o) + \frac{\partial}{\partial x}(\rho h_o u) + \frac{\partial}{\partial y}(\rho h_o v) + \frac{\partial}{\partial z}(\rho h_o w) = 0$$

Euler Equations - Non-Conservation Form

Continuity, momentum and energy equations in Cartesian coordinates, velocity components u, v, w (no body forces, no added heat), calorically perfect gas

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} + \gamma p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

Conservation and Non-Conservation Form

The governing equations on non-conservation form are not, although the name might give that impression, less physically accurate than the equations on conservation form. The nomenclature comes from CFD where the equations on conservation form are preferred.

Using the conservation form as a basis for a Finite-Volume Method (FVM) solver ensures conservation of mass, momentum and energy.

Conservation and Non-Conservation Form

- Conservative equations are equations that directly stems from conservation of flow quantities over a control volume
- The equations on non-conservation form are derived from the corresponding equations on conservation form using the chain rule for derivatives
- Thus the equations on non-conservation form do not stem directly from a conservation law - but aren't the two formulations still equivalent?
- Only for continuous solutions! The chain rule can only be used for continuous fields

Conservation and Non-Conservation Form

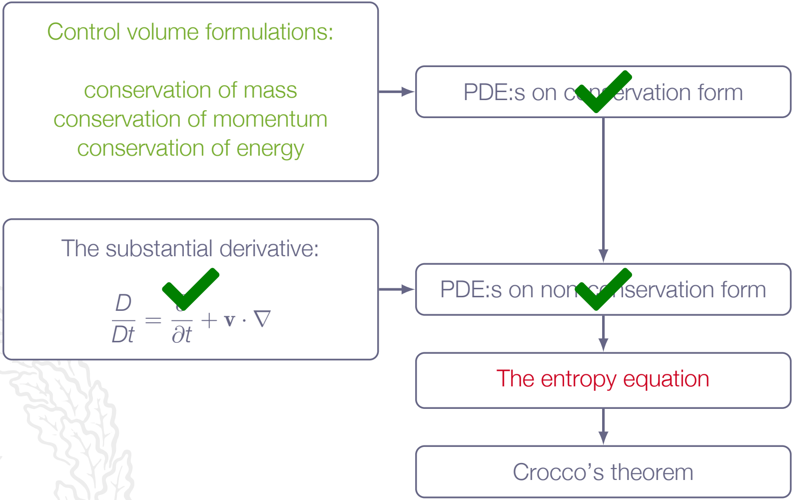
Conservation forms are useful for:

1. Numerical methods for compressible flow
2. Theoretical understanding of non-linear waves (shocks etc)
3. Provide link between integral forms (control volume formulations) and PDE:s

Non-conservation forms are useful for:

1. Theoretical understanding of behavior of numerical methods
2. Theoretical understanding of boundary conditions
3. Analysis of linear waves (aero-acoustics)

Roadmap - Differential Equations for Inviscid Flows



Chapter 6.5
The Entropy Equation

The Entropy Equation

From the first and second law of thermodynamics we have

$$\frac{De}{Dt} = T\frac{Ds}{Dt} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

which is called the entropy equation

The Entropy Equation

Compare the entropy equation

$$\frac{De}{Dt} = T\frac{Ds}{Dt} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

with the energy equation (inviscid flow):

$$\frac{De}{Dt} = \dot{q} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

we see that

$$T\frac{Ds}{Dt} = \dot{q}$$

The Entropy Equation

If $\dot{q} = 0$ (adiabatic flow) then

$$\frac{Ds}{Dt} = 0$$

i.e., entropy is constant for moving fluid element

Furthermore, if the flow is steady we have

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + (\mathbf{v} \cdot \nabla)s = (\mathbf{v} \cdot \nabla)s = 0$$

i.e., entropy is constant along streamlines

Roadmap - Differential Equations for Inviscid Flows

Control volume formulations:

conservation of mass
conservation of momentum
conservation of energy

The substantial derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

PDE:s on conservation form

PDE:s on non-conservation form

The entropy equation

Crocco's theorem

Chapter 6.6
Crocco's Theorem

Crocco's Theorem

"... a relation between gradients of total enthalpy, gradients of entropy, and flow rotation ..."

Crocco's Theorem

Momentum equation (no body forces)

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p$$

Writing out the substantial derivative gives

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p \Rightarrow \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p$$

First and second law of thermodynamics (energy equation)

$$dh = Tds + \frac{1}{\rho} dp$$

Replace differentials with a gradient operator

$$\nabla h = T\nabla s + \frac{1}{\rho}\nabla p \Rightarrow T\nabla s = \nabla h - \frac{1}{\rho}\nabla p$$

Crocco's Theorem

With pressure derivative from the momentum equation inserted in the energy equation we get

$$T\nabla s = \nabla h + \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}$$

$$h = h_o - \frac{1}{2}\mathbf{v}\cdot\mathbf{v} \Rightarrow \nabla h = \nabla h_o - \nabla(\frac{1}{2}\mathbf{v}\cdot\mathbf{v})$$

$$\nabla(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}) = \mathbf{v}\times(\nabla\times\mathbf{v}) + \mathbf{v}\cdot\nabla\mathbf{v}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$

$$\mathbf{A} = \mathbf{B} = \mathbf{v} \Rightarrow \nabla(\mathbf{v}\cdot\mathbf{v}) = 2[\mathbf{v}\cdot\nabla\mathbf{v} + \mathbf{v}\times(\nabla\times\mathbf{v})]$$

Crocco's Theorem

$$T\nabla s = \nabla h_o - \mathbf{v} \times (\nabla \times \mathbf{v}) - \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$$

$$\boxed{T\nabla s = \nabla h_o + \frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v})}$$

Note! $\nabla \times \mathbf{v}$ is the vorticity of the fluid

the rotational motion of the fluid is described by the angular velocity $\omega = \frac{1}{2}(\nabla \times \mathbf{v})$

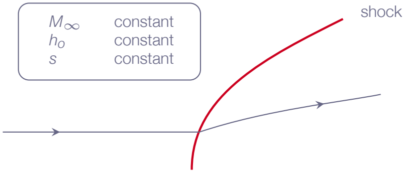
Crocco's Theorem

$$\boxed{T\nabla s = \nabla h_o + \frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v})}$$

"... when a steady flow field has gradients of total enthalpy and/or entropy Crocco's theorem dramatically shows that it is rotational ..."

Crocco's Theorem - Example

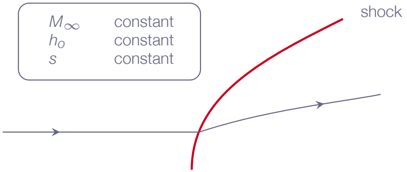
Curved stationary shock (steady-state flow)



- s is constant upstream of shock
- jump in s across shock depends on local shock angle
- s will vary from streamline to streamline downstream of shock
- $\nabla s \neq 0$ downstream of shock

Crocco's Theorem - Example

Curved stationary shock (steady-state flow)



- Total enthalpy upstream of shock
  - h_o is constant along streamlines
  - h_o is uniform
- Total enthalpy downstream of shock
  - h_o is uniform

$$\nabla h_o = 0$$

Crocco's Theorem - Example

Crocco's equation for steady-state flow:

$$T\nabla s = \nabla h_o - \mathbf{v} \times (\nabla \times \mathbf{v})$$

- $\mathbf{v} \times (\nabla \times \mathbf{v}) \neq 0$ downstream of a curved shock
- the rotation $\nabla \times \mathbf{v} \neq 0$ downstream of a curved shock

Explains why it is difficult to solve such problems by analytic means!

Roadmap - Differential Equations for Inviscid Flows

Control volume formulations:

conservation of mass
conservation of momentum
conservation of energy

The substantial derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

PDE:s on conservation form

PDE:s on non-conservation form

The entropy equation

Crocco's theorem