

# Compressible Flow - TME085

## Lecture 7

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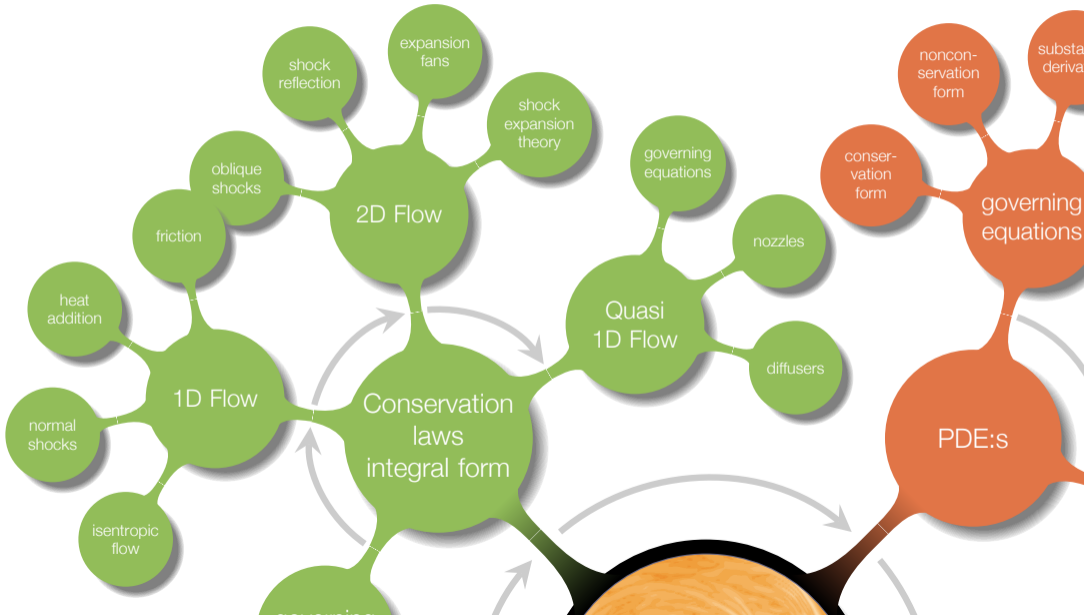


# Chapter 5

## Quasi-One-Dimensional Flow



# Overview

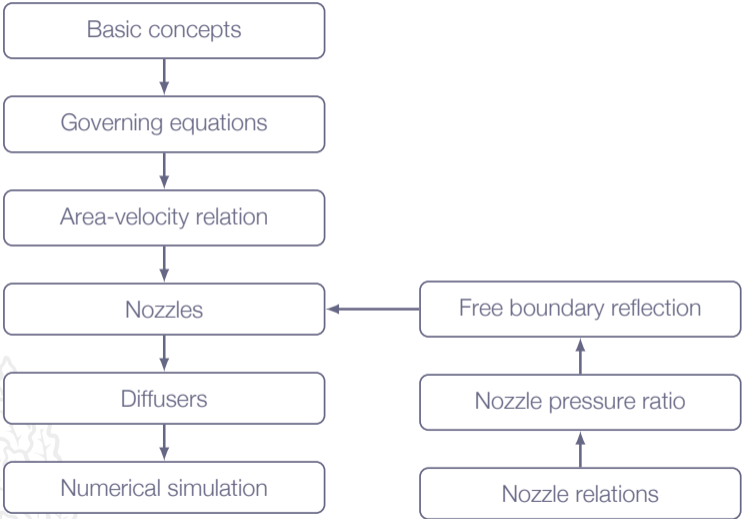


# Learning Outcomes

- 4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on
- 6 **Define** the special cases of calorically perfect gas, thermally perfect gas and real gas and **explain** the implication of each of these special cases
- 7 **Explain** why entropy is important for flow discontinuities
- 8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics
  - a 1D isentropic flow\*
  - b normal shocks\*
  - i detached blunt body shocks, nozzle flows
- 9 **Solve** engineering problems involving the above-mentioned phenomena (8a-8k)

*what does quasi-1D mean? either the flow is 1D or not, or?*

# Roadmap - Quasi-One-Dimensional Flow



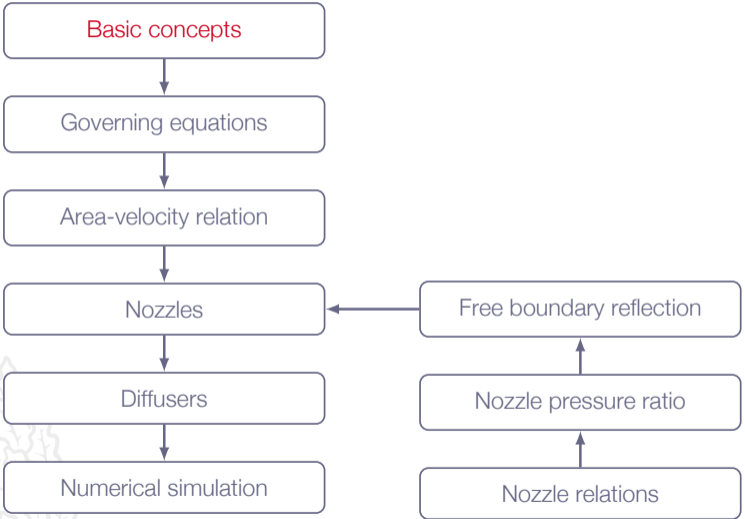
# Motivation

By extending the one-dimensional theory to quasi-one-dimensional, we can study important applications such as nozzles and diffusers

Even though the flow in nozzles and diffusers are in essence three dimensional we will be able to establish important relations using the quasi-one-dimensional approach



# Roadmap - Quasi-One-Dimensional Flow



# Quasi-One-Dimensional Flow

## Chapter 3 - One-dimensional steady-state flow

- ▶ overall assumption:
  - one-dimensional flow
  - constant cross section area
- ▶ applications:
  - normal shock
  - one-dimensional flow with heat addition
  - one-dimensional flow with friction

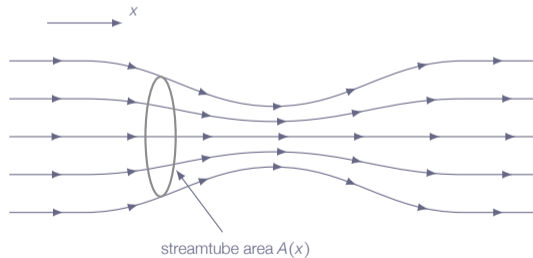
## Chapter 4 - Two-dimensional steady-state flow

- ▶ overall assumption:
  - two-dimensional flow
  - uniform supersonic freestream
- ▶ applications:
  - oblique shock
  - expansion fan
  - shock-expansion theory



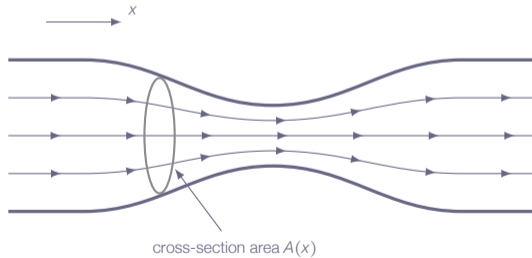
# Quasi-One-Dimensional Flow

- ▶ Extension of one-dimensional flow to allow **variations in streamtube area**
- ▶ Steady-state flow assumption still applied

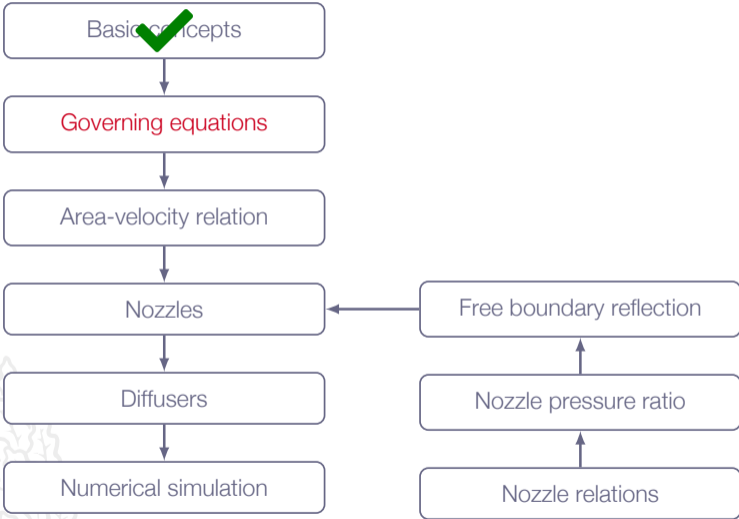


# Quasi-One-Dimensional Flow

Example: tube with variable cross-section area



# Roadmap - Quasi-One-Dimensional Flow



# Chapter 5.2

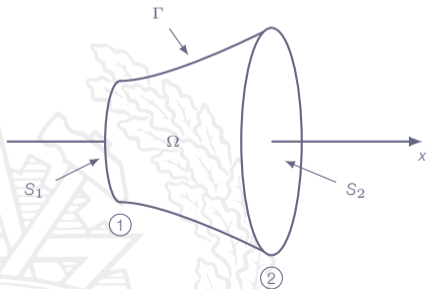
## Governing Equations



# Governing Equations

Introduce **cross-section-averaged flow quantities**  $\Rightarrow$   
all quantities depend on  $x$  only

$$A = A(x), \rho = \rho(x), u = u(x), p = p(x), \dots$$

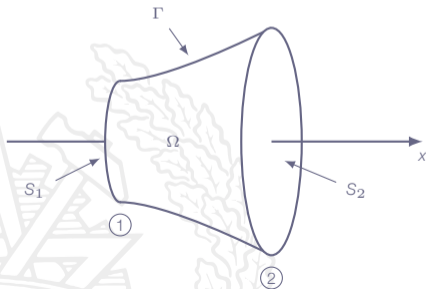


$\Omega$	control volume
$S_1$	left boundary (area $A_1$ )
$S_2$	right boundary (area $A_2$ )
$\Gamma$	perimeter boundary

$$\partial\Omega = S_1 \cup \Gamma \cup S_2$$

# Governing Equations - Assumptions

1. Inviscid flow (no boundary layers)
2. Steady-state flow (no unsteady effects)
3. No flow through  $\Gamma$  (control volume aligned with streamlines)



# Governing Equations - Conservation of Mass

$$\underbrace{\frac{d}{dt} \iiint_{\Omega} \rho d\mathcal{V}}_{=0} + \underbrace{\iint_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS}_{-\rho_1 u_1 A_1 + \rho_2 u_2 A_2} = 0$$

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

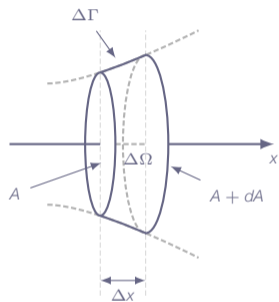


# Governing Equations - Conservation of Momentum

$$\underbrace{\frac{d}{dt} \iiint_{\Omega} \rho \mathbf{v} d\mathcal{V}}_{=0} + \iint_{\partial\Omega} [\rho(\mathbf{v} \cdot \mathbf{n})\mathbf{v} + p\mathbf{n}] dS = 0$$

$$\iint_{\partial\Omega} \rho(\mathbf{v} \cdot \mathbf{n})\mathbf{v} dS = -\rho_1 u_1^2 A_1 + \rho_2 u_2^2 A_2$$

$$\iint_{\partial\Omega} p\mathbf{n} dS = -p_1 A_1 + p_2 A_2 - \int_{A_1}^{A_2} p dA$$



$$(\rho_1 u_1^2 + p_1) A_1 + \int_{A_1}^{A_2} p dA = (\rho_2 u_2^2 + p_2) A_2$$



# Governing Equations - Conservation of Energy

$$\underbrace{\frac{d}{dt} \iiint_{\Omega} \rho e_o d\mathcal{V}}_{=0} + \iint_{\partial\Omega} [\rho h_o (\mathbf{v} \cdot \mathbf{n})] dS = 0$$

which gives

$$\rho_1 u_1 A_1 h_{o1} = \rho_2 u_2 A_2 h_{o2}$$

from continuity we have that  $\rho_1 u_1 A_1 = \rho_2 u_2 A_2 \Rightarrow$

$$h_{o1} = h_{o2}$$

# Governing Equations - Summary

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

$$(\rho_1 u_1^2 + p_1)A_1 + \int_{A_1}^{A_2} p dA = (\rho_2 u_2^2 + p_2)A_2$$

$$h_{o_1} = h_{o_2}$$



# Governing Equations - Differential Form

Continuity equation:

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2 \text{ or } \rho u A = c$$

where  $c$  is a constant  $\Rightarrow$

$$d(\rho u A) = 0$$



# Governing Equations - Differential Form

Momentum equation:

$$(\rho_1 u_1^2 + p_1)A_1 + \int_{A_1}^{A_2} p dA = (\rho_2 u_2^2 + p_2)A_2 \Rightarrow$$

$$d[(\rho u^2 + p)A] = p dA \Rightarrow$$

$$d(\rho u^2 A) + d(pA) = p dA \Rightarrow$$

$$\underbrace{u d(\rho u A)}_{=0} + \rho u A du + A dp + p dA = p dA \Rightarrow$$

$$\rho u A du + A dp = 0 \Rightarrow$$

$$\boxed{dp = -\rho u du}$$

Euler's equation

# Governing Equations - Differential Form

Energy equation:

$$h_{o_1} = h_{o_2} \Rightarrow dh_o = 0$$

$$h_o = h + \frac{1}{2}u^2 \Rightarrow$$

$$dh + udu = 0$$



# Governing Equations - Differential Form

Summary (valid for all gases):

$$d(\rho u A) = 0$$

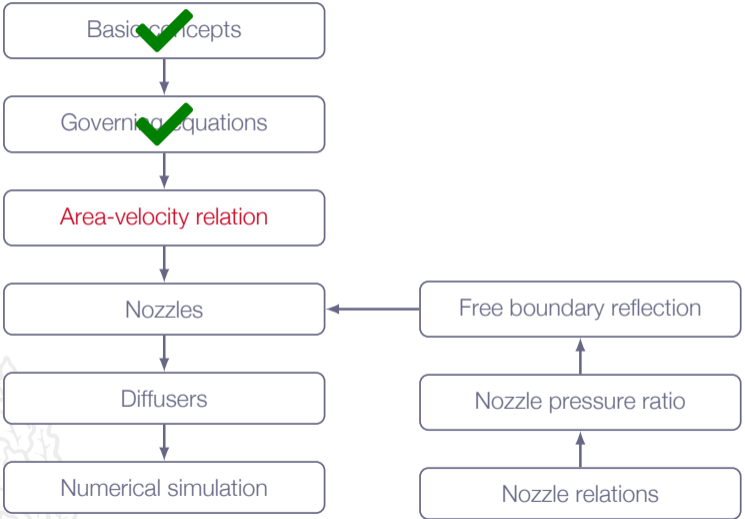
$$dp = -\rho u du$$

$$dh + u du = 0$$

Assumptions:

1. quasi-one-dimensional flow
2. inviscid flow
3. steady-state flow

# Roadmap - Quasi-One-Dimensional Flow



# Chapter 5.3

## Area-Velocity Relation





# Area-Velocity Relation

$$d(\rho u A) = 0 \Rightarrow u A d\rho + \rho A du + \rho u dA = 0$$

divide by  $\rho u A$  gives

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

Euler's equation:

$$dp = -\rho u du \Rightarrow \frac{dp}{\rho} = \frac{dp}{d\rho} \frac{d\rho}{\rho} = -u du$$

Assuming adiabatic, reversible (isentropic) process and the definition of speed of sound gives

$$\frac{dp}{d\rho} = \left( \frac{\partial p}{\partial \rho} \right)_s = a^2 \Rightarrow a^2 \frac{d\rho}{\rho} = -u du \Rightarrow \frac{d\rho}{\rho} = -M^2 \frac{du}{u}$$

# Area-Velocity Relation

Now, inserting the expression for  $\frac{d\rho}{\rho}$  in the rewritten continuity equation gives

$$(1 - M^2) \frac{du}{u} + \frac{dA}{A} = 0$$

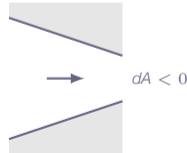
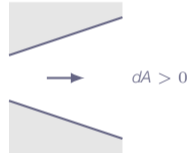
or

$$\frac{dA}{A} = (M^2 - 1) \frac{du}{u}$$

which is the **area-velocity relation**

# The Area-Velocity Relation

$$\frac{dA}{A} = \frac{du}{u} (M^2 - 1)$$



**Subsonic**  $M < 1$     **Supersonic**  $M > 1$

subsonic diffuser

$$du < 0$$

$$dp > 0$$

supersonic nozzle

$$du > 0$$

$$dp < 0$$

subsonic nozzle

$$du > 0$$

$$dp < 0$$

supersonic diffuser

$$du < 0$$

$$dp > 0$$

# The Area-Velocity Relation

$$\frac{du}{u}(M^2 - 1) = \frac{dA}{A}$$

What happens when  $M = 1$ ?

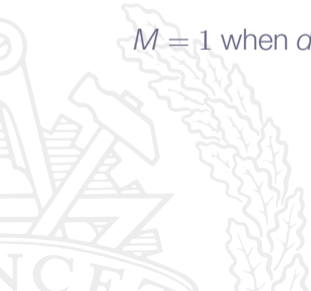


# The Area-Velocity Relation

$$\frac{du}{u}(M^2 - 1) = \frac{dA}{A}$$

What happens when  $M = 1$ ?

$M = 1$  when  $dA = 0$



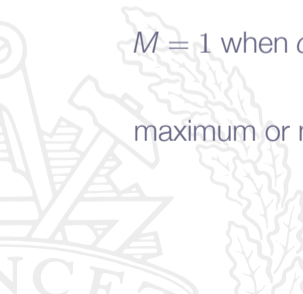
# The Area-Velocity Relation

$$\frac{du}{u}(M^2 - 1) = \frac{dA}{A}$$

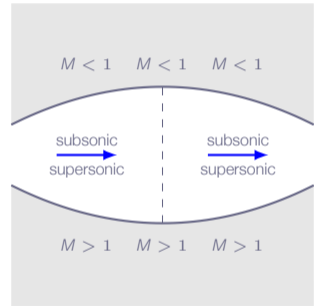
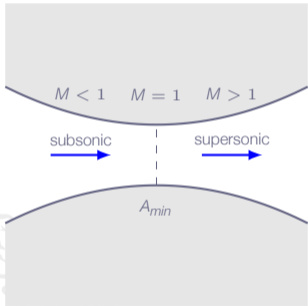
What happens when  $M = 1$ ?

$M = 1$  when  $dA = 0$

maximum or minimum area

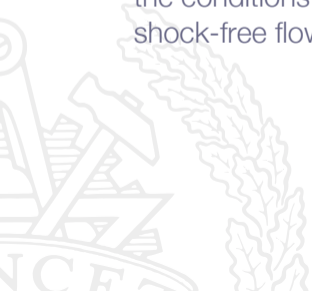


# The Area-Velocity Relation



# The Area-Velocity Relation

- ▶ A converging-diverging nozzle is the only possibility to obtain supersonic flow!
- ▶ A supersonic flow entering a convergent-divergent nozzle will slow down and, if the conditions are right, become sonic at the throat - hard to obtain a shock-free flow in this case





# Area-Velocity Relation

$$M \rightarrow 0 \Rightarrow \frac{dA}{A} = -\frac{du}{u}$$

$$\frac{dA}{A} + \frac{du}{u} = 0 \Rightarrow$$

$$\frac{1}{Au} [udA + Adu] = 0 \Rightarrow$$

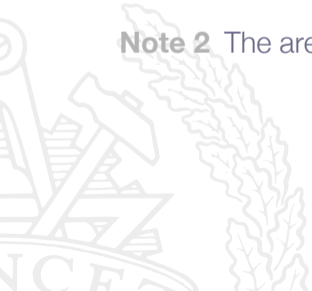
$$d(uA) = 0 \Rightarrow Au = c$$

where  $c$  is a constant

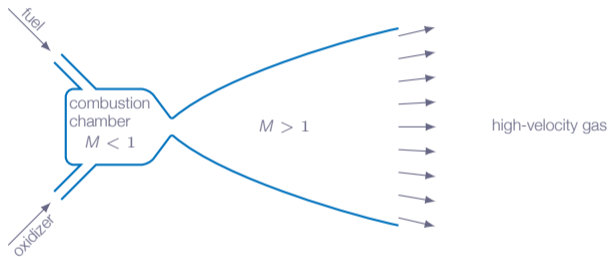
# Area-Velocity Relation

**Note 1** The area-velocity relation is only valid for isentropic flow  
not valid across a compression shock (due to entropy increase)

**Note 2** The area-velocity relation is valid for all gases

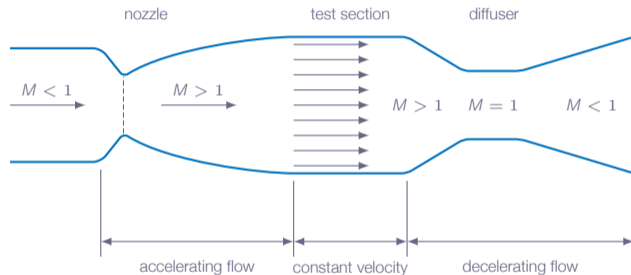


# Area-Velocity Relation Examples - Rocket Engine

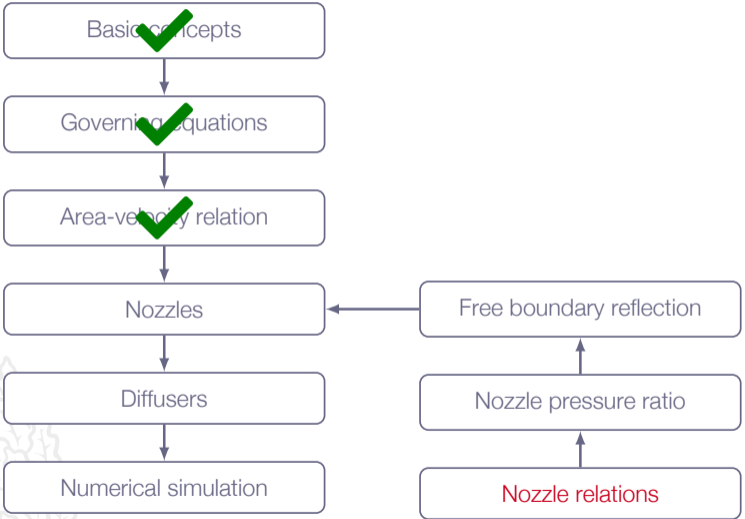


High-temperature, high-pressure gas in combustion chamber expand through the nozzle to very high velocities. Typical figures for a LH<sub>2</sub>/LOx rocket engine:  $p_o \sim 120$  [bar],  $T_o \sim 3600$  [K], exit velocity  $\sim 4000$  [m/s]

# Area-Velocity Relation Examples - Wind Tunnel



# Roadmap - Quasi-One-Dimensional Flow



# Chapter 5.4

## Nozzles



# Nozzle Flow with Varying Pressure Ratio

**time for rocket science!**



# Nozzle Flow - Relations

Calorically perfect gas assumed:

From Chapter 3:

$$\frac{T_o}{T} = \left(\frac{a_o}{a}\right)^2 = 1 + \frac{1}{2}(\gamma - 1)M^2$$

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{\frac{1}{\gamma-1}}$$





# Nozzle Flow - Relations

Critical conditions:

$$\frac{T_o}{T^*} = \left(\frac{a_o}{a^*}\right)^2 = \frac{1}{2}(\gamma + 1)$$

$$\frac{\rho_o}{\rho^*} = \left(\frac{T_o}{T^*}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_o}{\rho^*} = \left(\frac{T_o}{T^*}\right)^{\frac{1}{\gamma-1}}$$



# Nozzle Flow - Relations

$$M^{*2} = \frac{u^2}{a^{*2}} = \frac{u^2 a^2}{a^2 a^{*2}} = \frac{u^2 a^2 a_0^2}{a^2 a_0^2 a^{*2}} \Rightarrow$$

$$\frac{u^2}{a^2} = M^2$$

$$\frac{a^2}{a_0^2} = \left[ 1 + \frac{1}{2}(\gamma - 1)M^2 \right]^{-1}$$

$$\frac{a_0^2}{a^{*2}} = \frac{1}{2}(\gamma + 1)$$

$$\left. \begin{array}{l} \frac{u^2}{a^2} = M^2 \\ \frac{a^2}{a_0^2} = \left[ 1 + \frac{1}{2}(\gamma - 1)M^2 \right]^{-1} \\ \frac{a_0^2}{a^{*2}} = \frac{1}{2}(\gamma + 1) \end{array} \right\} \Rightarrow M^{*2} = M^2 \frac{\frac{1}{2}(\gamma + 1)}{1 + \frac{1}{2}(\gamma - 1)M^2}$$

# Nozzle Flow - Relations

For nozzle flow we have

$$\rho u A = c$$

where  $c$  is a constant and therefore

$$\rho^* u^* A^* = \rho u A$$

or, since at critical conditions  $u^* = a^*$

$$\rho^* a^* A^* = \rho u A$$

which gives

$$\frac{A}{A^*} = \frac{\rho^* a^*}{\rho u} = \frac{\rho^* \rho_0 a^*}{\rho_0 \rho u}$$

# Nozzle Flow - Relations

$$\frac{A}{A^*} = \frac{\rho^*}{\rho_0} \frac{\rho_0}{\rho} \frac{a^*}{u}$$

$$\left. \begin{aligned} \frac{\rho^*}{\rho_0} &= \left( \frac{T_0}{T^*} \right)^{\frac{-1}{\gamma-1}} \\ \frac{\rho_0}{\rho} &= \left( \frac{T_0}{T} \right)^{\frac{1}{\gamma-1}} \\ \frac{a^*}{u} &= \frac{1}{M^*} \end{aligned} \right\} \Rightarrow \frac{A}{A^*} = \frac{\left[ 1 + \frac{1}{2}(\gamma - 1)M^2 \right]^{\frac{1}{\gamma-1}}}{\left[ \frac{1}{2}(\gamma + 1) \right]^{\frac{1}{\gamma-1}} M^*}$$

## Nozzle Flow - Relations

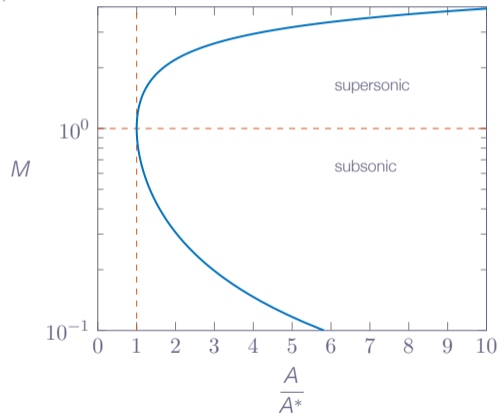
$$\left. \begin{aligned} \left(\frac{A}{A^*}\right)^2 &= \frac{\left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{\frac{2}{\gamma-1}}}{\left[\frac{1}{2}(\gamma + 1)\right]^{\frac{2}{\gamma-1}} M^{*2}} \\ M^{*2} &= M^2 \frac{\frac{1}{2}(\gamma + 1)}{1 + \frac{1}{2}(\gamma - 1)M^2} \end{aligned} \right\} \Rightarrow$$

$$\left(\frac{A}{A^*}\right)^2 = \frac{\left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{\frac{\gamma+1}{\gamma-1}}}{\left[\frac{1}{2}(\gamma + 1)\right]^{\frac{\gamma+1}{\gamma-1}} M^2}$$

which is the **area-Mach-number relation**

# The Area-Mach-Number Relation

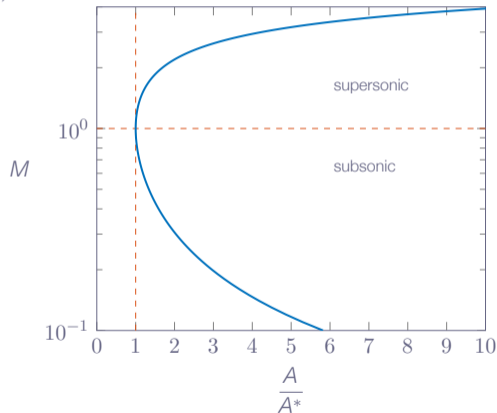
$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2} \left[ \frac{2 + (\gamma - 1)M^2}{\gamma + 1} \right]^{(\gamma+1)/(\gamma-1)}$$



# The Area-Mach-Number Relation

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2} \left[ \frac{2 + (\gamma - 1)M^2}{\gamma + 1} \right]^{(\gamma+1)/(\gamma-1)}$$

**Note!**  $\frac{A}{A^*} = \frac{\rho^* u^*}{\rho u}$



# Area-Mach-Number Relation

**Note 1** Critical conditions used here are those corresponding to **isentropic flow**. Do not confuse these with the conditions in the cases of one-dimensional flow with heat addition and friction

**Note 2** For quasi-one-dimensional flow, assuming inviscid steady-state flow, both **total and critical conditions are constant along streamlines** unless shocks are present (then the flow is no longer isentropic)

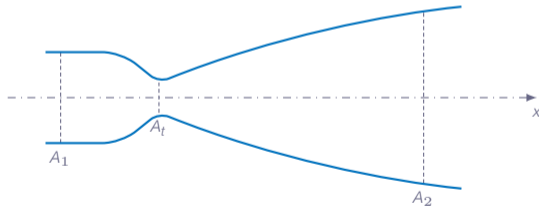
**Note 3** The derived area-Mach-number relation is **only valid for calorically perfect gas and for isentropic flow**. It is not valid across a compression shock



# Nozzle Flow

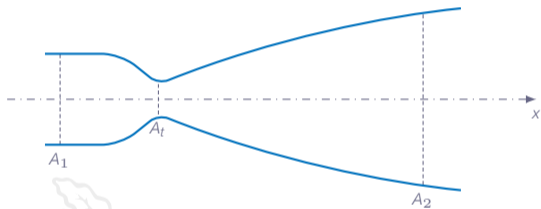
## Assumptions:

1. inviscid
2. steady-state
3. quasi-one-dimensional
4. calorically perfect gas



# The Area-Mach-Number Relation

Sub-critical (non-choked) nozzle flow

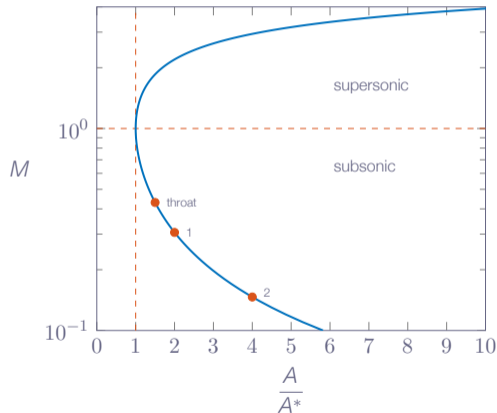


$M < 1$  at nozzle throat

$$A_t > A^*$$

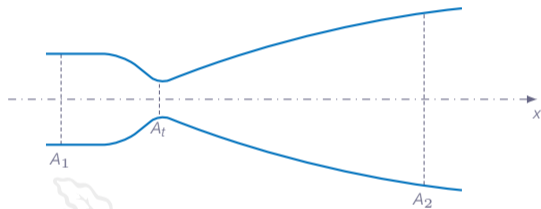
$$M_1 < 1$$

$$M_2 < 1$$



# The Area-Mach-Number Relation

Critical (choked) nozzle flow

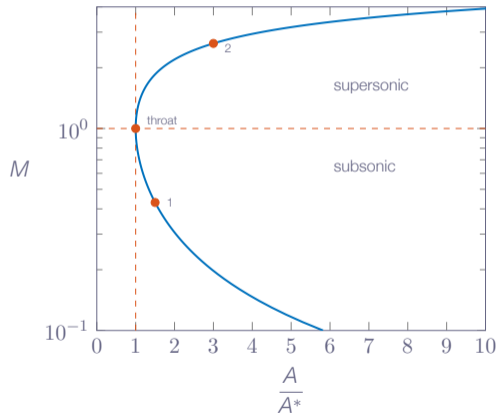


$M = 1$  at nozzle throat

$$A_t = A^*$$

$$M_1 < 1$$

$$M_2 > 1$$



# Nozzle Flow

Choked nozzle flow (no shocks):

- ▶  $A^*$  is constant throughout the nozzle
- ▶  $A_t = A^*$

$M_1$  given by the subsonic solution of

$$\left(\frac{A_1}{A^*}\right)^2 = \left(\frac{A_1}{A_t}\right)^2 = \frac{1}{M_1^2} \left[ \frac{2}{\gamma + 1} \left(1 + \frac{1}{2}(\gamma - 1)M_1^2\right) \right]^{\frac{\gamma+1}{\gamma-1}}$$

$M_2$  given by the supersonic solution of

$$\left(\frac{A_2}{A^*}\right)^2 = \left(\frac{A_2}{A_t}\right)^2 = \frac{1}{M_2^2} \left[ \frac{2}{\gamma + 1} \left(1 + \frac{1}{2}(\gamma - 1)M_2^2\right) \right]^{\frac{\gamma+1}{\gamma-1}}$$

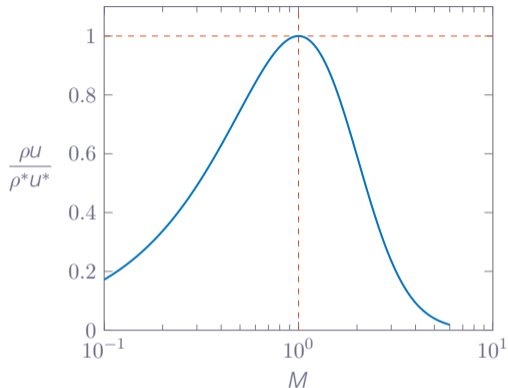
$M$  is uniquely determined everywhere in the nozzle, with subsonic flow upstream of throat and supersonic flow downstream of throat

# Nozzle Mass Flow

$$\rho u A = \rho^* A^* u^* \Rightarrow \frac{A^*}{A} = \frac{\rho u}{\rho^* u^*}$$

From the area-Mach-number relation

$$\frac{A^*}{A} = \begin{cases} < 1 & \text{if } M < 1 \\ 1 & \text{if } M = 1 \\ < 1 & \text{if } M > 1 \end{cases}$$



The maximum possible massflow through a duct is achieved when its throat reaches sonic conditions

# Nozzle Mass Flow

For a choked nozzle:

$$\dot{m} = \rho_1 u_1 A_1 = \rho^* u^* A^* = \rho_2 u_2 A_2$$

$$\left. \begin{aligned} \rho^* &= \frac{\rho^*}{\rho_0} \rho_0 = \left( \frac{2}{\gamma + 1} \right)^{\frac{1}{\gamma - 1}} \frac{\rho_0}{RT_0} \\ a^* &= \frac{a^*}{a_0} a_0 = \left( \frac{2}{\gamma + 1} \right)^{\frac{1}{2}} \sqrt{\gamma RT_0} \end{aligned} \right\} \Rightarrow$$

$$\dot{m} = \frac{\rho_0 A_t}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R} \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma + 1}{\gamma - 1}}}$$

# Nozzle Mass Flow

$$\dot{m} = \frac{\rho_o A_t}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R} \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma+1}{\gamma-1}}}$$

The **maximum mass flow** that can be sustained through the nozzle

Valid for quasi-one-dimensional, inviscid, steady-state flow and calorically perfect gas

**Note!** The massflow formula is valid even if there are shocks present downstream of throat!

# Nozzle Mass Flow

How can we increase mass flow through nozzle?

- ▶ increase  $p_o$
- ▶ decrease  $T_o$
- ▶ increase  $A_t$
- ▶ decrease  $R$   
(increase molecular weight, without changing  $\gamma$ )

