

Compressible Flow - TME085

Lecture 4

Niklas Andersson

Chalmers University of Technology
Department of Mechanics and Maritime Sciences
Division of Fluid Mechanics
Gothenburg, Sweden

`niklas.andersson@chalmers.se`

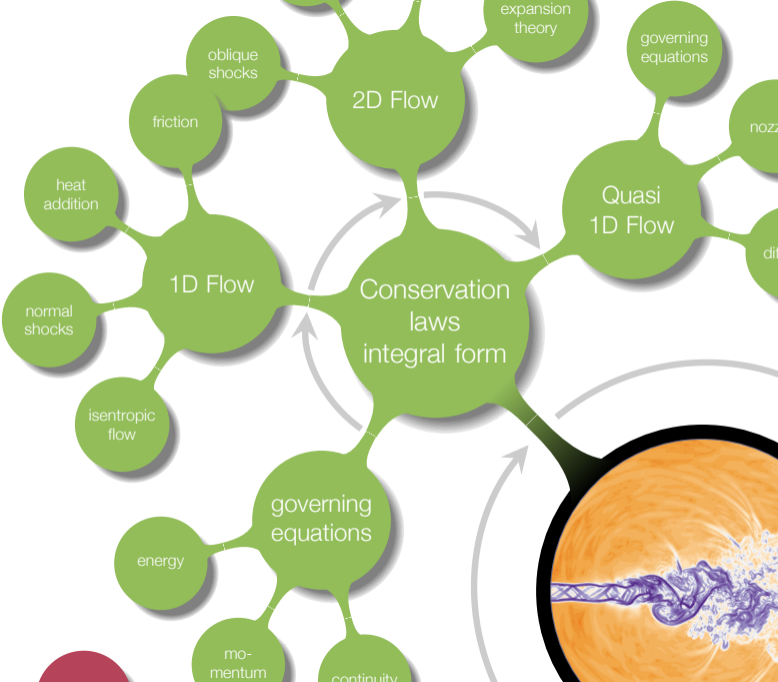


Chapter 3

One-Dimensional Flow



Overview

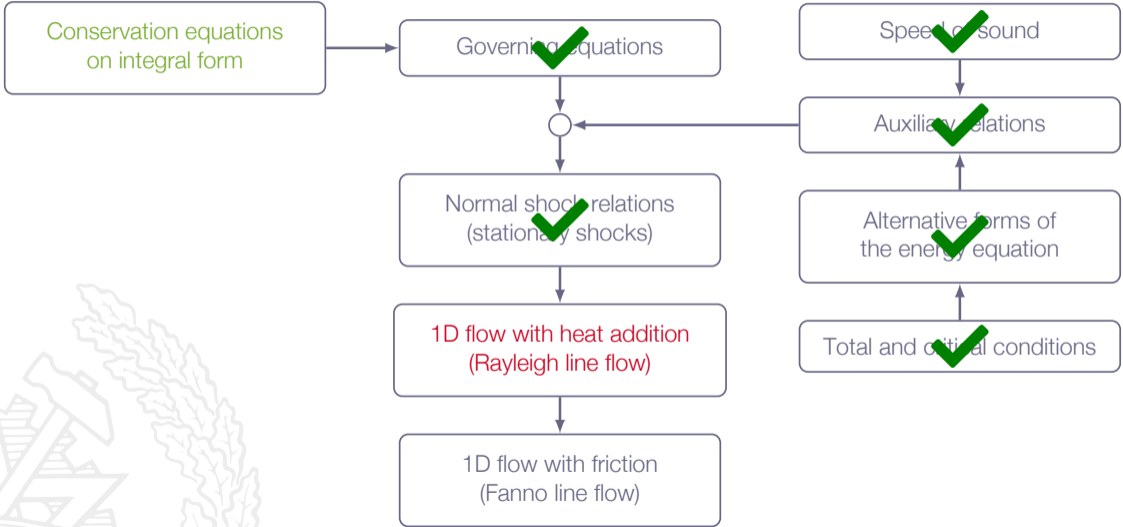


Learning Outcomes

- 4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on
- 5 **Explain** how thermodynamic relations enter into the flow equations
- 6 **Define** the special cases of calorically perfect gas, thermally perfect gas and real gas and **explain** the implication of each of these special cases
- 8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics
 - a 1D isentropic flow*
 - b normal shocks*
 - c 1D flow with heat addition*
 - d 1D flow with friction*

one-dimensional flows - isentropic and non-isentropic

Roadmap - One-dimensional Flow

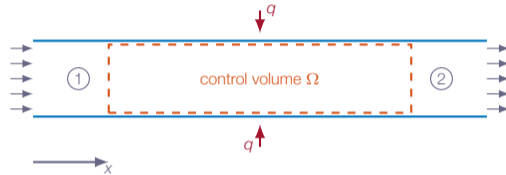


Chapter 3.8

One-Dimensional Flow with Heat Addition



One-Dimensional Flow with Heat Addition



Pipe flow:

- ▶ no friction
- ▶ 1D steady-state \Rightarrow all variables depend on x only
- ▶ q is the amount of heat per unit mass added between 1 and 2
- ▶ analyze by setting up a control volume between station 1 and 2

One-Dimensional Flow with Heat Addition

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 + q = h_2 + \frac{1}{2}u_2^2$$

Valid for all gases!

General gas \Rightarrow Numerical solution necessary

Calorically perfect gas \Rightarrow can be solved analytically

One-Dimensional Flow with Heat Addition

Calorically perfect gas ($h = C_p T$):

$$C_p T_1 + \frac{1}{2} u_1^2 + q = C_p T_2 + \frac{1}{2} u_2^2$$

$$q = \left(C_p T_2 + \frac{1}{2} u_2^2 \right) - \left(C_p T_1 + \frac{1}{2} u_1^2 \right)$$

$$C_p T_o = C_p T + \frac{1}{2} u^2 \Rightarrow$$

$$q = C_p (T_{o2} - T_{o1})$$

i.e. heat addition increases T_o downstream

One-Dimensional Flow with Heat Addition

Momentum equation:

$$p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2$$

$$\left\{ \rho u^2 = \rho a^2 M^2 = \rho \frac{\gamma p}{\rho} M^2 = \gamma p M^2 \right\}$$

$$p_2 - p_1 = \gamma p_1 M_1^2 - \gamma p_2 M_2^2 \Rightarrow$$

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}$$

Normal Shock Relations

We used the momentum equation to derive the relation for p_2/p_1 . In what way is this relation different than the one for normal shocks – the momentum equation is the same?



Normal Shock Relations

We used the momentum equation to derive the relation for p_2/p_1 . In what way is this relation different than the one for normal shocks – the momentum equation is the same?

Answer: There is no difference. If we would insert $M_2 = f(M_1)$ from the normal shock relations, we would end up with the normal shock relation for p_2/p_1 .

The relation for $M_2 = f(M_1)$ for normal shocks was derived assuming adiabatic flow

One-Dimensional Flow with Heat Addition

Ideal gas law:

$$T = \frac{p}{\rho R} \Rightarrow \frac{T_2}{T_1} = \frac{\rho_2}{\rho_1} \frac{\rho_1 R}{\rho_2 R} = \frac{\rho_2}{\rho_1} \frac{\rho_1}{\rho_2}$$

Continuity equation:

$$\rho_1 u_1 = \rho_2 u_2 \Rightarrow \frac{\rho_1}{\rho_2} = \frac{u_2}{u_1}$$

$$\frac{u_2}{u_1} = \frac{M_2 a_2}{M_1 a_1} = \frac{M_2 \sqrt{\gamma R T_2}}{M_1 \sqrt{\gamma R T_1}} \Rightarrow \frac{\rho_1}{\rho_2} = \frac{M_2}{M_1} \sqrt{\frac{T_2}{T_1}}$$

$$\frac{T_2}{T_1} = \left(\frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \right)^2 \left(\frac{M_2}{M_1} \right)^2$$

One-Dimensional Flow with Heat Addition

Calorically perfect gas, analytic solution:

$$\frac{T_2}{T_1} = \left[\frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \right]^2 \left(\frac{M_2}{M_1} \right)^2$$

$$\frac{\rho_2}{\rho_1} = \left[\frac{1 + \gamma M_2^2}{1 + \gamma M_1^2} \right] \left(\frac{M_1}{M_2} \right)^2$$

$$\frac{\rho_2}{\rho_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}$$



One-Dimensional Flow with Heat Addition

Calorically perfect gas, analytic solution:

$$\frac{\rho_{o2}}{\rho_{o1}} = \left[\frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \right] \left(\frac{1 + \frac{1}{2}(\gamma - 1)M_2^2}{1 + \frac{1}{2}(\gamma - 1)M_1^2} \right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_{o2}}{T_{o1}} = \left[\frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \right] \left(\frac{M_2}{M_1} \right)^2 \left(\frac{1 + \frac{1}{2}(\gamma - 1)M_2^2}{1 + \frac{1}{2}(\gamma - 1)M_1^2} \right)^{\frac{\gamma}{\gamma-1}}$$



One-Dimensional Flow with Heat Addition

Initially subsonic flow ($M < 1$)

- ▶ the Mach number, M , increases as more heat (per unit mass) is added to the gas
- ▶ for some limiting heat addition q^* , the flow will eventually become sonic $M = 1$

Initially supersonic flow ($M > 1$)

- ▶ the Mach number, M , decreases as more heat (per unit mass) is added to the gas
- ▶ for some limiting heat addition q^* , the flow will eventually become sonic $M = 1$

Note! The (*) condition in this context is not the same as the "critical" condition discussed for isentropic flow

One-Dimensional Flow with Heat Addition

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}$$

Calculate the ratio between the pressure at a specific location in the flow p and the pressure at sonic conditions p^*

$$p_1 = p, M_1 = M, p_2 = p^*, \text{ and } M_2 = 1$$

$$\frac{p^*}{p} = \frac{1 + \gamma M^2}{1 + \gamma}$$

One-Dimensional Flow with Heat Addition

$$\frac{T}{T^*} = \left[\frac{1 + \gamma}{1 + \gamma M^2} \right]^2 M^2$$

$$\frac{\rho_o}{\rho_o^*} = \left[\frac{1 + \gamma}{1 + \gamma M^2} \right] \left(\frac{2 + (\gamma - 1)M^2}{(\gamma + 1)} \right)^{\frac{\gamma}{\gamma - 1}}$$

$$\frac{\rho}{\rho^*} = \left[\frac{1 + \gamma M^2}{1 + \gamma} \right] \left(\frac{1}{M^2} \right)$$

$$\frac{T_o}{T_o^*} = \frac{(\gamma + 1)M^2}{(1 + \gamma M^2)^2} (2 + (\gamma - 1)M^2)$$

$$\frac{\rho}{\rho^*} = \frac{1 + \gamma}{1 + \gamma M^2}$$

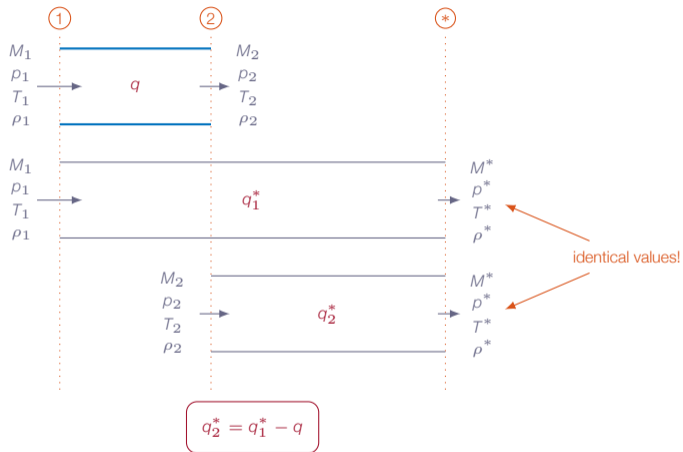
One-Dimensional Flow with Heat Addition

Amount of heat per unit mass needed to choke the flow:

$$q^* = C_p(T_o^* - T_o) = C_p T_o \left(\frac{T_o^*}{T_o} - 1 \right)$$



One-Dimensional Flow with Heat Addition



Note! for a given flow, the starred quantities are constant values

One-Dimensional Flow with Heat Addition

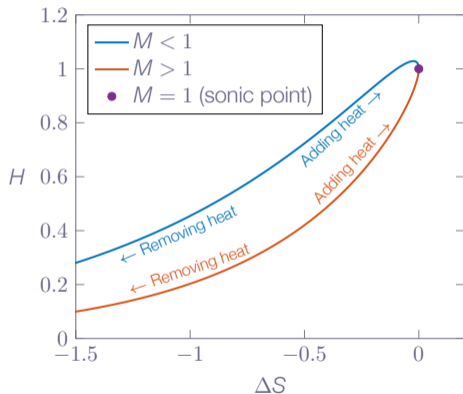


Lord Rayleigh 1842-1919
Nobel prize in physics 1904

Note! it is theoretically possible to heat an initially subsonic flow to reach sonic conditions and then continue to accelerate the flow by cooling

$$\Delta S = \frac{\Delta s}{C_p} = \ln \left[M^2 \left(\frac{\gamma + 1}{1 + \gamma M^2} \right)^{\frac{\gamma + 1}{\gamma}} \right]$$
$$H = \frac{h}{h^*} = \frac{C_p T}{C_p T^*} = \frac{T}{T^*} = \left[\frac{(\gamma + 1)M}{1 + \gamma M^2} \right]^2$$

Rayleigh curve ($\gamma = 1.4$)



One-Dimensional Flow with Heat Addition

And now, the million-dollar question ...



One-Dimensional Flow with Heat Addition

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Removing heat seems to reduce the entropy. Isn't that a violation of the second law of thermodynamics?!

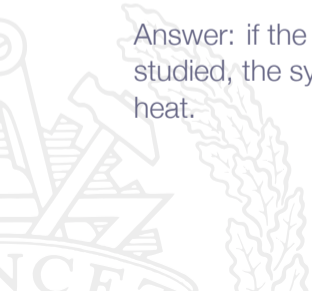


One-Dimensional Flow with Heat Addition

And now, the million-dollar question ...

Removing heat seems to reduce the entropy. Isn't that a violation of the second law of thermodynamics?!

Answer: if the heat source or sink would have been included in the system studied, the system entropy would increase both when adding and removing heat.



One-Dimensional Flow with Heat Addition

$M < 1$: Adding heat will

increase M
decrease p
increase T_o
decrease p_o
increase s
increase u
decrease ρ

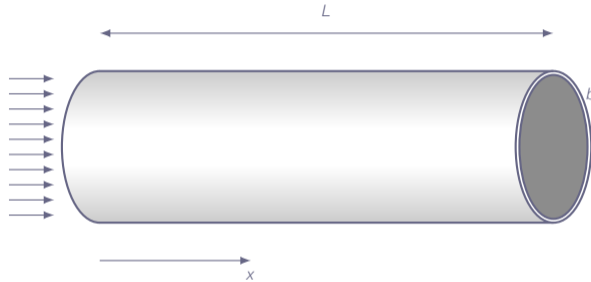
$M > 1$: Adding heat will

decrease M
increase p
increase T_o
decrease p_o
increase s
decrease u
increase ρ

Note! the flow is not isentropic, there will always be losses

One-Dimensional Flow with Heat Addition

Relation between added heat per unit mass (q) and heat per unit surface area and unit time (\dot{q}_{wall})



Pipe with arbitrary cross section (constant in x):

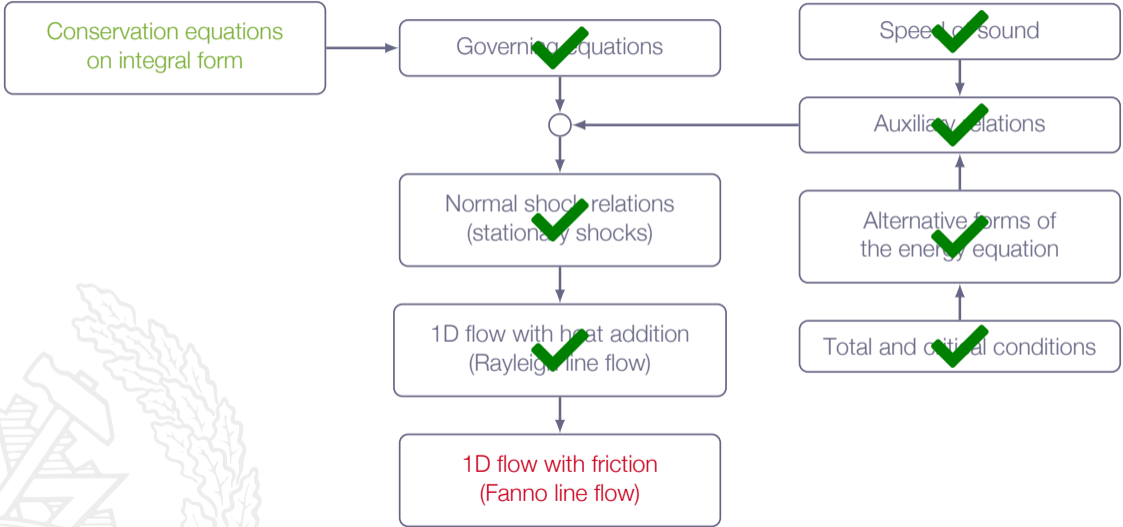
mass flow through pipe \dot{m}

axial length of pipe L

circumference of pipe $b = 2\pi r$

$$q = \frac{Lb\dot{q}_{wall}}{\dot{m}}$$

Roadmap - One-dimensional Flow



Chapter 3.9

One-Dimensional Flow with Friction

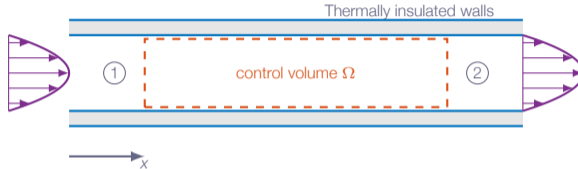


One-Dimensional Flow with Friction

inviscid flow with friction?!



One-Dimensional Flow with Friction



Pipe flow:

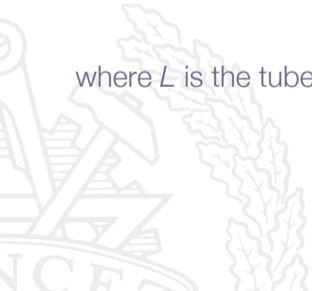
- ▶ adiabatic ($q = 0$)
- ▶ cross section area A is constant
- ▶ average all variables in each cross-section \Rightarrow only x -dependence
- ▶ analyze by setting up a control volume between station 1 and 2

One-Dimensional Flow with Friction

Wall-friction contribution in momentum equation

$$\oiint_{\partial\Omega} \tau_w dS = b \int_0^L \tau_w dx$$

where L is the tube length and b is the circumference



One-Dimensional Flow with Friction

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 - \frac{4}{D} \int_0^L \tau_w dx = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2} u_1^2 = h_2 + \frac{1}{2} u_2^2$$



One-Dimensional Flow with Friction

τ_w varies with the distance x and thus complicating the integration

Solution: let L shrink to dx and we end up with relations on differential form

$$d(\rho u^2 + p) = -\frac{4}{D}\tau_w dx \Leftrightarrow \frac{d}{dx}(\rho u^2 + p) = -\frac{4}{D}\tau_w$$



One-Dimensional Flow with Friction

From the continuity equation we get

$$\rho_1 u_1 = \rho_2 u_2 = \text{const} \Rightarrow \frac{d}{dx}(\rho u) = 0$$

Writing out all terms in the momentum equation gives

$$\frac{d}{dx}(\rho u^2 + p) = \rho u \frac{du}{dx} + u \underbrace{\frac{d}{dx}(\rho u)}_{=0} + \frac{dp}{dx} = -\frac{4}{D}\tau_w \Rightarrow \rho u \frac{du}{dx} + \frac{dp}{dx} = -\frac{4}{D}\tau_w$$

Common approximation for τ_w :

$$\tau_w = f \frac{1}{2} \rho u^2 \Rightarrow \rho u \frac{du}{dx} + \frac{dp}{dx} = -\frac{2}{D} \rho u^2 f$$

One-Dimensional Flow with Friction

Energy conservation:

$$h_{o1} = h_{o2} \Rightarrow \frac{d}{dx}h_o = 0$$



One-Dimensional Flow with Friction

Summary:

$$\frac{d}{dx}(\rho u) = 0$$

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = -\frac{2}{D} \rho u^2 f$$

$$\frac{d}{dx} h_o = 0$$

Valid for all gases!

General gas \Rightarrow Numerical solution necessary

Calorically perfect gas \Rightarrow Can be solved analytically (for constant f)

One-Dimensional Flow with Friction

Calorically perfect gas:

$$\int_{x_1}^{x_2} \frac{4f}{D} dx = \left[-\frac{1}{\gamma M^2} - \frac{\gamma + 1}{2\gamma} \ln \left(\frac{M^2}{1 + \frac{\gamma - 1}{2} M^2} \right) \right]_{M_1}^{M_2}$$



One-Dimensional Flow with Friction

Calorically perfect gas and adiabatic flow:

$$\frac{T_2}{T_1} = \frac{T_2}{T_{o2}} \frac{T_1}{T_2} \frac{T_{o1}}{T_1} = \{T_o = const\} = \frac{T_2}{T_o} \frac{T_o}{T_1} = \frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2}$$

Continuity:

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{a_1 M_1}{a_2 M_2} = \left\{ a = \sqrt{\gamma RT} \right\} = \sqrt{\frac{T_1}{T_2}} \left(\frac{M_1}{M_2} \right)$$

Perfect gas:

$$\frac{p_2}{p_1} = \{p = \rho RT\} = \frac{\rho_2 T_2}{\rho_1 T_1}$$

Total pressure:

$$\frac{p_{o2}}{p_{o1}} = \frac{p_{o2}}{p_2} \frac{p_1}{p_2} \frac{p_1}{p_{o1}}$$

One-Dimensional Flow with Friction

Calorically perfect gas:

$$\frac{T_2}{T_1} = \frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2}$$

$$\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2} \left[\frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2} \right]^{-1/2}$$

$$\frac{p_2}{p_1} = \frac{M_1}{M_2} \left[\frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2} \right]^{1/2}$$

$$\frac{p_{o2}}{p_{o1}} = \frac{M_1}{M_2} \left[\frac{2 + (\gamma - 1)M_2^2}{2 + (\gamma - 1)M_1^2} \right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

One-Dimensional Flow with Friction

Initially subsonic flow ($M_1 < 1$)

- ▶ M_2 will increase as L increases
- ▶ for a critical length L^* , the flow at point 2 will reach sonic conditions, *i.e.* $M_2 = 1$

Initially supersonic flow ($M_1 > 1$)

- ▶ M_2 will decrease as L increases
- ▶ for a critical length L^* , the flow at point 2 will reach sonic conditions, *i.e.* $M_2 = 1$

Note! The (*) condition in this context is not the same as the "critical" condition discussed for isentropic flow

One-Dimensional Flow with Friction

$$\frac{T}{T^*} = \frac{(\gamma + 1)}{2 + (\gamma - 1)M^2}$$

$$\frac{\rho}{\rho^*} = \frac{1}{M} \left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1} \right]^{1/2}$$

$$\frac{p}{p^*} = \frac{1}{M} \left[\frac{\gamma + 1}{2 + (\gamma - 1)M^2} \right]^{1/2}$$

$$\frac{p_o}{p_o^*} = \frac{1}{M} \left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1} \right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

see Table A.4

One-Dimensional Flow with Friction

and

$$\int_0^{L^*} \frac{4f}{D} dx = \left[-\frac{1}{\gamma M^2} - \frac{\gamma + 1}{2\gamma} \ln \left(\frac{M^2}{1 + \frac{\gamma - 1}{2} M^2} \right) \right]_M^1$$

where L^* is the tube length needed to change current state to sonic conditions

Let \bar{f} be the average friction coefficient over the length $L^* \Rightarrow$

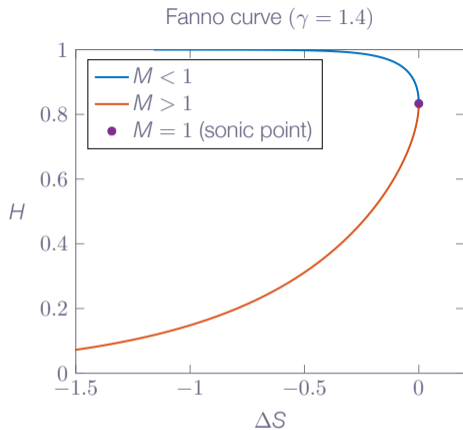
$$\frac{4\bar{f}L^*}{D} = \frac{1 - M^2}{\gamma M^2} + \frac{\gamma + 1}{2\gamma} \ln \left(\frac{(\gamma + 1)M^2}{2 + (\gamma - 1)M^2} \right)$$

Turbulent pipe flow $\rightarrow \bar{f} \sim 0.005$ ($Re > 10^5$, roughness $\sim 0.001D$)

One-Dimensional Flow with Friction

$$H = \frac{h}{h_o} = \frac{C_p T}{C_p T_o} = \frac{T}{T_o} = \left[1 + \frac{\gamma - 1}{2} M^2 \right]^{-1}$$

$$\Delta S = \frac{\Delta s}{C_p} = \ln \left[\left(\frac{1}{H} - 1 \right)^{\frac{\gamma-1}{2\gamma}} \left(\frac{2}{\gamma-1} \right)^{\frac{\gamma-1}{2\gamma}} \left(\frac{\gamma+1}{2} \right)^{\frac{\gamma+1}{2\gamma}} (H)^{\frac{\gamma+1}{2\gamma}} \right]$$



One-Dimensional Flow with Friction

$M < 1$: Friction will

increase M
decrease p
decrease T
decrease p_o
increase s
increase u
decrease ρ

$M > 1$: Friction will

decrease M
increase p
increase T
decrease p_o
increase s
decrease u
increase ρ

Note! the flow is not isentropic, there will always be losses

Roadmap - One-dimensional Flow

