# Compressible Flow - TME085

## Lecture 2

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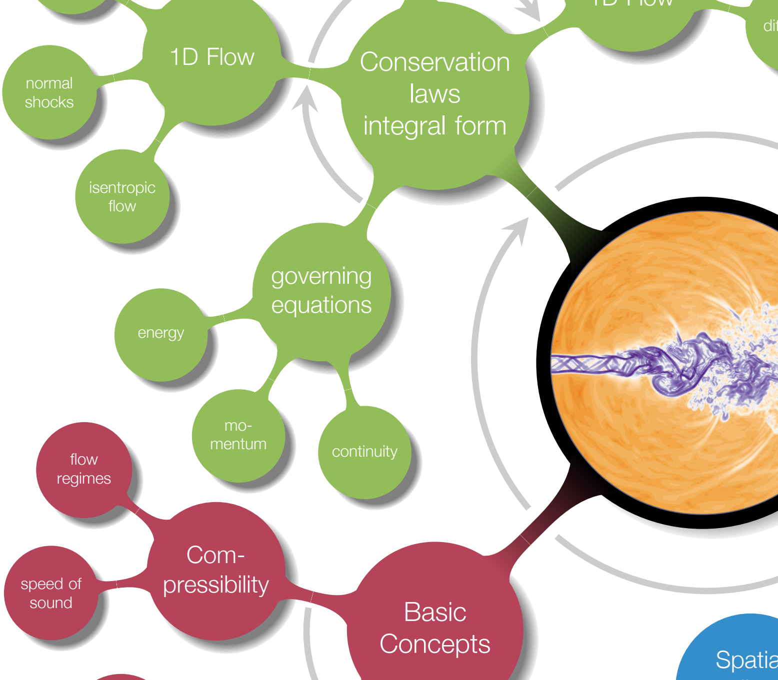
# Chapter 2
# Integral Forms of the Conservation Equations for Inviscid Flows

# Overview



1D Flow
normal shocks
isentropic flow
Conservation laws integral form
governing equations
energy
mo-mentum
continuity
flow regimes
speed of sound
Com-pressibility
Basic Concepts
Spatia

# Learning Outcomes

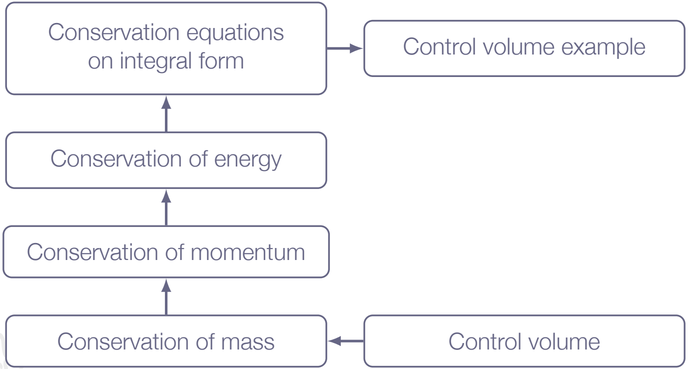
4 Present at least two different formulations of the governing equations for compressible flows and explain what basic conservation principles they are based on

5 Explain how thermodynamic relations enter into the flow equations

7 Explain why entropy is important for flow discontinuities

*equations, equations and more equations*

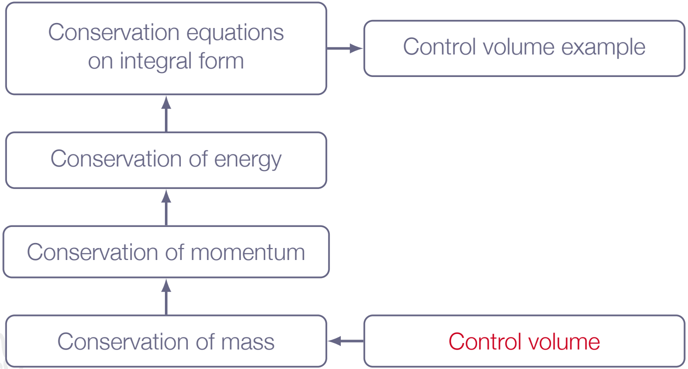
# Roadmap - Integral Relations



# Motivation

We need to formulate the basic form of the governing equations for compressible flow before we get to the applications

# Roadmap - Integral Relations



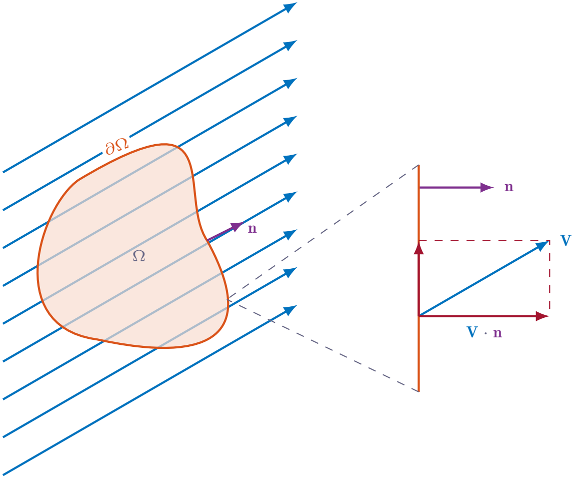
# Integral Forms of the Conservation Equations

Conservation principles:

- conservation of mass
- conservation of momentum (*Newton's second law*)
- conservation of energy (*first law of thermodynamics*)

# Integral Forms of the Conservation Equations

The control volume approach:




Notation:

$\Omega$: fixed control volume

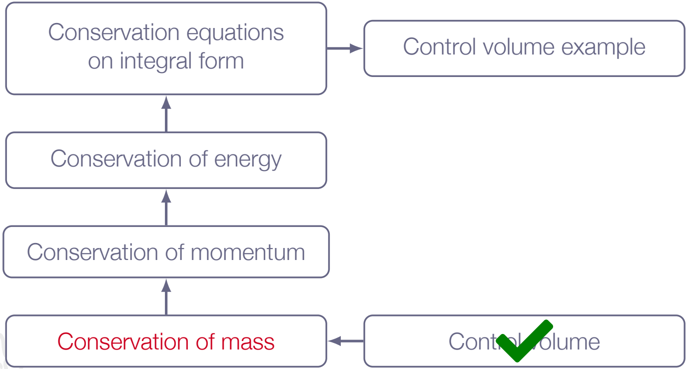
$\partial\Omega$: boundary of $\Omega$

$\mathbf{n}$: outward facing unit normal vector

$\mathbf{v}$: fluid velocity

$V = |\mathbf{v}|$

# Roadmap - Integral Relations



# Chapter 2.3
# Continuity Equation

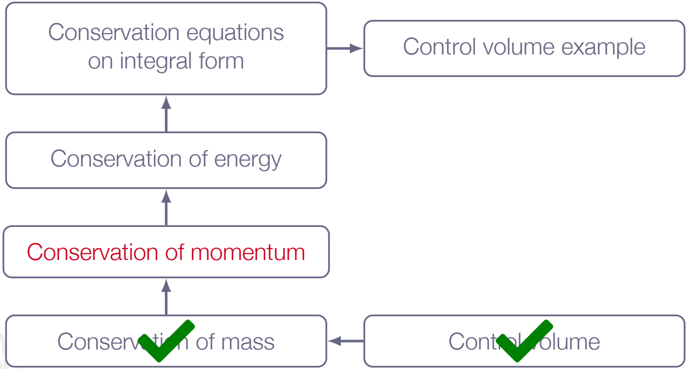
# Continuity Equation

Conservation of mass:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho d\mathscr{V}}_{\text{rate of change of total mass in }\Omega} + \underbrace{\oiint\limits_{\partial\Omega}\rho\mathbf{v}\cdot\mathbf{n}dS}_{\text{net mass flow out from }\Omega} = 0$$

**Note!** notation in the text book $\mathbf{n}\cdot dS = d\mathbf{S}$

# Roadmap - Integral Relations



# Chapter 2.4
# Momentum Equation

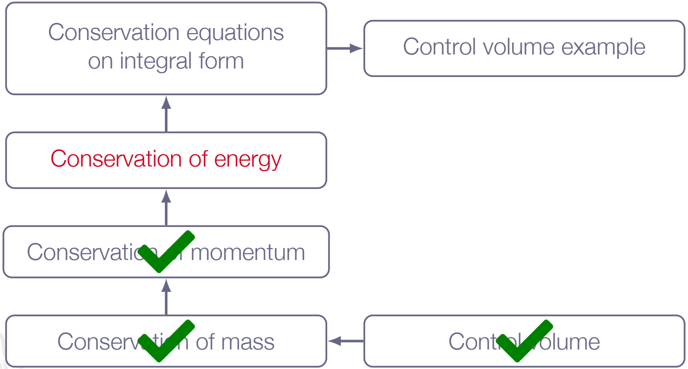
# Momentum Equation

Conservation of momentum:

$$\underbrace{\frac{d}{dt}\iiint_{\Omega} \rho \mathbf{v} d\mathscr{V}}_{\text{rate of change of total momentum in } \Omega} + \underbrace{\oiint_{\partial\Omega} \left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v} + p\mathbf{n}\right] dS}_{\text{net momentum flow out from } \Omega \text{ plus surface force on } \partial\Omega \text{ due to pressure}} = \underbrace{\iiint_{\Omega} \rho \mathbf{f} d\mathscr{V}}_{\text{rate of momentum generation due to forces inside } \Omega}$$

**Note!** friction forces due to viscosity are not included here. To account for these forces, the term $-(\tau \cdot \mathbf{n})$ must be added to the surface integral term.
The body force, $f$, is force per unit mass.

# Roadmap - Integral Relations



# Chapter 2.5
# Energy Equation

# Energy Equation

Conservation of energy:

$$\underbrace{\frac{d}{dt}\iiint_{\Omega} \rho e_o d\mathscr{V}}_{\text{rate of change of total internal energy in } \Omega} + \underbrace{\oiint_{\partial\Omega} \left[\rho e_o (\mathbf{v}\cdot\mathbf{n}) + p\mathbf{v}\cdot\mathbf{n}\right] dS}_{\text{net flow of total internal energy out from } \Omega \text{ plus work due to surface pressure on } \partial\Omega} = \underbrace{\iiint_{\Omega} \rho\mathbf{f}\cdot\mathbf{v} d\mathscr{V}}_{\text{work due to forces inside } \Omega}$$

where

$$\rho e_o = \rho\left(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) = \rho\left(e + \frac{1}{2}v^2\right)$$

is the total internal energy

# Energy Equation

The surface integral term may be rewritten as follows:

$$\oiint_{\partial\Omega} \left[\rho\left(e + \frac{1}{2}v^2\right)(\mathbf{v}\cdot\mathbf{n}) + p\mathbf{v}\cdot\mathbf{n}\right] dS$$

$$\Leftrightarrow$$

$$\oiint_{\partial\Omega} \left[\rho\left(e + \frac{p}{\rho} + \frac{1}{2}v^2\right)(\mathbf{v}\cdot\mathbf{n})\right] dS$$

$$\Leftrightarrow$$

$$\oiint_{\partial\Omega} \left[\rho\left(h + \frac{1}{2}v^2\right)(\mathbf{v}\cdot\mathbf{n})\right] dS$$

# Energy Equation

Introducing total enthalpy

$$h_o = h + \frac{1}{2}v^2$$

we get

$$\frac{d}{dt}\iiint\limits_{\Omega} \rho e_o d\mathscr{V} + \oiint\limits_{\partial\Omega} [\rho h_o \mathbf{v}\cdot\mathbf{n}]\, dS = \iiint\limits_{\Omega} \rho\mathbf{f}\cdot\mathbf{v} d\mathscr{V}$$

# Energy Equation

**Note 1:** to include friction work on $\partial\Omega$, the energy equation is extended as

$$\frac{d}{dt}\iiint\limits_{\Omega} \rho e_o d\mathscr{V} + \oiint\limits_{\partial\Omega} \left[\rho h_o \mathbf{v}\cdot\mathbf{n} - (\tau\cdot\mathbf{n})\cdot\mathbf{v}\right] dS = \iiint\limits_{\Omega} \rho\mathbf{f}\cdot\mathbf{v} d\mathscr{V}$$

**Note 2:** to include heat transfer on $\partial\Omega$, the energy equation is further extended

$$\frac{d}{dt}\iiint\limits_{\Omega} \rho e_o d\mathscr{V} + \oiint\limits_{\partial\Omega} \left[\rho h_o \mathbf{v}\cdot\mathbf{n} - (\tau\cdot\mathbf{n})\cdot\mathbf{v} + \mathbf{q}\cdot\mathbf{n}\right] dS = \iiint\limits_{\Omega} \rho\mathbf{f}\cdot\mathbf{v} d\mathscr{V}$$

where $\mathbf{q}$ is the heat flux vector

# Energy Equation

**Note 3:** the force $\mathbf{f}$ inside $\Omega$ may be a distributed body force field

Examples:

- gravity
- Coriolis and centrifugal acceleration terms in a rotating frame of reference

# Energy Equation

**Note 4:** there may be objects inside $\Omega$ which we choose to represent as sources of momentum and energy.

For example, there may be a solid object inside $\Omega$ which acts on the fluid with a force $\mathbf{F}$ and performs work $\dot{W}$ on the fluid

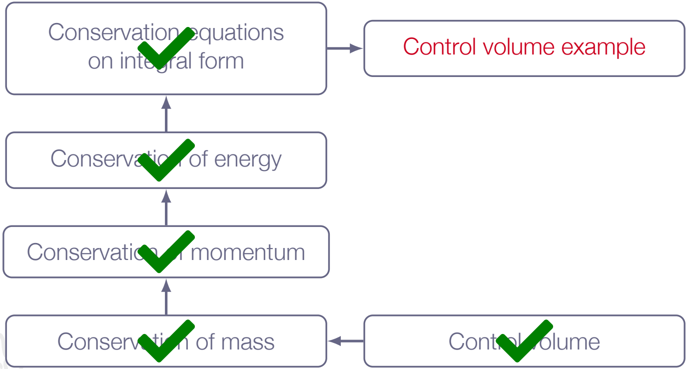
Momentum equation:

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho\mathbf{v}d\mathscr{V}+\oiint\limits_{\partial\Omega}\left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}+p\mathbf{n}\right]dS=\iiint\limits_{\Omega}\rho\mathbf{f}d\mathscr{V}+\mathbf{F}$$

Energy equation:

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho e_o d\mathscr{V}+\oiint\limits_{\partial\Omega}\left[\rho h_o\mathbf{v}\cdot\mathbf{n}\right]dS=\iiint\limits_{\Omega}\rho\mathbf{f}\cdot\mathbf{v}d\mathscr{V}+\dot{W}$$

# Roadmap - Integral Relations



# Integral Equations - Applications

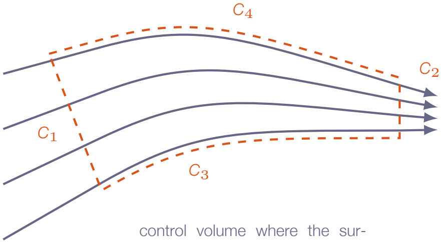
How can we use control volume formulations of conservation laws?

- Let $\Omega \to 0$: In the limit of vanishing volume the control volume formulations give the Partial Differential Equations (PDE:s) for mass, momentum and energy conservation (see Chapter 6)

- Apply in a ”smart” way $\Rightarrow$ Analysis tool for many practical problems involving compressible flow (see Chapter 2, Section 2.8)

# Integral Equations - Applications

Example: Steady-state adiabatic inviscid flow




control volume where the surfaces $C_1$ and $C_2$ are normal to the flow and $C_3$ and $C_4$ are parallel to the stream lines

# Integral Equations - Applications

Conservation of mass:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega} \rho d\mathscr{V}}_{=0} + \underbrace{\oint\!\!\!\oint\limits_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS}_{-\rho_1 v_1 A_1 + \rho_2 v_2 A_2} = 0$$

Conservation of energy:

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega} \rho e_o d\mathscr{V}}_{=0} + \underbrace{\oint\!\!\!\oint\limits_{\partial\Omega} [\rho h_o \mathbf{v} \cdot \mathbf{n}]\, dS}_{-\rho_1 h_{o_1} v_1 A_1 + \rho_2 h_{o_2} v_2 A_2} = 0$$

# Integral Equations - Applications

Conservation of mass:

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2$$

Conservation of energy:

$$\rho_1 h_{o_1} v_1 A_1 = \rho_2 h_{o_2} v_2 A_2$$

$$\Leftrightarrow$$

$$h_{o_1} = h_{o_2}$$

Total enthalpy $h_o$ is conserved along streamlines in steady-state adiabatic inviscid flow

# Roadmap - Integral Relations