Compressible Flow TME085

Unsteady Wave Motion

Moving Normal Shock Relations

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Stationary Normal Shock

The starting point is the governing equations for stationary normal shocks (repeated here for convenience).

$$\rho_1 u_1 = \rho_2 u_2 \tag{1}$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \tag{2}$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2 \tag{3}$$

Moving Normal Shock

Shock moving to the right with the constant speed W into a gas that is standing still. Moving with the shock, we would see a gas velocity ahead of the shock $u_1 = W$, and the gas behind the shock moves to the right with the velocity $u_2 = W - u_p$. Now, let's insert u_1 and u_2 in the stationary shock relations 1 - 3.

$$\rho_1 W = \rho_2 (W - u_p) \tag{4}$$

$$\rho_1 W^2 + p_1 = \rho_2 (W - u_p)^2 + p_2 \tag{5}$$

$$h_1 + \frac{1}{2}W^2 = h_2 + \frac{1}{2}(W - u_p)^2 \tag{6}$$

Rewriting Eqn. 4

$$(W - u_p) = W\frac{\rho_1}{\rho_2} \tag{7}$$

Inserting Eqn. 7 in Eqn. 5 gives

$$p_1 + \rho_1 W^2 = p_2 + \rho_2 W^2 \left(\frac{\rho_1}{\rho_2}\right)^2 \Rightarrow p_2 - p_1 = \rho_1 W^2 \left(1 - \frac{\rho_1}{\rho_2}\right)$$

$$W^2 = \frac{p_2 - p_1}{\rho_2 - \rho_1}\left(\frac{\rho_2}{\rho_1}\right) \tag{8}$$

From the continuity equation 4, we get

$$W = (W - u_p)\left(\frac{\rho_2}{\rho_1}\right) \tag{9}$$

Inserting Eqn. 9 in Eqn. 8 gives

$$(W - u_p)^2 = \frac{p_2 - p_1}{\rho_2 - \rho_1} \left(\frac{\rho_1}{\rho_2} \right) \tag{10}$$

Now, let's insert Eqns. 8 and 10 in the energy equation (Eqn. 6).

$$h_1 + \frac{1}{2} \left[\frac{p_2 - p_1}{\rho_2 - \rho_1} \left(\frac{\rho_2}{\rho_1} \right) \right] = h_2 + \frac{1}{2} \left[\frac{p_2 - p_1}{\rho_2 - \rho_1} \left(\frac{\rho_1}{\rho_2} \right) \right] \tag{11}$$

$$h = e + \frac{p}{\rho} \tag{12}$$

$$e_1 + \frac{p_1}{\rho_1} + \frac{1}{2} \left[\frac{p_2 - p_1}{\rho_2 - \rho_1} \left(\frac{\rho_2}{\rho_1} \right) \right] = e_2 + \frac{p_2}{\rho_2} + \frac{1}{2} \left[\frac{p_2 - p_1}{\rho_2 - \rho_1} \left(\frac{\rho_1}{\rho_2} \right) \right] \tag{13}$$

which can be rewritten as

$$e_2 - e_1 = \frac{p_1 + p_2}{2} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \tag{14}$$

Eqn 14 is the same Hugoniot equation as we get for a stationary normal shock. The Hugoniot equation is a relation of thermodynamic properties over a shock. As the shock in the unsteady case is moving with a constant velocity, the frame of reference moving with the shock is an inertial frame and thus the same physical relations apply in the moving shock case as in the stationary shock case. The fact that the Hugoniot relation does not include any velocities or Mach numbers but only thermodynamic properties, the relation will be unchanged for a moving shock.

Moving Shock Relations

For a calorically perfect gas we have $e = C_v T$. Inserted in the Hugoniot relation above this gives

$$C_v(T_2 - T_1) = \frac{p_1 + p_2}{2} (\nu_1 - \nu_2) \tag{15}$$

where $\nu = 1/\rho$

Now, using the ideal gas law $T = p\nu/R$ and $C_v/R = 1/(\gamma - 1)$ gives

$$\left(\frac{1}{\gamma - 1} \right) (p_2\nu_2 - p_1\nu_1) = \frac{p_1 + p_2}{2} (\nu_1 - \nu_2)$$

$$\Leftrightarrow$$

$$p_2\left(\frac{\nu_2}{\gamma-1}-\frac{\nu_1-\nu_2}{2}\right)=p_1\left(\frac{\nu_1}{\gamma-1}+\frac{\nu_1-\nu_2}{2}\right)$$

From this result, we can derive a relation for the pressure ratio over the shock as a function of density ratio

$$\frac{p_2}{p_1}=\frac{\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{\nu_1}{\nu_2}\right)-1}{\left(\frac{\gamma+1}{\gamma-1}\right)-\left(\frac{\nu_1}{\nu_2}\right)} \tag{16}$$

$\nu = RT/p$ and thus

$$\frac{\nu_1}{\nu_2}=\frac{T_1}{T_2}\frac{p_2}{p_1} \tag{17}$$

Eqn. 20 in Eqn. 18 gives

$$\frac{p_2}{p_1}=\frac{\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{T_1}{T_2}\frac{p_2}{p_1}\right)-1}{\left(\frac{\gamma+1}{\gamma-1}\right)-\left(\frac{T_1}{T_2}\frac{p_2}{p_1}\right)} \tag{18}$$

Now, we can get a relation for calculation of the temperature ratio over the moving shock as function of the shock pressure ratio

$$\frac{T_2}{T_1}=\frac{p_2}{p_1}\left[\frac{\left(\frac{\gamma+1}{\gamma-1}\right)+\left(\frac{p_2}{p_1}\right)}{1+\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right)}\right] \tag{19}$$

Once again using the ideal gas law

$$\frac{\rho_2}{\rho_1}=\frac{\left(\frac{\gamma+1}{\gamma-1}\right)+\left(\frac{p_2}{p_1}\right)}{1+\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right)} \tag{20}$$

Going back to the momentum equation

$$p_2 - p_1 = \rho_1 W^2 \left(1 - \frac{\rho_1}{\rho_2}\right) = \{W = M_s a_1\} = \rho_1 M_s^2 a_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right)$$

with $a_1^2 = \gamma p_1/\rho_1$, we get

$$\frac{p_2}{p_1} = \gamma M_s^2 \left(1 - \frac{\rho_1}{\rho_2}\right) + 1 \tag{21}$$

From the normal shock relations, we have

$$\frac{\rho_1}{\rho_2} = \frac{2 + (\gamma - 1)M_s^2}{(\gamma + 1)M_s^2} \tag{22}$$

Eqn. 22 in 21 gives

$$\frac{p_2}{p_1} = 1 + \left(\frac{2\gamma}{\gamma + 1}\right)(M_s^2 - 1) \tag{23}$$

or

$$M_s = \sqrt{\left(\frac{\gamma + 1}{2\gamma}\right)\left(\frac{p_2}{p_1} - 1\right) + 1} \tag{24}$$

Eqn. 24 with $M_s = W/a_1$

$$W = a_1 \sqrt{\left(\frac{\gamma + 1}{2\gamma}\right)\left(\frac{p_2}{p_1} - 1\right) + 1} \tag{25}$$

Induced Flow Behind Moving Shock

Let's try to find a relation for calculation of the induced velocity behind the moving shock. Once again, the starting point is the continuity equation for moving shocks (Eqn. 4) repeated here for convenience

$$\rho_1 W = \rho_2(W - u_p)$$

The induced velocity appears on the right side of the continuity equation

$$W(\rho_1 - \rho_2) = -\rho_2 u_p$$

$$u_p = W\left(1 - \frac{\rho_1}{\rho_2}\right) \tag{26}$$

From before we have a relation for W as a function of pressure ratio and one for ρ_1/ρ_2, also as a function of pressure ratio.

Eqn. 26 togheter with Eqns. 25 and 20 gives

$$u_p = a_1 \underbrace{\sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_2}{p_1} - 1\right) + 1}}_{I} \underbrace{\left[1 - \frac{\left(\frac{\gamma+1}{\gamma-1}\right) + \left(\frac{p_2}{p_1}\right)}{1 + \left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right)}\right]}_{II} \tag{27}$$

The equation subsets I and II can be rewritten as:

Term I:

$$\sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_2}{p_1} - 1\right) + 1} = \sqrt{\frac{\gamma+1}{2\gamma}\left[\left(\frac{p_2}{p_1}\right) + \left(\frac{\gamma-1}{\gamma+1}\right)\right]}$$

Term II:

$$\left[1 - \frac{\left(\frac{\gamma+1}{\gamma-1}\right) + \left(\frac{p_2}{p_1}\right)}{1 + \left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right)}\right] = \frac{1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\frac{\left(\frac{2\gamma}{\gamma+1}\right)}{\left(\frac{\gamma-1}{\gamma+1}\right) + \left(\frac{p_2}{p_1}\right)}$$

With the rewritten terms I and II implemented, Eqn. 27 becomes

$$u_p = \frac{a_1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\sqrt{\frac{\left(\frac{2\gamma}{\gamma+1}\right)}{\left(\frac{\gamma-1}{\gamma+1}\right) + \left(\frac{p_2}{p_1}\right)}} \tag{28}$$

Since the region behind the moving shock is region 2, the induced flow Mach number is obtained as

$$M_p = \frac{u_p}{a_2} = \frac{u_p}{a_1}\frac{a_1}{a_2} = \frac{u_p}{a_1}\sqrt{\frac{\gamma R T_1}{\gamma R T_2}} = \frac{u_p}{a_1}\sqrt{\frac{T_1}{T_2}}$$

With up/a_1 from Eqn. 28 and T_1/T_2 from Eqn. 19

$$M_p = \frac{1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left(\frac{\left(\frac{2\gamma}{\gamma+1}\right)}{\left(\frac{\gamma-1}{\gamma+1}\right) + \left(\frac{p_2}{p_1}\right)}\right)^{1/2}\left(\frac{1 + \left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right)}{\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right) + \left(\frac{p_2}{p_1}\right)^2}\right)^{1/2} \tag{29}$$

There is a theoretical upper limit for the induced Mach number M_p

$$\lim_{p_2/p_1 \to \infty} M_p\left(\frac{p_2}{p_1}\right) = \sqrt{\frac{2}{\gamma(\gamma-1)}}$$

As can be seen, at the upper limit the induced Mach number is a function of γ and for air ($\gamma = 1.4$) we get

$$\lim_{p_2/p_1 \to \infty} M_p\left(\frac{p_2}{p_1}\right) \simeq 1.89$$