Compressible Flow TME085

Unsteady Wave Motion

Finite Non-Linear Waves

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Non-Linear One-Dimensional Flow

Starting point: the governing flow equations on partial differential form

Continuity equation:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \tag{1}$$

Momentum equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{2}$$

Any thermodynamic property can be expressed as a function of two other thermodynamic properties. This means that we can get density as a function of pressure and entropy: $\rho = \rho(p, s)$ and therefore

$$d\rho = \left(\frac{\partial \rho}{\partial p}\right)_s dp + \left(\frac{\partial \rho}{\partial s}\right)_p ds$$

Assuming isentropic flow $ds = 0$ gives

$$d\rho = \left(\frac{\partial \rho}{\partial p}\right)_s dp$$

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \left(\frac{\partial \rho}{\partial p}\right)_s \frac{\partial p}{\partial t} = \frac{1}{a^2}\frac{\partial p}{\partial t} \\ \frac{\partial \rho}{\partial x} &= \left(\frac{\partial \rho}{\partial p}\right)_s \frac{\partial p}{\partial x} = \frac{1}{a^2}\frac{\partial p}{\partial x} \end{aligned} \tag{3}$$

Now, insert 3 in 1 gives

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho a^2\frac{\partial u}{\partial x} = 0 \tag{4}$$

Dividing 4 by ρa gives

$$\frac{1}{\rho a}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right) + a\frac{\partial u}{\partial x} = 0 \tag{5}$$

A slightly modified form of the momentum equation is obtained by multiplying and dividing the last term by a

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho a}\left(a\frac{\partial p}{\partial x}\right) = 0 \tag{6}$$

If the continuity equation on the form 5 is added to the momentum equation on the form 6, we get

$$\left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right] + \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right] = 0 \tag{7}$$

If, instead, the continuity equation on the form 5 is subtracted from the momentum equation on the form 6, we get

$$\left[\frac{\partial u}{\partial t} + (u-a)\frac{\partial u}{\partial x}\right] + \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u-a)\frac{\partial p}{\partial x}\right] = 0 \tag{8}$$

Since $u = u(x,t)$, we have from the definition of a differential

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}\frac{dx}{dt}dt \tag{9}$$

Now, let $dx/dt = u + a$

$$du = \frac{\partial u}{\partial t}dt + (u+a)\frac{\partial u}{\partial x}dt = \left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right]dt \tag{10}$$

which is the change of u in the direction $dx/dt = u + a$

In the same way

$$dp = \frac{\partial p}{\partial t}dt + \frac{\partial p}{\partial x}dx = \frac{\partial p}{\partial t}dt + \frac{\partial p}{\partial x}\frac{dx}{dt}dt \tag{11}$$

and thus, in the direction $dx/dt = u + a$

$$dp = \frac{\partial p}{\partial t}dt + (u+a)\frac{\partial p}{\partial x}dt = \left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right] dt \tag{12}$$

If we go back and examine Eqn. 7, we see that Eqns. 10 and 12 appear in the equation and thus it can now be rewritten as follows

$$\frac{du}{dt} + \frac{1}{\rho a}\frac{dp}{dt} = 0 \Rightarrow du + \frac{dp}{\rho a} = 0 \tag{13}$$

Eqn. 13 applies along a C^+ characteristic, i.e., a line in the direction $dx/dt = u + a$ in xt-space and is called the compatibility equation along the C^+ characteristic. If we instead chose a C^- characteristic, i.e., a line in the direction $dx/dt = u - a$ in xt-space, we get

$$du = \left[\frac{\partial u}{\partial t} + (u-a)\frac{\partial u}{\partial x}\right] dt \tag{14}$$

$$dp = \left[\frac{\partial p}{\partial t} + (u-a)\frac{\partial p}{\partial x}\right] dt \tag{15}$$

which can be identified as subsets of Eqn. 8 and thus

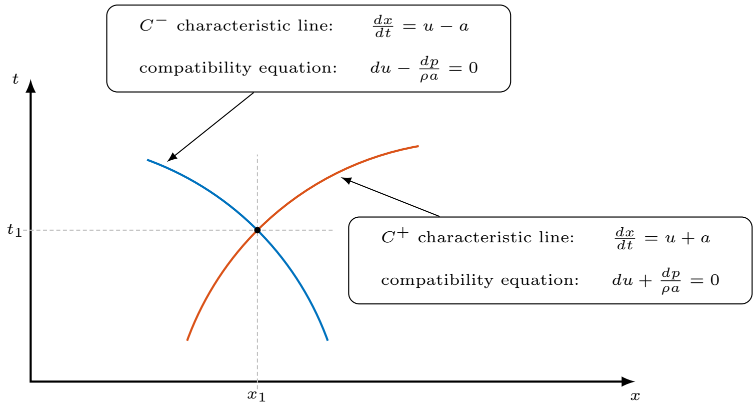
$$\frac{du}{dt} - \frac{1}{\rho a}\frac{dp}{dt} = 0 \Rightarrow du - \frac{dp}{\rho a} = 0 \tag{16}$$

Eqn. 16 applies along a C^- characteristic, i.e., a line in the direction $dx/dt = u - a$ in xt-space and is called the compatibility equation along the C^- characteristic.

So, what we have done now is that we have have found paths through a point (x_1,t_1) along which the governing partial differential equations Eqns. 7 and 8 reduces to the ordinary differential equations 13 and 16. The C^+ and C^- characteristic lines are physically the paths of right- and left-running sound waves in the xt-plane.

Riemann Invariants

If the compatibility equations are integrated along respective characteristic line, i.e., integrate 13 along the C^+ characteristic and 16 along the C^- characteristic, we get the Riemann invari-



Figur 1: Characteristic lines through a point (x_1,t_1)

ants J^+ and J^-.

$$J^+ = u + \int \frac{dp}{\rho a} = const \tag{17}$$

$$J^- = u - \int \frac{dp}{\rho a} = const \tag{18}$$

The Riemann invariants are constants along the associated characteristic line.

We have assumed isentropic flow and thus we may use the isentropic relations

$$p = C_1 T^{\gamma/(\gamma-1)} = C_2 a^{2\gamma/(\gamma-1)} \tag{19}$$

where C_1 and C_2 are constants. Differentiating Eqn. 19 gives

$$dp = C_2 \left(\frac{2\gamma}{\gamma - 1}\right) a^{[2\gamma/(\gamma-1)-1]} da \tag{20}$$

Now, if we further assume the gas to be calorically perfect

$$a^2 = \gamma RT = \frac{\gamma p}{\rho} \Rightarrow \rho = \frac{\gamma p}{a^2} \tag{21}$$

Eqn. 19 in 21 gives

$$\rho = C_2 \gamma a^{[2\gamma/(\gamma-1)-2]} \tag{22}$$

and thus

$$J^+ = u + \int \frac{C_2 \left(\frac{2\gamma}{\gamma-1}\right) a^{[2\gamma/(\gamma-1)-1]}}{C_2 \gamma a^{[2\gamma/(\gamma-1)-2]} a} da = u + \left(\frac{2}{\gamma-1}\right) \int da$$

$$J^+ = u + \frac{2a}{\gamma-1} \tag{23}$$

$$J^- = u - \frac{2a}{\gamma-1} \tag{24}$$

Eqns. 23 and 24 are the Riemann invariants for a calorically perfect gas. The Riemann invariants are constants along C^+ and C^- characteristics and if the situation shown in Fig. 2 appears, that fact can be used to calculate the flow velocity and speed of sound in the location (x_1,t_1).

$$J^+ + J^- = u + \frac{2a}{\gamma-1} + u - \frac{2a}{\gamma-1} = 2u \Rightarrow u = \frac{1}{2}(J^+ + J^-) \tag{25}$$

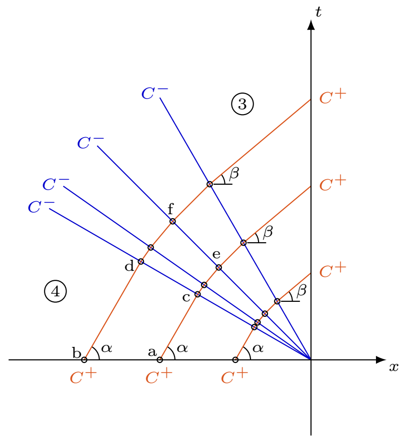
$$J^+ = u + \frac{2a}{\gamma-1} = \frac{1}{2}(J^+ + J^-) + \frac{2a}{\gamma-1} \Rightarrow a = \frac{\gamma-1}{4}(J^+ - J^-) \tag{26}$$

Expansion Wave

The expansion wave propagation into the driver section in a shock tube can be described using characteristic lines.

The expansion is propagating into stagnant fluid in region four (the driver section), which means that the flow properties ahead of the expansion wave are constant.

$$J_a^+ = J_b^+$$



Figur 2: Expansion fan centered at $(x, t) = (0.0, 0.0)$

J^+ invariants constant along C^+ characteristics

$$J_a^+ = J_c^+ = J_e^+$$

$$J_b^+ = J_d^+ = J_f^+$$

Since $J_a^+ = J_b^+$ this also implies $J_e^+ = J_f^+$. In fact, since the flow properties ahead of the expansion are constant, all C^+ lines will have the same J^+ value.

J^- invariants constant along C^- characteristics

$$J_c^- = J_d^-$$

$$J_e^- = J_f^-$$

$$\left.\begin{array}{l} u_e = \frac{1}{2}(J_e^+ + J_e^-) \\ u_f = \frac{1}{2}(J_f^+ + J_f^-) \\ J_e^- = J_f^- \\ J_e^+ = J_f^+ \end{array}\right\} \Rightarrow u_e = u_f \Rightarrow a_e = a_f$$

Due to the fact the J^+ is constant in the entire expansion region, u and a will be constant along each C^- line.

The constant J^+ value can be used to obtain relations for the variation of flow properties through the expansion region. Evaluation of the J^+ invariant at any position within the expansion region should give the same value as in region 4.

$$u + \frac{2a}{\gamma - 1} = u_4 + \frac{2a_4}{\gamma - 1} = 0 + \frac{2a_4}{\gamma - 1}$$

and thus

$$\frac{a}{a_4} = 1 - \frac{\gamma - 1}{2}\left(\frac{u}{a_4}\right) \tag{27}$$

Eqn. 27 and $a = \sqrt{\gamma RT}$ gives

$$\frac{T}{T_4} = \left[1 - \frac{\gamma - 1}{2}\left(\frac{u}{a_4}\right)\right]^2 \tag{28}$$

Using isentropic relations, we can get pressure ratio and density ratio

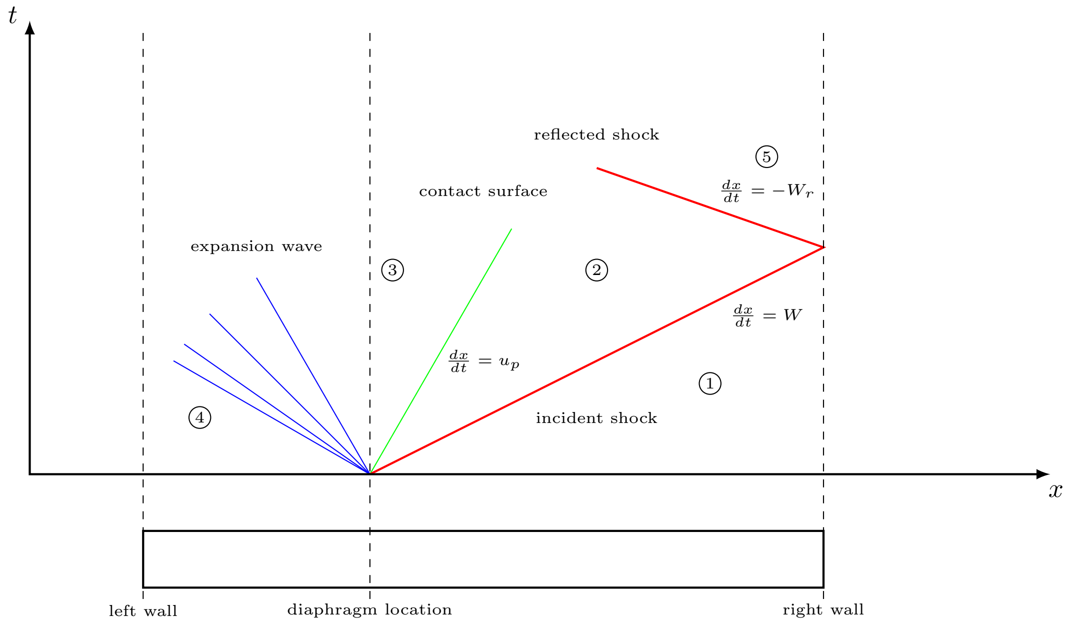
$$\frac{p}{p_4} = \left[1 - \frac{\gamma - 1}{2}\left(\frac{u}{a_4}\right)\right]^{2\gamma/(\gamma-1)} \tag{29}$$

$$\frac{\rho}{\rho_4} = \left[1 - \frac{\gamma - 1}{2}\left(\frac{u}{a_4}\right)\right]^{2/(\gamma-1)} \tag{30}$$

Shock Tube

From the analysis of the incident shock, we have a relation for the induced flow behind the shock

$$u_2 = u_p = \frac{a_1}{\gamma_1}\left(\frac{p_2}{p_1} - 1\right)\left(\frac{\left(\frac{2\gamma_1}{\gamma_1 + 1}\right)}{\left(\frac{\gamma_1 - 1}{\gamma_1 + 1}\right) + \left(\frac{p_2}{p_1}\right)}\right)^{1/2} \tag{31}$$



Figur 3: traveling waves in a shock tube

The velocity in region 3 can be obtained from the expansion relations

$$\frac{p_3}{p_4} = \left[1 - \frac{\gamma_4 - 1}{2}\left(\frac{u_3}{a_4}\right)\right]^{2\gamma_4/(\gamma_4-1)} \tag{32}$$

Solving for u_3 gives

$$u_3 = \frac{2a_4}{\gamma_4 - 1}\left[1 - \left(\frac{p_3}{p_4}\right)^{(\gamma_4-1)/(2\gamma_4)}\right] \tag{33}$$

There is no change in pressure or velocity over the contact surface, which means $u_2 = u_3$ and $p_2 = p_3$.

$$u_2 = \frac{2a_4}{\gamma_4 - 1}\left[1 - \left(\frac{p_2}{p_4}\right)^{(\gamma_4-1)/(2\gamma_4)}\right] \tag{34}$$

Now, we have two ways of calculating u_2. Setting Eqn. 31 equal to Eqn. 34 leads to the shock tube relation

$$\frac{p_4}{p_1} = \frac{p_2}{p_1} \left\{ 1 - \frac{(\gamma_4 - 1)(a_1/a_4)(p_2/p_1 - 1)}{\sqrt{2\gamma_1 \left[2\gamma_1 + (\gamma_1 + 1)(p_2/p_1 - 1) \right]}} \right\}^{-2\gamma_4/(\gamma_4 - 1)} \tag{35}$$