Compressible Flow TME085

Quasi-One-Dimensional Flow

The Area-Mach-Number Relation

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Governing Equations

The governing equations for quasi-one-dimensional form are the continuity, momentum, and energy equations on differential form where a term accounting for the variation in cross-section area is added in the momentum equation.

$$d(\rho uA) = 0 \tag{1}$$

$$dp = -\rho u du \tag{2}$$

$$dh + udu = 0 \tag{3}$$

The Area-Velocity Relation

Starting point - the continuity equation (Eqn. 1):

$$d(\rho uA) = 0 \Rightarrow \rho uA = const$$

This applies everywhere in the nozzle and therefore the sonic conditions can be used as a reference

$$\rho uA = \rho^* u^* A^* = \{u^* = a^*\} = \rho^* a^* A^*$$

divide by $\rho u A^*$ gives

$$\frac{\rho^*}{\rho}\frac{a^*}{u} = \frac{A}{A^*}$$

$a^*/u = 1/M^*$ but ρ^*/ρ is unknown

$$\frac{\rho^*}{\rho} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}$$

and thus

$$\frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{1}{M^*} = \frac{A}{A^*} \tag{4}$$

Using the isentropic relations, we get

$$\frac{\rho^*}{\rho_o} = \frac{1}{\left[\frac{1}{2}(\gamma - 1)\right]^{1/(\gamma-1)}} \tag{5}$$

$$\frac{\rho_o}{\rho} = \left[1 + \frac{1}{2}(\gamma + 1)M^2\right]^{1/(\gamma-1)} \tag{6}$$

Eqns. 5 and 6 in Eqn. 4 gives

$$\frac{A}{A^*} = \frac{1}{M^*}\left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1}\right]^{1/(\gamma-1)} \tag{7}$$

What remains now is to replace M^*

$$M^{*2} = \frac{u^2}{a^{*2}} = \frac{u^2}{a^2}\frac{a^2}{a^{*2}} = \frac{u^2}{a^2}\frac{a^2}{a_o^2}\frac{a_o^2}{a^{*2}} = M^2\frac{a^2}{a_o^2}\frac{a_o^2}{a^{*2}} \tag{8}$$

For a calorically perfect gas $a = \sqrt{\gamma RT}$, which gives

$$\frac{a^2}{a_o^2} = \frac{T}{T_o} = \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-1} \tag{9}$$

$$\frac{a_o^2}{a^{*2}} = \frac{T_o}{T^*} = \frac{1}{2}(\gamma + 1) \tag{10}$$

Eqns. 9 and 10 in Eqn. 8 gives

$$M^{*2} = \frac{(\gamma + 1)M^2}{2 + (\gamma - 1)M^2} \tag{11}$$

Now, rewrite Eqn. 7 as

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^{*2}}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{2/(\gamma-1)} \tag{12}$$

and insert M^{*2} from Eqn. 11

$$\left(\frac{A}{A^*}\right)^2 = \frac{2+(\gamma-1)M^2}{(\gamma+1)M^2}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{2/(\gamma-1)} \Rightarrow$$

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{1+2/(\gamma-1)} \Rightarrow$$

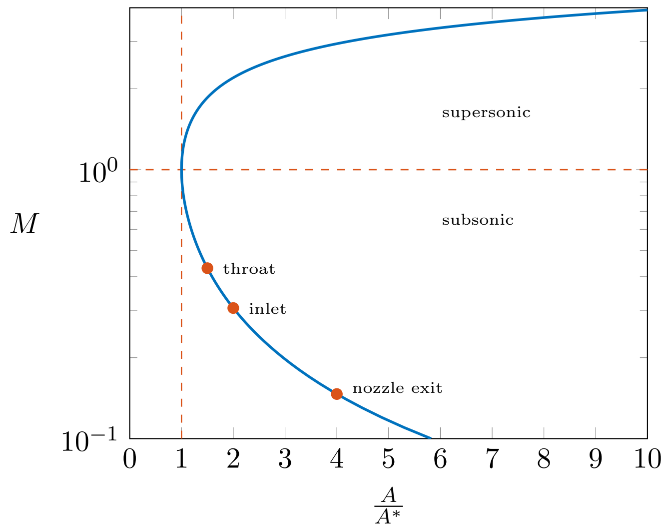
$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{(\gamma+1)/(\gamma-1)} \tag{13}$$

which is the area-Mach-number relation.

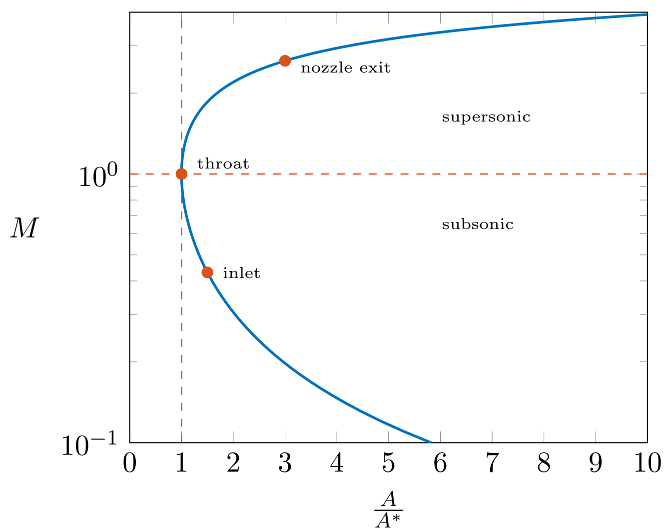
For a nozzle flow, the area-Mach-number relation gives the Mach number, M, at any location inside the nozzle as a function of the ratio between the local cross-section area, A, and the throat area at choked conditions, A^*.

$$M = f\left(\frac{A}{A^*}\right)$$

Due to the assumptions made in the derivation, the area-Mach-number relation is only valid for isentropic flows of calorically perfect gases. This means that it cannot be used throughout the divergent part of a convergent-divergent nozzle in case there is a shock within the nozzle. It can, however, be used both upstream and downstream of the shock. Note that A^* will change over the shock.



Figur 1: Area-Mach-number relation - subsonic nozzle flow



Figur 2: Area-Mach-number relation - supersonic nozzle flow