Compressible Flow TME085

One-Dimensional Flow with Friction

Details on the derivation of the momentum equation for Fanno flows

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From chapter 3.9 we have the following expression for the momentum equation for one-dimensional flow with friction (equation (3.95))

$$dp + \rho u du = -\frac{1}{2}\rho u^2 \frac{4f dx}{D} \quad (3.95)$$

For cases dealing with calorically perfect gas, (3.95) can be recast completely in terms of Mach number using the following relations

speed of sound: $a^2 = \gamma p/\rho$

the definition of Mach number: $M^2 = u^2/a^2$

the ideal gas law for thermally perfect gas: $p = \rho RT$

the continuity equation: $\rho u = const$

energy equation: $c_p T + u^2/2 = const$

1 Continuity equation

We start with the continuity equation which for one-dimensional steady flows reads

$$\rho u = const \tag{1}$$

Differentiating (1) gives

$$d(\rho u) = 0. \Leftrightarrow \rho du + u d\rho = 0. \tag{2}$$

If $u \neq 0.$ we can divide by ρu which gives us

$$\frac{du}{u} + \frac{d\rho}{\rho} = 0. \tag{3}$$

Now, if we divide and multiply the first term in (3) by $2u$ and use the chain rule for derivatives we get

$$\frac{d(u^2)}{2u^2} + \frac{d\rho}{\rho} = 0. \tag{4}$$

2 Energy equation

For an adiabatic one-dimensional flow we have that

$$c_pT + \frac{u^2}{2} = const \tag{5}$$

If we differentiate (5) we get

$$c_pdT + \frac{1}{2}d(u^2) = 0. \tag{6}$$

We replace c_p with $\gamma R/(\gamma - 1)$ and multiply and divide the first term with T which gives us

$$\frac{\gamma RT}{(\gamma - 1)}\frac{dT}{T} + \frac{1}{2}d(u^2) = 0. \tag{7}$$

Now, divide by $\gamma RT/(\gamma - 1)$ and multiply and divide the second term by u^2 gives

$$\frac{dT}{T} + \frac{(\gamma - 1)}{2}M^2\frac{d(u^2)}{u^2} = 0. \tag{8}$$

We want to remove the dT/T-term in (8). From the definition of Mach number we have that

$$a^2M^2 = u^2 \tag{9}$$

which we can rewrite using the expression for speed of sound ($a^2 = \gamma RT$) according to

$$\gamma RTM^2 = u^2 \tag{10}$$

Differentiating (10) gives us

$$\gamma RM^2dT + \gamma RTd(M^2) = d(u^2) \tag{11}$$

Now, if we divide (11) by γRTM^2 and use $a^2 = \gamma RT$ and $a^2M^2 = u^2$ we get

$$\frac{dT}{T} + \frac{d(M^2)}{M^2} = \frac{d(u^2)}{u^2} \tag{12}$$

Equation (12) may now be used to replace the dT/T-term in equation (8)

$$-\frac{d(M^2)}{M^2} + \frac{d(u^2)}{u^2} + \frac{(\gamma-1)}{2}M^2\frac{d(u^2)}{u^2} = 0. \tag{13}$$

which can be rewritten according to

$$\frac{d(u^2)}{u^2} = \left[1 + \frac{(\gamma-1)}{2}M^2\right]^{-1}\frac{d(M^2)}{M^2} \tag{14}$$

Using the chain rule for derivatives, the last term may be rewritten according to

$$\frac{d(M^2)}{M^2} = 2M\frac{dM}{M^2} = 2\frac{dM}{M}$$

which gives

$$\frac{d(u^2)}{u^2} = 2\left[1 + \frac{(\gamma-1)}{2}M^2\right]^{-1}\frac{dM}{M} \tag{15}$$

3 The ideal gas law

For a perfect gas the ideal gas law reads

$$p = \rho RT \tag{16}$$

Differentiating (16) gives:

$$dp = \rho RdT + RTd\rho \tag{17}$$

If $p \neq 0.$, we can divide (20) by p which gives

$$\frac{dp}{p} = \frac{dT}{T} + \frac{d\rho}{\rho} \tag{18}$$

which can be rearranged according to

$$\left[\frac{dp}{p} - \frac{d\rho}{\rho}\right] = \frac{dT}{T} \tag{19}$$

Now, inserting dT/T from equation (8) gives

$$\left[\frac{dp}{p} - \frac{d\rho}{\rho}\right] + \frac{(\gamma-1)}{2}M^2\frac{d(u^2)}{u^2} = 0. \tag{20}$$

The $d\rho/\rho$-term can be replaced using equation (4)

$$\frac{dp}{p} + \frac{d(u^2)}{2u^2} + \frac{(\gamma-1)}{2}M^2\frac{d(u^2)}{u^2} = 0. \tag{21}$$

Collect terms and rewrite gives

$$\frac{dp}{p} + \left[\frac{1+(\gamma-1)M^2}{2}\right]\frac{d(u^2)}{u^2} = 0. \tag{22}$$

4 Momentum equation

By combining the above derived relations and the momentum equation on the form given by (3.95), we can get an expression where the friction force is a function of Mach number only

For convenience equation (3.95) is written again here

$$dp + \rho u du = -\frac{1}{2}\rho u^2 \frac{4f dx}{D} \; (3.95)$$

if $u \neq 0.$, we can divide by $0.5\rho u^2$ which gives

$$2\frac{dp}{\rho u^2} + 2\frac{\rho u du}{\rho u^2} = -\frac{4f dx}{D} \tag{23}$$

using $M^2 = u^2/a^2$, $a^2 = \gamma p/\rho$ and the chain rule in (23) gives

$$\frac{2}{\gamma M^2}\frac{dp}{p} + \frac{d(u^2)}{u^2} = -\frac{4f dx}{D} \tag{24}$$

From equation (22) we can get a relation that expresses the pressure derivative term, dp/p, in terms of Mach number and $d(u^2)/u^2$. Inserting this in (24) gives

$$\frac{2}{\gamma M^2}\left\{-\left[\frac{1+(\gamma-1)M^2}{2}\right]\frac{d(u^2)}{u^2}\right\} + \frac{d(u^2)}{u^2} = -\frac{4f dx}{D} \tag{25}$$

collecting terms and rearranging gives

$$\frac{M^2-1}{\gamma M^2}\frac{d(u^2)}{u^2} = \frac{4f dx}{D} \tag{26}$$

if we now use equation (15) to get rid of the $d(u^2)/u^2$-term we end up with an expression corresponding to equation (3.96)

$$\frac{4f dx}{D} = \frac{2}{\gamma M^2}(1-M^2)\left[1+\frac{(\gamma-1)}{2}M^2\right]^{-1}\frac{dM}{M} \quad (3.96)$$