Compressible Flow TME085

Oblique Shocks and Expansion Waves

The Prandtl-Meyer Function

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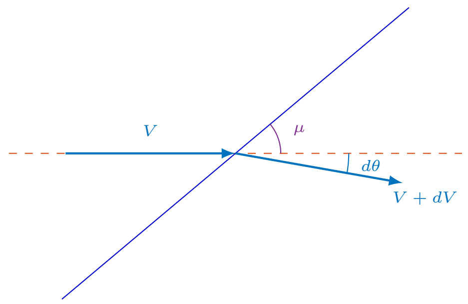


Figur 1: Mach wave flow turning

Prandtl-Meyer Function

A single Mach wave has a insignificant effect on the flow passing it but an expansion region constitutes an infinite number of Mach waves and the integrated effect is significant. The net turning of the flow by a single Mach wave is depicted schematically in Fig. 1. It can be shown geometrically that

$$d\theta = \sqrt{M^2 - 1}\frac{dV}{V} \tag{1}$$

Since Eqn. 1 is derived from the flow turning geometry with the assumption that the net flow tuning is small, it is valid for all gas models.

To get the integrated effect of all Mach waves in the expansion region, we integrate Eqn. 1 over the expansion region

$$\int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \sqrt{M^2 - 1}\frac{dV}{V} \tag{2}$$

To be able to do the integration, we need to rewrite it

$$V = Ma \Rightarrow \ln V = \ln M + \ln a$$

Differentiate to get

$$\frac{dV}{V} = \frac{dM}{M} + \frac{da}{a} \tag{3}$$

Each Mach wave is isentropic and thus the expansion is an isentropic process, which means that we can use the adiabatic energy equation

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2 \tag{4}$$

For a calorically perfect gas $a = \sqrt{\gamma R T}$ and $a_o = \sqrt{\gamma R T_o}$ and thus

$$\frac{T_o}{T} = \left(\frac{a_o}{a}\right)^2 \Rightarrow$$

$$\left(\frac{a_o}{a}\right)^2 = 1 + \frac{\gamma - 1}{2} M^2 \tag{5}$$

Solve for a gives

$$a = a_o \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1/2} \tag{6}$$

Differentiate Eqn 6 to get

$$da = a_o \left(-\frac{1}{2}\right) \left(\frac{\gamma - 1}{2}\right) 2M \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-3/2} dM \tag{7}$$

Eqn. 6 in Eqn. 7 gives

$$\frac{da}{a} = -\left(\frac{\gamma - 1}{2}\right) \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1} M dM \tag{8}$$

From Eqn. 9, we have

$$\frac{dV}{V} = \frac{dM}{M} + \frac{da}{a}$$

With da/a from Eqn. 8, we get

$$\frac{dV}{V} = \frac{dM}{M} - \left(\frac{\gamma-1}{2}\right)\left(1+\frac{\gamma-1}{2}M^2\right)^{-1} M dM$$

$$\frac{dV}{V} = dM\left[\frac{\left(1+\frac{\gamma-1}{2}M^2\right) - \left(\frac{\gamma-1}{2}\right)M^2}{\left(1+\frac{\gamma-1}{2}M^2\right)M}\right] = \left(1+\frac{\gamma-1}{2}M^2\right)^{-1}\frac{dM}{M}$$

Now, insert dV/V in Eqn. 2 to get

$$\int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \sqrt{M^2-1}\left(1+\frac{\gamma-1}{2}M^2\right)^{-1}\frac{dM}{M} \tag{9}$$

The integral on the right hand side of Eqn. 9 is the Prandtl-Meyer function, which is usually denoted ν. The Prandtl-Meyer function evaluated for Mach number M becomes

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1} \tag{10}$$

and thus the net turning of the flow can be calculated as

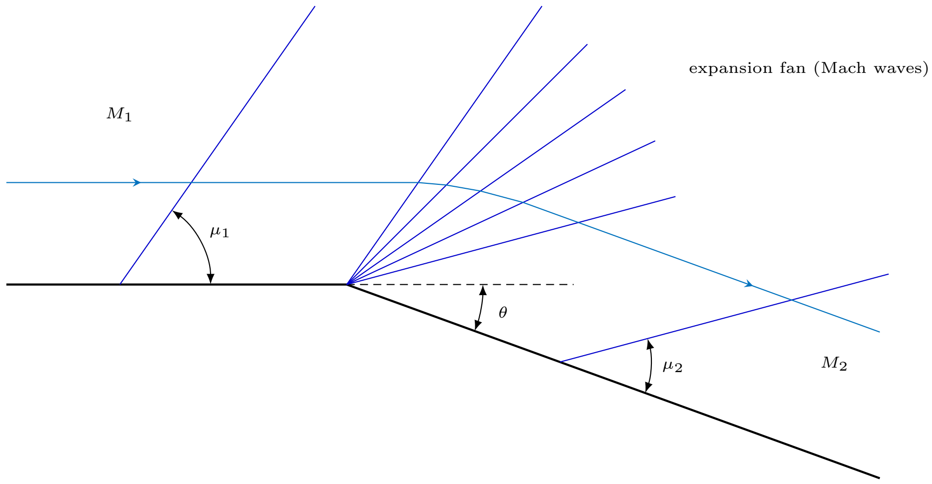
$$\theta_2 - \theta_1 = \nu(M_2) - \nu(M_1) \tag{11}$$

Solving Problems using the Prandtl Meyer Function

A typical problem is one where we know the net flow turning and the upstream flow conditions and want to calculate the flow conditions downstream of the expansion region. An example of such a problem is given in Fig. 2.

A problem of that type can be solved as follows:

1. Calculate νM_1 using Eqn. 10 or tabulated values
2. Calculate $\nu(M_2)$ as $\nu(M_2) = \theta_2 - \theta_1 + \nu(M_1)$
3. Calculate M_2 from the known νM_2 using Eqn. 10 or tabulated values



Figur 2: Expansion corner with known net flow turning