Compressible Flow TME085

Compressible Inviscid Fluid Flow Equations

Integral Form

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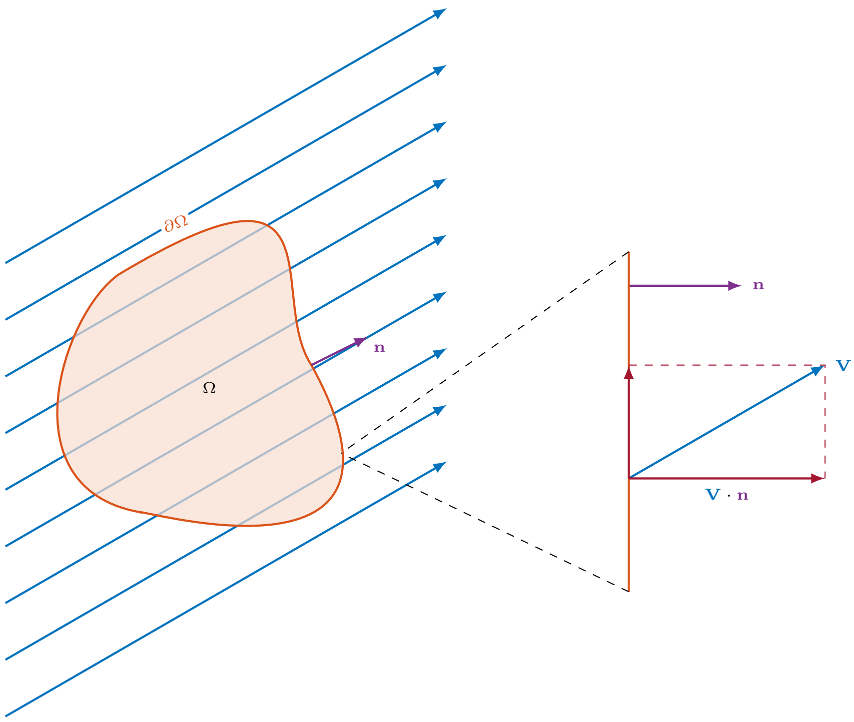

Figur 1: Generic control volume

Governing Equations

The governing equations stems from mass conservation, conservation of momentum and conservation of energy

The Continuity Equation

"Mass can be neither created nor destroyed, which implies that mass is conserved"

The net massflow into the control volume Ω in Fig. 1 is obtained by integrating mass flux over the control volume surface $\partial\Omega$

$$-\oiint_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS$$

Now, let's consider a small infinitesimal volume $d\mathscr{V}$ inside Ω. The mass of $d\mathscr{V}$ is $\rho d\mathscr{V}$. Thus, the mass enclosed within Ω can be calculated as

$$\iiint_{\Omega} \rho d\mathscr{V}$$

The rate of change of mass within Ω is obtained as

$$\frac{d}{dt} \iiint_{\Omega} \rho d\mathscr{V}$$

Mass is conserved, which means that the rate of change of mass within Ω must equal the net flux over the control volume surface.

$$\frac{d}{dt} \iiint_{\Omega} \rho d\mathscr{V} = -\oiint_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS$$

or

$$\frac{d}{dt} \iiint_{\Omega} \rho d\mathscr{V} + \oiint_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS = 0 \tag{1}$$

which is the integral form of the continuity equation.

The Momentum Equation

"The time rate of change of momentum of a body equals the net force exerted on it"

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}$$

What type of forces do we have?

- Body forces acting on the fluid inside Ω
  - gravitation
  - electromagnetic forces
  - Coriolis forces

- Surface forces: pressure forces and shear forces

Body forces inside Ω:

$$\iiint_\Omega \rho \mathbf{f} d\mathscr{V}$$

Surface force on $\partial\Omega$:

$$-\oiint_{\partial\Omega} p\mathbf{n} dS$$

Since we are considering inviscid flow, there are no shear forces and thus we have the net force as

$$\mathbf{F} = \iiint_\Omega \rho \mathbf{f} d\mathscr{V} - \oiint_{\partial\Omega} p\mathbf{n} dS$$

The fluid flowing through Ω will carry momentum and the net flow of momentum out from Ω is calculated as

$$\oiint_{\partial\Omega} (\rho\mathbf{v} \cdot \mathbf{n} dS)\mathbf{v} = \oiint_{\partial\Omega} (\rho\mathbf{v} \cdot \mathbf{n})\mathbf{v} dS$$

Integrated momentum inside Ω

$$\iiint_\Omega \rho \mathbf{v} d\mathscr{V}$$

Rate of change of momentum due to unsteady effects inside Ω

$$\frac{d}{dt} \iiint_\Omega \rho \mathbf{v} d\mathscr{V}$$

Combining the rate of change of momentum, the net momentum flux and the net forces we get

$$\frac{d}{dt} \iiint_\Omega \rho \mathbf{v} d\mathscr{V} + \oiint_{\partial\Omega} (\rho\mathbf{v} \cdot \mathbf{n})\mathbf{v} dS = \iiint_\Omega \rho \mathbf{f} d\mathscr{V} - \oiint_{\partial\Omega} p\mathbf{n} dS$$

combining the surface integrals, we get

$$\frac{d}{dt}\iiint_{\Omega}\rho\mathbf{v}d\mathscr{V}+\oiint_{\partial\Omega}\left[(\rho\mathbf{v}\cdot\mathbf{n})\mathbf{v}+p\mathbf{n}\right]dS=\iiint_{\Omega}\rho\mathbf{f}d\mathscr{V} \tag{2}$$

which is the momentum equation on integral form.

The Energy Equation

"Energy can be neither created nor destroyed; it can only change in form"

$$E_1 + E_2 = E_3$$

E_1: Rate of heat added to the fluid in Ω from the surroundings

- heat transfer
- radiation

E_2: Rate of work done on the fluid in Ω

E_3: Rate of change of energy of the fluid as it flows through Ω

$$E_1 = \iiint_\Omega \dot{q}\rho d\mathscr{V}$$

where $\dot{q}$ is the rate of heat added per unit mass

The rate of work done on the fluid in Ω due to pressure forces is obtained from the pressure force term in the momentum equation.

$$E_{2_{pressure}} = -\oiint_{\partial\Omega} (p\mathbf{n}dS)\cdot\mathbf{v} = -\oiint_{\partial\Omega} p\mathbf{v}\cdot\mathbf{n}dS$$

The rate of work done on the fluid in Ω due to body forces is

$$E_{2_{body\ forces}} = \iiint_\Omega (\rho\mathbf{f}d\mathscr{V})\cdot\mathbf{v} = \iiint_\Omega \rho\mathbf{f}\cdot\mathbf{v}d\mathscr{V}$$

$$E_2 = E_{2_{pressure}} + E_{2_{body\ forces}} = -\oiint_{\partial\Omega} p\mathbf{v}\cdot\mathbf{n}dS + \iiint_\Omega \rho\mathbf{f}\cdot\mathbf{v}d\mathscr{V}$$

The energy of the fluid per unit mass is the sum of internal energy e (molecular energy) and the kinetic energy $V^2/2$ and the net energy flux over the control volume surface is calculated by the following integral

$$\oiint_{\partial\Omega} (\rho \mathbf{v} \cdot \mathbf{n} dS) \left(e + \frac{V^2}{2} \right)$$

Analogous to mass and momentum, the total amount of energy of the fluid in Ω is calculated as

$$\iiint_{\Omega} \rho \left(e + \frac{V^2}{2} \right) d\mathscr{V}$$

The time rate of change of the energy of the fluid in Ω is obtained as

$$\frac{d}{dt} \iiint_{\Omega} \rho \left(e + \frac{V^2}{2} \right) d\mathscr{V}$$

Now, E_3 is obtained as the sum of the time rate of change of energy of the fluid in Ω and the net flux of energy carried by fluid passing the control volume surface.

$$E_3 = \frac{d}{dt} \iiint_{\Omega} \rho \left(e + \frac{V^2}{2} \right) d\mathscr{V} + \oiint_{\partial\Omega} (\rho \mathbf{v} \cdot \mathbf{n} dS) \left(e + \frac{V^2}{2} \right)$$

With all elements of the energy equation defined, we are now ready to finally compile the full equation

$$\frac{d}{dt} \iiint_{\Omega} \rho \left(e + \frac{V^2}{2} \right) d\mathscr{V} + \oiint_{\partial\Omega} \left[\rho \left(e + \frac{V^2}{2} \right) (\mathbf{v} \cdot \mathbf{n}) + p \mathbf{v} \cdot \mathbf{n} \right] dS = \iiint_{\Omega} \rho \mathbf{f} \cdot \mathbf{v} d\mathscr{V} + \iiint_{\Omega} \dot{q} \rho d\mathscr{V} \tag{3}$$

The surface integral in the energy equation may be rewritten as

$$\oiint_{\partial\Omega} \left[\rho \left(e + \frac{V^2}{2} \right) (\mathbf{v} \cdot \mathbf{n}) + p \mathbf{v} \cdot \mathbf{n} \right] dS = \oiint_{\partial\Omega} \rho \left[e + \frac{p}{\rho} + \frac{V^2}{2} \right] (\mathbf{v} \cdot \mathbf{n}) dS$$

and with the definition of enthalpy $h = e + p/\rho$, we get

$$\oiint_{\partial\Omega} \rho \left[h + \frac{V^2}{2} \right] (\mathbf{v} \cdot \mathbf{n}) dS$$

Furthermore, introducing total internal energy e_o and total enthalpy h_o defined as

$$e_o = e + \frac{1}{2}V^2$$

and

$$h_o = h + \frac{1}{2}V^2$$

the energy equation is written as

$$\frac{d}{dt}\iiint_{\Omega} \rho e_o d\mathscr{V} + \oiint_{\partial\Omega} \rho h_o(\mathbf{v} \cdot \mathbf{n})dS = \iiint_{\Omega} \rho \mathbf{f} \cdot \mathbf{v} d\mathscr{V} + \iiint_{\Omega} \dot{q}\rho d\mathscr{V} \tag{4}$$

Summary

The integral form of the governing equations for inviscid compressible flow has been derived

Continuity:

$$\frac{d}{dt}\iiint_{\Omega} \rho d\mathscr{V} + \oiint_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS = 0$$

Momentum:

$$\frac{d}{dt}\iiint_{\Omega} \rho \mathbf{v} d\mathscr{V} + \oiint_{\partial\Omega} \left[(\rho \mathbf{v} \cdot \mathbf{n})\mathbf{v} + p\mathbf{n}\right] dS = \iiint_{\Omega} \rho \mathbf{f} d\mathscr{V}$$

Energy:

$$\frac{d}{dt}\iiint_{\Omega} \rho e_o d\mathscr{V} + \oiint_{\partial\Omega} \rho h_o(\mathbf{v} \cdot \mathbf{n})dS = \iiint_{\Omega} \rho \mathbf{f} \cdot \mathbf{v} d\mathscr{V} + \iiint_{\Omega} \dot{q}\rho d\mathscr{V}$$