Compressible Flow TME085

Basic Concepts

Isentropic Relations

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Entropy

First law of thermodynamics:

$$de = \delta q - \delta w \tag{1}$$

For a reversible process: $\delta w = pd(1/\rho)$ and $\delta q = Tds$

$$de = Tds - pd\left(\frac{1}{\rho}\right) \tag{2}$$

Enthalpy is defined as: $h = e + p/\rho$ and thus

$$dh = de + pd\left(\frac{1}{\rho}\right) + \left(\frac{1}{\rho}\right)dp \tag{3}$$

Eliminate de in Eqn. 2 using Eqn. 3

$$Tds = dh - \cancel{pd\left(\frac{1}{\rho}\right)} - \left(\frac{1}{\rho}\right)dp + \cancel{pd\left(\frac{1}{\rho}\right)}$$

$$ds = \frac{dh}{T} - \frac{dp}{\rho T}$$

Using $dh = C_pT$ and the equation of state $p = \rho RT$, we get

$$ds = C_p\frac{dT}{T} - R\frac{dp}{p} \tag{4}$$

Integrating Eqn. 4 gives

$$s_2 - s_1 = \int_1^2 C_p\frac{dT}{T} - R\ln\left(\frac{p_2}{p_1}\right) \tag{5}$$

For a calorically perfect gas, C_p is constant (not a function of temperature) and can be moved out from the integral and thus

$$s_2 - s_1 = C_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) \tag{6}$$

An alternative form of Eqn. 6 is obtained by using $de = C_v dT$ Eqn. 2, which gives

$$s_2 - s_1 = \int_1^2 C_v \frac{dT}{T} - R\ln\left(\frac{\rho_2}{\rho_1}\right) \tag{7}$$

Again, for a calorically perfect gas, we get

$$s_2 - s_1 = C_v \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{\rho_2}{\rho_1}\right) \tag{8}$$

Isentropic Relations

Adiabatic and reversible processes, i.e., isentropic processes implies $ds = 0$ and thus Eqn. 6 reduces to

$$\frac{C_p}{R}\ln\left(\frac{T_2}{T_1}\right) = \ln\left(\frac{p_2}{p_1}\right)$$

$$\frac{C_p}{R} = \frac{\gamma}{\gamma - 1}$$

$$\frac{\gamma}{\gamma - 1}\ln\left(\frac{T_2}{T_1}\right) = \ln\left(\frac{p_2}{p_1}\right) \Rightarrow$$

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)} \tag{9}$$

In the same way, Eqn. 8 gives

$$\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{1/(\gamma-1)} \tag{10}$$

Eqn. 11 and Eqn. 10 constitutes the isentropic relations

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)} \tag{11}$$